# Volume-Conservative Nonlinear Flood Routing

Roland K. Price[1]

**Abstract:** The nonlinear Muskingum and diffusion wave methods for flood routing are deduced from a conservative modified-advection equation as an approximation of the one-dimensional Saint-Venant equations. The modified-advection equation has the kinematic wave speed and attenuation parameter as functions of discharge, which can be derived for the particular averaged geometry for the cross section and hydraulic properties of the reach. The nonlinear Muskingum and diffusion wave methods also include any uniformly distributed time-dependent lateral inflow along the river and are valid for any Froude number. The methods are discussed in relation to corresponding methods derived recently for Muskingum routing and for diffusion wave routing. Numerical experiments on a synthetic river including extensive flood plains show that the proposed nonlinear Muskingum method is highly accurate compared with a full solution of the Saint-Venant equations. The deterioration of accuracy with decreasing bed gradient is highlighted for cases with and without lateral inflow.

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## Introduction

It has proven difficult to develop nonlinear versions of the Muskingum and diffusion wave (DW) methods for routing flows in rivers such that the parameters for the methods vary with discharge and volume is conserved. A number of papers have appeared in recent years that address this problem [see, for example, Ponce and Chaganti (1994); Cappelaere (1997); Perumal (1994); Todini (2007)]. The first and the last papers begin with the Muskingum method and attempt to develop nonlinear forms of the Muskingum parameters while acknowledging the insights of the one-dimensional (1D) Saint-Venant (SV) equations. Apollov et al. 1964 (see Perumal 1992) and Cunge (1969) related to the Muskingum parameters, assumed constant, to terms from the SV equations. Price (1973) developed a nonlinear Muskingum (NLM) method based on an advection-diffusion equation (which needed a downstream boundary condition), which was not, however, volume conservative. It introduced the nonlinear kinematic wave speed and attenuation parameter as functions of discharge. These two functions were derived from the normal-depth equation applied to free surface flow in a river reach with a uniform cross section and bed gradient. Subsequently Price (1985) developed a modified-advection equation (which does not need a downstream boundary condition) based on the SV equations under the strict assumption that the water surface gradient defined relative to the bed gradient is small (but not negligible). This equation also makes no assumption about the Froude number, $F$.

By assuming that the inertia terms in the SV equations can be ignored (equivalent to $F^2$ being small), Cappelaere (1997) developed an advection-diffusion equation (with a downstream boundary condition) as the basis of a DW method. By adopting a step fractioning technique with a characteristic solution for the advection equation and a Crank-Nicholson finite difference approach for the diffusion equation, he showed that his method is both accurate compared with a numerical solution of the SV equations and acceptably volume conservative. Analysis of the Cappelaere equations shows, however, that they are the equivalent of a version of the modified-advection equation of Price (1985), which includes the downstream boundary condition. This analysis is introduced below and carried out in Appendix I.

Starting with the basic Muskingum equations and comparing them with the SV equations, Todini (2007) developed a NLM method, which was demonstrated to be mass conservative. The analysis presented herein shows that NLM is in fact a solution of the modified-advection equation of Price (1985), which can therefore be regarded as the basis for both the NLM and DW methods.

Volume-conservative finite difference schemes for the modified-advection equation without and with a downstream boundary condition are used to demonstrate the effectiveness of the NLM and DW methods. Numerical experiments with the methods applied to a synthetic river with a complex cross section including flood plains confirm that the methods are highly accurate when compared with a full solution of the SV equations for cases without and with lateral inflow. The degradation of accuracy of the methods with bed gradient, $s_0$, is derived, providing guidelines on their application. However, the tests highlight the potential for violating the 1D assumption on the recession of the flood hydrograph when using these and similar methods if there is extensive inundation of associated flood plains.

[1]Emeritus Professor, Dept. of Hydroinformatics and Knowledge Management, UNESCO-IHE, P.O. Box 3015, 2601 DA, Delft, The Netherlands. E-mail: r.price@unesco-ihe.org

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## Nonlinear Modified-Advection Equations for Flood Routing

Consider a reach of river with a uniform but arbitrary cross section, which can include a main channel with associated flood plains and a uniform bed gradient, $s_0$, and boundary roughness. We follow Apollov et al. (1964) and Cunge (1969) by seeking an approximation to the SV equations such that the surface gradient, $\partial y/\partial x$, of the flow defined relative to $s_0$ is small (but not negligible)

$$\varepsilon = \max \text{ or } \min\left(\frac{1}{s_0}\frac{\partial y}{\partial x}\right) \text{such that } |\varepsilon| \ll 1 \tag{1}$$

where $\varepsilon$=surface slope number, $y$=depth (m), and $x$=distance from the upstream boundary (m). Using this assumption (while not making any equivalent assumption about $F$), Price (1985) showed that

$$\frac{\partial Q}{\partial t} + c_0\frac{\partial Q}{\partial x} + c_0\frac{\partial}{\partial t}\left(\frac{a_0}{c_0^2}\frac{\partial Q}{\partial x}\right) = c_0 q - c_0\frac{\partial}{\partial t}\left(\frac{Qq}{2gAs_0}\right) \tag{2}$$

where $Q$=discharge ($m^3/s$), $c_0(Q)$=kinematic wave speed (m/s), $a_0(Q)$=attenuation parameter ($m^2/s$), $q$=uniform lateral inflow ($m^2/s$), $g$=acceleration due to gravity ($m/s^2$), $A(Q)$=wetted cross-sectional area in the channel at normal depth ($m^2$), and $t$ =time variable (s). The parameters, $c_0(Q)$ and $a_0(Q)$, are both single valued continuous functions of discharge and are independent of time or space derivatives of $Q$ and are not necessarily monotonically increasing functions of $Q$. They are evaluated for normal-depth flow

$$c_0(Q) = \frac{dQ}{dA} \tag{3}$$

$$a_0(Q) = \frac{Q}{2s_0B}\left\{1 - \frac{B}{gA}\left(c_0 - \frac{Q}{A}\right)^2\right\} \tag{4}$$

where $B(Q)$=surface width at normal-depth flow (m). The development of Eq. (2) is based on the original SV equations using perturbation theory and does not follow the approach of Cunge (1969) in matching numerical with physical diffusion, though the second-order derivative term in Eq. (2) acts as a diffusion term.

