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Technical Notes

FLOOD ROUTING BY THE MUSKINGUM METHOD

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ABSTRACT

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The coefficients k and x in the linear Muskingum equation are determined by the least-squares method (LSM). A routing scheme is proposed for situations where the storage and weighted flow relationship is nonlinear. Although the routing coefficients in the nonlinear equation are not determined by LSM, an alternative working method is proposed. Another simple method for nonlinear flood routing is proposed which is called herein the segmented-curve method. The coefficients in this method are determined by LSM. This method involves only a slightly more work than is required in the linear Muskingum method which is inaccurate in nonlinear problems.

INTRODUCTION

The Muskingum equation is frequently used for routing of floods in river channels. This equation is written as:

$$S = k[Ix + (1-x)O] \tag{1}$$

where S is the storage within the routing reach, I, the rate of inflow at the upstream end, O, the rate of outflow from the downstream end, x, a weighting factor, and k the gradient of the storage vs. the weighted flow curve. Eq.1 is solved with the continuity equation which is written as

$$I - O = \mathrm{d}S/\mathrm{d}t \tag{2}$$

Solution of eqs.1 and 2 can be algebraically obtained if I can be expressed in algebraic terms. Such a solution is presented by Diskin (1967) and also in Dooge (1973). Generally I is available only in discrete form i.e. the data of inflow at a certain interval of time are available. A solution is then obtained using finite quantities instead of the differential in eq.2. Eq.2 is thus expressed as:

$$\tfrac{1}{2}[(I_1 + I_2) - (O_1 + O_2)] = \Delta S/\Delta t = (S_2 - S_1)/\Delta t \tag{3}$$

The storage values in eq.3 are to be determined from eq.1. However, to use eq.1, the coefficients k and x need to be determined before hand. These are determined by using historical flood data. It is known that x varies between 0.0 and 0.5. A tentative value of x is assumed and the historical data are plotted S vs. $[Ix + (1 - x)O]$. The data generally plot in the form of a loop (sometimes there are two or more loops). The value of x for which the width of the loop is reduced to the minimum is believed to be the correct value of x. This trial- and-error procedure can easily be replaced by the least-squares method (LSM) which is described herein later. The slope of the line for the correct value of x determines the value of k.

It should be recognised that the storage vs. the weighted flow relationship is not always and essentially linear as is implied in eq.1. If the relationship is discovered to be nonlinear, using eq.1 may not be justified and may indeed introduce considerable error. A nonlinear form of relationship is frequently quoted in text books, e.g. Linsley et al. (1949) and Chow (1959), and is of the following type:

$$S = k[xI^n + (1 - x)O^n] \tag{4}$$

where n is an exponent. Although, purely from a theoretical viewpoint, such a relationship can be fitted to the storage curves, this method has seldom been used as such for flood routing. An alternative nonlinear method is proposed herein. The situation may indeed, in some cases at least, be still more complicated. The coefficients x and k may themselves be variable with respect to the weighted flow. Awareness of this complication already exists but there are very few, if any, methods available by which the variation of these coefficients may be accounted for. An approximate method is proposed herein by which the variation can partially be accounted for.

LEAST-SQUARES METHOD (LSM)

The storage S in eq.1 is the absolute storage. The practical values of storage that are normally available are the relative values. According to eq.1, if absolute values of storage are plotted against the corresponding weighted flow, $[xI + (1 - x)O]$, values, a straight line passing through the origin can be drawn through the mean positions of the plotted points. If the storage values are relative, the straight line will not pass through the origin. In view of this observation, eq.1 is modified as follows:

$$S = k[xI + (1 - x)O] + \sigma \tag{5}$$

where S is now the relative storage and σ the difference between the relative and the absolute storages. To simplify notation in eq.5, let $kx = A$ and $k(1 - x) = B$, so that:

