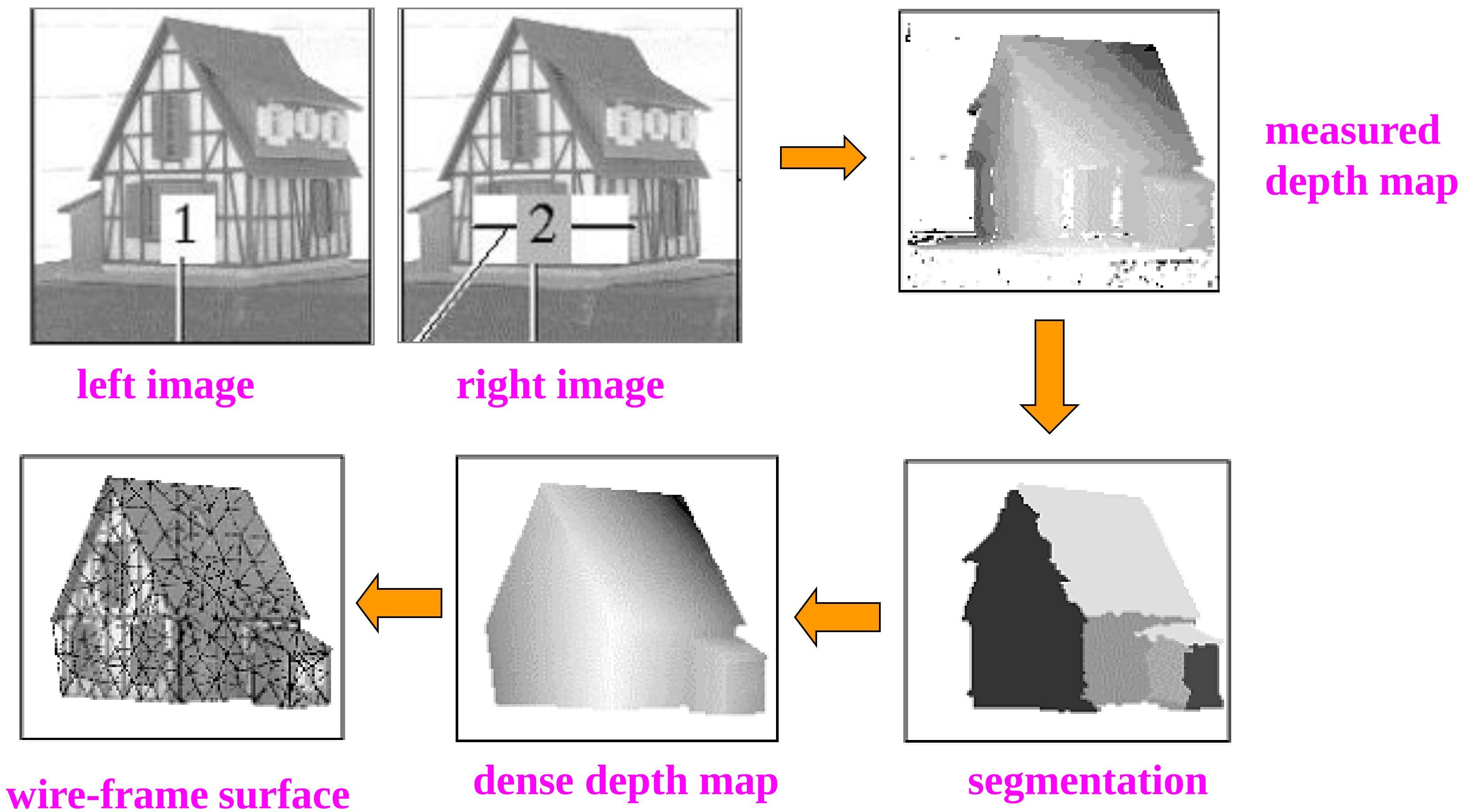
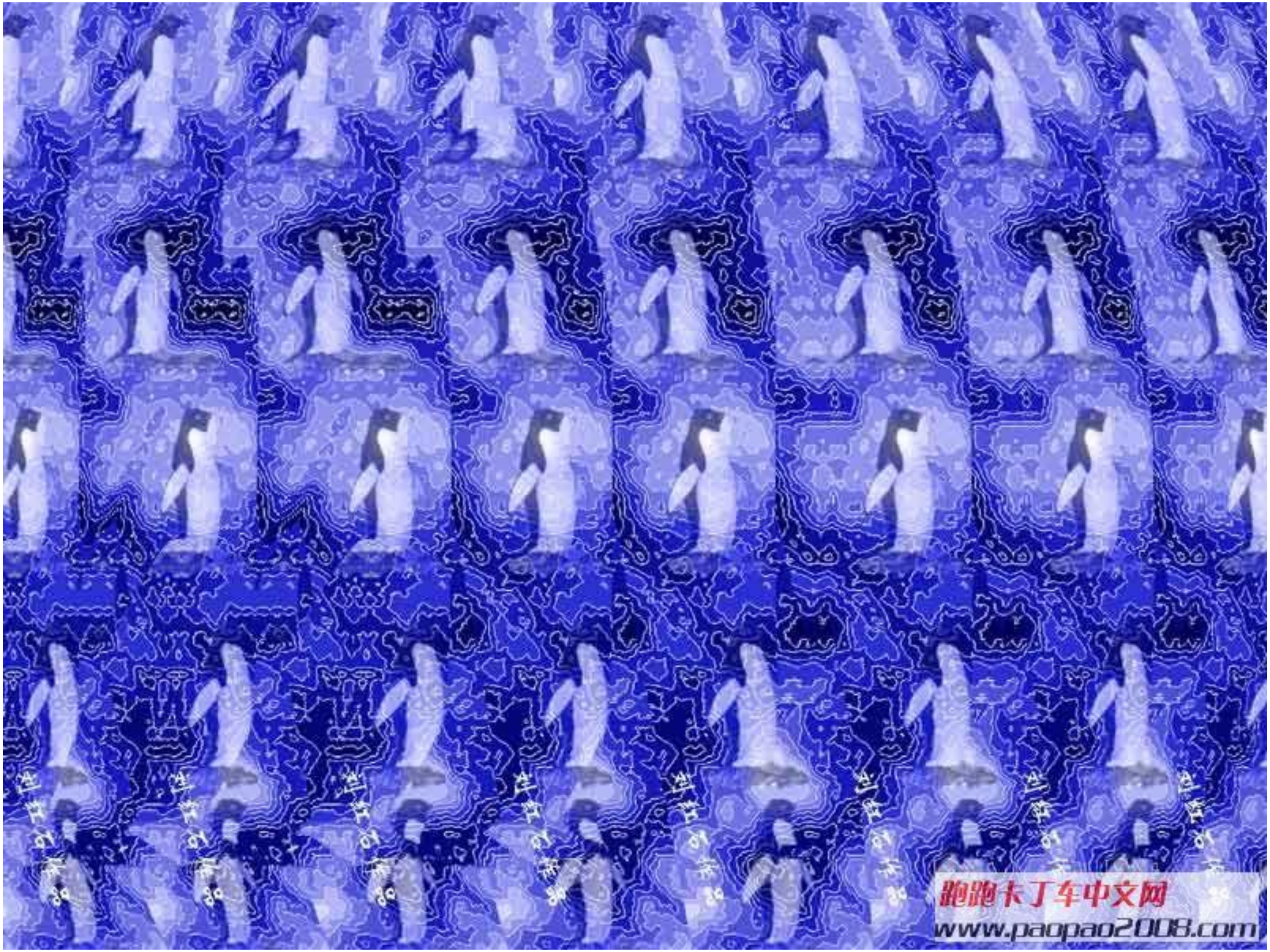
Chapter6 Image Reconstruction

