2DME20 - Non-linear optimization - Planar actuators project

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November 8, 2023

1. Questions

1. For a chosen position $p_{x_0} = 0$ mm we need to find a current vector i that satisfies Eq.1.

$$w_{\text{desired}} = \Gamma(p_x, p_z)i = \Gamma(0,1)i = [20, 20, 0]^T \quad (1)$$

   Where $\Gamma \in \mathbb{R}^{3\times K}$, $i \in \mathbb{R}^{K\times 1}$ and $w_{\text{desired}} \in \mathbb{R}^{3\times 1}$.

   Putting matrix $\Gamma(0,1)$ in reduced row echelon form we find the following matrix:

$$\Gamma(0,1) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 & 1 & -3 & 2 & 2 & -5 & 3 & 3 & -7 & 4 & 4 & -9 & 5 & 5 & -11 \\ 0 & 1 & 0 & -2 & 2 & 1 & -4 & 3 & 2 & -6 & 4 & 3 & -8 & 5 & 4 & -10 & 6 & 5 & -12 \\ 0 & 0 & 1 & -2 & 1 & 2 & -4 & 2 & 3 & -6 & 3 & 4 & -8 & 4 & 5 & -10 & 5 & 6 & -12 \end{bmatrix}$$

   We see that $\Gamma(0,1)$ has a column rank of $r = 3$, and has $K - r = 19 - 3 = 16$ linearly dependent columns. So we have 16 free variables, which leads us to the conclusion that $\Gamma(0,1)$ is underdetermined and thus has an infinite number of solutions i.

2. To prove that there is always a solution i for any p_x it is sufficient to prove that the RREF of Γ has 3 pivots for any p_x. Since then we can always construct the other 16 columns by taking a linear combination of the pivot columns. We will construct a submatrix $\Gamma_s \in \mathbb{R}^{3\times 3}$ considering of the first 3 columns of Γ with p_x, τ and c_k with $k \in \{1,2,3\}$ as symbols. Using the symbolic math toolbox in MATLAB we calculate the RREF of Γ_s, which gives the identity matrix $\text{RREF}(\Gamma_s) = \text{I}$. Hence there are always 3 pivots for any p_x and we conclude the existence of i does not depend on p_x.

3. We previously established that the system $\Gamma(0,1)i|w_{\text{desired}}$ is underdetermined. The total general solution for such a system has the form:

$$i_1 = i_0 + Nv \quad (2)$$

   Where:

   - $N \in \mathbb{R}^{K\times K-3}$ is the null space s.t. $\Gamma(0,1)i = 0$.
   - $i_0 \in \mathbb{R}^K$ is any solution to Eq. (1).
   - $v \in \mathbb{R}^{K-3}$ is any real-valued vector.

   The null space N can be computed using the `null(A )` function in MATLAB. We then sample any vector v and verify $\Gamma \cdot (i_0 + Nv) = w_{\text{desired}}$.

4. We define the optimization problem:

$$\begin{aligned} \min_i \quad & \frac{1}{2} i^T R i \\ \text{s.t.} \quad & \Gamma i = w_{\text{desired}} \end{aligned} \quad (3)$$

   This is a QP with linear equality constraints. The problem is convex, because R is a positive diagonal matrix and the equality constraint is affine. Now we set up the Lagrange function and find the optimal commutation vector i using Lagrange multipliers:

$$L(i,\mu) = \frac{1}{2} i^T R i + \mu^T(\Gamma i - w_{\text{desired}}) \quad (4)$$

   KKT conditions:

$$\nabla L(i,\mu) = Ri + \mu^T \Gamma = 0 \quad (5)$$

$$\Gamma i - w_{\text{desired}} = 0 \quad (6)$$

   Where $\mu \in R^{3\times 1}$. This results in the following system of equations:

$$\begin{bmatrix} R & \Gamma^T \\ \Gamma & \vec{0} \end{bmatrix} \begin{bmatrix} i \\ \mu \end{bmatrix} = \begin{bmatrix} \vec{0} \\ w_{\text{desired}} \end{bmatrix} \quad (7)$$

   This is an equation of the form $Ax = b$ which we can directly solve for i by using the `A\b` command in MATLAB. The solution i is then the first K elements of x and the last 3 elements are the Lagrange multipliers.