$$S = AI + BO + \sigma \tag{6}$$

The problem is now reduced to determining A, B and σ which will fix the position of the straight line such that the width of the loop is minimised. Described in this manner, the problem becomes quite simple. Let the deviation between the predicted and the actual values of S be denoted by δ so that:

$$\delta_1 = S_1 - (AI_1 + BO_1 + \sigma), \qquad \delta_2 = S_2 - (AI_2 + BO_2 + \sigma)$$

and so on. It is sought to minimise:

$$\sum_1^N \delta^2 = \delta_1^2 + \delta_2^2 + \ldots \delta_N^2$$

where N is the number of available data. Using the usual procedure, the following normal equations are obtained from which the values of A, B and σ can be obtained:

$$\sum_1^N S - A\sum_1^N I - B\sum_1^N O - N\sigma = 0 \tag{7}$$

$$\sum_1^N SI - A\sum_1^N I^2 - B\sum_1^N IO - \sigma\sum_1^N I = 0 \tag{8}$$

$$\sum_1^N OS - A\sum_1^N IO - B\sum_1^N O^2 - \sigma\sum_1^N O = 0 \tag{9}$$

Once the values of A and B are determined, k and x are calculated from the following equations:

$$x/(1-x) = A/B \tag{10}$$

$$k = A + B \tag{11}$$

The preceding method is now applied to a specific example. Linsley et al. (1949) have tabulated the inflow and outflow data for the Sewickley—Wheeling reach of the Ohio river. Their data are used here for determining x, k and σ. The storage values are in units of 1,000 acre ft. (1 acre ft. equals 1,233 m^3.) and weighted flow in units of 100,000 c.f.s. (1 c.f.s. equals 0.028 m^3/s) (Fig.1). The calculated optimum values of the constants are $A = 0.610$, $B = 1.203$, $k = 1.812$, $x = 0.337$ and $\sigma = -243.734$.

LSM was used in solving several other typical problems and the results were invariably found to be wholly satisfactory.

NONLINEAR STORAGE—FLOW RELATIONSHIP

The storage—flow relationship is not always expressible by the linear formula, eq.1, as noted earlier. If the element of nonlinearity is appreciable, it is proposed that the following equation may be fitted to the data:

$$S = \alpha[Ix + (1 - x)O]^m + \sigma \tag{12}$$

where m is an exponent and α is a coefficient. The coefficient k of eq.1 is obviously rendered variable in eq.12 while x is kept constant. The constants α, x, m and σ can not be obtained by LSM because the resulting normal equations are implicit in character and require a tedious trial-and-error solution. It is suggested that the value of x should be determined by trial and error. Having determined a suitable value of x, a plot of S against $[xI + (1 - x)O]$ should be