Eq. (2) is hyperbolic, has a single characteristic with a positive gradient, and has a first-order derivative in $x$. It therefore requires only one upstream space boundary condition. Consequently, the equation is incapable of taking downstream influences into account. This is a result of the flow being at normal depth to zeroth order, no matter what the value of $F$. The second-order derivative term in Eq. (2) can be modified by replacing $\partial Q/\partial t$ with $c_0 q - c_0\partial Q/\partial x + O(\varepsilon)$ to give

$$\frac{\partial Q}{\partial t} + c_0\frac{\partial Q}{\partial x} - c_0\frac{\partial}{\partial x}\left(\frac{a_0}{c_0}\frac{\partial Q}{\partial x}\right) = c_0 q\left[1 - \frac{\partial}{\partial x}\left(\frac{a_0}{c_0}\right)\right] - c_0\frac{\partial}{\partial t}\left(\frac{Qq}{2gAs_0}\right) \tag{5}$$

correct to $O(\varepsilon^2)$. This is the basic equation for the DW method, which does require a second space boundary condition downstream. Cappelaere (1997) deduced a similar equation by first assuming that the inertia terms in the SV equations are negligible (and therefore $F^2$ is small). Cappelaere's advection-diffusion equation appears to include all of the physics of this noninertial approximation. To obtain an advection-diffusion equation that is a function of $Q$ alone the so-called correction factor $[1-(1/s_0)\,\partial y/\partial x]^{1/2}$ is modified by replacing $\partial y/\partial x$ with $(1/Bc)\,\partial Q/\partial x$. This assumes that $\varepsilon \ll 1$, which means that the correction factor is approximately unity. Using perturbation theory it can be shown (Appendix I) that to first order in $\varepsilon$, Cappelaere's equation is equivalent to Eq. (5).

In order to conserve volume with finite-difference representations of Eqs. (2) and (5), the equations are rewritten in conservative form

$$\frac{\partial}{\partial t}\int_0^Q \frac{dQ'}{c_0(Q')} + \frac{\partial Q}{\partial x} + \frac{\partial}{\partial t}\left[\frac{a_0(Q)}{c_0^2(Q)}\frac{\partial Q}{\partial x}\right] = q - \frac{\partial}{\partial t}\left(\frac{Qq}{2gAs_0}\right) \tag{6}$$

From Eq. (3)

$$\int_0^Q \frac{dQ'}{c_0(Q')} = A(Q) = \frac{Q}{v(Q)} \tag{7}$$

where $v(Q)$=velocity at normal depth (m/s). Eq. (6) can therefore be written as

$$\frac{\partial}{\partial t}\left(\frac{Q}{v}\right) + \frac{\partial Q}{\partial x} + \frac{\partial}{\partial t}\left(\frac{a_0}{c_0^2}\frac{\partial Q}{\partial x}\right) = q - \frac{\partial}{\partial t}\left(\frac{Qq}{2gAs_0}\right) \tag{8}$$

Adopting the minimum of four grid points in space-time and the midpoint rule to conserve volume (that is, to conserve the fluxes into the grid cell $(j\Delta x, n\Delta t), [(j+1)\Delta x, n\Delta t], [j\Delta x, (n+1)\Delta t], [(j+1)\Delta x, (n+1)\Delta t]$) yields

$$C_1Q_{j+1}^{n+1} + C_2Q_j^{n+1} + C_3Q_{j+1}^n + C_4Q_j^n + \tfrac{1}{2}C_0(q^{n+1}+q^n) = 0 \tag{9}$$

where

$$C_0 = -2\Delta t$$

$$C_1 = \frac{1}{v_{j+1}^{n+1}} + \frac{\Delta t}{\Delta x} + \frac{2}{\Delta x}\left(\frac{a_0}{c_0^2}\right)_{j+(1/2)}^{n+1} + \frac{q^{n+1}}{2gs_0A_{j+1}^{n+1}}$$

$$C_2 = \frac{1}{v_j^{n+1}} - \frac{\Delta t}{\Delta x} - \frac{2}{\Delta x}\left(\frac{a_0}{c_0^2}\right)_{j+(1/2)}^{n+1} + \frac{q^{n+1}}{2gs_0A_j^{n+1}}$$

$$C_3 = \frac{-1}{v_{j+1}^n} + \frac{\Delta t}{\Delta x} - \frac{2}{\Delta x}\left(\frac{a_0}{c_0^2}\right)_{j+(1/2)}^{n} - \frac{q^n}{2gs_0A_{j+1}^n}$$

$$C_4 = \frac{-1}{v_j^n} - \frac{\Delta t}{\Delta x} + \frac{2}{\Delta x}\left(\frac{a_0}{c_0^2}\right)_{j+(1/2)}^{n} - \frac{q^n}{2gs_0A_j^n} \tag{10}$$

These are similar to the equations derived by Price (1985) but differ in the inclusion of $v$. Here the suffix $n+(1/2)$ denotes a function evaluated for values of $Q$ as an average of values at times $n\Delta t$ and $(n+1)\Delta t$; the subscript $j+(1/2)$ is defined similarly. Eq. (9) can be solved for $Q_{j+1}^{n+1}$ using the iterative Newton-Raphson procedure.

The truncation error for the finite-difference equation is second order in $\Delta x$ and $\Delta t$ (see Appendix II). Because the difference scheme is centered, there is no first-order numerical diffusion term to control any numerical instability induced by the advection term. This device is not needed in Eq. (8) because the second-order derivative term provides sufficient (physically based) diffusion to control any instability. What is important, however, is that the second-order truncation error is significantly smaller than the order of the second-order derivative term, that is, $\Delta x \ll L\varepsilon^{1/2}$ and $\Delta t \ll T\varepsilon^{1/2}$, where $L$=length scale of flood disturbance (m) and $T$=time scale of flood disturbance (s). This is generally the case for floods in rivers with $s_0 < 0.002$.