5. Solving Eq. 7 we obtain:

$$i_{\text{com}} = 10^{-3} \times \begin{bmatrix} 64 & 34 & -141 & 105 & \cdots & 92 & 42 & -133 & 53 \end{bmatrix}^T$$

   Which for the selected position $p_x = 0$ gives a minimum power of $P_{i_{com}} = i_{\text{com}}^T R i_{\text{com}} = 1.1789W$.

6. Figure 1 shows the current distributions i_{com} for different values of p_x. Figure 2 shows the value of $P_{i_{com}}$ obtained for different values of p_x. We notice that the power plot is more or less symmetric, which is expected. We also notice that the closer we get to the edge coils (which have higher resistance) the further the power consumption increases. Hence $P_{i_{com}}$ depends on p_x. This can also be seen in the distribution of the currents i_{com}, which is very narrow around $p_x = 0$ but disperses the closer we get to the edge.

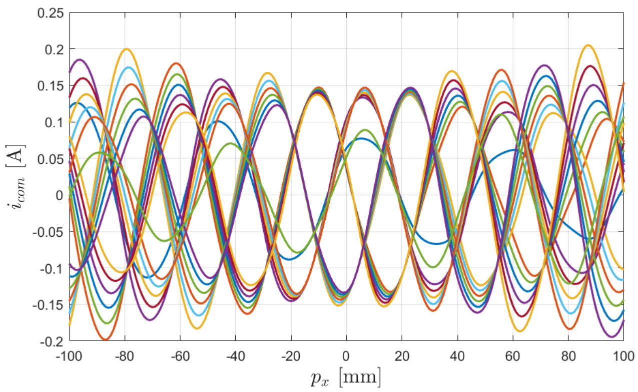

Figure 1. Current distributions i_{com} for different values of p_x. Each colour represents the current through one of K coils for p_x.

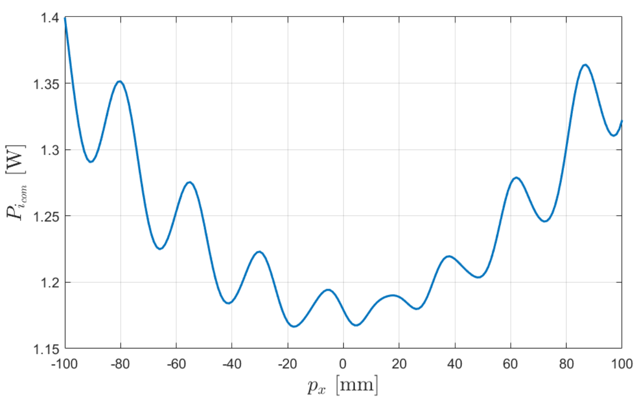

Figure 2. $P_{i_{com}}$ for different values of p_x.

7. We define the minimax optimization problem:

$$\begin{aligned} \min \quad & \max_{k=1,\dots,K} |i_k| \\ \text{s.t.} \quad & \Gamma i = w_{\text{desired}} \end{aligned} \tag{8}$$

This can not be solved straightforwardly, the absolute value is not differentiable in the minimum. So we will