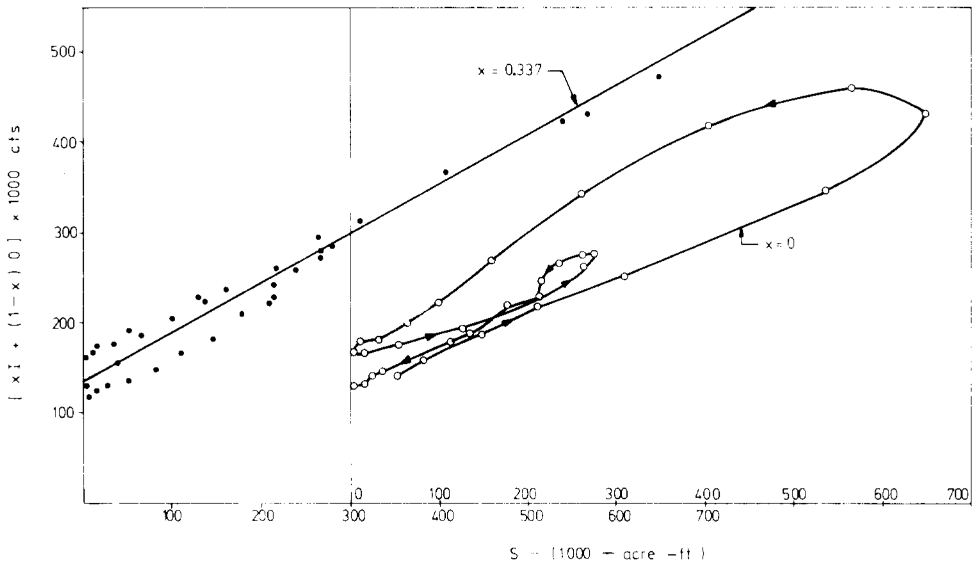

Fig.1. Plot of storage vs. weighted flow for the Sewickley—Wheeling reach of the Ohio river after Linsley et al. (1949).

prepared and a smooth curve should be drawn through the mean positions of the plotted data. The method of three points should be used for determining the constants α, m and σ. Select three suitable points on the curve covering the whole range of the data values. Let these points be designated:

$[S_1, (xI_1 + (1 - x)O_1)]$, $[S_2, (xI_2 + (1 - x)O_2)]$, and $[S_3, (xI_3 + (1 - x)O_3)]$

Introducing these values in eq.12

$$S_1 = \alpha[I_1 x + (1 - x)O_1]^m + \sigma \tag{13}$$

$$S_2 = \alpha[I_2 x + (1 - x)O_2]^m + \sigma \tag{14}$$

$$S_3 = \alpha[I_3 x + (1 - x)O_3]^m + \sigma \tag{15}$$

The following results can now be obtained from eqs.13—15.

$$\frac{\log\left(\dfrac{S_3 - \sigma}{S_2 - \sigma}\right)}{\log\left(\dfrac{S_2 - \sigma}{S_1 - \sigma}\right)} = \frac{\log\left(\dfrac{xI_3 + (1-x)O_3}{xI_2 + (1-x)O_2}\right)}{\log\left(\dfrac{xI_2 + (1-x)O_2}{xI_1 + (1-x)O_1}\right)} \tag{16}$$

$$\log\left(\frac{S_3 - \sigma}{S_2 - \sigma}\right) = m \log\left(\frac{xI_3 + (1-x)O_3}{xI_2 + (1-x)O_2}\right) \tag{17}$$

The coefficient σ can be determined from eq.16 by trial and error. The exponent m can then be determined from eq.17. The coefficient α can be determined using the calculated values of σ and m in any of the eqs.13—15.

The proposed method is now explained by applying it to a specific example. This example is taken from Wilson (1974). Using known data of I, O and S, eq.5 was first tried using eqs.7—9. The values of the constants were determined as $x = 0.254$, $k = 4.611$, and $\sigma = -102.640$. The units of I and O are in m^3/s while those of S are (1/4 day) m^3/s. The straight-line fit is shown in Fig.2. The value of x determined in three trials by Wilson is 0.25. An average line was then drawn by Wilson arbitrarily which gave a value of 6.0 for k.

However, it is apparent from Fig.2 that the plotted points of the actual data seem to lie on a curve rather than a straight line. It was therefore tried to fit a nonlinear equation of the type of eq.12. The values of α, m and σ were determined by the method outlined above and were found to be 0.010, 2.347 and −2.000, respectively. The empirically fitted curve is shown in Fig.3. The fit is quite satisfactory and is definitely an improvement on the straight-line fit of Fig.2.