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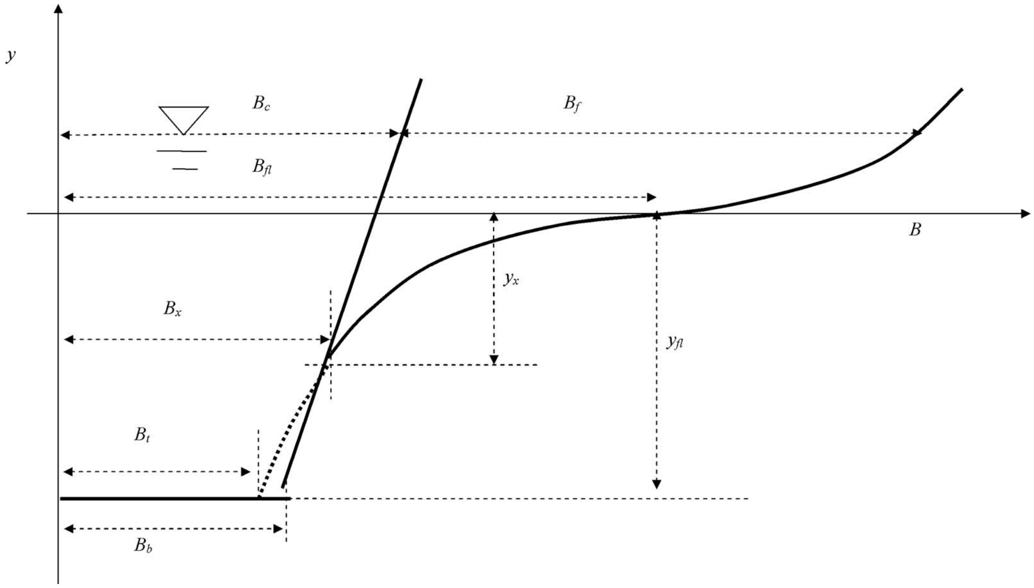


**Fig. 1.** Semicross section for synthetic river

Eq. (8) can now be rewritten as

$$\frac{\partial}{\partial t}\left[\frac{Q}{v}+\frac{a_0(Q)}{c_0^2(Q)}\frac{\partial Q}{\partial x}+\frac{Qq}{2gAs_0}\right]+\frac{\partial Q}{\partial x}=q \tag{11}$$

such that the storage, $S$, can be defined as

$$S=\int_0^x\left[\frac{Q}{v}+\frac{a_0(Q)}{c_0^2(Q)}\frac{\partial Q}{\partial x}+\frac{Qq}{2gAs_0}\right]dx \tag{12}$$

Using the midpoint rule again, together with the trapezium rule for the integral

$$S_{j+(1/2)}^n=\frac{\Delta x}{v_{j+(1/2)}^n}\left\{Q_j^n\left[\frac{1}{2}-\frac{1}{\Delta x}\left(\frac{a_0 v}{c_0^2}\right)_{j+(1/2)}^n+\frac{v_{j+(1/2)}^n q^n}{4gs_0A_j^n}\right]+Q_{j+1}^n\left[\frac{1}{2}+\frac{1}{\Delta x}\left(\frac{a_0 v}{c_0^2}\right)_{j+(1/2)}^n+\frac{v_{j+(1/2)}^n q^n}{4gs_0A_{j+1}^n}\right]\right\} \tag{13}$$

Comparing this expression for $S_{j+(1/2)}^n$ with the traditional Muskingum formulation [see, for example, Todini (2007)], the NLM parameters become

$$K_{j+(1/2)}^n=\frac{\Delta x}{v_{j+(1/2)}^n} \tag{14}$$

$$\theta_{j+(1/2)}^n=\frac{1}{2}-\frac{1}{\Delta x}\left(\frac{a_0 v}{c_0^2}\right)_{j+(1/2)}^n \tag{15}$$

$$\phi_j^n=\frac{v_{j+(1/2)}^n q^n}{4gs_0A_j^n} \tag{16}$$

such that

$$S_{j+(1/2)}^n=K_{j+(1/2)}^n\{Q_j^n[\theta_{j+(1/2)}^n+\phi_j^n]+Q_{j+1}^n[1-\theta_{j+(1/2)}^n+\phi_{j+1}^n]\} \tag{17}$$

This NLM method is volume conservative. It includes a contribution from the inertia terms in the SV equations and from a uniform time-dependent lateral inflow. The method of Todini (2007) is essentially the same as the finite-difference formulation in Eqs. (9) and (10), except that Todini does not include allowance for the inertia terms or any lateral inflow (see Appendix III). Todini's method is therefore a numerical solution of the analytical Eq. (8) (with $q=0$).

## Test of the Nonlinear Routing Methods for a Synthetic River

A straightforward and rigorous test of the NLM and DW methods is to reproduce floods without and with lateral inflow in a synthetic river and to compare them with the corresponding flood flows generated by a full solution of the SV equations.

The synthetic river consists of a reach 100 km long with a uniform (semi) cross section, as shown in Fig. 1. The main channel is trapezoidal with semibed width $B_b$, side slope $s$, and semisurface width $B_c$. There are associated berms or flood plains to the left and right of the channel. See Appendix IV for the formulas that define the synthetic channel.