- Preview
- 6.1 Introduction
- 6.2 Reconstruction by Fourier Inversion
- 6.3 Reconstruction by convolution and backprojection
- 6.4 Finite series-expansion

Preview

Reconstruction by Stereoscopic image pair

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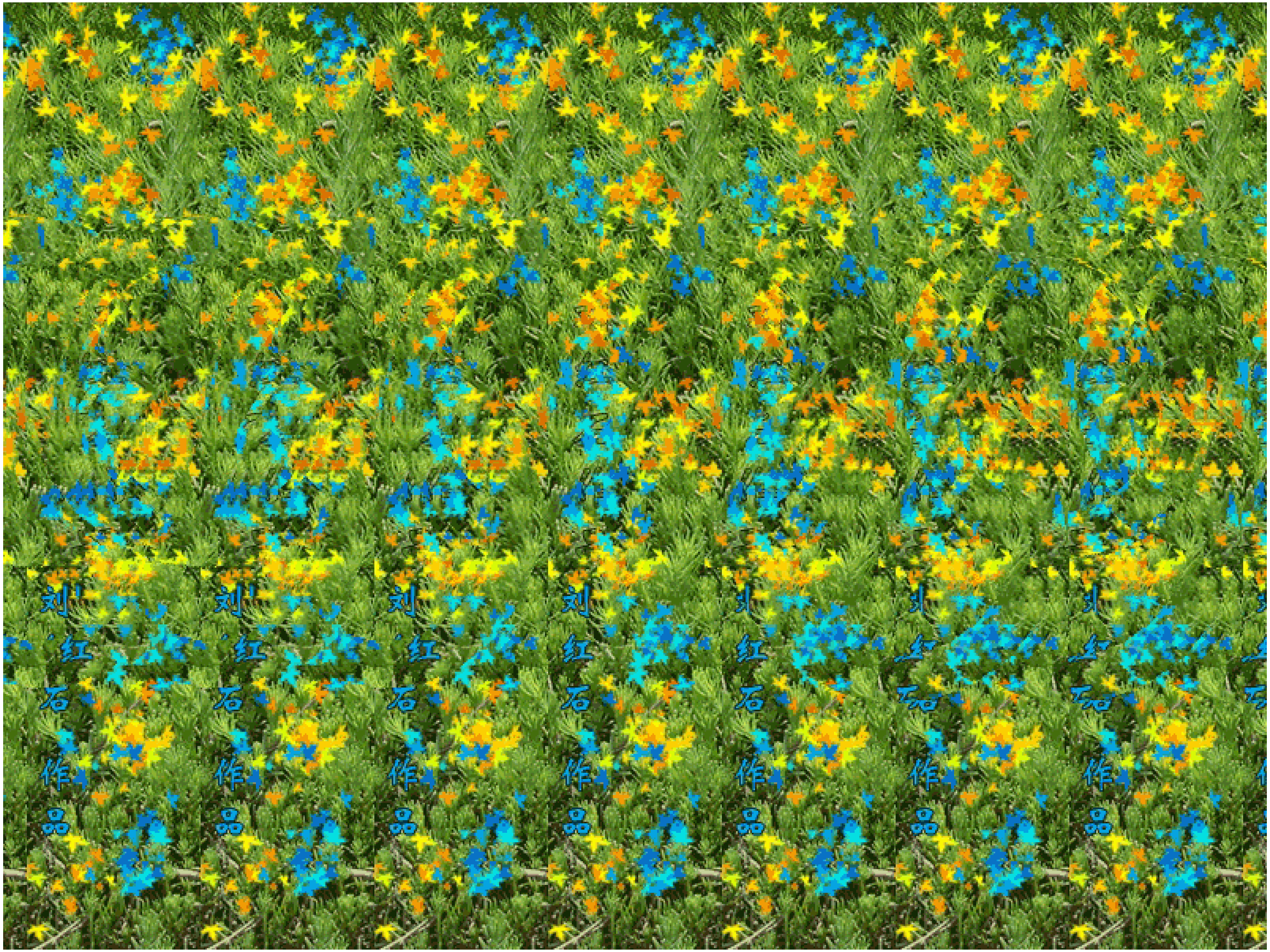
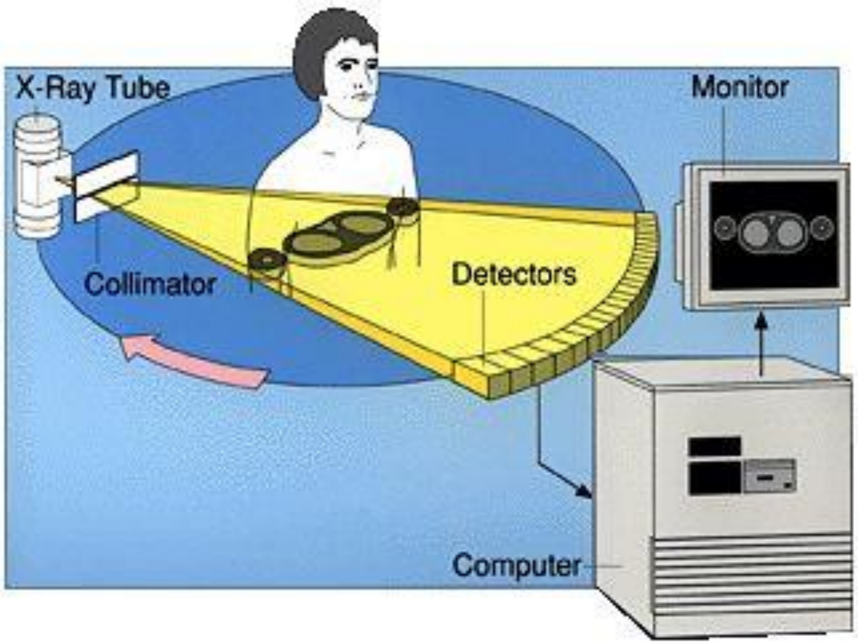