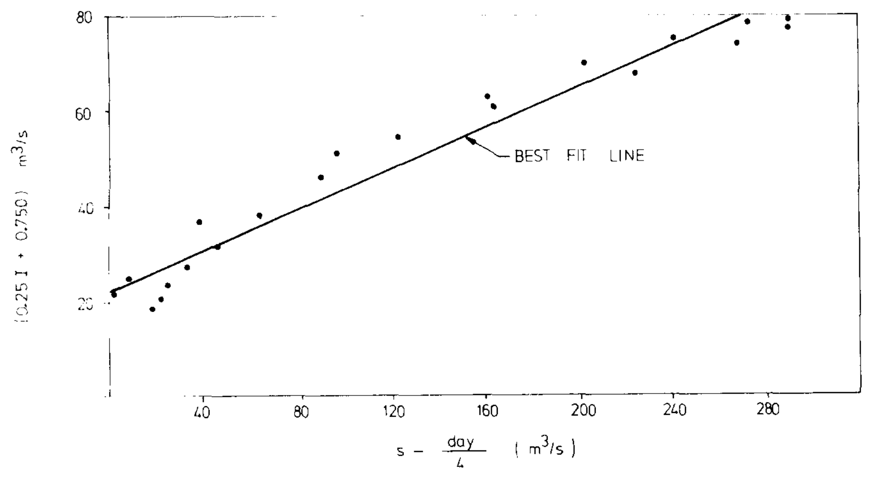

Fig.2. Fitting a straight line by LSM to the data from Wilson (1974).

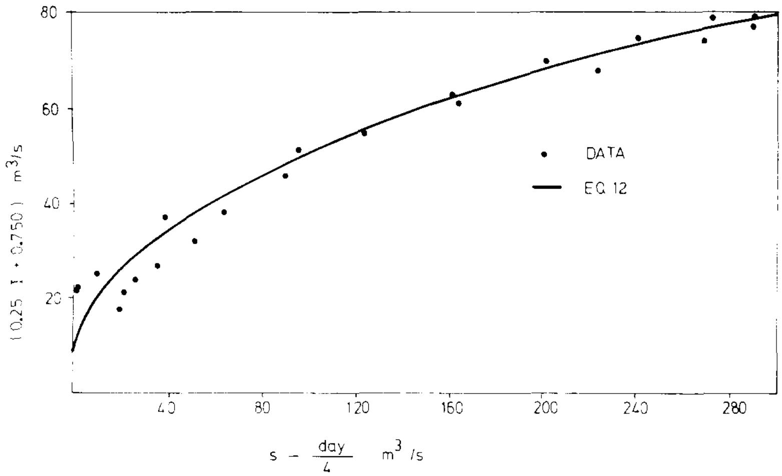

Fig.3. Fitting a curve to the data from Wilson (1974).

The value of x used for the nonlinear relationship is the same which gave the best straight-line fit. It may be argued that a more suitable value of x for the nonlinear relationship may be different from the one obtained for the straight-line fit. While this may be true, the same value was nonetheless used because it also agreed with the value used by Wilson which was obtained by him in three trials.

In addition to the weakness of the trial-and-error method required for determining the value of x in the above method, there is another objection which

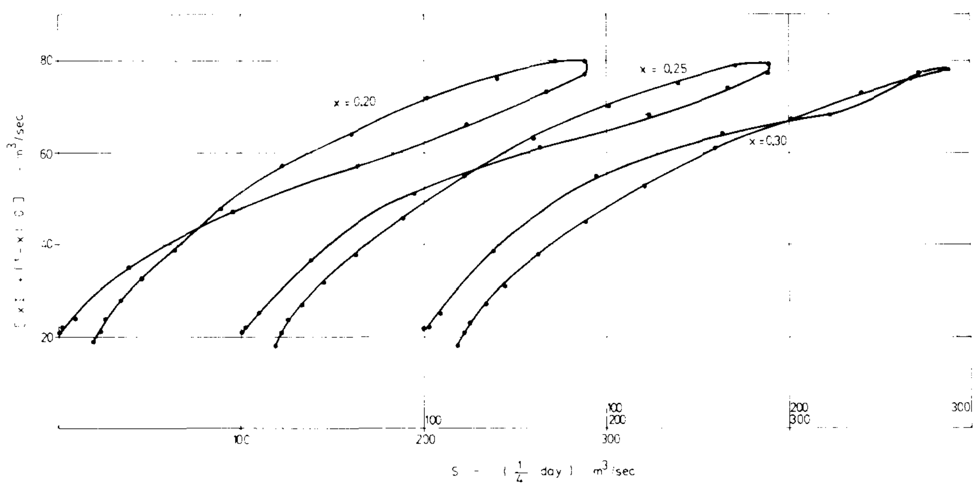

Fig.4. Plot of storage vs. weighted flow from Wilson (1974) to show the variability of the weighting factor, x.

