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A conservative finite elements approach to overland flow: the control volume finite element formulation

P. Di Giammarco*, E. Todini, P. Lamberti

Institute for Hydraulic Construction, University of Bologna, Viale Risorgimento 2, 40136 Bologna, Italy

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Abstract

This paper proposes the control volume finite element (CVFE) method, a locally conservative formulation of the better known finite elements (FE) approach, to deal more effectively with overland flow. The two-dimensional overland flow problem is introduced and several approaches available in the literature are briefly reported. The partial differential equations describing overland and channel processes are then presented and a consistent two-dimensional formulation for the head losses is introduced. The derivation of the CVFE discrete formulation is preceded by a discussion on the classical integrated finite difference (IFD) and FE approaches, together with their advantages and disadvantages. The CVFE formulation is shown to improve on both approaches, resulting in a better representation of the gradients than that of the IFD approach and, in contrast to the FE, allows for the conservation of mass at local scale. The preliminary results obtained with a CVFE computer code are presented and compared with analytical solutions. Finally, several computational aspects are discussed, such as the formulation of the time integration, the representation of the water volumes pertaining to each node of the space discretization mesh and the explicit imposition of the local mass balance.

1. Introduction

The study of two-dimensional flow over a land surface refers either to flood plain inundation or overland flow resulting from intense rainfall. The mechanism governing overland and channel processes is characterized by the presence of a free surface, the elevation of which may vary in time and space. In addition, there is interdependence between water level, conveyance, discharge, channel bed slope and free surface slope (Chow, 1964; Dooge, 1986). Both the overland flow and channel processes can in general be described by the combined use of the mass conservation equation and

* Corresponding author.

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the momentum equation expressed in simplified form, i.e. in which the contribution of the inertial and kinetic terms is ignored. The resulting system of differential equations is therefore of the parabolic type.

Unless the water overtops dikes and spreads over the flood plain, rendering the one-dimensional description inadequate, the one-dimensional assumption is reasonably accurate for describing channel flow. The literature contains a large number of one-dimensional numerical models suitable for describing channel routing. They differ according to the accuracy required and the degree of approximation of the complete equations, the solution methods and the particular discontinuities in the propagation which they are able to handle (sudden geometrical changes, presence of dams, bridges, etc.). Quasi-two-dimensional models have also found application in the study of flood events; in these, the flooded area is described by a series of reservoirs and channels whose position may correspond more or less to their exact spatial arrangement in the area in question (Cunge et al., 1980).

In parallel with this, a large number of numerical solutions for two-dimensional models have been developed which can broadly be distinguished according to the type of spatial discretization employed: the mesh can be structured (with nodes arranged on a straight-line grid or adapted to the boundaries by appropriate geometrical transformations) or non-structured (irregular and composed of triangles, quadrangles or polygons). Structured meshes are usually associated with finite difference equation (FD) discretizations; non-structured meshes are generally found in integrated finite differences (IFD) or in finite element (FE) discretizations. Hromadka et al. (1985), among others, developed a finite difference model based on an integrated finite difference version of the nodal domain integration model, in which the discretization in polygons (control volumes) is based on an irregular triangular discretization. Alternative solutions for the two-dimensional overland flow problem using the FE approach have been given by Galland et al. (1991) and Katopodes (1980, 1984).

Mixed one–two-dimensional (1–2-D) approaches (i.e. two-dimensional (2-D) flood plain and one-dimensional (1-D) channel flow models dynamically linked by matching their respective boundary conditions) are used to describe flood plain inundation problems, to deal with the whole domain by defining a unique mathematical problem to be solved at each time step. The mixed 1–2-D approach also affords approximation of bends, expansions and contractions of the river bed, flow breakouts, and the general main flood carrying area. Furthermore, when rivers are characterized by accentuated meandering, a flood may overtop the banks and change the preferential flow direction, originally described with the 1-D approach, thus becoming fully 2-D. As a consequence, a major advantage of a mixed 1–2-D description is that the construction of the model does not require a priori knowledge of the main direction of propagation, which may vary in time, provided that the model is supplied with the necessary topographic information. Anselmo et al. (1996) used a mixed 1–2-D model, also described by Todini and Venutelli (1991), for verifying the design of the Montalto di Castro Power Plant.

The scope of this paper is to present a mixed 1–2-D overland flow model based upon the control volume finite element (CVFE) discretization method (originally introduced in heat transfer and flow in porous media calculations (Patankar, 1980)), which can handle isotropic and anisotropic problems. This last feature is required, when dealing with 2-D overland flow modelling, for two reasons. First, in highly cultivated lands, the characteristics of propagation can be affected by furrows and drainage systems so that the flow will show a preferential path which should be modelled using anisotropic roughness patterns. Second, at the scale of the finite discretization mesh, the overland flow problem will always be anisotropic, as will subsequently be demonstrated. This aspect has been taken into account in the derivation of the model equations, and in the sequel it will be shown how it can be modelled. Moreover, this derivation can be easily transferred to the problem of flow in porous media in which anisotropy is a non-negligible aspect that heavily affects propagation.

2. Overland and channel processes

The equations describing overland flow are the well-known De Saint Venant equations (Abbott, 1979):

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}+\frac{\partial (vh)}{\partial y}=q \tag{1}$$

$$\frac{\partial (uh)}{\partial t}+\frac{\partial (u^2h)}{\partial x}+\frac{\partial (uvh)}{\partial y}+gh\left(S_{fx}+\frac{\partial H}{\partial x}\right)=0 \tag{2}$$

$$\frac{\partial (vh)}{\partial t}+\frac{\partial (v^2h)}{\partial y}+\frac{\partial (uvh)}{\partial x}+gh\left(S_{fy}+\frac{\partial H}{\partial y}\right)=0 \tag{3}$$

where $H(x,y,t)$ is water surface elevation above an horizontal datum, $h(x,y,t)$ is local water depth, t are time, x,y are horizontal cartesian coordinates, $u(x,y,t)$, $v(x,y,t)$ are flow velocities in x and y directions (depth averaged flow velocities), $q(x,y,t)$ is net distributed input (for instance precipitation or lateral inflows), $S_{fx}(x,y,t), S_{fy}(x,y,t)$ are friction slope in x and y directions, and g is acceleration due to gravity.

In accordance with Xanthopoulos and Koutitas (1976), Hromadka et al. (1985), De Vries et al. (1986) and Labadie (1992), in 2-D overland flow the use of the diffusive approximation — in which the inertial terms are ignored over the gravitational terms, friction and pressure heads — permits the salient aspects of the phenomenon under study to be reproduced. As regards the channel flow, there is general agreement on the validity of the diffusion approximation except for very steep waves travelling on flat channels (Fread, 1985, personal communication, 1990; Todini and Bossi, 1987; Lamberti and Pilati, 1996). Hromadka (1985) concluded that it is acceptable for low-to-moderate velocity–flow regimes, Henderson (1966) considered inertia terms to be negligible in most cases, and Ahn et al. (1993) argued that such simplification induces errors between 5% and 10%, which are still negligible in real-world applications when compared with the uncertainty affecting geometrical and hydrological measurements.