刘红石作品

Preview

CT reconstruction

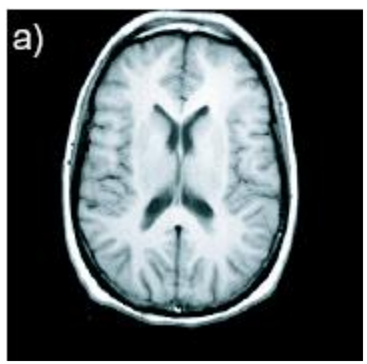


Preview

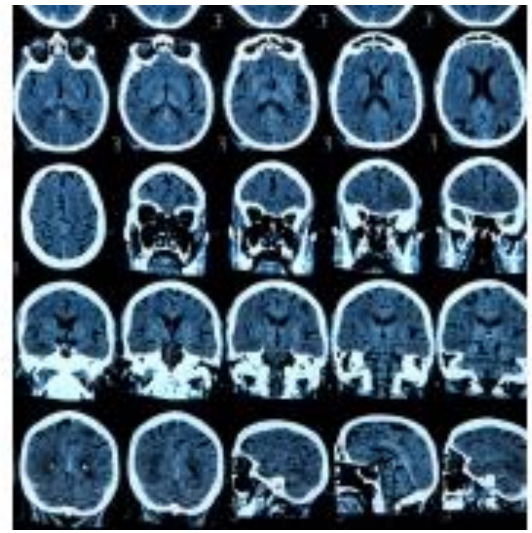
CT reconstruction

CT machine

TCT



MRI



6.1 Introduction

6.1.1 Overview of Computed Tomography (CT)

1895.12.22

1895年德国物理学家伦琴(Wilhelm Conrad Roentgen)发现了X射线

1964年美国物理学家科马克（Allan Macleod Cormack）在数学上证明了X射线吸收量与不同的器官、组织的密度之间的关系。

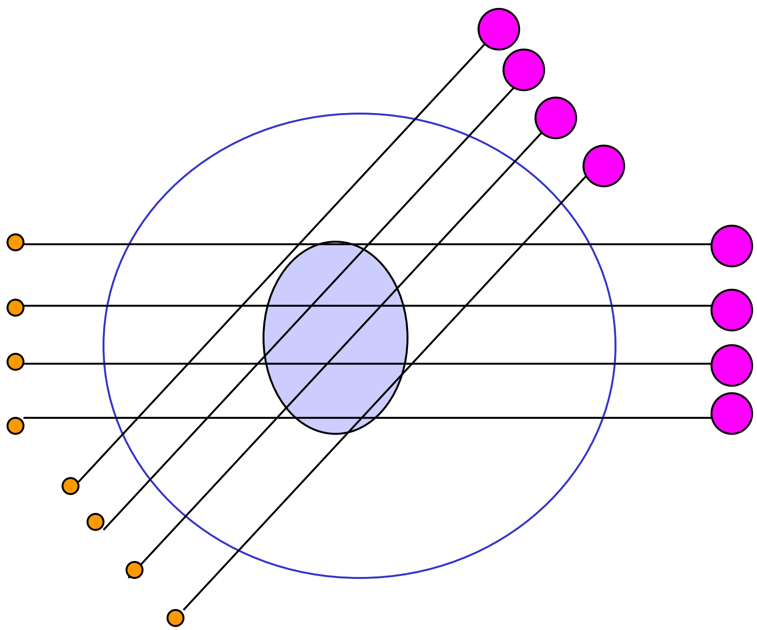
1924-1998

1919-2004

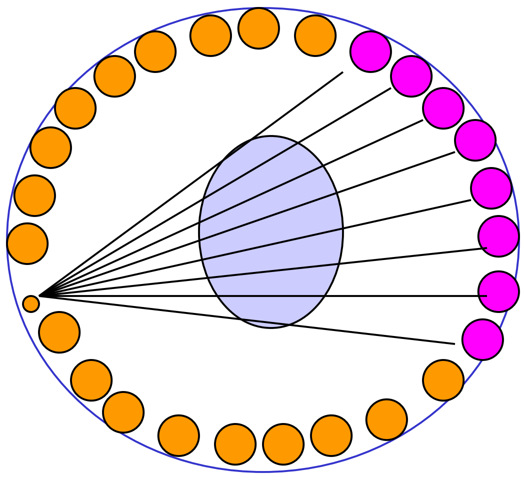
1972年英国工程师豪斯菲尔德（Godfrey Newbold Hounsfield, 电力和音乐仪器有限公司)发明了第一台CT机。