can be raised against the proposed nonlinear relationship. Eq.12 takes care of the variability of k but assumes x to remain constant. This assumption is certainly not valid in several cases. The example considered above appears to be one of such cases. In Fig.4, storage vs. weighted flow diagram is plotted for three different values of x, after Wilson (1974). It may be observed that for $x = 0.2$, the lower loop is quite narrow while the upper loop is much wider. For $x = 0.3$, the upper loop is narrowed to almost a straight line while the lower loop is considerably widened. For $x = 0.25$, the middle portion of the curve is narrower than the upper and lower portions. It is obvious that x is really a variable in this case. Similar situation may exist in numerous other cases. A wholly satisfactory method should account for the continuous variation of x and k but unfortunately such an analytical method is still not available. An approximate method is proposed in the following.

SEGMENTED-CURVE METHOD

The whole available data are subdivided into several groups. Such groups can be formed on the basis of the storage values. It is then assumed that k and x remain constant in each subgroup but may vary from subgroup to subgroup. The normal equations, eqs.7—9, are used in each subgroup and the relevant values of k, x and σ are determined for each subgroup. With the known values of these constants, the straight lines can be drawn on a plot S vs. $[xI + (1-x)O]$ On this diagram, a segmented curve is obtained in place of a smooth curve. If the number of subgroups is increased, depending on the number of available data, the segmented curve can be sufficiently smoothened. This method may be preferable to using eq.12 because the method is free from the tedium of trial and error. It will be seen later that the computation of routing by the segmented-curve method (SCM) is also considerably easier than using eq.12, although care is required while moving from one segment of the curve into another. The method is now explained by solving the same example from Wilson which was used to illustrate the application of eq.12.

The data were divided into three subgroups. The calculated values of k, x and σ for each subgroup together with the ranges of storage values are given in Table I. The results are plotted in Fig.5.

TABLE I

Numerical values of the coefficients in the segmented curve method (SCM)

Subgroup	k (¼ day)	x	σ ((¼ day) m^3/s)	Range of storage ((¼ day) m^3/s)
I	2.717	0.080	−46.058	0—38
II	3.948	0.246	−90.432	45—163
III	8.228	0.293	−359.572	201—289

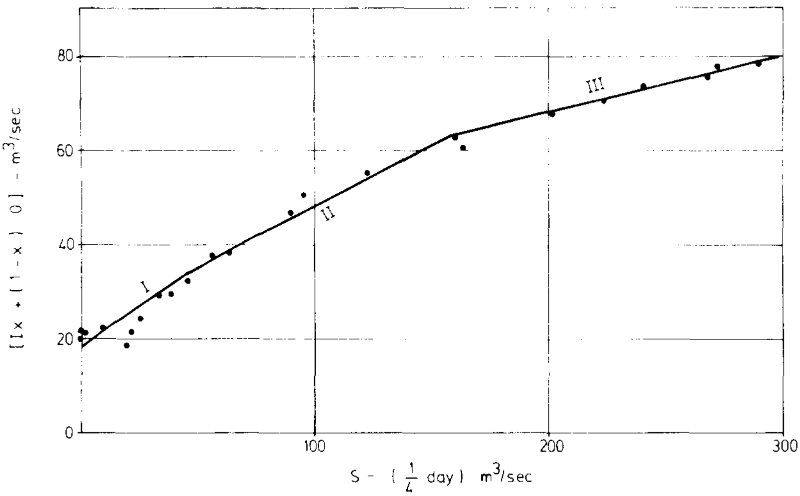

Fig.5. Fitting a segmented curve to the data from Wilson (1974) by LSM.