Using the diffusion approach, Eqs. (2) and (3) can be replaced by the following system of parabolic differential equations:

$$\left(S_{fx} + \frac{\partial H}{\partial x}\right) = 0 \tag{4}$$

$$\left(S_{fy} + \frac{\partial H}{\partial y}\right) = 0 \tag{5}$$

In addition, applying the Manning–Stricker law to the description of the friction slopes that appear in the preceding equations, the relation between the velocity and water depth components can be obtained as

$$S_{fx} = \frac{n_x^2}{h^{4/3}} |\boldsymbol{w}| \boldsymbol{w} \times \boldsymbol{i} = \frac{n_x^2}{h^{4/3}} (u^2 + v^2)^{1/2} u$$

$$S_{fy} = \frac{n_y^2}{h^{4/3}} |\boldsymbol{w}| \boldsymbol{w} \times \boldsymbol{j} = \frac{n_y^2}{h^{4/3}} (u^2 + v^2)^{1/2} v$$

where $\boldsymbol{w} = \boldsymbol{u}_i + \boldsymbol{v}_j$ is the velocity vector, n_x, n_y are the Manning roughness coefficients in the directions x and y, respectively, and therefore

$$|\boldsymbol{w}|^2 = u^2 + v^2 = h^{4/3} \left(\frac{S_{fx}^2}{n_x^4} + \frac{S_{fy}^2}{n_y^4} \right)^{1/2}$$

Lastly, the replacement of the preceding relation in the expressions for S_{fx} and S_{fy} determines the following expressions for the components of the velocity vector:

$$u = -\frac{\partial H}{\partial x} \frac{h^{2/3}}{n_x^2} \frac{1}{\left[\left(\frac{\partial H}{\partial x} \right)^2 \frac{1}{n_x^4} + \left(\frac{\partial H}{\partial y} \right)^2 \frac{1}{n_y^4} \right]^{1/4}} \tag{6}$$

$$v = -\frac{\partial H}{\partial y} \frac{h^{2/3}}{n_y^2} \frac{1}{\left[\left(\frac{\partial H}{\partial x} \right)^2 \frac{1}{n_x^4} + \left(\frac{\partial H}{\partial y} \right)^2 \frac{1}{n_y^4} \right]^{1/4}} \tag{7}$$

Eqs. (6) and (7) differ from those normally used (see Hromadka et al., 1985; Reitano, 1992a,b) in that the equations presented here have been derived on the assumption of two-dimensional flow in anisotropic conditions and considering the x and y axes to coincide with the main anisotropy directions of the propagation domain under study.

When dealing with one-dimensional channel unsteady flow, the problem can be simplified, by assuming one of the two coordinates to coincide with the channel longitudinal abscissa, so that the momentum in the orthogonal direction may be regarded as negligible. Eqs. (1) and (2) (or (4)), the mass and momentum balance equations, can still be used, assuming x as the longitudinal abscissa and by considering null all the derivatives in the y direction.

3. Spatial discretization methods

Given that Eqs. (1), (2) and (3) are written for an infinitesimal spatial domain, an infinite number of equations would be required to solve the problem on a finite domain. The alternative is to define a number of simple finite elementary domains and to analytically integrate the equations over these elementary domains to lump their effect and to reduce the dimensionality to a more or less large but finite set of balance equations to be integrated in time. The elementary domain, well known in physics as the control volume, is the finite volume (in one, two or three dimensions) resulting from the spatial integration of the infinitesimal domain over which the differential equations are usually defined. Most physical principles, such as conservation of mass, energy and momentum, can be directly applied, in finite terms, to the control volume.

In reality, not all the space discretization methods conventionally used, such as the FD method for regular structured meshes or the IFD method and the FE method for non-structured meshes, take full advantage of the control volume principle. For instance, the original derivation of the FD did not specifically refer to a control volume; the method was sought as a mere rewriting of the differential equations in terms of incremental ratios, with no integration over the finite elementary domains. Integration of the equations over the elementary domains can be found in both the IFD and the FE approaches. Nevertheless, although the IFD elementary nodal domain can be viewed as a real control volume, on which the scalar quantities such as mass and energy can be imposed, the FE method will only impose mass and energy conservation over a polygon formed by all the finite elements linked to a specific node; inside the polygon these quantities will only be balanced on the average and not specifically in each finite elementary domain.

The CVFE approach presented here addresses and solves the discretization problem, by defining a double mesh: the first one is the usual triangular mesh of the FE over which the momentum equations can be lumped, and the second one is the typical IFD polygonal mesh, resulting from the Dirichlet tessellation, over which mass balance (and energy balance, when needed) equations can be lumped.

Although, without loss of generality, the numerical experiment presented here will deal mainly with a regular grid mesh to compare the results with an available FD solution based on SHE (Bathurst, 1986), the paper will mainly focus on discretization methods using non-structured irregular meshes, which are better suited to describing the high variability and irregularity of the terrain elevation and slopes.

3.1. The IFD approach

The IFD method has been extensively used in the solution of two-dimensional overland flow problems. In broad terms, the main advantages of this method lie in the simplicity of the finite difference approximation used to evaluate the gradients and in its flexibility. Indeed, one-, two- and three-dimensional problems can all be handled simply. Another, by no means negligible advantage, is that the quantities deriving from the discretization and, specifically, the exchanges between the various nodes, lend themselves well to simple physical interpretation. These advantages are common to all the control-volume based methods.

The greatest drawback in using the IFD method in the simulation of flow and transport problems lies in its inability to represent flows deriving from tensor quantities: the reason is that, as is well known, in the IFD method the momentum equations are lumped over 1-D domains and therefore the gradients are calculated along a line perpendicular to a given surface — the interface between the elements — whereas in reality the problem is 2-D, for which the tensor quantities may present non-zero tangential components at this point. For the case of anisotropic problems, a more correct evaluation of the flow terms was proposed by De Marsily (1986), who suggested calculating the gradients on the basis of three rather than just two points. It should, however, be noted that the procedure is not easy to automate and that a less than careful choice of the third point may result in errors in the mass balances.

Ferraresi and Marinelli (1996) have suggested a different correction of the IFD method, to allow the reproduction of flows deriving from tensor quantities, and at the same time preserving the simplicity of the integrated finite difference approach. The proposed method borrows from the finite element theory the concept of 2-D interpolation on the discretization elements. Unfortunately, implementation of the correction causes the system matrix to lose some of the advantages it enjoys in normal integrated finite difference schemes. Specifically, after the tangential contributions owing to anisotropic conditions have been introduced, the symmetry of the system matrix is no longer guaranteed, except in those cases where the triangles forming the basis for the construction of the polygons are equilateral. Further details on the application of the IFD method have been given by Hromadka et al. (1985), Narasimhan and Witherspoon (1976), De Marsily (1986), Todini and Venutelli (1991), and Di Giammarco et al. (1994).

3.2. The FE approach

Although the use of the FE method offers undeniable general advantages in 2-D overland flow problems, including considerable precision in the reproduction of the spatial characteristics of the domain under study, the immediacy of the physical interpretation of the flux quantities present in the discretization equations is lost. Moreover, in parabolic problems, the usual linear finite element schemes based upon triangular elements do not conserve mass at the local level (i.e. on each element), the conservation of mass being guaranteed only on a set of polygons each of which surrounds a node and which constitutes a partition of the domain under study; shapes and sizes of the polygons depend on the value taken by the variable at the nodal points and they cannot therefore be defined a priori (De Marsily, 1986; Fung et al., 1991). In addition, although the problem to be resolved in 2-D overland flow modelling is similar to that of flow in porous media, there are, however, aspects, such as the greater magnitude of the gradients and the greater velocities reached, which pose enormous problems for the traditional finite element method, in which the number of unknowns — the heads — is taken to be equal to the number of nodes in the grid. Therefore, in the most common numerical solutions of the 2-D overland flow problem, methods which present explicitly as unknowns not only the heads at the nodes but also the components of the velocity vector, in two right-angled directions taken as a frame of reference, are generally used (Katopodes, 1980; Bechteler et al., 1992). In practical terms, this requirement trebles the number of unknowns in the system of equations to be solved when one wishes to use an implicit solution method. Many solutions proposed in the literature use not so much an implicit time integration method in the true sense but rather explicit methods of the conventional or predictor–corrector type. However, it should be stressed that implicit integration methods can only be avoided if extremely small time steps are used, bearing in mind the spatial dimensions of the discretization meshes generally used and the propagation celerity of the perturbations involved, given the more or less marked steepness of the gradients and, especially, the limited heights of the water levels, which may even be zero when starting from dry soil.