We set $B_t$=18.0 m, $B_b$=20.0 m, $B_{\mathrm{fl}}$=150.0 m, $y_{\mathrm{fl}}$=5.0 m, and $s$=0.5. The Manning coefficient for the channel is set at 0.035 s/m$^{1/3}$ and for the flood plain at 0.075 s/m$^{1/3}$. The bed gradient, $s_0$, takes values in the range 0.0001–0.001. Fig. 2 shows

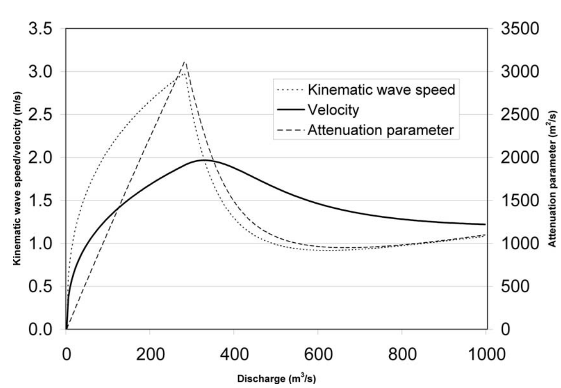


**Fig. 2.** $c_0$, $v$, and $a_0$ for the synthetic river with $s_0$=0.001

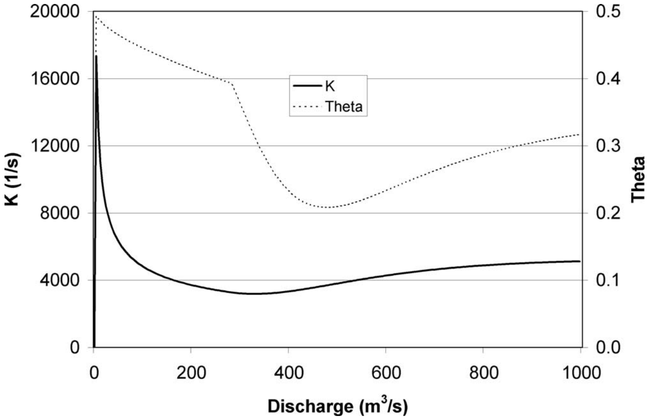


**Fig. 3.** Variation of the NLM parameters for the synthetic river with $s_0=0.001$

the resulting curves for $c_0$, $v$, and $a_0$ for $s_0=0.001$. The curve for $c_0$ has a well-defined peak and a pronounced decrease as $Q$ increases above bank. As might be expected the curve for $v$ is considerably smoother. The corresponding functions for $K(Q)$ and $\theta(Q)$ are shown in Fig. 3. The highly nonlinear forms for $c_0$ and $a_0$, reflecting significant overbank flows, are a much more severe test of the accuracy of the routing methods than those considered by most writers [with the exception of Cappelaere (1997)] in testing their methods.

The upstream flood event and uniform lateral inflow for the synthetic reach are defined by the formulas in Appendix IV with $Q_{\text{base}}=100.0\ \text{m}^3/\text{s}$, $Q_{\text{peak}}=800.0\ \text{m}^3/\text{s}$, $T_Q=24.0\ \text{h}$, and $\beta_Q=5.0$. Similarly, $q_{\text{base}}=0.0002\ \text{m}^2/\text{s}$, $q_{\text{peak}}=0.05\ \text{m}^2/\text{s}$, $T_q=6.0\ \text{h}$, $t_q=36.0\ \text{h}$, and $\beta_q=1.0$.

Given the geometry of the reach and the upstream discharge hydrograph, the discharges at the downstream boundary are generated using a four-point implicit and iterative (nonlinear) finite-difference solution of the SV equations. The downstream boundary condition imposed is the normal-depth curve relating the discharge to the depth for the uniform cross section and bed gradient, adjusted according to the Jones' formula (Jones 1916). The space and time increments in the calculations with the SV equations are 1,000 m and 225 s, respectively. Tables for $A(y)$ and $B(y)$ were generated at 0.001 m increments in $y$ for the parameterized cross section. In turn, we derived tables for $A(Q)$ and $B(Q)$ by interpolation on the tables for $A(y)$ and $B(y)$. These tables are used to define tables for $c_0$ and $a_0$. The same is done for the conveyance, $F(y)$, defined by

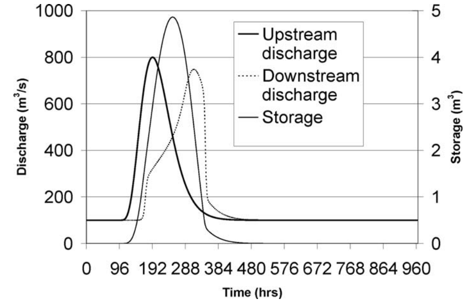


**Fig. 4.** Upstream and downstream discharge hydrographs for the synthetic river ($q=0$)

$$F(y) = Q/s_0^{1/2} \tag{18}$$

Correspondingly, we generate numerical solutions of the NLM method based on Eq. (10) and of the DW method. The latter uses a time and space centered Crank-Nicholson finite-difference scheme with six grid points and a free downstream boundary condition at an extended distance of 200 km. The resulting nonlinear equations are solved using Newton-Raphson and double sweep algorithms. We maintain the accuracy of the numerical solutions by satisfying the Courant condition

$$\frac{\Delta x}{\Delta t} \approx \overline{c_0} \tag{19}$$

where $\overline{c_0}$=average value of the kinematic wave speed.

## Simulations for Synthetic Channel

The upstream and downstream hydrographs for the river with $s_0=0.001$ with $q=0$ are shown in Fig. 4. The corresponding hydrographs generated by the approximate solutions are too close to be distinguished graphically. Table 1 shows the Nash-Sutcliffe (NS) performance criterion (Nash and Sutcliffe 1970) for the downstream hydrographs defined by the solutions of the SV equations and the NLM method for different experiments due to changes in the bed gradient. The NS is defined by

**Table 1.** Results with NLMC for the Synthetic River with Different Bed Gradients ($q=0$)

| Bed gradient | NS | Downstream peak discharge (SV) ($\text{m}^3/\text{s}$) | Peak discharge error (%) | Volume error (%) | Surface gradient number (maximum) | Surface gradient number (minimum) |
|---|---|---|---|---|---|---|
| 0.0001 | 0.96388 | 326.143 | −9.27330 | −0.01209 | 1.47610 | −0.30911 |
| 0.0002 | 0.99145 | 463.057 | −3.52270 | 0.00006 | 0.74885 | −0.19589 |
| 0.0003 | 0.99652 | 553.852 | −1.28380 | 0.00009 | 0.92026 | −0.14529 |
| 0.0004 | 0.99807 | 615.035 | −0.38156 | 0.00005 | 0.89215 | −0.11064 |
| 0.0005 | 0.99873 | 657.449 | −0.00408 | 0.00009 | 0.82489 | −0.08548 |
| 0.0006 | 0.99906 | 687.568 | 0.15837 | 0.00057 | 0.75846 | −0.07773 |
| 0.0007 | 0.99921 | 709.603 | 0.21722 | −0.00095 | 0.70971 | −0.07439 |
| 0.0008 | 0.99943 | 726.001 | 0.23695 | −0.00029 | 0.55195 | −0.06950 |
| 0.0009 | 0.99950 | 738.447 | 0.23616 | 0.00085 | 0.50706 | −0.06481 |
| 0.001 | 0.99947 | 748.152 | 0.21679 | −0.00074 | 0.48451 | −0.12693 |