NUMERICAL EXAMPLE ON FLOOD ROUTING

Using the numerical values of the various coefficients determined in the first example from Wilson (1974), the inflow flood of the second example from Wilson is required to be routed. The flood will be routed by the three methods considered herein, namely: (1) using a linear relationship between storage and the weighted flow; (2) using nonlinear relationship of eq.12; and (3) using SCM. The results will then be compared with one another and with those of Wilson.

(1) Best-fit line. Using $x = 0.25$ and $k = 4.611$ in eq.1, the routing equation is written as:

$$O_2 = \frac{3.342\, I_1 - 1.342\, I_2 + 5.880\, O_1}{7.880} \tag{18}$$

The initial values I_1 and O_1 are known so that O_2 can be calculated. For the next calculation, the calculated value of O_2 becomes the initial value of O, so that the next value of O can be similarly computed. In this way the flood is completely routed. The results are given in Table II and are plotted in Fig.6.

(2) Eq.12. Substituting the calculated values of the coefficients α and m in eq.12 and using it together with eq.3, the following result is obtained:

$$\tfrac{1}{2}[(I_1 + I_2) - (O_1 + O_2)] + 0.010\,[0.25 I_1 + 0.75 O_1]^{2.347} =$$

$$0.010\,[0.25 I_2 + 0.75 O_2]^{2.347} \tag{19}$$

The values of I_1, I_2 and O_1, will be known for each step of routing, so that O_2 can be calculated by trial and error. The calculated results are given in Table II and are plotted in Fig.6.

(3) SCM. The following routing equations are obtained for each segment.

segment I: $$O_2 = (1.446I_1 + 0.554I_2 + 3.988O_1)/5.988 \tag{20}$$

segment II: $$O_2 = (2.942I_1 - 0.942I_2 + 4.954O_1)/6.954 \tag{21}$$

segment III: $$O_2 = (5.822I_1 - 3.822I_2 + 10.634O_1)/12.634 \tag{22}$$

These equations can be used within the respective segments. Some modification will be needed when the successive flows occur in two different segments. It is thus advisable to keep a record of the weighted flow values to determine the segment(s) within which the current-routing values occur. Although such a sophistication was not used in the calculations presented herein, it is easy to

TABLE II

Comparison of computed results of flood routing by various methods

No.	Time (h)	I (m^3/s)	O (m^3/s)			
			eq.12	SCM*[1]	BSLF*[2]	Wilson (1974)
1	0	31	31	31	31	31
2	6	50	34.9	32.8	27.8	27.2
3	12	86	30.8	32.8	27.3	24.6
4	18	123	37.8	43.1	35.9	29.5
5	24	145	48.7	49.1	54.2	43.8
6	30	150	61.8	62.8	76.4	63.0
7	36	144	74.5	78.4	96.1	81.6
8	42	128	86.7	93.6	111.0	97.3
9	48	113	95.2	103.6	117.9	106.4
10	54	95	101.8	110.5	119.7	111.4
11	60	79	105.2	112.9	116.2	111.3
12	66	65	105.9	111.8	109.1	107.6
13	72	55	103.6	107.4	99.7	101.1
14	78	46	99.7	101.8	89.9	93.7
15	84	40	93.7	94.8	79.7	85.3
16	90	35	87.1	87.6	70.5	77.3
17	96	31	79.3	80.5	62.2	69.6
18	102	27	70.8	73.9	54.9	62.7
19	108	25	61.5	60.7	48.2	56.0
20	114	24	52.1	50.6	42.5	50.0
21	120	23	43.3	43.0	38.0	45.0
22	126	22	35.8	37.4	34.3	40.8

*[1] Segmented-curve method.
*[2] Best straight-line fit.