Lastly, it may be noted that, in the application of linear finite elements, in which triangular elements are used for the solution of flow problems, discontinuities of flow are present at the boundaries of the triangles (Patankar, 1980; De Marsily, 1986). Therefore the sides of the triangles used in the finite elements should not be used as the boundaries of a control volume. Moreover, as already pointed out, if the values of the heads and of the flow are evaluated at the same grid points, lack of conservation in the scalar quantities, particularly that of mass, with possible spatial oscillations in the solution, can result. Therefore, control volumes need to be explicitly introduced in the linear FE method also to avoid the undesirable oscillations in the solution which appear when all the variables (heads and flows) are calculated at the same grid points (Patankar, 1980). Greater details of the FE method have been given by Zienkiewicz (1971), Gallagher et al. (1975), Pinder and Gray (1977), Huyakorn and Pinder (1983) and Dhatt and Touzot (1984).

3.3. The CVFE approach

The introduction of the CVFE method not only permits a physical interpretation of the quantities that describe the flow terms (an aspect which is not altogether clearly defined in the traditional finite element method), but it also allows the problem to be addressed by considering just the heads at the nodes as unknowns, thus permitting rapidly converging implicit schemes to be developed. Below, in addition to illustrating the application of the method to overland flow propagation, attention is drawn to the points the CVFE method has in common with the traditional IFD and FE methods, showing how it may be regarded as a synthesis of these two methods which combines the undeniable advantages of both. In particular, the CVFE method is locally conservative (like the IFD method) and permits an accurate description of complex geometries and also of pronounced heterogeneities and anisotropies (like the FE method). As previously mentioned, this paper does not consider the finite difference methods, because, if they are intended to be conservative, on the basis of staggered grids, then in practice they can be regarded as the application of IFD methods on regular grids.

Also presented is a numerical example which has been chosen not so much to highlight the characteristics peculiar to the CVFE method in the field of overland flow, which mostly derive from its theoretical definition, as to verify the computer code which has been developed. In the example given, in fact, a comparison of the solutions obtained is made with various numerical methods, including that obtained by Bathurst (1986), and with an available theoretical solution.

4. The CVFE discretized equations for overland flow

The replacement of Eqs. (6) and (7) in Eq. (1), on the assumption that x and y are the main anisotropy directions, leads to the following system of partial differential equations of the parabolic type:

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial H}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial H}{\partial y}\right)+q=\frac{\partial H}{\partial t} \tag{8}$$

where

$$k_x=\frac{1}{n_x^2}\frac{1}{\gamma(\nabla H)}h^{5/3},\qquad k_y=\frac{1}{n_y^2}\frac{1}{\gamma(\nabla H)}h^{5/3} \tag{9a}$$

and

$$\gamma(\nabla H)=\left[\left(\frac{\partial H}{\partial x}\right)^2\frac{1}{n_x^4}+\left(\frac{\partial H}{\partial y}\right)^2\frac{1}{n_y^4}\right]^{1/4} \tag{9b}$$

Once the initial and boundary conditions have been assigned (assigned head (Dirichlet), assigned discharge (von Neumann) or mixed conditions), the problem is fully defined in mathematical terms and the solution is obtained by the integration of Eq. (8).

The CVFE method, like any other control volume based method and, hence, like the IFD method, is based on assigning the conservation of mass to each control volume, the shape and position of which is known with precision. In this case, attention is focused on a defined volume, fixed in space — the control volume — whose shape and position is chosen arbitrarily but is invariable in time, and the balance equations are written for each such volume.

The spatial discretization of the domain can be obtained by making use of triangular elements. In particular, the choice of linear triangular elements results in a linear approximation of the soil altitude inside each triangle as well as in constant values of the Manning's coefficients in each triangle. Fig. 1 shows an example of the breakdown of the domain into polygonal cells (Dirichlet tessellation or Voronoi diagrams (see Sloan, 1977; Gold, 1991), better known in hydrology as Thiessen polygons), the mean properties of which are represented at point i inside the element itself.

The boundaries of the control volumes are formed by the perpendicular bisectors of the sides of triangles forming the mesh. It follows that the points C_1 and C_2 in Fig. 2 correspond to the circumcentres of the two triangular elements. The exchanges between the nodes I and J are then calculated along the sides $\overline{C_I m_{IJ}}$ (belonging to

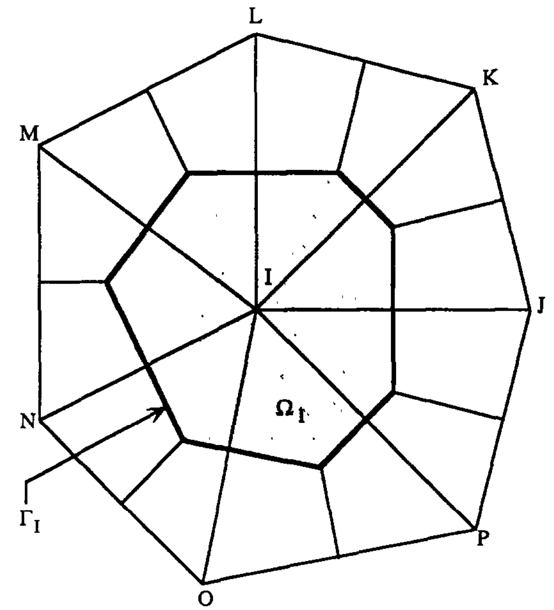

Fig. 1. Control volume finite element domain discretization.

the element IJK) and $\overline{m_{IJ}C_2}$ (belonging to the element IPJ). The procedure followed in the construction of the polygons ensures maximum precision in calculating the exchanges, as the boundaries of the control volume lie at the midpoint of the link between nodes I and J. The flexibility of the triangular element discretization structure permits the accurate spatial reproduction of the domain, even in the case of complex geometries. This feature turns out to be of particular interest for convergent hillslope area faces and to represent sharp gradients. The dual definition of the polygons (control volumes) which act as the basis for determining the exchanges between adjoining nodes (and at which the balance equations are defined) ensures the immediate physical interpretation of the quantities present in the discretization equation. Further details of the CVFE method have been given by Patankar (1980), and an example of its application to the simulation of flow in porous media has been developed by Fung et al. (1991).

Starting from Eq. (8), the relevant discretization equation will be obtained with reference to just the node I and its control volume (Ω_I); extension to the overall domain (Ω) will be accomplished by assembling the contributions for all the nodes. The procedure illustrated is legitimate insofar as the control volumes constitute a partition of the whole domain.

The first step in the application of the CVFE method is the integration of Eq. (8) on the surface domain Ω_I bounded by the curve Γ_I (see Fig. 1 for the meaning of the symbols):

$$\int_{\Omega_I}\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial H}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial H}{\partial y}\right)\right]\mathrm{d}\Omega_I+\int_{\Omega_I}q\mathrm{d}\Omega_I=\frac{\partial}{\partial t}\int_{\Omega_I}H\mathrm{d}\Omega_I \tag{10}$$

The application of the divergence theorem and the reformulation of the term on the right-hand side gives

$$\underbrace{\int_{\Gamma_I}\left[\left(k_x\frac{\partial H}{\partial x}\right)\boldsymbol{i}+\left(k_y\frac{\partial H}{\partial y}\right)\boldsymbol{j}\right]\times \boldsymbol{n}_{\Gamma_I}\mathrm{d}\Gamma_I}_{\mathfrak{I}} + q_I\Omega_I = \frac{\partial V_I}{\partial t} \tag{11}$$

where $\times$ signifies a scalar product, $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors of the x and y coordinates, $\boldsymbol{n}_{\Gamma_I}$ is the unit normal to the line Γ_I, V_I is the volume of water contained in the control volume to which node I belongs, $\partial V_I/\partial t$ is the volume time variation and q_I is the mean value of the external discharge entering the surface domain Ω_I.