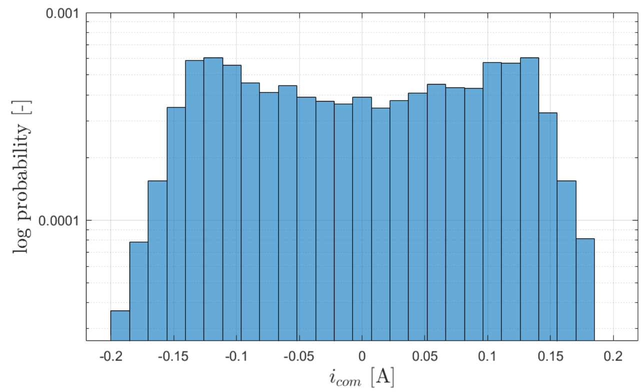

Figure 3. Log probability plot of i_k for $p_x \in [-100, 100]$ when using the objective of minimizing the absorbed power.

use two tricks: the absolute value trick and the minimax trick, to transform this non-linear problem into an equivalent linear problem which we can easily solve:

$$\begin{aligned} \min \quad & M \\ \text{s.t.} \quad & \Gamma i = w_{\text{desired}} \\ & M \geq i_k && \forall i \in 1, 2, \dots K \\ & M \geq -i_k && \forall i \in 1, 2, \dots K \end{aligned} \tag{9}$$

Which is a simple LP with equality and inequality constraints.

From Figure 1 we see that the edges have a larger variance in current amplitude and are thus more interesting to look at for this particular optimization target. So we solve this problem for $p_{x_0} = 80$ mm. We obtain:

$$i_{\text{com}} =$$

$$10^{-3} \times \begin{bmatrix} -3 & -115 & 115 & -115 & \cdots & 115 & -115 & 115 \end{bmatrix}^T$$

With $M = \max_{k=1,\dots,K} |i_k| = 0.115A$, which is the minimal achievable amplitude. This is an improvement from the situation in Figure 1, where a current amplitude of $0.2A$ was reached at $p_{x_0} = 80$ mm.

8. Figure 4 shows the current distributions for the minimax objective. There is some variation, but the minimal achievable amplitude does not seem to have any dependence on p_x. The current envelope has a more or less constant amplitude with respect to p_x.

9. We define the optimization problem:

$$\begin{aligned} \min \quad & \sum_{k=1}^{K} |i_k| \\ \text{s.t.} \quad & \Gamma i = w_{\text{desired}} \end{aligned} \tag{10}$$

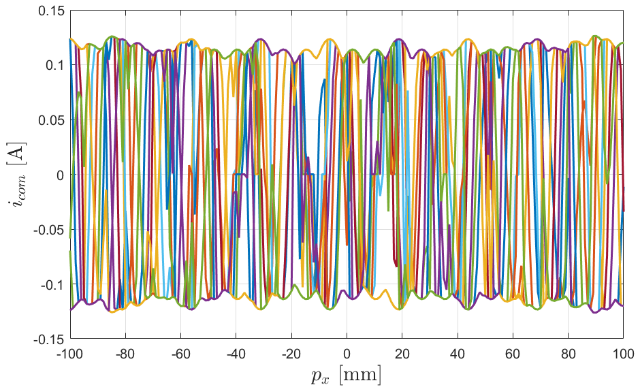

Figure 4. Minimax current distributions i_k for different p_x.

We will use the absolute values trick so that:

$$i_k = i_k^+ - i_k^- \tag{11}$$

$$|i_k| = i_k^+ + i_k^- \tag{12}$$

$$i_k^+, i_k^- \geq 0 \tag{13}$$

Which gives the transformed LP:

$$\begin{aligned} \min_i \quad & \sum_{k=1}^{K} i_k^+ + \sum_{k=1}^{K} i_k^- = 1^T i_k^+ + 1^T i_k^- \\ \text{s.t.} \quad & \Gamma(i^+ - i^-) = w_{\text{desired}} \\ & -i_k^+ \leq 0 \qquad \forall i \in 1, 2, ...K \\ & -i_k^- \leq 0 \qquad \forall i \in 1, 2, ...K \end{aligned} \tag{14}$$

