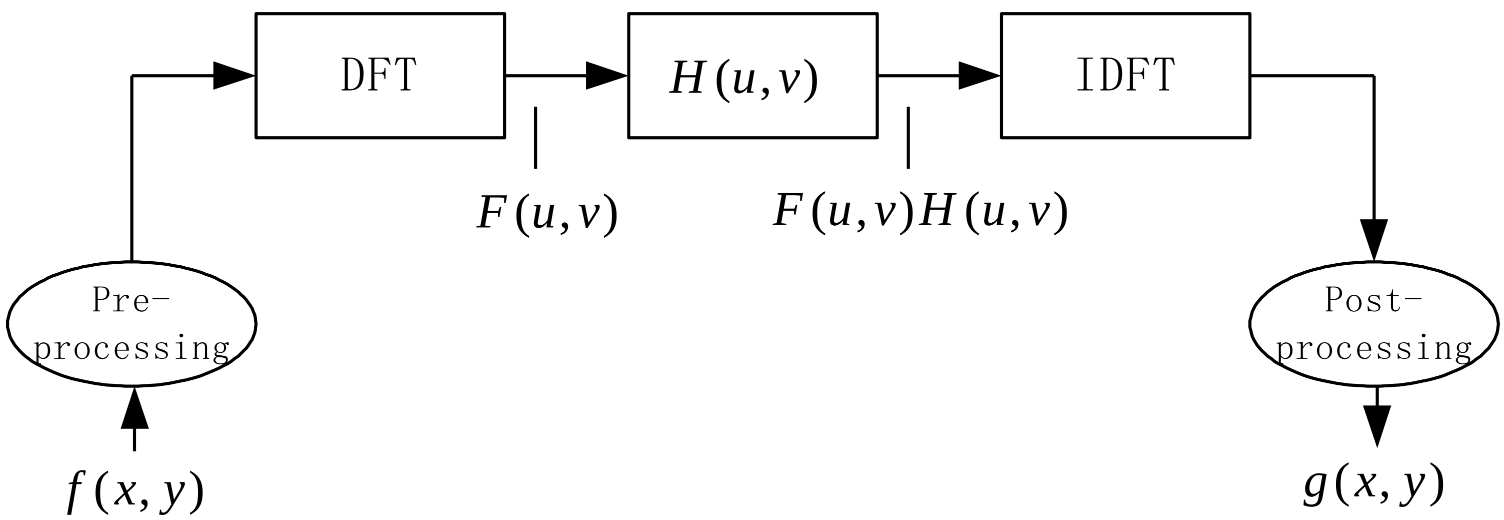
Chapter3 Image Transforms

- Preview
- 3.1General Introduction and Classification
- 3.2 The Fourier Transform and Properties
- 3.3 Other Separable Image Transforms
- 3.4 Hotelling Transform

Chapter3 Image Transforms

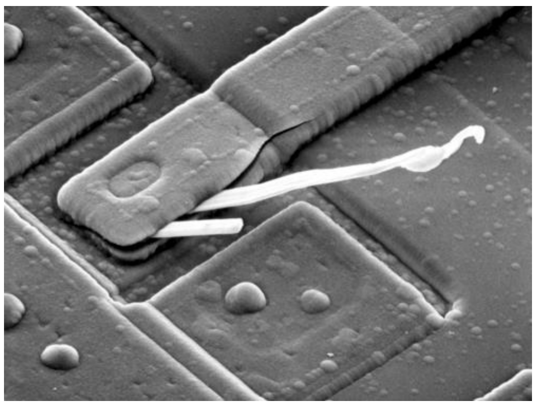
Preview

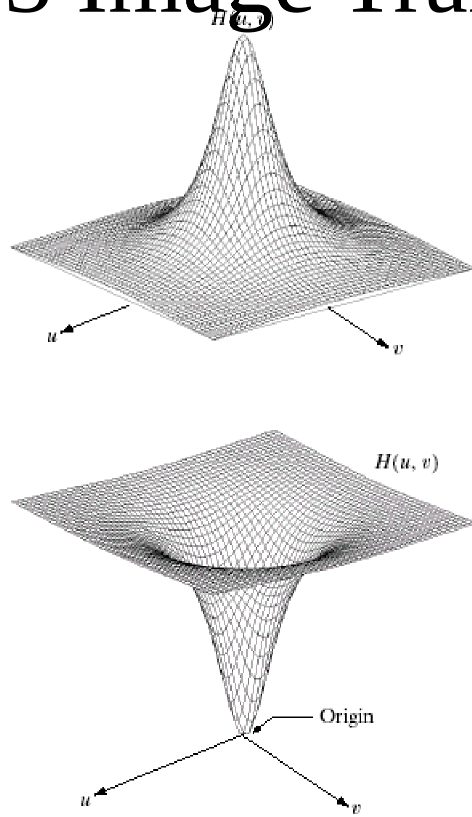
General steps of operation in frequency domain