In Eq. (11), the line integral $\mathfrak{I}$ represents the total discharge crossing the boundary Γ_I of the domain Ω_I for all the elements sharing the same node I, and the right-hand-side term represents the exchanges with the external world. As the triangles surrounding node I contain portions of the control volume and its boundary, the total value of the exchange can be obtained on the basis of the contribution furnished by the portion G of the control volume indicated in Fig. 2. Linear functions of the following type are used as shape functions:

$$\Phi_i(x,y) = \frac{1}{2A}(a_i y + b_i x + c_i), \qquad i = I,J,K$$

$$a_k = x_j - x_i$$

$$b_i = y_j - y_k, \qquad i,j,k = I,J,K$$

$$c_i = x_j y_k - x_k y_j$$

where A is the area of the triangle IJK in Fig. 2 and a_k and b_i are obtained from anticyclic permutations of the x coordinates and cyclic permutations of the y coordinates, respectively. Each variable defined inside the triangular element may in this way be assigned on the basis of its known value at the nodal points. For example, the head inside the triangular element may be expressed by the relation

$$H(x,y) = \sum_i \Phi_i(x,y)H_i, \qquad i = I,J,K \tag{12}$$

In the event that $n_x = n_y = n$ and, therefore, $k_x = k_y$, the analytical solution of the line integral in (11), limited to just the generic element e identified by nodes IJK, gives the value of the discharge crossing the boundaries $\overline{C_I m_{IJ}}$ and $\overline{C_I m_{IK}}$:

$$Q_I^e = -[(T_{IJ}^e + T_{IK}^e)H_I - T_{IJ}^e H_J - T_{IK}^e H_K] \tag{13}$$

where the letter e identifies the triangular element with vertices I, J, K, and where the conveyances T_{IJ}^e, T_{IK}^e along the directions IJ, IK are defined respectively as

$$T_{IJ}^e = \phi(I,J)\frac{1}{\gamma(\nabla H)}\frac{1}{n^2}\frac{h_{C_1}^{8/3} - h_{m_{IJ}}^{8/3}}{h_{C_1} - h_{m_{IJ}}}$$

$$T_{IK}^e = \phi(I,K)\frac{1}{\gamma(\nabla H)}\frac{1}{n^2}\frac{h_{C_1}^{8/3} - h_{m_{IK}}^{8/3}}{h_{C_1} - h_{m_{IK}}}$$

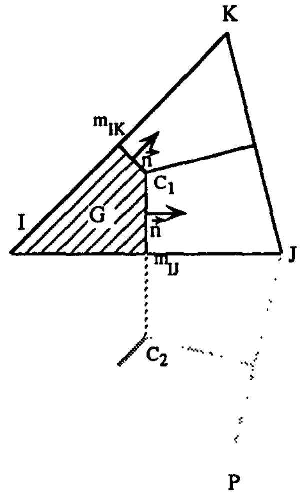

Fig. 2. Mesh elements involved in calculation of flow between node I and J.

with $\phi(I,J)$ and $\phi(I,K)$ written in terms of the adopted shape functions and of the geometrical characteristics of the triangular elements:

$$\phi(I,J) = \frac{3}{4}A\frac{\Phi_K(C_1)}{a_K^2 + b_K^2}$$

$$\phi(I,K) = \frac{3}{4}A\frac{\Phi_J(C_1)}{a_J^2 + b_J^2}$$

As the gravitational field is dissipative and a negative conveyance value would imply local generation of energy, the conveyances must obey the following physical constraints:

$$\left.\begin{aligned} T_{IJ}^e \geqslant 0 \\ T_{IK}^e \geqslant 0 \end{aligned}\right\} \quad \forall I, J, K$$

These conditions can be assured by the appropriate choice of the shapes of the triangles; if the basic triangular element is acute, the $\phi(I,J)$ and $\phi(I,K)$ functions always have a positive value, thus ensuring the positive value of the coefficients of the conveyance. This latter result is valid in isotropic situations, whereas, when roughness is also anisotropic, slightly different results apply.

At this point, it is of the utmost importance to note that at the scale of a triangular element, the 2-D overland unsteady flow is in practice always anisotropic. In fact, even in the case of isotropy at the micro-scale, after integrating the equations over the finite domain, the problem becomes anisotropic in finite terms in that the three resulting conveyances will differ from each other, in each triangular element, as a consequence of the non-constant water elevation. This is analogous to defining a symmetrical anisotropic conveyance tensor at the scale of the finite element, where the three components of the conveyance T^e_{IJ}, T^e_{IK}, T^e_{JK} can be defined in relation to the three independent tensorial quantities T^e_{xx}, T^e_{xy}, T^e_{yy}, by a simple plane geometry linear transformation.

If the flows for nodes J and K are also considered, the expression containing the contributions for one single element in the mesh can be obtained in matrix form as

$$\begin{vmatrix} Q^e_I \\ Q^e_J \\ Q^e_K \end{vmatrix} = -\begin{vmatrix} (T^e_{IJ}+T^e_{IK}) & -T^e_{IJ} & -T^e_{IK} \\ -T^e_{IJ} & (T^e_{IJ}+T^e_{JK}) & -T^e_{JK} \\ -T^e_{IK} & -T^e_{JK} & (T^e_{IK}+T^e_{JK}) \end{vmatrix} \begin{vmatrix} H_I \\ H_J \\ H_K \end{vmatrix} \tag{14}$$

where the letter e identifies again the triangular element with vertices I, J, K. It should be noted that the rank of the matrix of conveyances is at most two, in analogy with the rank of the conveyance tensor that may be defined over the element.

In addition, as previously mentioned, the conveyance matrix is symmetric, thus satisfying the condition laid down by the physical problem: equality of the discharge flowing between nodes I and K and between nodes K and I, and each diagonal term has the property of assuming a value equal to the sum, in absolute value, of the extra-diagonal terms in the relevant row or column. The last step entails the assembly of all the elements connected to node I and the assembly of all the nodes in the domain, to obtain the following expression for the exchanges between all the nodes in the domain:

$$\{Q\} = \begin{vmatrix} Q_1 \\ Q_2 \\ \cdots \\ Q_{n-1} \\ Q_n \end{vmatrix} = -\begin{vmatrix} \sum\limits_{\substack{i=1\\ i\neq 1}}^{n} T_{1i} & -T_{12} & \cdots & -T_{1(n-1)} & -T_{1n} \\ -T_{12} & \sum\limits_{\substack{i=1\\ i\neq 2}}^{n} T_{2i} & \cdots & -T_{2(n-1)} & -T_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -T_{1(n-1)} & -T_{2(n-1)} & \cdots & \sum\limits_{\substack{i=1\\ i\neq n-1}}^{n} T_{(n-1)i} & -T_{(n-1)n} \\ -T_{1n} & -T_{2n} & \cdots & -T_{(n-1)n} & \sum\limits_{\substack{i=1\\ i\neq n}}^{n} T_{ni} \end{vmatrix} \begin{vmatrix} H_1 \\ H_2 \\ \cdots \\ H_{n-1} \\ H_n \end{vmatrix}$$

$$= -[T]\{H\} \tag{15}$$

where Q_i represents the algebraic sum of flows entering in the ith nodal control volume from the relevant elements, and T_{ij} denotes the total conveyance between nodes I and J contributed by the two elements that share the same IJ connection, with T_{ij} equal to zero if the two nodes I and J are not directly connected. The matrix T is structurally and formally identical to the one used by the IFD, or by the FE; the basic difference lies in the different values obtained for the conveyances.