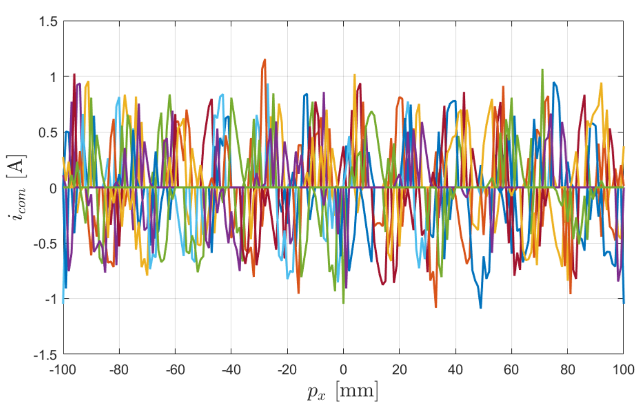

Figure 5. Sparse current distributions i_k for different p_x.

From Figure 5 we note that the current distributions are flat ($i_k = 0$) for a significant portion of the graph. This indicates a set of sparse current distributions, which is confirmed by inspecting the log probability of Figure 6 and comparing it to Figure 3. When we inspect the individual results for a given p_x we find that all but 3 values are zero, which is consistent with our results from question 3 which stated we have $K - 3 = 16$ free variables. Hence we require at minimum 3 actuators, but which 3 depends on the position p_x which is clear from Figure 5 and Figure 11.

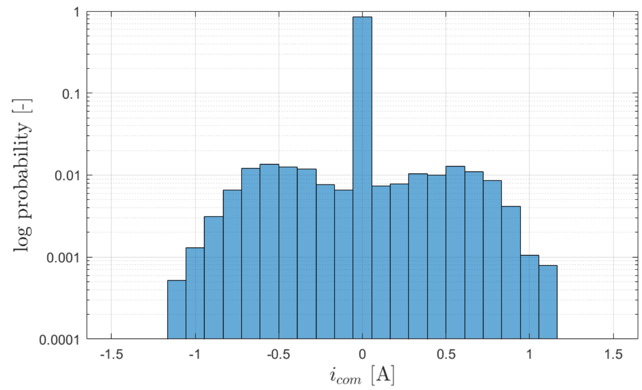

Figure 6. Log probability plot of i_k for $p_x \in [-100, 100]$ when using the objective of minimizing the number of active coils.

From the theory of regularization and methods like lasso regression we know that the L_1-norm promotes sparsity, due to the gradient of the L_1-norm being either 1, -1 or 0 when $i_k \to 0$. Sparsity is observed here since many i_k's are exactly zero. Finally we also notice that the maximum magnitude of i_k significantly increased. This is due to the fact we only use the absolute minimum number of actuators at any given p_x, but the required amount of work remains the same. Hence the current for each active actuator must increase to still achieve w_{desired}.

10. We use the following differential equations to calculate the time-dependent wrench vector $w(t) = \Gamma(p_x(t), p_z(t))i(t)$.

$$m\frac{\mathrm{d}^2 p_x}{\mathrm{d}t^2} = F_x(t) + n_x(t) \tag{15}$$

$$m\frac{\mathrm{d}^2 p_z}{\mathrm{d}t^2} = F_z(t) - mg + n_z(t) \tag{16}$$

We then discretize these second-order ODE's using central differences in p_x and p_z:

$$F_x[k] = \frac{m}{\Delta t^2}\left(p_x[k+1] - 2p_x[k] + p_x[k-1]\right) + n_x[k] \tag{17}$$

$$F_z[k] = \frac{m}{\Delta t^2}\left(p_z[k+1] - 2p_z[k] + p_z[k-1]\right) + mg + n_z[k] \tag{18}$$

Where $\Delta t = \frac{1}{N} = \frac{1}{1000}$ is the sampling interval, with $N = 1000$ being the number of samples.

Table 1. Max power and current for each optimization objective.