Chapter3 Image Transforms

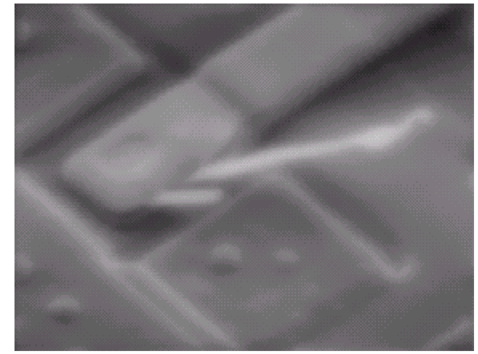
Preview

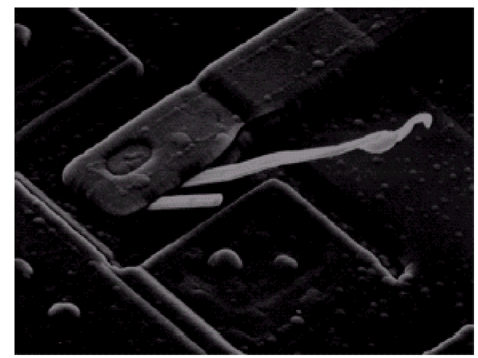


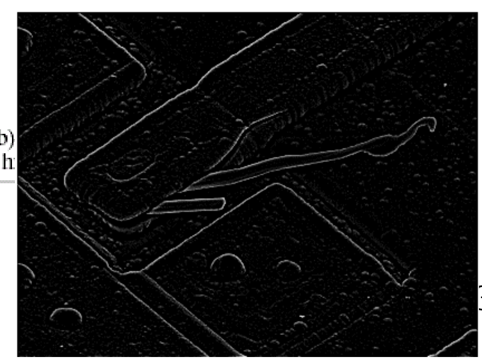


a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b)
(c) A two-dimensional highpass filter function. (d) Result of h