Thus, after discretization in space by means of the CVFE approach, Eq. (11) can be written in vector matrix form as

$$\frac{\partial\{V\}}{\partial t} = [\Omega]\{q\} - [T]\{H\} \tag{16}$$

where all the vectors and matrices have the dimensionality of the nodal domain.

4.1. Space discretization of storage

As far as the calculation of the unsteady state term in Eq. (16) is concerned, two calculation methods, known as the 'lumped' and 'distributed' mass matrices formulations, can be used (Wood and Calver, 1990). Generally speaking, the volume contained inside the domain of node I is a function of the heads at all the nodes surrounding node I:

$$V_I = f(H_I, H_l), \qquad l = \{\text{set of nodes surrounding node } I\}$$

and therefore

$$\frac{\partial V_I}{\partial t} = \frac{\partial V_I}{\partial H_I}\frac{\partial H_I}{\partial t} + \sum_l \frac{\partial V_l}{\partial H_l}\frac{\partial H_l}{\partial t} \tag{17}$$

The approach based on Eq. (17) constitutes a distributed representation of the volumetric changes and is also called the consistent formulation; in this approach, the volume variation at node I depends on the head variations at all the nodes surrounding node I and the positive contributions are added to the existing off-diagonal non-null elements. Unfortunately, although it provides for a more accurate calculation of the storage term, the consistent formulation results in the loss of diagonal dominance of the system matrix T in that the original off-diagonal elements are all negative. In the lumped formulation the second summation term in Eq. (17) is usually ignored and the term that takes account of the storage at the node is concentrated solely in the diagonal elements of the matrix, which are already positive. In this case, the system matrix is therefore of the dominant diagonal type, and its structure offers major advantages in computational terms as it permits the use of rapid convergence solution algorithms with the system of algebraic equations.

A similar computational advantage of the lumped formulation over the consistent type lies in the integration methods based on the traditional finite element approach. Neuman (1975), in the analysis of a problem regarding flow in an unsaturated porous medium conducted with finite element integration based on the Galerkin method, encountered convergence problems in the case of the consistent approximation; these problems were eliminated when the lumped approach was used. Huyakorn and Pinder (1983) also report that, for flow in porous media, the lumped approach generally produces less accurate but more stable solutions. Recently, Wood and Calver (1990) compared the results derived from the two approaches and did not find any substantial differences in convergence, although they noted a considerable increase in the accuracy of the solution obtained using the consistent approach.

To enhance the reproduction of the storage term, while retaining the advantages of the lumped formulation, a correction is developed here and applied to the method for calculating the variation in the volume stored in the element. The basic idea is to evaluate correctly the storage terms at the nodes (i.e. according to the water levels at all the connected nodes) and to assign these values solely to the diagonal term of the system matrix. For the purposes of the calculation, the variation over time of the volume contained in the domain to which node I belongs is expressed in the form

$$\frac{\partial V_I}{\partial t} = \Omega_I'(H_l)\frac{\partial H_I}{\partial t}, \qquad l = \{\text{set of nodes surrounding node } I\}$$

where the control volume surface domain is no longer regarded as fixed but as varying with changes in the piezometric head of the nodes surrounding the node under study.

The stored volume calculation procedure used here proves particularly useful in the case of depletion of the water stored in the elements and, although it generates a limited increase in the number of iterations and therefore increases computation time, it achieves greater precision in the evaluation of the levels and the exchanges between the nodes.

4.2. Integration in time

The time discretization was initially performed here using a centred implicit scheme (Crank–Nicholson with $\alpha = 0.5$). The implicit scheme was preferred over the explicit type because, as is known, it permits the use of larger time discretization steps without causing problems of stability. Accordingly, by discretizing in time Eq. (16), the following expression is obtained:

$$\left[\alpha[T] + \left[\frac{\Omega'}{\Delta t}\right]\right]\{H\}^{t+\Delta t} = \left[-(1-\alpha)[T] + \left[\frac{\Omega'}{\Delta t}\right]\right]\{H\}^{t} + [\Omega]((1-\alpha)\{q\}^{t} + \alpha\{q\}^{t+\Delta t}) \tag{18}$$

where Δt is the time step used in the simulation.

To speed up the convergence towards the system solution it was decided to furnish estimates, at the start of each time step, of the values of the water levels at the end of it. Various methods for predicting the head values, based on interpolation and extrapolation techniques, were analysed. The comparison was made with the results obtained without the help of any forecasting mechanism in terms of either the correctness of the value obtainable at the end of each time step or the number of iterations required to achieve the solution.

The results of the tests showed that the greatest reduction in the number of iterations is obtained by using estimates of the values of the water levels at the start of each time step based on a parabolic extrapolation method which takes account, at the nth time step, of the last three solutions obtained. In the case of linear extrapolation based on the knowledge of the most recent two solutions, the average number of iterations is reduced by approximately 20%, and, where parabolic extrapolation is adopted, a reduction of approximately 40% in the number of iterations is achieved (with maxima of approximately 60%).

The advantages deriving from the introduction of a quadratic method in the approximation of time derivatives compared with the traditional Crank–Nicholson linear model are currently being evaluated. In theoretical terms, the advantages of this method lie mainly in a correct approximation up to the second order compared with a linear approximation, such as the Crank–Nicholson one, even though the latter has second-order terms that become negligible.

4.3. Solution of the system of algebraic equations

The assignment of the initial and boundary conditions is reflected in a partition of the heads' vector at the nodes: some of the values of H are in fact assigned and therefore combine to form part of the known term of Eq. (18). This equation is then transformed into a system of equations which, expressed in matrix form, becomes

$$[B]\{H\}^{t+\Delta t} = [A]\{H\}^{t} + \{C\} \tag{19}$$

The problem of the integration of Eq. (8) has therefore been transformed into the search for the solution to the non-linear system of algebraic equations in the unknowns H. If Nn is the total number of nodal points in the domain and N_0 is the number of nodes in which the head is assigned, the vector H will have $(Nn - N_0)$ unknowns and the system matrix will be a sparse symmetrical matrix of rank $(Nn - N_0)$, with only $(Nn - N_0) + 2N_c$ non-null elements, where $N_c = Nn + N_e - 1$ is the number of topological connections among nodes and N_e is the number of triangular elements.

The non-linearity of the starting equation calls for the recursive calculation of the values of T (functions of the local values of h) and suggests the adoption of iterative solution methods. At each time step a certain number of equation systems must therefore be resolved and the efficiency of the solvers to be used for the system (18) becomes crucial. The knowledge of the topological structure of the system matrix must therefore be used to the fullest advantage. The system matrix originating from the CVFE has the same properties that characterize the matrix deriving from the IFD method and, therefore, the same considerations apply to the solution method: (1) the solver must take into account sparsity, symmetry and positive definiteness; (2) the modified conjugate gradient method has been found to be the most efficient iterative method with respect to memory requirements (Ajiz and Jennings, 1984; Todini and Pilati, 1987); (3) some direct methods are the most efficient from the point of view of computational time requirements, provided an appropriate sorting of elements is performed.