6.1 Introduction

6.1.1 Overview of Computed Tomography (CT)



parallel-beam



fan-beam

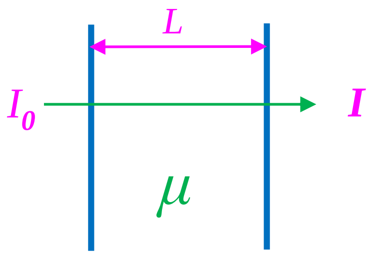
6.1 Introduction

6.1.2 Physical basis of projection

Let us consider the simplest case, a single block of *homogeneous* tissue and a monochromatic beam of X-rays. The linear attenuation coefficient μ is defined by

$$I = I_0 e^{-\mu L}$$

Barkla，1917



where

L is the length of the block

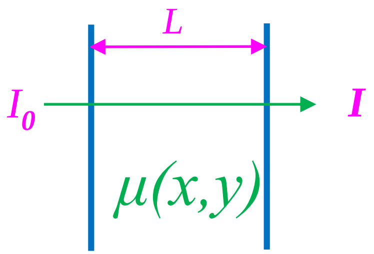
I and I_0 are incident and attenuated intensities of the X-ray, respectively

6.1 Introduction

6.1.2 Physical basis of projection

Let $\mu(x,y)$ denote the sectional attenuation variation. For an infinitely thin beam of monochromatic X-rays, the detected intensity of the X-ray along a straight line ***L*** is expressed as

$$I = I_0 e^{-\int_L \mu(x,y)dl}$$



$$\Downarrow$$

$$-\ln \frac{I}{I_0} = \int_L \mu(x,y)dl$$

6.1 Introduction

6.1.3 Purpose

Reconstruction $\mu(x,y)$ from its projections

6.1.4 Methods

Fourier Inversion (frequency domain)

convolution and backprojection (spatial domain)

Finite series-expansion (spatial domain)

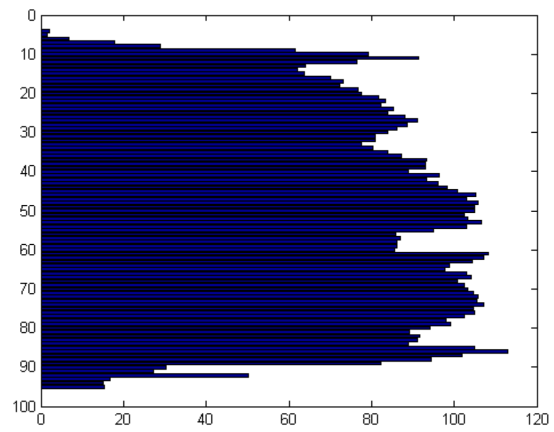
6.2 Reconstruction by Fourier Inversion

6.2.1 Mathematical expression of projection

Projection in x- and y-axial

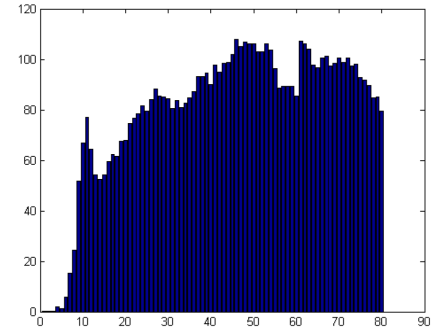
$$p(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

$$p(y) = \int_{-\infty}^{\infty} f(x, y)dx$$



Projection in arbitrary direction

$$p_\theta(t) = \int_{-\infty}^{\infty} f(x, y)ds$$



6.2 Reconstruction by Fourier Inversion

6.2.1 Mathematical expression of projection

where

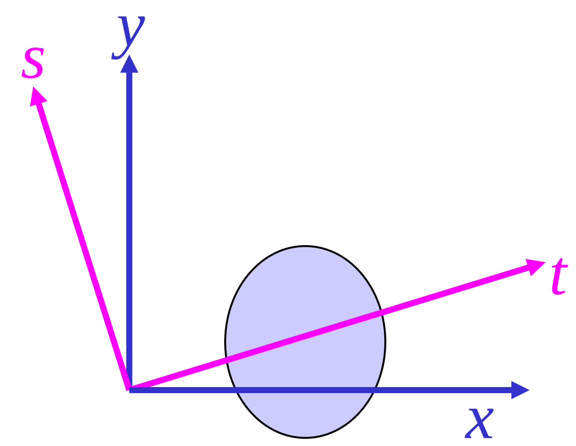
$$t = y\sin\theta + x\cos\theta$$

$$s = y\cos\theta - x\sin\theta$$

$$\Downarrow$$

$$x = t\cos\theta - s\sin\theta$$

$$y = t\sin\theta + s\cos\theta$$



$$\Downarrow$$

$$p_\theta(t) = \int_{-\infty}^{\infty} f(x,y)\,ds$$

$$= \int_{-\infty}^{\infty} f(t\cos\theta - s\sin\theta,\, t\sin\theta + s\cos\theta)\,ds$$