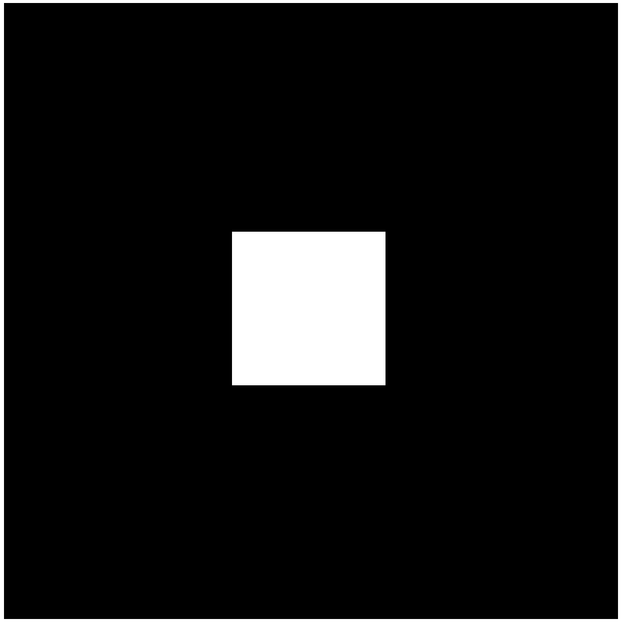


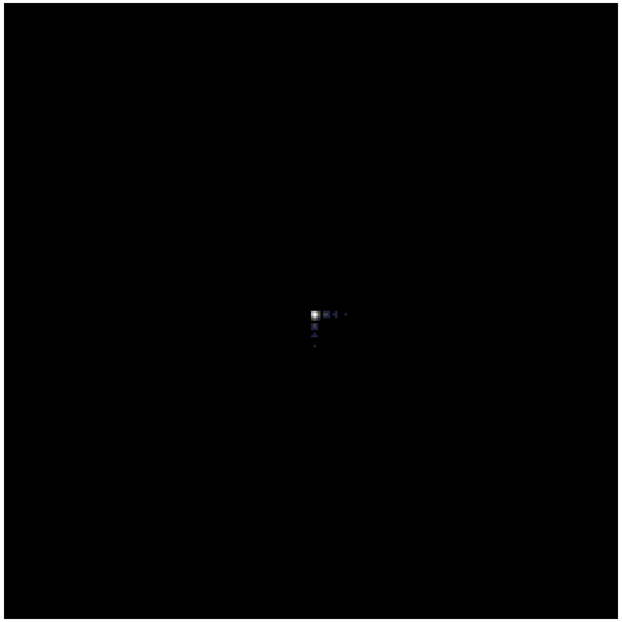
Lowpass filter
And highpass filter

Chapter3 Image Transforms

Preview

Directly display DFT coefficients

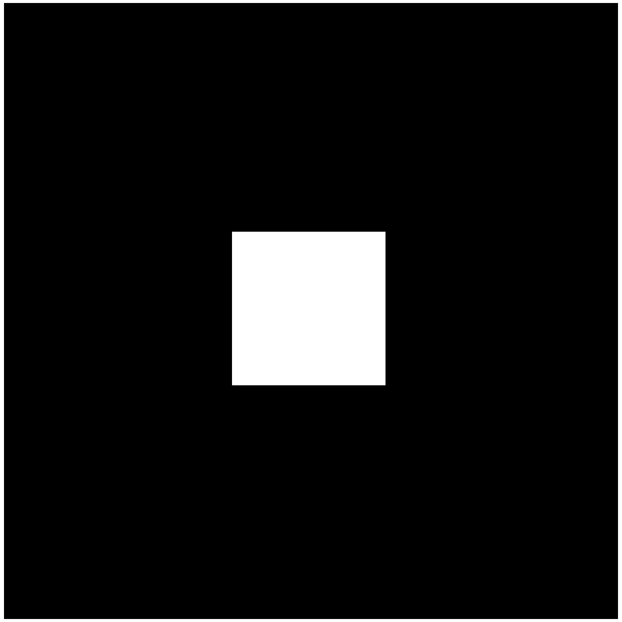


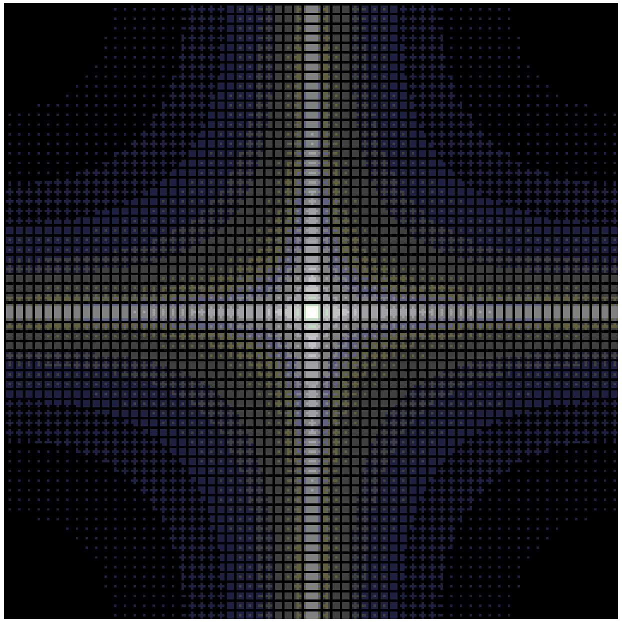


Chapter3 Image Transforms

Preview

Display DFT coefficients after log operate

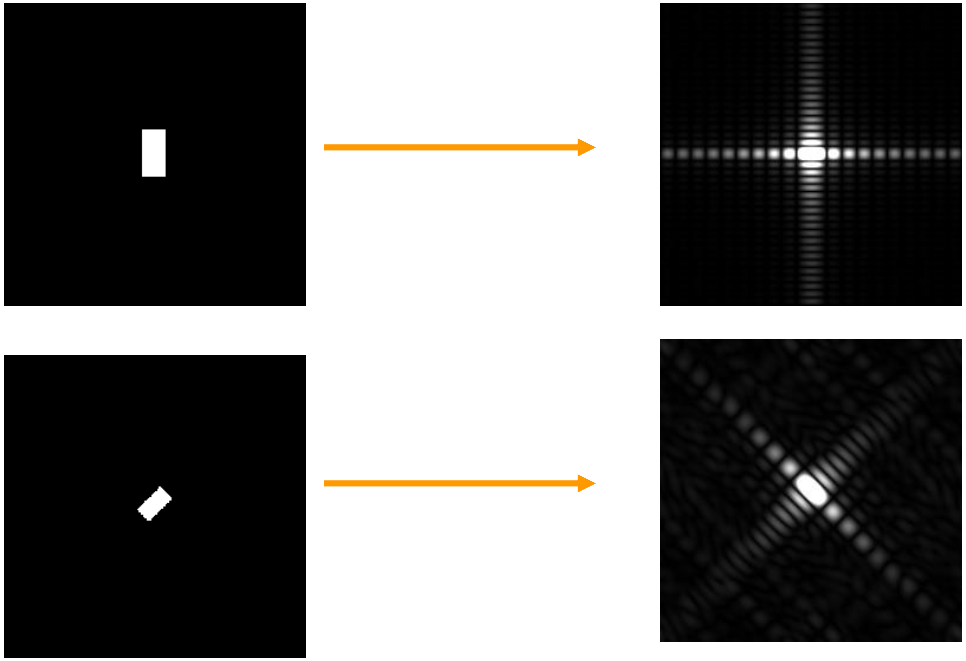




Chapter3 Image Transforms

Preview

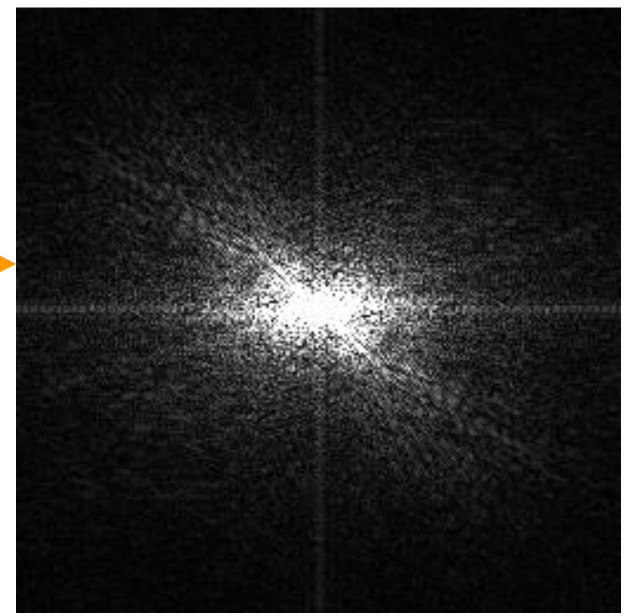
Rotational properties of DFT



Chapter3 Image Transforms

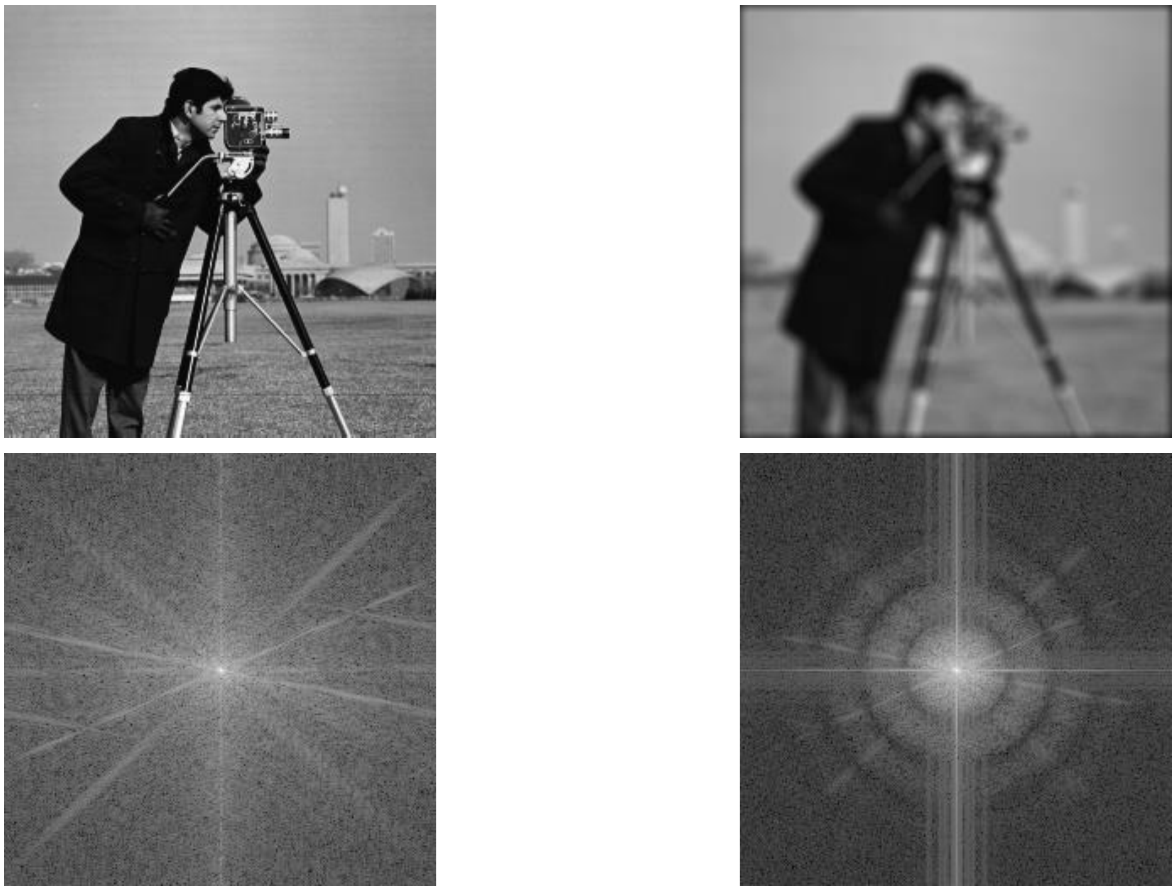
Preview

Examples of DFT



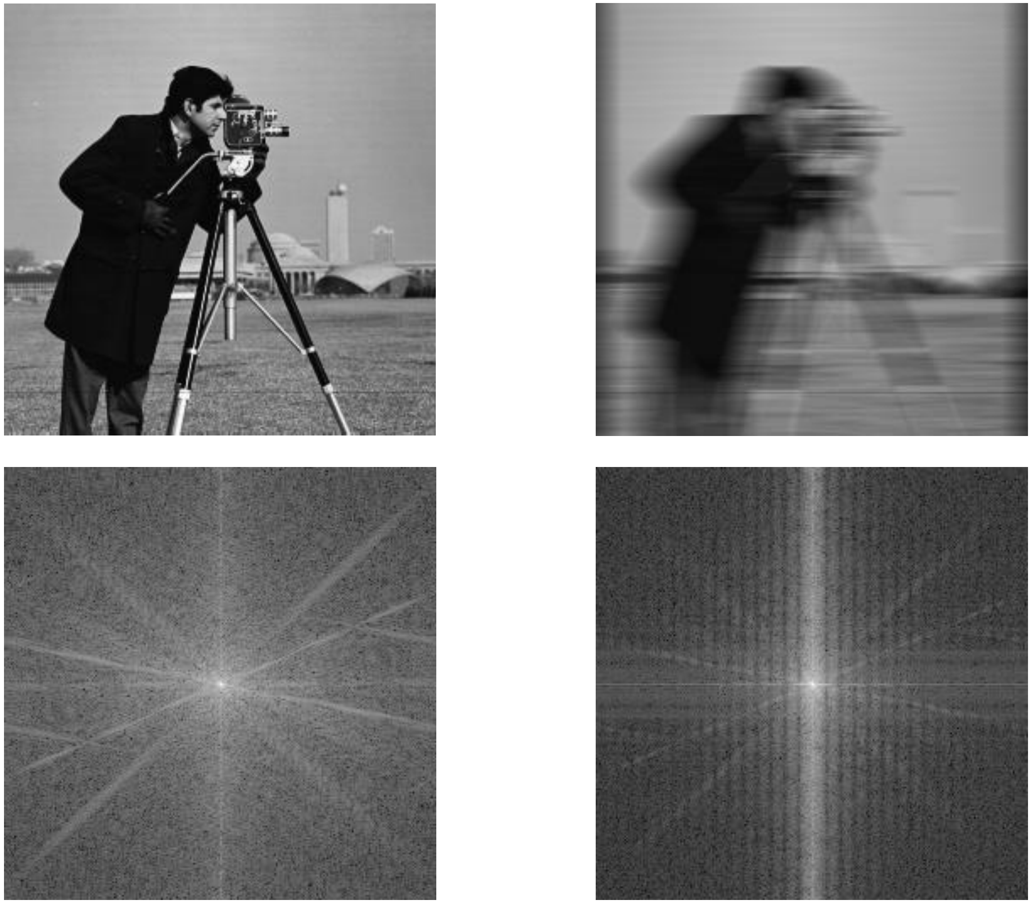
Chapter3 Image Transforms

Preview Blurred image and its DFT



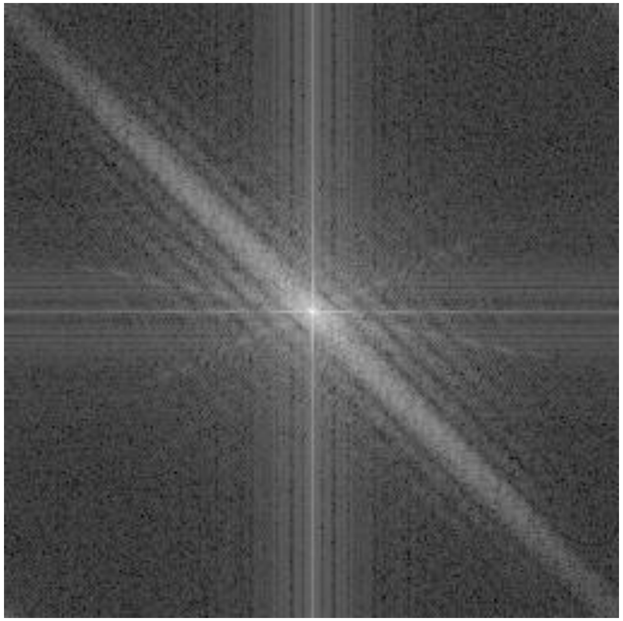
Chapter3 Image Transforms

Preview Blurred image and its DFT



Chapter3 Image Transforms

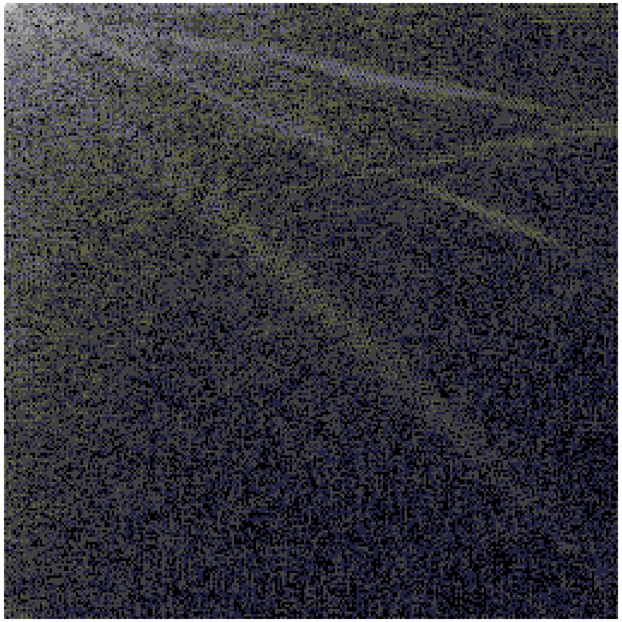