At present, it has been found that the most suitable solution methods are the modified conjugate gradient method and a direct method developed by Duff and Reid (1982) — available in Harwell library (MA27) — which is particularly effective for solving systems of algebraic equations with a sparse system matrix and large dimensions. With regard to this latter method, it may be noted that the direct methods based on a re-sorting of the system matrix save computational time and, although they require a larger memory allocation than the iterative methods, they require only a limited increase in memory storage and are therefore suitable for solving problems

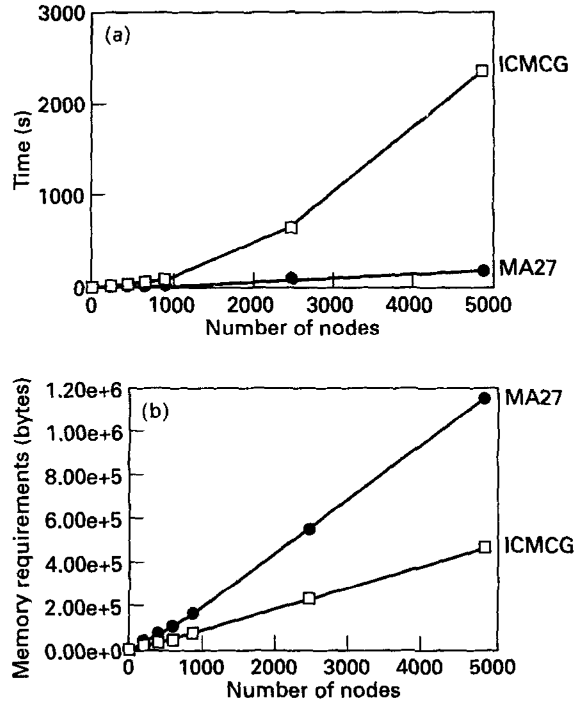

Fig. 3. (a) Comparison of time requirements; (b) comparison of size requirements. (Redrawn from Salgado et al. (1995).)

of major proportions (Duff and Reid, 1982; Duff et al., 1986; Salgado et al., 1995). Figs. 3(a) and 3(b) show a comparison, in the case of a matrix derived from a similar problem, between MA27 and a modified conjugate gradient method (ICMCG) (Todini and Venutelli, 1991).

5. Numerical experiments and results

In this phase of the research, which is devoted more to the verification of the computer program under development than to enhancing the specific properties of the CVFE method, which are theoretically proven, the mathematical model was tested to compare its results with those deriving from an analytical solution. The need to use an analytical solution limited the choice of possible cases. An approximate analytical solution was found for the V-shaped schematic catchment (see Overton and Brakensiek (1973) and Stephenson and Meadows (1986) for further details) illustrated in Figs. 4 and 5. The simulations were made on the basis of a test case with a single slope on the plane and a single slope in the channel (Fig. 4) as well as in the case of a double slope on the plane and a single slope in the channel (Fig. 5). The depth of the channel in the single slope problem varies from 1 m at the upstream end to 20 m at the downstream end, and, in the double slope case, the channel has a

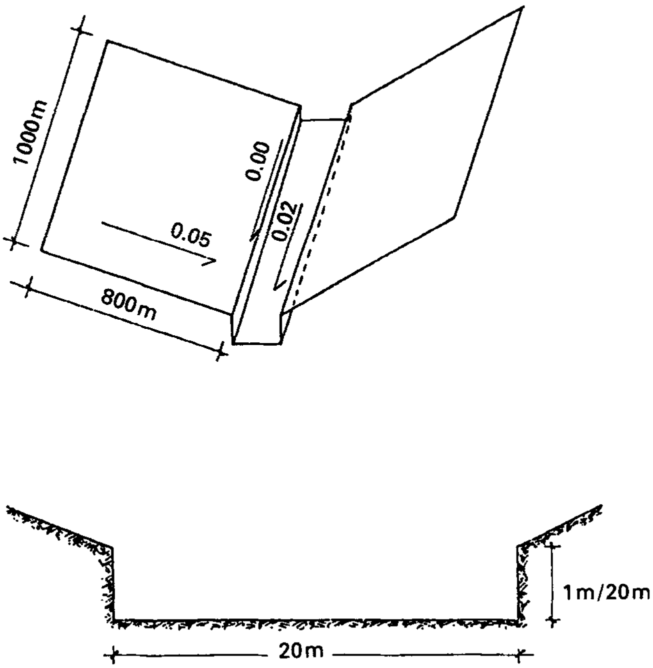

Fig. 4. Level V-catchment.

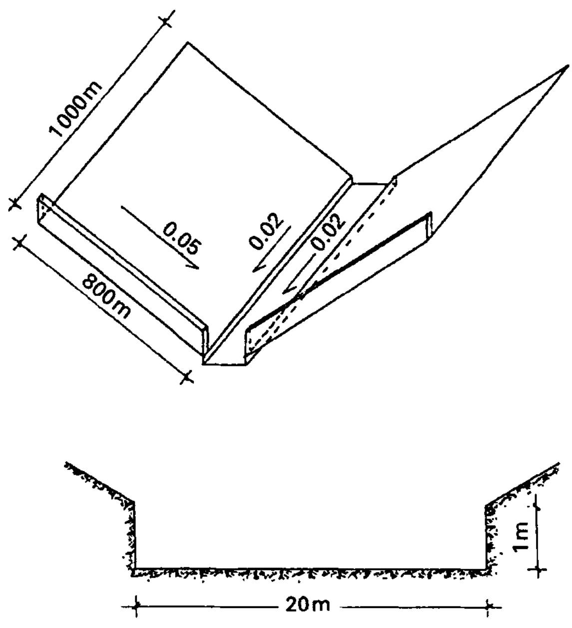

Fig. 5. Tilted V-catchment.

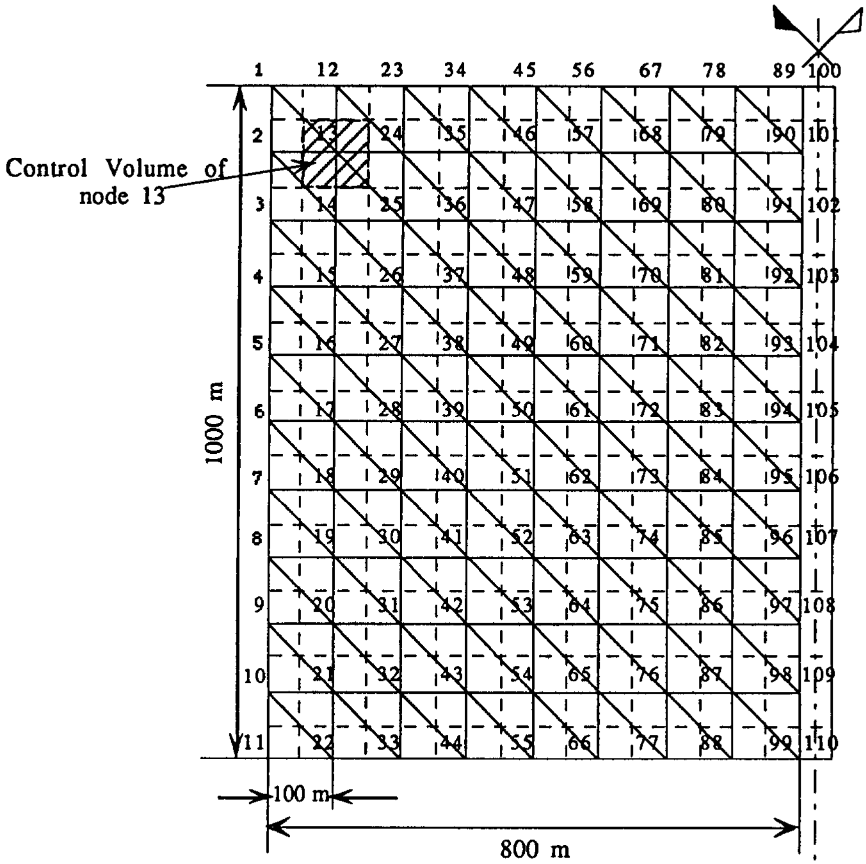

Fig. 6. V-catchment: discretization scheme for CVFE method. Continuous lines: triangular element boundaries; dotted lines: control volume boundaries.

constant depth of 1 m, which in any case is sufficient to avoid backwater effects on the valley side.