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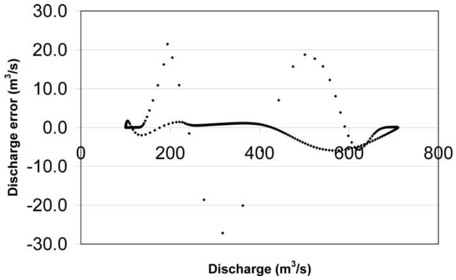


**Fig. 5.** Errors in the predicted downstream discharges with the NLM method plotted against the corresponding SV discharges for $s_0 = 0.0007$ ($q = 0$)

$$\mathrm{NS} = 1 - \frac{\sum_1^N (Q_{dn}^n - Q_J^n)^2}{\sum_1^N (Q_{dn}^n - \bar{Q}_{dn})^2} \tag{20}$$

where $Q_{dn}^n$ and $Q_J^n$ = solutions for the downstream discharge from the SV and NLM equations, respectively, and $\bar{Q}_{dn}$ = average downstream discharge for the SV solution. The NS is sensitive to any lag even if the shapes of the hydrographs are in agreement. In these experiments the standard time and space steps in the NLM and DW solutions are 1,800 s and 2,000 m, respectively.

The volume error at the downstream boundary is very small in all cases. The volume stored at any instant along the reach is given by

$$S^{n+1} = \sum_1^{J-1} S_{j+(1/2)}^{n+1}$$

This is also evaluated for each case. In all cases the discharge returns to its initial value along the reach, which implies that there is no residual volume remaining. The errors in the predicted $Q$ are plotted against the $Q$ generated by the SV solution for $s_0 = 0.000\,7$ (see Fig. 5). This reveals a complex hysteresis in which there is a very small overall loss in volume on the rise in the hydrograph while the gains and losses in the recession are much more significant. This behavior is attributed to the considerable steepening of the hydrograph on the recession, which is reflected in the minimum (negative) and maximum (positive) values of $\varepsilon$ recorded for each run. Here the negative value refers to the rise in the hydrograph and the positive to the recession. The fact that the positive value of $\varepsilon$ is considerably greater than the negative value is due to the comparatively short duration of the hydrograph and the extensive nature of the flood plains in the example. Consequently, there is considerable potential in this case for a breakdown in the 1D assumption in which the water surface across a section is assumed to be continuous and horizontal. Such a possibility is one of the dangers of using the NLM (and DW) method for a river with extensive flood plains, though the same problem arises when using the SV equations for this situation. Strictly, a two-dimensional model should be used in such circumstances.

As can be expected, the volume error with the NLM method deteriorates as $s_0$ becomes smaller. This trend, together with the observation that the channel dimensions are likely to be larger with a smaller $s_0$ with a similar flood regime, indicates that, in practice, the error in the peak discharge is accentuated as $s_0$ becomes smaller.

To complete the analysis of the nonlinear model (for $q = 0$), experiments were made by changing the space step while keeping the time step constant. Again, the volume error (Fig. 6) is very small. The optimal space step is on the order of 6,250 m for a time step of 1,800 s. It is clear that the solution deteriorates very strongly if the space step is too large but there is considerable scope for varying the space step such that the Courant condition is approximately satisfied.

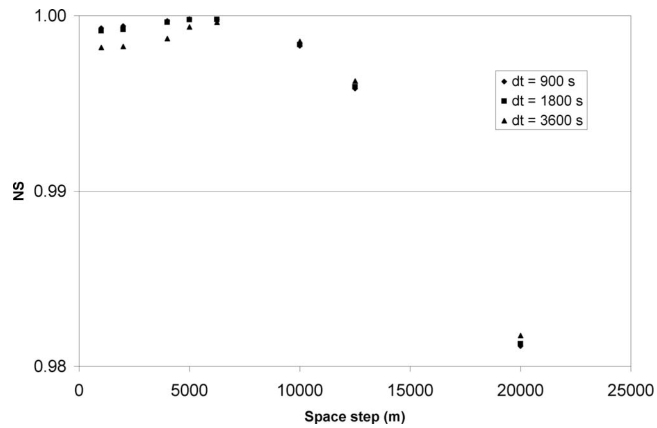


**Fig. 6.** Dependence of NS on space step for $s_0 = 0.0007$ ($q = 0$)

The depth $y$ can be calculated from the normal-depth relationship for the discharge $Q + (a_0/c_0)\,\partial Q/\partial x$. Fig. 7 shows the loop rating curve for the event with $s_0 = 0.000\,2$.

Experiments carried out with $q \neq 0$ are summarized in Table 2. The downstream hydrograph predicted by SV for $s_0 = 0.001$ is shown in Fig. 8. Again, the accuracy of the NLM method is very good, indicating that this numerical solution is very reliable and can be used with confidence under conditions that do not violate its basic assumptions.

Finally, results with the DW method for $q = 0$ are shown in Table 3. As with the NLM method, the NS values indicate that the DW is of similar accuracy, though the accuracy does not deteriorate with the decrease in $s_0$. This is probably due to the better accommodation of the steepness of the recession of the hydrograph by the DW method. The results from the DW method with $q \neq 0$ are not reported here but are similar to those for the NLM method.

## Conclusions

Reliable nonlinear and volume-conservative versions of the Muskingum and DW methods for routing flood flows in rivers can be based on an analytical modified-advection equation, first derived by Price (1985) as an approximation to the SV equations.