Preview Blurred image and its DFT



Chapter3 Image Transforms

Preview

Examples of DCT



3.1 General Introduction and Classification

3.1.1 introduction

- Image transforms are the bases of image processing and analysis

- This chapter deals with two-dimensional transforms and their properties

- Image transforms are used in image enhancement, restoration, reconstruction, encoding and description

3.1 General Introduction and Classification

3.1.1 introduction

Definition1: if X is an *N-by-1* vector and T is an *N-by-N* matrix then:

$$y_i = \sum_{j=0}^{N-1} t_{i,j} x_j$$

polynomial expression

matrix expression

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} t_{0,0} & t_{0,1} & \cdots & t_{0,N-1} \\ t_{1,0} & t_{1,1} & \cdots & t_{1,N-1} \\ \vdots & & \ddots & \\ t_{N-1,0} & t_{N-1,1} & \cdots & t_{N-1,N-1} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} \longrightarrow Y = TX$$

3.1 General Introduction and Classification

3.1.1 introduction

Definition2: inversion $X = T^{-1}Y$

If *rank(T)*= *N* then it is a *linear* transform

If $T^{-1} = T^{*t}$ then it is a *Unitary* transform $TT^{*t} = TT^{-1} = I$

If $T^{-1} = T^{t}$ then it is a *orthogonal* transform $TT^{t} = TT^{-1} = I$

3.1 General Introduction and Classification

3.1.1 introduction

Example: 1-D Discrete Fourier Transform (DFT)

$$F(u) = \sum_{x=0}^{N-1} f(x) e^{-j\frac{2\pi}{N}ux} \longrightarrow F = Tf$$

$$f(x) = \frac{1}{N} \sum_{u=0}^{N-1} F(u) e^{j\frac{2\pi}{N}ux} \longrightarrow f = T^{-1}F$$

$$T^{-1} = T^{*t}$$

It is a *Unitary* transform

3.1 General Introduction and Classification

3.1.1 introduction

Definition3: 2-D transformation

$$y_{m,n} = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} x_{i,j} \Phi(i, j, m, n)$$

$\Phi(i, j, m, n)$ Can be thought of as a *MN-by-MN* block matrix have *M* rows of *M* blocks, each of which is an *N-by-N* matrix

3.1 General Introduction and Classification

3.1.1 introduction

Definition4: if $\Phi(i,j,m,n)$ can be separated into the product of rowwise and columnwise component function, that is

$$\Phi(i,j,m,n)=T_r(i,m)T_c(j,n)$$

then the transformation is called *separable*

$$y_{m,n}=\sum_{i=0}^{M-1}\left[\sum_{j=0}^{N-1}x_{i,j}T_c(j,n)\right]T_r(i,m)$$

3.1 General Introduction and Classification

3.1.1 introduction

Definition5: if two component are identical:

$$\Phi(i, j, m, n) = T(i, m)T(j, n)$$

then the transformation is called *symmetric*

$$y_{m,n} = \sum_{i=0}^{M-1} \left[\sum_{j=0}^{N-1} x_{i,j} T(j,n) \right] T(i,m)$$

or

$$Y = TXT$$

3.1 General Introduction and Classification

3.1.1 introduction

Example: 2-D Discrete Fourier Transform (DFT)