Because in the single slope V-catchment there is no gradient in the valley direction, the discharge flows perpendicularly to the channel. By contrast, in the double slope V-catchment, both the valley side and the channel are characterized by a slope extending from upstream to downstream; in this case, overland flow is diagonal to the channel. Propagation in the channel was computed together with propagation in the plane, although the former was assumed to be one-dimensional and the channel propagation equations were discretized using an integrated finite difference approach into which the CVFE method degenerates for one-dimensional problems. This dual equation discretization method (necessary for treating real-world cases, where the overall behaviour can frequently be one- or two-dimensional depending upon the water levels), was made possible thanks to the above-mentioned similarities in the system matrices deriving from the CVFE and the IFD method.

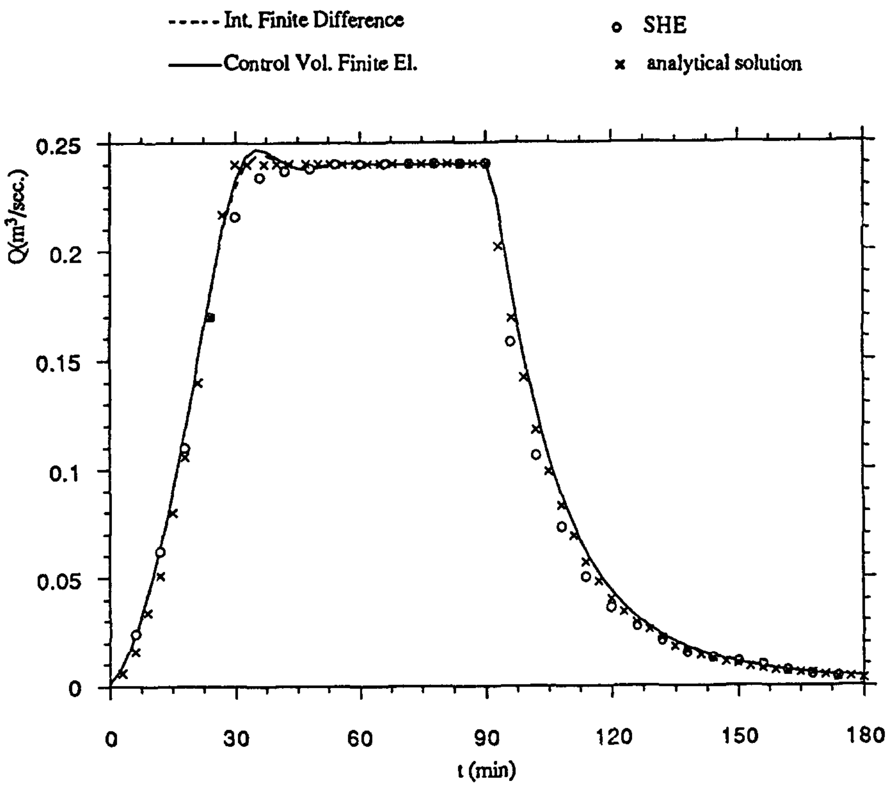

Fig. 7. Model results on the plain side of the level V-catchment.

In the CVFE approach to both the single slope V-catchment and the double slope catchment, a discretization with a 'uniform' triangular element grid was used together with control volumes measuring 100 m × 100 m in most cases. The mesh in the channel was given a rectangular shape (20 m × 100 m). Both of the spatial discretization methods are illustrated in Fig. 6, and were used to allow for a comparison with results available from an original version of the SHE model based on a conventional finite difference scheme (Bathurst, 1986).

The initial and boundary conditions were set as follows: both the valley side and the channel were considered initially dry and only downstream boundary condition were imposed. Given that the channel depth of 1 m avoided backwater effects on the valley side, the downstream boundary conditions were set in the form of critical depth both at the edge between the valley side and the channel and at the downstream end of the channel. The simulation was then conducted using a rainfall input with an intensity of 10.8 mm h^{-1} and a duration of 1.5 h. The precipitation was assumed not to affect the channel. Values for the Manning's coefficients of $n_v = 0.015$ for the valley side and $n_c = 0.15$ for the channel were assigned which, although unrealistic in natural conditions, were assumed so as to obtain a similar characteristic concentration time both

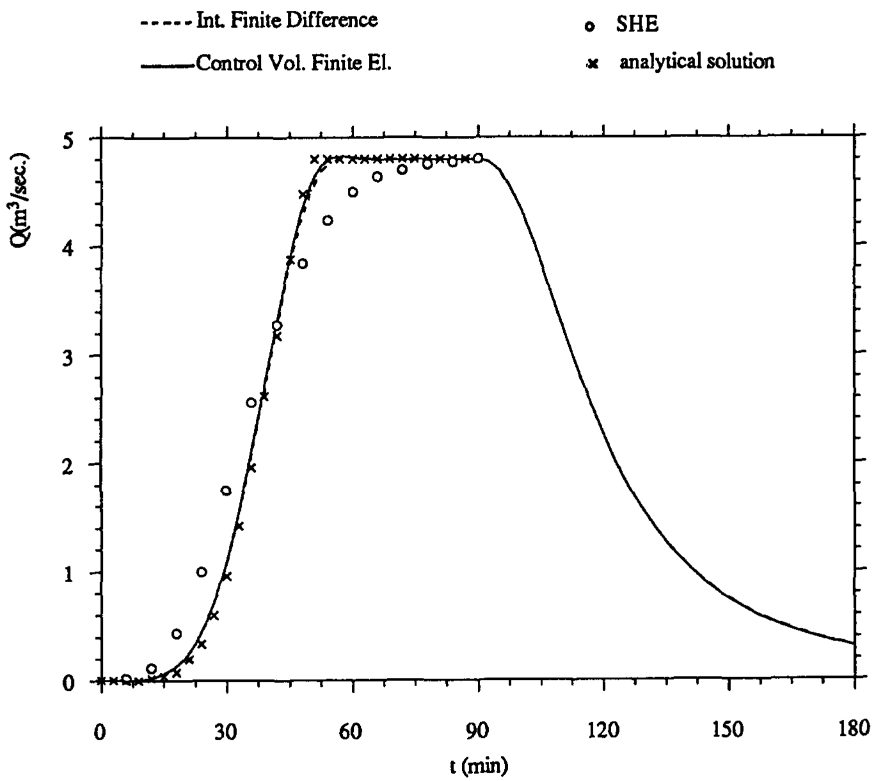

Fig. 8. Discharge at the outlet of the channel in the level V-catchment (both the analytical solution and the FD solutions are not available for the recession).

in the valley side and in the channel, and again to allow for the above-mentioned comparison.

The results for the SHE model were obtained using a grid of squares with a side length of 100 m. For overland flow, the scheme followed was explicit, whereas an implicit scheme was used for the simulation of propagation in the channel (Abbott et al., 1986). The discretization scheme used in the IFD method employs the same type of mesh. Figs. 7 and 8 show a comparison between the kinematic analytical solution, the numerical solution obtained with the SHE model, an IFD model and the CVFE model with the single slope catchment configuration.