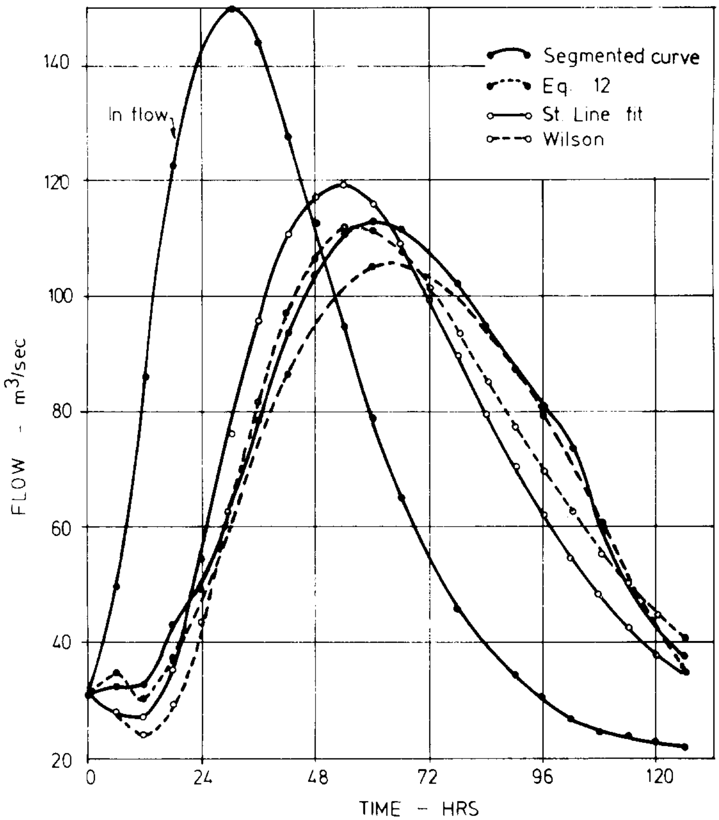

Fig.6. Routing of a flood from Wilson (1974) by the methods of linear LSM, nonlinear LSM (eq. 12), and SCM.

visualise that the need for it may frequently arise. The principle is therefore explained in general terms. Consider that the initial value within a routing calculation occurs in segment *II* (Fig.7) while the end-value occurs in segment *III*. In the routing equation, eq.3, clearly S_2 is to be evaluated by using the relevant Muskingum equation for segment *III* and S_1 from the corresponding equation in segment *II*.

The results of calculations are given in Table II and are plotted in Fig.6.

DISCUSSION OF RESULTS

The routing results by different methods are given in Table II together with Wilson's results and are plotted in Fig.6 for comparison. The results using eq.12 and SCM are identical. The largest discrepancy occurs near the peak which can be easily explained. The largest flood magnitude which was required to be routed was greater than the flood used in determining the numerical values of

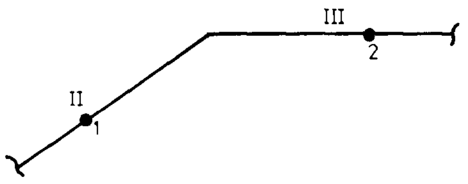

Fig.7. Sketch to show the use of SCM moving from one segment to the other.

the constants in the storage vs. weighted flow relationships. These relationships were therefore extrapolated for routing. The calculated discrepancy in the routed flood peaks is due to extrapolation. It is expected that the two nonlinear methods would function satisfactorily if the coefficients are determined from historic flood. The discrepancy between Wilson's and the best-fit straight-line results is larger. These computations can at best be regarded as approximate results.

SUMMARY AND CONCLUSIONS

The constants of the Muskingum equation can be deduced by LSM which obviates the trial and error and the element of subjectivity which is inherent in the traditional method. Two methods of developing nonlinear relationships between storage and the weighted flow are presented. Trial-and-error solution is required using eq.12 while SCM is a straightforward method which does not require any trial and error at any stage of the solution. The results from the two nonlinear methods were identical for the numerical example considered in the paper.

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