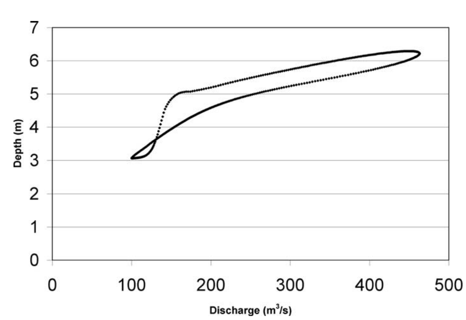


**Fig. 7.** Loop rating curve for $s_0 = 0.0002$ ($q = 0$)



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**Table 2.** Results with NLMC for the Synthetic River with Different Bed Gradients ($q \neq 0$)

| Bed gradient | NS | Downstream peak discharge (SV) ($m^3/s$) | Peak discharge error (%) | Volume error (%) | Surface gradient number (maximum) | Surface gradient number (minimum) |
|---|---|---|---|---|---|---|
| 0.0001 | 0.97808 | 362.745 | −8.39910 | −0.01034 | 1.62620 | −0.22066 |
| 0.0002 | 0.99545 | 548.820 | −3.01310 | −0.00727 | 0.74340 | −0.15746 |
| 0.0003 | 0.99818 | 649.719 | −1.09870 | −0.00727 | 0.90529 | −0.11753 |
| 0.0004 | 0.99895 | 718.572 | −0.35099 | −0.00740 | 0.90995 | −0.08875 |
| 0.0005 | 0.99930 | 767.336 | −0.03804 | −0.00775 | 0.80338 | −0.06779 |
| 0.0006 | 0.99945 | 803.107 | 0.08303 | −0.00713 | 0.78672 | −0.05215 |
| 0.0007 | 0.99954 | 830.159 | 0.13480 | −0.00722 | 0.73052 | −0.04047 |
| 0.0008 | 0.99963 | 851.168 | 0.17307 | −0.00659 | 0.62915 | −0.03165 |
| 0.0009 | 0.99971 | 868.098 | 0.16589 | −0.00891 | 0.53644 | −0.07619 |
| 0.001 | 0.99971 | 881.830 | 0.17181 | −0.00659 | 0.49644 | −0.02433 |

The approximation assumes the water surface gradient defined relative to the bed gradient is small (but not negligible) compared to the bed gradient. The resulting NLM and DW methods also include an allowance for the inertia terms in the SV equations, with no assumption made about the magnitude of the Froude number. Any time-dependent lateral inflow that is uniformly distributed along the river reach is formally included in both methods.

It is shown that the DW equation of Cappelaere (1997) is of the same order and equivalent to the modified-advection equation of Price (1985). Similarly, the MTC method of Todini (2007) is also a solution of this equation.

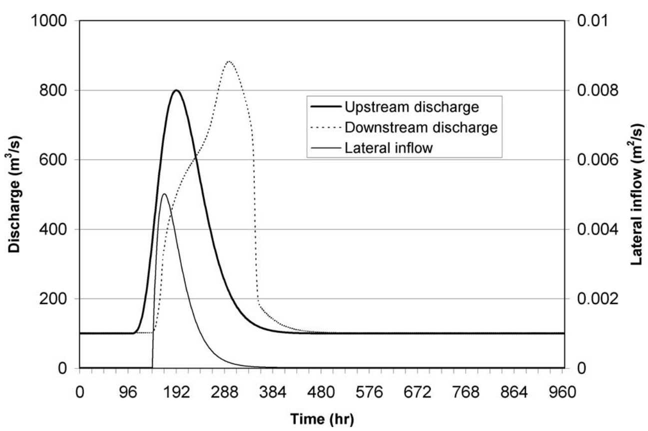


**Fig. 8.** Upstream and downstream discharge hydrographs for the synthetic river ($q \neq 0$)

Applying the NLM method to a synthetic river with a compound cross section that has a highly nonlinear kinematic wave speed curve accurately reproduces the peak discharges and volumes of the downstream discharge hydrographs both without and with a dynamically varying uniform lateral inflow. There is a degradation of the resulting percentage errors with decreasing bed gradient, as would be expected from the initial basic assumption involved in the derivation of the nonlinear routing equation. However, the numerical experiments confirm the validity of the resulting method. Similar conclusions can be made for the nonlinear DW method.

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## Appendix I. Equivalence of Cappelaere's Diffusion Wave Equation and the Advection-Diffusion Equation

Cappelaere (1997) formulated his DW equation as follows (using the notation of this note):

$$\frac{\partial Q}{\partial t} + c\left(y, \frac{\partial y}{\partial x}\right)\frac{\partial Q}{\partial x} - a\left(y, \frac{\partial y}{\partial x}\right)\frac{\partial^2 Q}{\partial x^2} = c\left(y, \frac{\partial y}{\partial x}\right)q \qquad (21a)$$

**Table 3.** Results with NLDW for the Synthetic River with Different Bed Gradients ($q=0$)

| Bed gradient | NS | Downstream peak discharge (SV) ($m^3/s$) | Peak discharge error (%) | Volume error (%) | Surface gradient number (maximum) | Surface gradient number (minimum) |
|---|---|---|---|---|---|---|
| 0.0001 | 0.99445 | 326.143 | 1.13700 | 0.53941 | 1.54470 | −0.48757 |
| 0.0002 | 0.99961 | 463.057 | −0.21111 | 0.06341 | 0.71293 | −0.25258 |
| 0.0003 | 0.99985 | 553.852 | −0.21674 | −0.00815 | 0.51295 | −0.16854 |
| 0.0004 | 0.99986 | 615.035 | −0.09047 | −0.01588 | 0.54242 | −0.12132 |
| 0.0005 | 0.99986 | 657.449 | 0.02819 | −0.01317 | 0.54404 | −0.09052 |
| 0.0006 | 0.99985 | 687.568 | 0.12335 | −0.00927 | 0.53361 | −0.07956 |
| 0.0007 | 0.99982 | 709.603 | 0.15637 | −0.00812 | 0.49929 | −0.07596 |
| 0.0008 | 0.99980 | 726.001 | 0.20316 | −0.00396 | 0.47459 | −0.07032 |
| 0.0009 | 0.99972 | 738.447 | 0.21844 | −0.00299 | 0.47770 | −0.06590 |
| 0.001 | 0.99964 | 748.152 | 0.21280 | −0.00296 | 0.46383 | −0.05987 |