Objective	**max P[W]**	**max $\mathbf{i_k}$[A]**
min $i^T Ri$	5850	9.3587
min $\|\|i\|\|_\infty$	9281	8.6264
min $\|\|i\|\|_1$	31183	52.1894

Note that the discretized second derivative not only requires the current and the previous position of p_x and p_z, but also the next. We will assume the next position is always available. This doesn't seem unreasonable given that our trajectory is planned well ahead of time, bar any real time corrections. The discrete wrench vector then becomes:

$$w[k] = \Gamma(p_x[k], p_z[k])i[k] = \begin{bmatrix} F_x[k] \\ F_z[k] \\ 0 \end{bmatrix} \tag{19}$$

We define a reference track: $r(t) := (r_x(t), 0, r_z(t))$ for $0 \leq t \leq 1$. We will set a reference $r_z(t) = 2$, which simply keeps the height of the translator stationary. For $r_x(t)$ we define the following function adapted from the trigonometric form of a square wave [2]:

$$r_x(t) := -\frac{A}{\arctan\left(\frac{1}{\delta}\right)} \arctan\left(\frac{\cos(2\pi t f)}{\delta}\right) + A \tag{20}$$

Where:

- A is the amplitude of the square wave.
- δ is a smoothness factor defining the ramp rate.
- f is the frequency in Hz.

This function is suitable for our application because it is smooth and second-order differentiable everywhere. It also easy to control the frequency and ramp rate to simulate different conditions. For our simulation we will use $A = 0.5$,$f = 1$ and $\delta = 0.1$ which gives the curve in Figure 7.

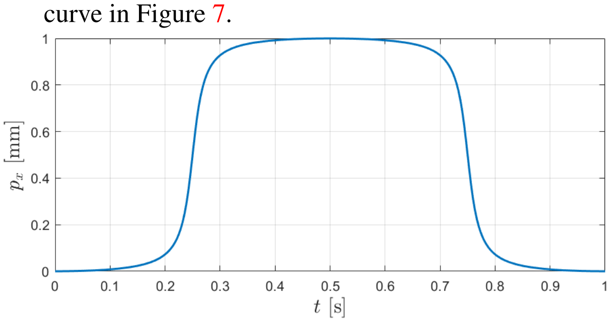

Figure 7. p_x-time plot of the reference track $r_x(t)$.

Optimization target 1: Minimize absorbed power

For our first optimization target we obtain the optimal current distribution i_{com} at every time step by minimizing the absorbed power $P_{i_{com}}$, see Eq. (3). Figure 8 shows the time domain plot of this objective. From Eq. (16) we know that the force $F_x(t)$ is proportional to the second derivative of p_x. So we expect our power consumption to follow this proportionality. And this is also what we see in our time domain plot: when the second derivative is largest our power consumption significantly increases, and this is strongly related to the steepness of the ramp. Also note that in the linear portion of the reference track, when the second derivative is small, the absorbed power drops almost to zero.

Comparing both curves of Figure 8 we conclude we have attained the optimization target of interest. The value of $P_{i_{com}}$ is at its lowest for each position p_x, which is consistent with the findings presented in Table 1.

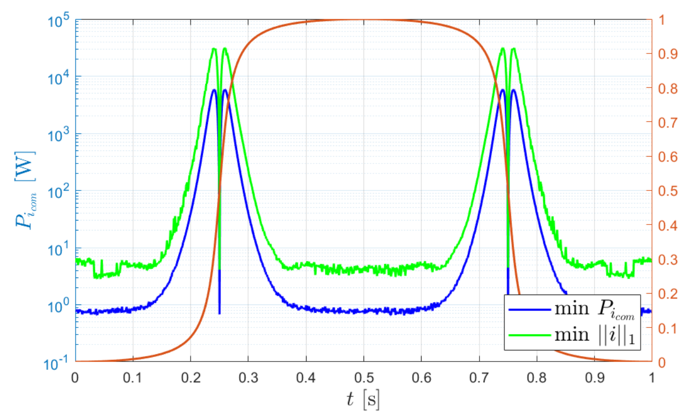

Figure 8. Comparing $P_{i_{com}}$ for both objectives relative to $r_x(t)$.