It should be noted that there is no comparison with the results derived from the kinematic analytical solution at the catchment outlet section for the descending wave phase. This is due to the fact that the analytical solution adopted requires an instantaneous interruption in the channel inflow, whereas in this case the inflow from the plane diminishes gradually. As can be seen in Figs. 7 and 8, the results obtained with the CVFE method and the IFD method are very close to those of the kinematic analytical solution. An improvement can also be noted over the solution obtained with the SHE finite difference scheme, as reported by Bathurst (1986), particularly for

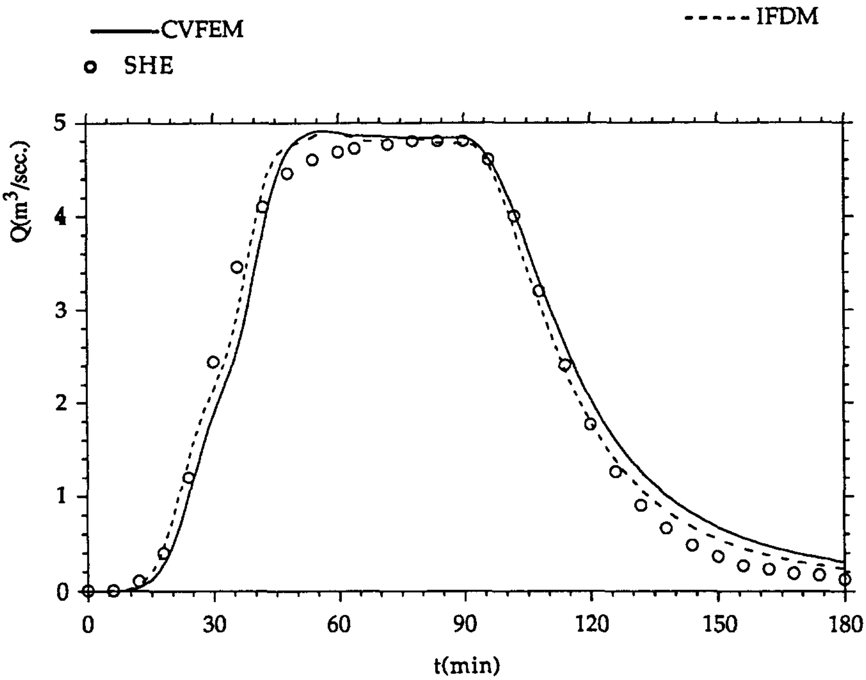

Fig. 9. Discharge at the outlet of the channel in the tilted V-catchment.

the channel outflow in the case of the single slope catchment. However, a fully implicit solution scheme for both overland and channel flow has since been implemented within the new flow, sediment and contaminant modelling system SHETRAN (Ewen, 1995), and further comparisons with SHETRAN results will be made in the future.

With regard to the double slope catchment, a 'barrier' is present at the downstream end of the plane, which creates a local increase in the water level. This barrier is responsible for the differences between the solutions obtained with the IFD and the CVFE schemes. In fact, in the latter scheme, the nodes are positioned exactly at the boundary and are therefore more sensitive to a rise in the water level. In the IFD scheme, the different positions of the nodes, at the centroids of the grid elements, attenuates this effect. The results obtained are shown in Fig. 9 and are compared with those obtained with the SHE model.

A more correct comparison with the approximate analytical solution in the double slope catchment case can be obtained using a simulation for a single strip in the maximum slope direction of the catchment (see Fig. 10). In this case, a more suitable polygonal control volume was used. The CVFE, IFD and the analytical solution for the discharge flowing from the plane into the channel show in this case that the CVFE scheme achieves greater accuracy in the reproduction of the two-dimensional nature of the flow compared with the IFD solution. It can in fact be seen in Fig. 11 that the CVFE solution is closer to the analytical solution.

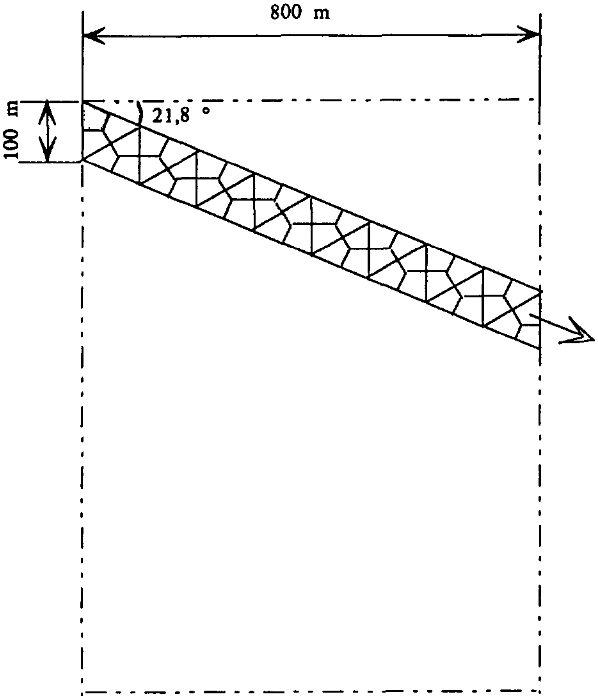

Fig. 10. Discretization scheme for the comparison of CVFE method, IFD method and analytical results in the tilted V-catchment.

6. Conclusions

The introduction of the CVFE method for solving two-dimensional overland-flow problems can be viewed as an attempt at merging the advantages of being both conservative at local scale (as in the IFD method) and capable of representing spatial domains characterized by complex and irregular geometry (as in the FE method). From the point of view of the conservation of the scalar quantities, such as mass and energy, the CVFE scheme can also be thought of as an extension of the classical FD conservative staggered grid approach (Mesinger and Arakawa, 1976) to irregular domains.

The ease with which 1-D and 2-D internal boundaries can be defined as well as the external boundaries, together with the previously mentioned schematization advantages, makes the CVFE method extremely useful and flexible not only for overland flow studies but also for flood plain modelling. In particular, the possibility of easily imposing as well as physically interpreting fluxes along lines and boundaries makes the method attractive for applications where the physical interpretation of the results is needed. The simulation results obtained with the developed computer program show a very accurate reproduction of the analytical solution, which was unfortunately available for a regular V-catchment domain only.

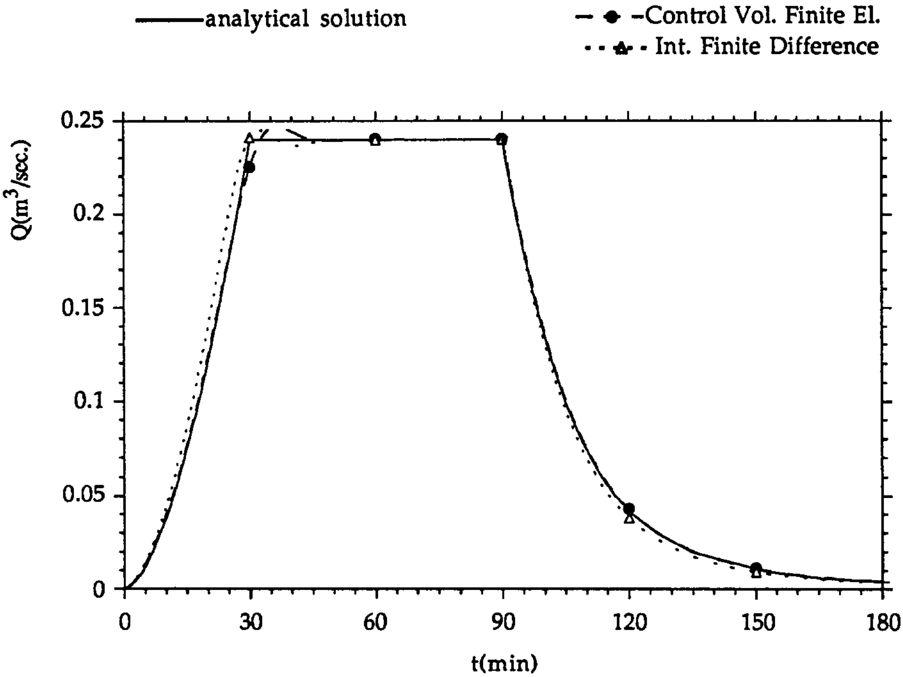

Fig. 11. Results for a strip of the tilted valley side.

At present, the introduction of all the special hydraulic representations of elements such as roads, culverts, dikes, etc., to extend the program to the solution of flood plain inundation problems, is under way, and a comparison with the results of the available IFD program (Di Giammarco et al., 1994; Anselmo et al., 1996) will be performed on the basis of real-world cases.

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