6.2 Reconstruction by Fourier Inversion

6.2.2 Fourier Slice Theorem

If *$F(u,v)$* is the Fourier Transform of *$f(x,y)$*, *$p_\theta(t)$* is the projection of *$f(x,y)$* in *θ –direction*, and the *$S_\theta(\omega)$* is the Fourier transform of *$p_\theta(t)$* then, *$S_\theta(\omega)$* is a slice of *$F(u,v)$* in the *θ –direction*

$$S_\theta(\omega) = F(\omega, \theta)$$

where $F(\omega,\theta)$ is the polar coordinate expression of $F(u,v)$

6.2 Reconstruction by Fourier Inversion

6.2.2 Fourier Slice Theorem: proof

When $\theta=0$

$$p(x)=\int_{-\infty}^{\infty} f(x,y)dy \Rightarrow P(u)=\int_{-\infty}^{\infty} p(x)\exp(-2\pi jux)dx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dy\exp(-2\pi jux)dx$$

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-2\pi j(ux+vy)]dxdy$$

Let $v=0$ $\Rightarrow$ $F(u,0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp(-2\pi jux)dxdy$

$$\therefore \quad F(u,0)=P(u)$$

In other words, the Fourier transform of the vertical projection of an image is the horizontal radial profile of the 2D Fourier transform of the image.

6.2 Reconstruction by Fourier Inversion

6.2.2 Fourier Slice Theorem :proof

In general

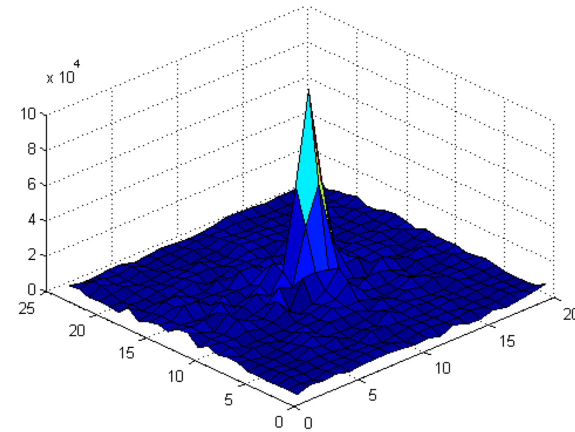
By the nature of the Fourier transform, if an image ***f*****(*x*,*y*)** is rotated by an angle with respect to the ***x*** axis, the Fourier transform ***F*****(*u*,*v*)** will be correspondingly rotated by the same angle with respect to the ***u*** axis.

$$S_\theta(\omega) = F(\omega, \theta)$$

6.2 Reconstruction by Fourier Inversion

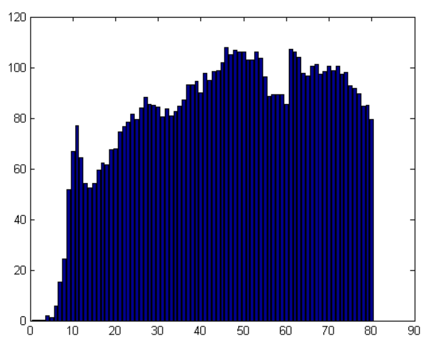
6.2.2 Fourier Slice Theorem

2D-DFT

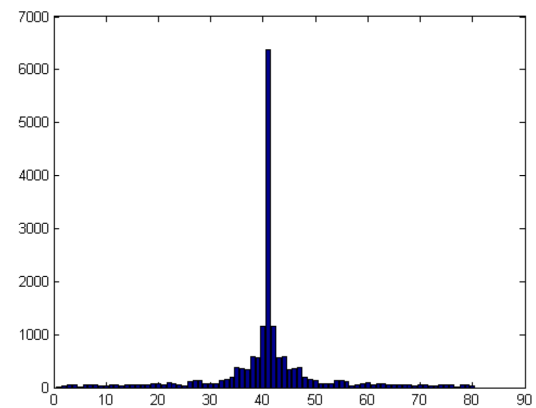


$$F(u,0) = P(u)$$

projection



1D-DFT



6.2 Reconstruction by Fourier Inversion

6.2.3 Arithmetic of Fourier inversion

Step1: calculate the projection's DFT $S_\theta(\omega)$ in θ_m –direction, $m=0,1, M-1$

Step2: combine $S_\theta(\omega)$ into $F(\omega,\theta)$

Step3: interpolation

Step4: 2D-IDFT

6.3 Reconstruction by convolution and backprojection

6.3.1 Parallel-Beam Reconstruction

With the inverse Fourier transform, an image *f(x,y)* can be expressed as

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)\exp\left[2\pi j(ux+vy)\right]dudv$$

Let

$$u=\omega\cos\theta \qquad v=\omega\sin\theta$$

we have

$$f(x,y)=\int_{0}^{2\pi}\int_{0}^{\infty}F(\theta,\omega)\exp\left[2\pi j(x\cos\theta+y\sin\theta)\omega\right]\omega d\omega d\theta$$

Because

$$F(\theta+\pi,\omega)=F(\theta,-\omega)$$

we have

$$f(x,y)=\int_{0}^{\pi}\int_{-\infty}^{\infty}F(\theta,\omega)\left|\omega\right|\exp\left[2\pi j(x\cos\theta+y\sin\theta)\omega\right]d\omega d\theta$$

6.3 Reconstruction by convolution and backprojection

6.3.1 Parallel-Beam Reconstruction

Using the Fourier slice theorem, we have

$$f(x,y)=\int_0^{\pi}\int_{-\infty}^{\infty}S_\theta(w)\left|\omega\right|\exp\left[2\pi j(x\cos\theta+y\sin\theta)\omega\right]d\omega d\theta$$

let $t=x\cos\theta+y\sin\theta$

$$f(x,y)=\int_0^{\pi}\int_{-\infty}^{\infty}S_\theta(w)\left|\omega\right|\exp\left[2\pi j\omega t\right]d\omega d\theta$$

$$S_\theta(\omega)\left|\omega\right| \Leftrightarrow p_\theta(t)*h(t)$$

where

$$h(t)=F^{-1}(\left|\omega\right|) \Rightarrow f(x,y)=\int_0^{\pi}d\theta\int_{-\infty}^{\infty}p_\theta(t)h(t-\tau)d\tau$$