Optimization target 2: minimize active coils

For the second optimization target we minimize the number of active coils, see Eq. (10). This is an interesting objective to compare because it contrasts with our power minimization target, they are in fact competing objectives: one makes optimal use of free variables, and the other removes them from the problem entirely. The green curve in Figure 8 shows the power consumption while tracking the reference signal. We notice that the absorbed power is much higher for this optimization objective.

By minimizing the number of active coils we no longer have any free variables, at every time step we use exactly 3 coils. Which coils we use depends on the position p_x. By minimizing the number of active coils we also significantly increase the peak current in the system, shown in Figure 9.

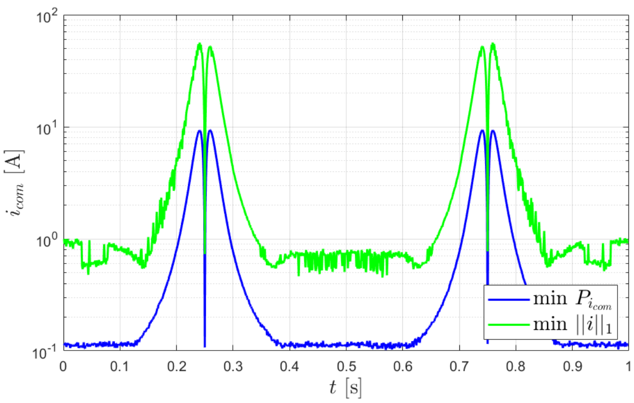

Figure 9. Comparing max $|i_k|$ for the two objectives.

This is an important parameter to take into account when dimensioning the power supply. Lastly, for our second optimization objective we are also interested in the total number of coil activations across the entire reference trajectory. Figure 10 shows that the number of coil activations ($i_k \neq 0$) for the second objective is much lower, while for the power minimization objective it is always maximal. This result is expected, since the L2 norm does not promote sparsity while the L1 norm does. We even notice some coils are never activated at all, this may help us recognize redundant coils which we may want to remove.

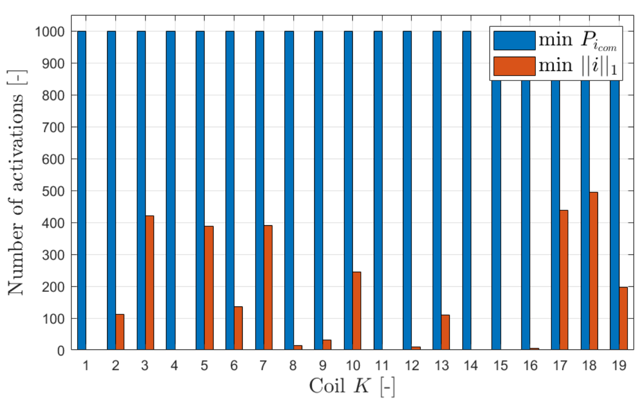

Figure 10. Total number of activations for each coil $1 \ldots K$.

Scaling to 2D movements

Scaling this system up to 2D movements is relatively simple. We can add another force component F_y and the torque T_x to the wrench vector to calculate the force in the y direction. The formulas are very similar, the coupling matrix now becoming $\Gamma(p_x(t), p_y(t), p_z(t))$. The wrench vector now becomes:

$$\vec{w} = \begin{bmatrix} F_x & F_y & F_z & T_x & T_y & T_z \end{bmatrix}^\top \tag{21}$$

Where we should now take into account the 2D position of each coil. This is discussed in detail in a (TU/e) paper by van Lierop et al. [1] which provides a lot of insight in how to design such a system. We should also analyse whether in this 2D case the number of free variables (coils) is still sufficient to obtain enough optimization freedom. Figure 11 shows that for a reference track which moves the translator from $p_x = -100$mm to $p_x = 100$mm the set of active coils $|i_k| > 0.1[A]$ is highly dependent on the position p_x. This makes sense, because our optimization objective expresses the goal of finding the combination of coil current magnitudes which produces the requires force most efficiently. Intuitively, if we imagine the act of balancing a plate, it becomes clear that the coils located nearest to the plate's edges and directly beneath its center will have to exert the greatest force. The same will hold in 2D, and should guide our optimization strategy.