Separable and *Symmetric Unitary* transform

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\exp\left[\frac{-j2\pi ux}{N}\right]\sum_{y=0}^{N-1} f(x,y)\exp\left[\frac{-j2\pi vy}{N}\right]$$

lets

$$W_M = \exp(j2\pi / M)$$

$$W_N = \exp(j2\pi / N)$$

then

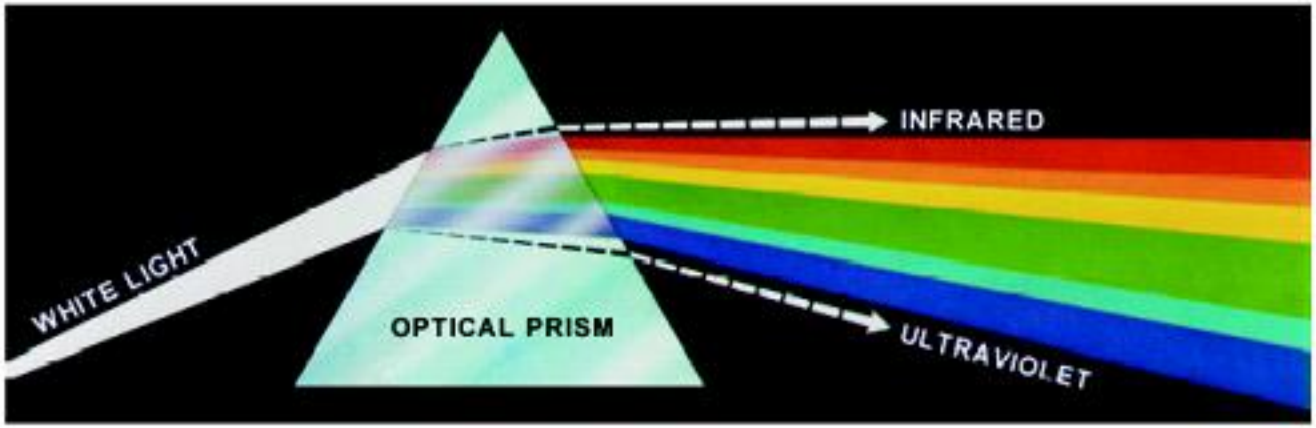
3.1 General Introduction and Classification

3.1.2 classification

- image transforms
  - separable transforms
    - DFT and its properties
    - others
      - DCT
      - Hadamard Transform
      - Walsh Transform
      - Slant Transform
      - Haar Transform
  - statistical transforms: Hotelling transform