6.3 Reconstruction by convolution and backprojection

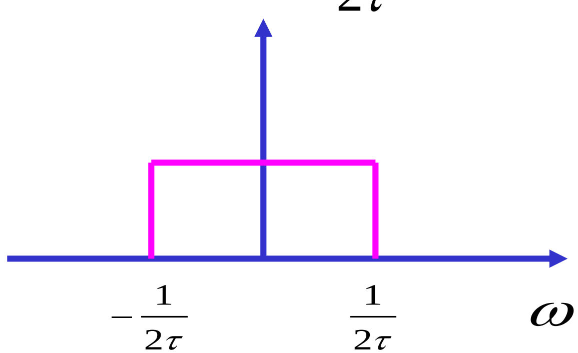
6.3.1 Parallel-Beam Reconstruction

Note that $h(t)$ does not exist in an ordinary sense, but $S_\theta(\omega)$ is essentially bandlimited, $h(t)$ can be accurately evaluated within the maximum bandwidth of $S_\theta(\omega)$.

$$h(t) = F^{-1}(|\omega|) \quad = \int_{-\omega c}^{\omega c} \omega \exp(2\pi j\omega t) d\omega$$

If $\quad \omega c = \dfrac{1}{2\tau}$

$$h(t) = \frac{1}{2\tau^2}\left(\frac{\sin(2\pi t/2\tau)}{2\pi t/2\tau}\right) - \frac{1}{4\tau^2}\left(\frac{\sin(2\pi t/2\tau)}{2\pi t/2\tau}\right)^2$$



6.3 Reconstruction by convolution and backprojection

6.3.2 Algorithm of Fourier inversion

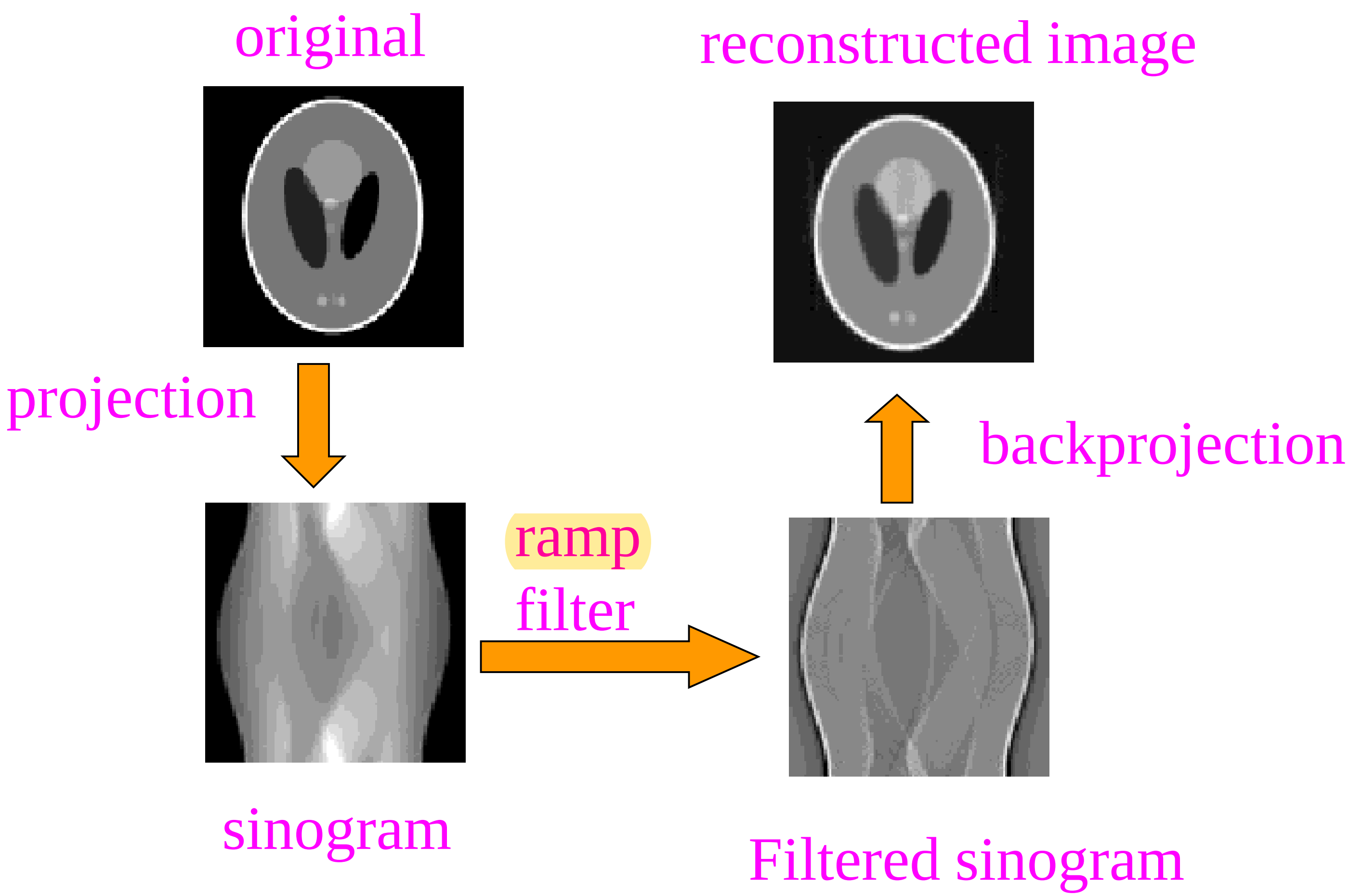
Step1: applying the filter $h(t)$ to $p_\theta(t)$, get $p'_\theta(t)$,