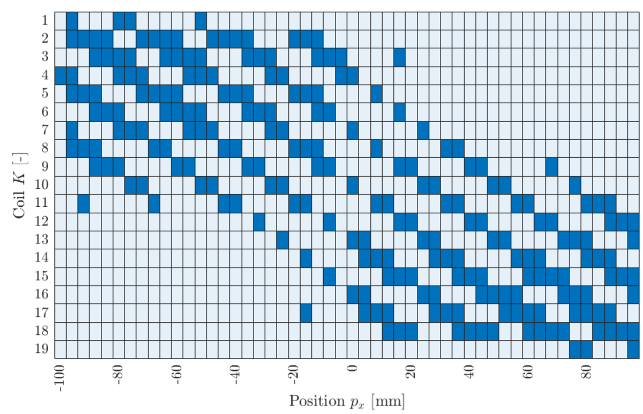

Figure 11. Coil activation as a function of p_x for $r_x = 200t - 100$ using the minimization of absorbed power strategy.

Computational complexity

As to the computational complexity, we have seen that all optimization targets can be expressed by either solving a linear system directly or by solving an LP. Many efficient, scalable and proven algorithms exist for these problems. So for the optimization problems considered in this work it should be feasible to optimize the solver to meet real-time requirements on a given hardware specification. We can reduce the computational complexity somewhat by making use of the null space defined in Eq. (2).

Instead of a constrained problem in K dimensions we can solve an unconstrained problem in $K - 3$ dimensions, which is easier to solve. For example, the power minimization objective from Eq. (3) can be rewritten as:

$$Q(i) = \min_{i} \frac{1}{2} i^T R i \tag{22}$$

$$Q(v) = \min_{v} \frac{1}{2} (i_0 + Nv)^T R (i_0 + Nv) \tag{23}$$

Setting $\nabla Q(v) = 0$ gives:

$$\nabla Q(v) = N^T R(i_0 + Nv) = N^T R i_0 + N^T R N v \tag{24}$$

$$v = -(N^T R N)^{-1} N^T R i_0 \tag{25}$$

With i_0 any vector s.t. $\Gamma i_0 = w_{\text{desired}}$. We then compute:

$$P_{i_{com}} = (i_0 + Nv)^T R (i_0 + Nv) \tag{26}$$

Which for $w_{\text{desired}} = [20, 20, 0]^T$ and $p_x = 0$ gives $P_{i_{com}} = 1.1789W$, which is the same minimum we found by solving Eq. (7) in question 5. This involves inverting a 16×16 matrix, so to not lose any computational efficiency we may also solve it iteratively.

Error between the reference and the actual position

In our simulation we have not taken into account any limitations of the coils to produce the required force so we can follow a reference track of any form. In the current configuration the error is fully dependent on the noise term. In a realistic situation we will be limited by our electronics to generate the required ramp current. So especially when dealing with a reference track with very steep slopes the current would rise exponentially. This must be taken into account when both designing the actuator electronics and the reference track. For optimal control we may want to consider some combination of previously designed criteria.

References

[1] JW Jansen, CMM Van Lierop, Elena A Lomonova, and André JA Vandenput. Magnetically levitated planar actuator with moving magnets. *IEEE Transactions on Industry Applications*, 44(4):1108–1115, 2008. 5

[2] Wikipedia. Trigonometric form of a square wave. https://en.wikipedia.org/wiki/Square_wave, 2023. [Online; accessed 02-Nov-2023]. 4