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$$c=\frac{\bar{c}(y)}{2}\left\{\mathrm{COR}\left[1+\frac{\bar{Q}}{\bar{a}(y)}\frac{d\bar{a}(y)}{d\bar{Q}}\right]+\frac{1}{\mathrm{COR}}\left[1-\frac{\bar{Q}}{\bar{a}(y)}\frac{d\bar{a}(y)}{d\bar{Q}}\right]\right\} \quad (21b)$$

$$\bar{Q}(y)=\frac{Q}{\mathrm{COR}} \quad (21c)$$

$$a=\frac{\bar{a}(y)}{\mathrm{COR}}=\frac{\bar{Q}}{2Bs_0\mathrm{COR}} \quad (21d)$$

$$\mathrm{COR}=\sqrt{1-\frac{1}{s_0}\frac{\partial y}{\partial x}} \quad (21e)$$

where COR=correction factor. The term $\partial y/\partial x$ in the expression for the correction factor needs to be replaced by a term involving $Q$. Cappelaere uses

$$\frac{1}{s_0}\frac{\partial y}{\partial x}\approx\frac{2\bar{a}}{\bar{c}Q}\frac{\partial Q}{\partial x} \quad (22)$$

which arises from the differentiation in Eq. (21$c$) with respect to $x$ under the assumption that COR$\sim$1, or $\varepsilon \ll 1$. Rewriting Eq. (21$e$) including $\varepsilon$ to indicate the first-order term, then

$$\mathrm{COR}=\left(1-\frac{\varepsilon}{2s_0}\frac{\partial y}{\partial x}\right)+O(\varepsilon^2)=\left(1-\frac{\varepsilon\bar{a}}{\bar{c}\bar{Q}}\frac{\partial Q}{\partial x}\right)+O(\varepsilon^2) \quad (23)$$

Eq. (21$c$) can now be rewritten as

$$\bar{Q}(y)=Q\left(1+\frac{\varepsilon\bar{a}}{\bar{c}\bar{Q}}\frac{\partial Q}{\partial x}\right)+O(\varepsilon^2) \quad (24)$$

From Eq. (24), $y$=function of the expression on the right hand side of the equation. Therefore, Eq. (21$b$) becomes

$$c=\frac{\bar{c}(y)}{2}\left\{\left(1-\frac{\varepsilon\bar{a}}{\bar{c}\bar{Q}}\frac{\partial Q}{\partial x}\right)\left[1+\frac{\bar{Q}}{\bar{a}(y)}\frac{d\bar{a}(y)}{d\bar{Q}}\right]+\left(1+\frac{\varepsilon\bar{a}}{\bar{c}\bar{Q}}\frac{\partial Q}{\partial x}\right)\left[1-\frac{\bar{Q}}{\bar{a}(y)}\frac{d\bar{a}(y)}{d\bar{Q}}\right]\right\}+O(\varepsilon^2) \quad (25)$$

or

$$c=c_0(Q)\left[1+\frac{\varepsilon a_0}{c_0^2}\frac{dc_0}{dQ}\frac{\partial Q}{\partial x}\right]\left[1-\frac{\varepsilon}{c_0}\frac{da_0}{dQ}\frac{\partial Q}{\partial x}\right]+O(\varepsilon^2)$$
$$=c_0\left[1-\varepsilon\left(\frac{1}{c_0}\frac{da_0}{dQ}-\frac{a_0}{c_0^2}\frac{dc_0}{dQ}\right)\frac{\partial Q}{\partial x}\right]+O(\varepsilon^2) \quad (26)$$

and Eq. (21$d$) becomes

$$a=\bar{a}(y)+O(\varepsilon)=\frac{Q}{2B(Q)s_0}+O(\varepsilon) \quad (27)$$

It is comparatively straightforward to show that

$$\frac{\partial Q}{\partial t}+c\frac{\partial Q}{\partial x}-\varepsilon a\frac{\partial^2 Q}{\partial x^2}=cq \quad (28)$$

with Eqs. (26) and (27) give

$$\frac{\partial Q}{\partial t}+c_0\frac{\partial Q}{\partial x}-\varepsilon c_0\frac{\partial}{\partial x}\left(\frac{a_0}{c_0}\frac{\partial Q}{\partial x}\right)=c_0q\left[1-\frac{\partial}{\partial x}\left(\frac{a_0}{c_0}\right)\right]+O(\varepsilon^2) \quad (29)$$

## Appendix II. Truncation Error of the Nonlinear Muskingum Method

The truncation error takes the form

$$\frac{\Delta t^2}{24}\frac{\partial^3}{\partial t^3}\left(\frac{Q}{v}\right)+\frac{\Delta x^2}{8}\frac{\partial^3}{\partial t\,\partial x^2}\left(\frac{Q}{v}\right)+\frac{\Delta x^2}{24}\frac{\partial^3 Q}{\partial x^3}+\frac{\Delta t^2}{8}\frac{\partial^3 Q}{\partial x\,\partial t^2}$$
$$+\frac{\Delta x^2}{24}\frac{\partial}{\partial t}\left(\frac{a}{c^2}\right)\frac{\partial^3 Q}{\partial x^3}+\frac{\Delta t^2}{8}\left[\frac{\partial^3 Q}{\partial x\,\partial t^2}\frac{\partial}{\partial t}\left(\frac{a}{c^2}\right)+\frac{\partial^2 Q}{\partial x\,\partial t}\frac{\partial^2}{\partial t^2}\left(\frac{a}{c^2}\right)\right] \quad (30)$$