Step2: calculate the integration

$$f(x,y)=\int_0^\pi p_\theta^{'}(t)d\theta$$

6.3 Reconstruction by convolution and backprojection

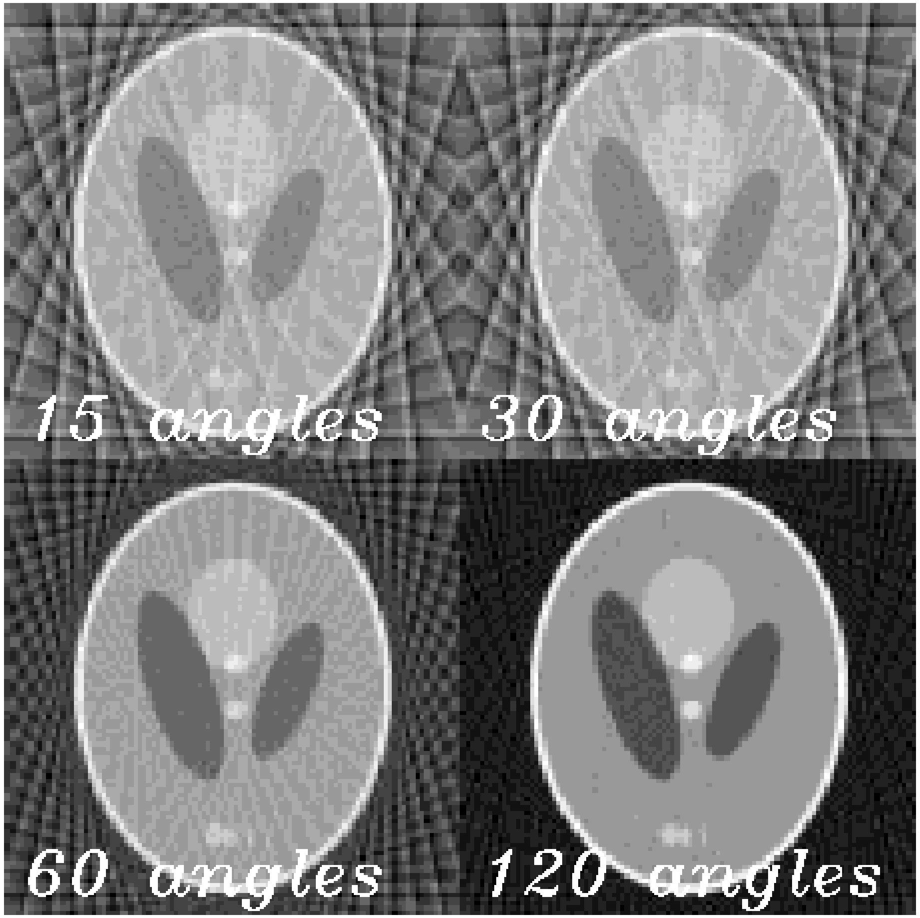
6.3.3 Experimental results



6.3 Reconstruction by convolution and backprojection

6.3.4 Practical problems: Aliasing - Insufficient angular sampling

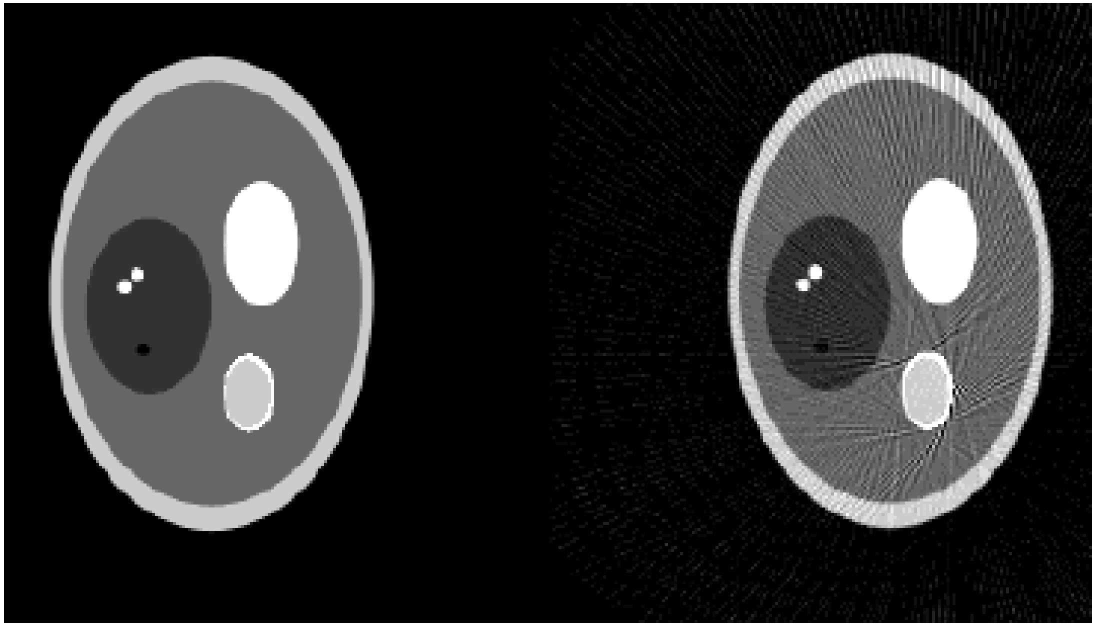
- reducing scanning time
- reducing patient dose



6.3 Reconstruction by convolution and backprojection

6.3.4 Practical problems: Aliasing - Insufficient radial sampling

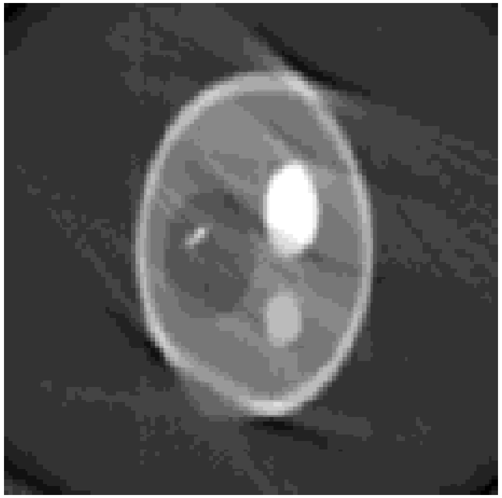
occurs when there is a sharp intensity change caused by, for example, bones.



6.3 Reconstruction by convolution and backprojection

6.3.4 Practical problems:Motion artifact

caused by patient motion, such as respiration and heart beat, during data acquistion



6.4 Finite series-expansion

6.4.1 Expression of Discrete projection

A 2-D array can be expressed as a vector

$$f(x, y) = \{ f_0, f_1 \cdots f_{N-1} \}$$

The projection of *i*-th ray is

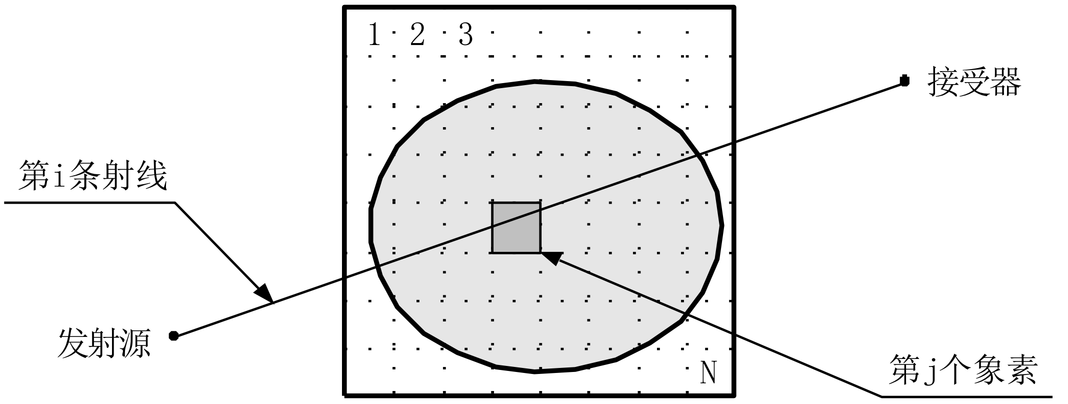
$$p_i = \sum_{j=0}^{N-1} w_{i,j} f_j \qquad i = 0, 1, \cdots M - 1$$

where

$$w_{i,j} = \begin{cases} 1 & \text{if the } i\text{-th ray cross the } j\text{-th point} \\ 0 & \text{else} \end{cases}$$

6.4 Finite series-expansion

6.4.1 Expression of Discrete projection



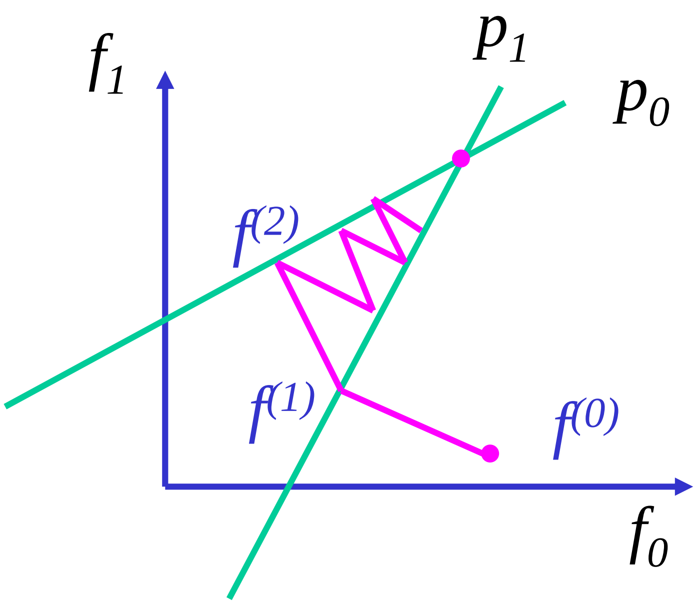
$$\begin{cases} w_{00} f_0 + w_{01} f_1 + \cdots + w_{0(N-1)} f_{N-1} = p_0 \\ w_{10} f_0 + w_{11} f_1 + \cdots + w_{1(N-1)} f_{N-1} = p_1 \\ \vdots \\ w_{(M-1)0} f_0 + w_{(M-1)1} f_1 + \cdots + w_{(M-1)(N-1)} f_{N-1} = p_{M-1} \end{cases}$$

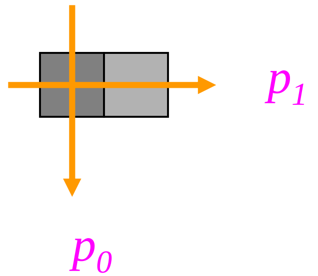
6.4 Finite series-expansion

6.4.2 Solution by iteration

f(x,y) can be seen as a point in a *N*-D space, each projection equation is a super-plane of this *N*-D space. If that equation set has a unique solution, then all the super-planes intersect in a point

For example *N*=2





$$\begin{cases} w_{00} f_0 + w_{01} f_1 = p_0 \\ w_{10} f_0 + w_{11} f_1 = p_1 \end{cases}$$

6.4 Finite series-expansion

6.4.2 Solution by iteration

$$\Rightarrow \quad f^{(1)} = f^{(0)} - \frac{w_0 f^{(0)} - p_0}{w_0 \bullet w_0} w_0$$

$$\Rightarrow \quad f^{(k+1)} = f^{(k)} - \frac{w_k f^{(k)} - p_k}{w_k \bullet w_k} w_k$$

6.4.3 Arithmetic of finite series-expansion

Step1: given an initial estimation $f^{(0)}, k=0$

Step2: adjust $f^{(k)}$ by a projection equation to get $f^{(k+1)}, k=k+1$

Step3: repeat step2 until the adjust value less than δ

Homework

5.1. 简述CT 发明过程。

5.14. 试证明投影定理

END