## Appendix III. Todini's MCT Method as a Solution of the Advection-Diffusion Equation

To show that Todini's method (Todini 2007) is based on the advection-diffusion equation, let $q=0$ and define the finite-difference scheme for the first term $\partial(Q/v)/\partial t$ in Eq. (8) as the "difference of the averages" rather than the "average of the differences." In this case, Eq. (10) becomes

$$C_1=\frac{1}{v_{j+(1/2)}^{n+1}}\left[1+\frac{\Delta t}{\Delta x}v_{j+(1/2)}^{n+1}+\frac{2}{\Delta x}\left(\frac{a_0 v}{c_0^2}\right)_{j+(1/2)}^{n+1}\right]$$

$$C_2=\frac{1}{v_{j+(1/2)}^{n+1}}\left[1-\frac{\Delta t}{\Delta x}v_{j+(1/2)}^{n+1}-\frac{2}{\Delta x}\left(\frac{a_0 v}{c_0^2}\right)_{j+(1/2)}^{n+1}\right]$$

$$C_3=\frac{1}{v_{j+(1/2)}^{n}}\left[-1+\frac{\Delta t}{\Delta x}v_{j+(1/2)}^{n}-\frac{2}{\Delta x}\left(\frac{a_0 v}{c_0^2}\right)_{j+(1/2)}^{n}\right]$$

$$C_4=\frac{1}{v_{j+(1/2)}^{n}}\left[-1-\frac{\Delta t}{\Delta x}v_{j+(1/2)}^{n}+\frac{2}{\Delta x}\left(\frac{a_0 v}{c_0^2}\right)_{j+(1/2)}^{n}\right] \quad (31)$$

Expressing Eq. (31) in terms of the cell and Reynolds numbers as in Todini's and noting Eq. (7)

$$C_1=(1+C^{*n+1}+D^{*n+1})$$

$$C_2=(1-C^{*n+1}-D^{*n+1})$$

$$C_3=(-1+C^{*n}-D^{*n})\frac{C^{*n+1}}{C^{*n}}$$

$$C_4=(-1-C^{*n}+D^{*n})\frac{C^{*n+1}}{C^{*n}} \quad (32)$$

where

$$C^{*n+1}=\frac{\Delta t}{\Delta x}v_{j+(1/2)}^{n+1}$$

$$D^{*n+1}=\frac{2}{\Delta x}\left(\frac{a_0 v}{c_0^2}\right)_{j+(1/2)}^{n+1} \quad (33)$$

and $C^{*n}$ and $D^{*n}$ are defined similarly. Comparison of Eq. (32) with Eq. (52) of Todini (2007) (as corrected on page 1,783) and assuming that $B$ in Todini's work is also a function of time, shows that the formulations are identical. Todini's method is therefore a finite-difference solution of Eq. (8).



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## Appendix IV. Parameters Used in Tests on the Synthetic River

The semiwidth $B$ of the synthetic river channel is defined by

$$B = B_{fl} + \frac{(B_t - B_{fl})\tanh(ky)}{\tanh(-ky_{fl})} \tag{34}$$

and where $B_t$=semiwidth(m) when $y=-y_{fl}$. The actual semibed width of the channel is $B_b > B_t$. $B_x$=semiwidth(m) of the channel at $y=-y_x$ where

$$B_x = B_{fl} + \frac{(B_t - B_{fl})\tanh(-ky_x)}{\tanh(-ky_{fl})} = B_b + s(y_{fl} - y_x) \tag{35}$$

Here $B=B_{fl}$ where $y=0$. The constant $k$ can be defined by

$$k = \frac{1}{2y_{fl}}\log\left(\frac{1+\lambda}{1-\lambda}\right) \tag{36}$$

where $\lambda$=parameter that typically has the value 0.999.

We define the upstream discharge hydrograph for the synthetic river reach by

$$Q(0,t) = Q_{base} \quad \text{for} \quad t < t_Q$$

$$Q(0,t) = Q_{base} + (Q_{peak} - Q_{base})\left\{\frac{(t-t_Q)}{T_Q}\exp\left[1 - \frac{(t-t_Q)}{T_Q}\right]\right\}^{\beta_Q} \quad \text{for} \quad t_Q < t \tag{37}$$

where $Q_{base}$=base flow $(m^3/s)$ and $Q_{peak}$=peak flow $(m^3/s)$, and $t_Q$=start(h) of the flow event with $T_Q$=time-to-peak (h). $\beta_Q$ =index.

Similarly, we can define a uniform lateral inflow along the reach by

$$q(t) = q_{base} \quad \text{for} \quad t < t_q$$

$$q(t) = q_{base} + (q_{peak} - q_{base})\left\{\frac{(t-t_q)}{T_q}\exp\left[1 - \frac{(t-t_q)}{T_q}\right]\right\}^{\beta_q} \quad \text{for} \quad t_q < t \tag{38}$$

Here, $q_{base}$=base lateral inflow $(m^2/s)$ and $q_{peak}$=peak lateral inflow $(m^2/s)$ and $t_q$=start of the lateral inflow event (h) with $T_q$ =time-to-peak (h). $\beta_q$=index.

## Notation

*The following symbols are used in this technical note:*

$A(Q)$ = wetted cross section area, $m^2$;
$a_0$ = attenuation parameter, $m^2/s$;
$B(Q)$ = surface width, m;
$C_0, C_1, C_2, C_3, C_4$ = finite difference functions;
$c_0$ = kinematic wave speed, m/s;
$g$ = acceleration due to gravity, m/s/s;
$J$ = total number of space steps along reach;
$j$ = counter for number of space steps;
$K$ = Muskingum parameter, s;
$L$ = length scale of flood disturbance, m;
$n$ = counter for number of time steps;
$Q$ = discharge, $m^3/s$;
$Q_j^n$ = $Q$ defined at $(j\Delta x, n\Delta t)$, $m^3/s$;
$q$ = lateral inflow, $m^2/s$;
$S$ = storage, $m^3$;
$s_0$ = bed gradient;
$T$ = time scale of flood disturbance, s;
$t$ = time, s;
$x$ = distance, m;
$y$ = flow depth, m;
$\Delta t$ = time step, s;
$\Delta x$ = space step, m;
$\varepsilon$ = surface slope number;
$\theta$ = Muskingum parameter;
$v(Q)$ = velocity, m/s; and
$\varphi$ = Muskingum parameter for lateral inflow.

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