3.2 Fourier Transform and Properties

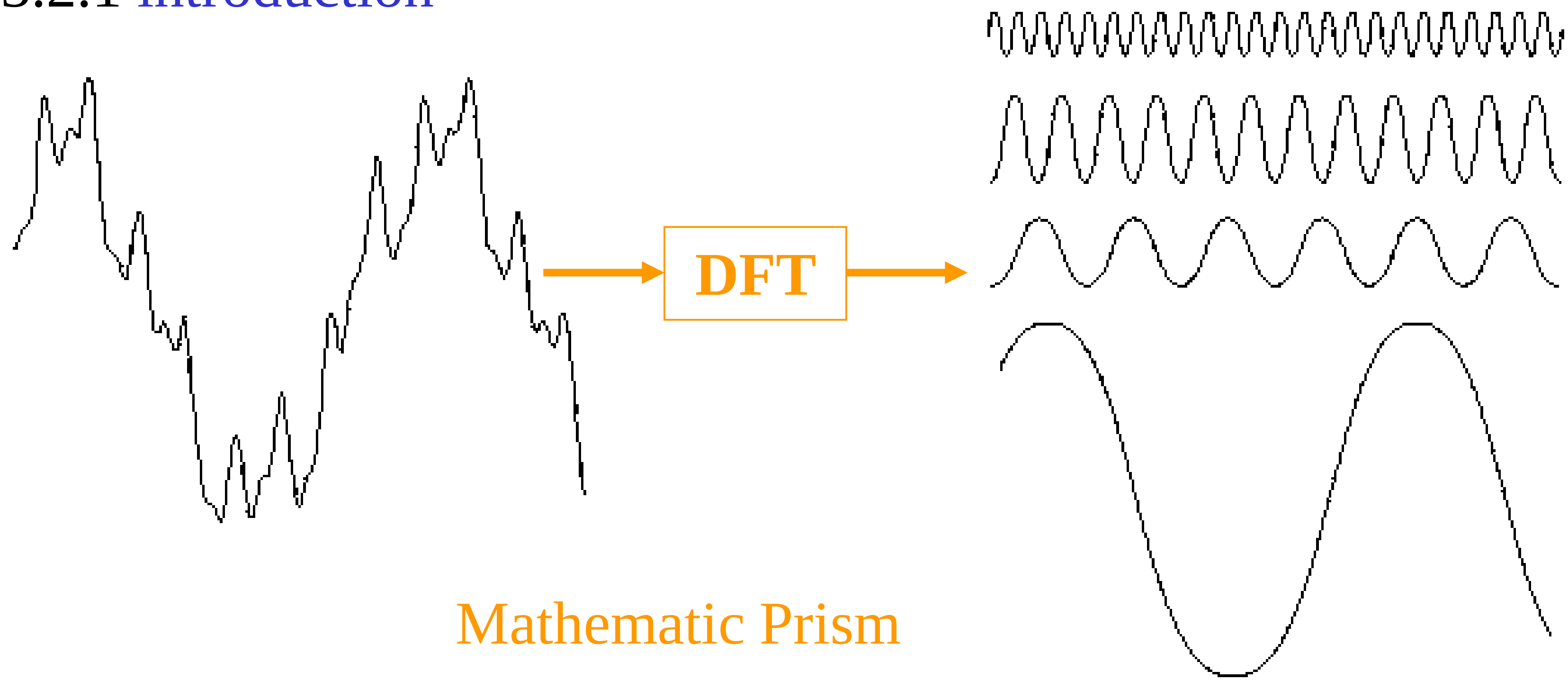
3.2.1 introduction



Optical Prism

3.2 Fourier Transform and Properties

3.2.1 introduction



3.2 Fourier Transform and Properties

3.2.2 definitions: 1-D CFT

The One-Dimensional Continuous Fourier Transform and its Inverse

$$F(u)=\int_{-\infty}^{+\infty} f(x)e^{-j2\pi ux}dx$$

$$f(x)=\int_{-\infty}^{+\infty} F(u)e^{j2\pi xu}du$$

3.2 Fourier Transform and Properties

3.2.2 definitions:1-D DFT

The One-Dimensional Discrete Fourier Transform and its Inverse

$$F(u) = \sum_{x=0}^{N-1} f(x) e^{-j\frac{2\pi}{N}ux} \qquad u = 0, 1, \cdots N-1$$

$$f(x) = \frac{1}{N} \sum_{u=0}^{N-1} F(u) e^{j\frac{2\pi}{N}ux} \qquad x = 0, 1, \cdots N-1$$

3.2 Fourier Transform and Properties

3.2.2 definitions :1-D DFT

Other expressions:

$$F(u) = R(u) + jI(u)$$

or

$$F(u) = |F(u)| \exp\left[j\phi(u) \right]$$

Euler's formula:

$$\exp\left[-j2\pi ux \right] = \cos 2\pi ux - j\sin 2\pi ux$$

3.2 Fourier Transform and Properties

3.2.2 definitions: spectrum, Phase angle Power spectrum

Magnitude or spectrum

$$|F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2}$$

Phase angle or phase spectrum

$$\phi(u) = \arctan\left[I(u)/R(u)\right]$$

Power spectrum
（Spectral density)

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

3.2 Fourier Transform and Properties

3.2.2 definitions:1-D DFT example

If a signal is expressed as $f(x)=\{2,3,4,4\}$, its DFT are:

$$F(0)=\sum_{x=0}^{3} f(x)\exp(0)= f(0)+ f(1)+ f(2)+ f(3)=13$$

$$F(1)=\sum_{x=0}^{3} f(x)\exp(-j2\pi x/4)=2e^{0}+3e^{-j\pi/2}+4e^{-j\pi}+4e^{-j3\pi/2}=-2+j$$

$$F(2)=\sum_{x=0}^{3} f(x)\exp(-j4\pi x/4)=2e^{0}+3e^{-j\pi}+4e^{-2j\pi}+4e^{-j3\pi}=-1$$

$$F(3)=\sum_{x=0}^{3} f(x)\exp(-j6\pi x/4)=2e^{0}+3e^{-j3\pi/2}+4e^{-j3\pi}+4e^{-j9\pi/2}=-2-j$$

3.2 Fourier Transform and Properties

3.2.2 definitions:1-D DFT example

And the Fourier spectra are :

$$|F(0)| = 13$$

$$|F(1)| = [(-2)^2 + 1^2]^{1/2} = \sqrt{5}$$

$$|F(2)| = [(-1)^2]^{1/2} = 1$$

$$|F(3)| = [(-2)^2 + (-1)^2]^{1/2} = \sqrt{5}$$

$$\phi(0) = 0$$

$$\phi(1) = 0.85\pi$$

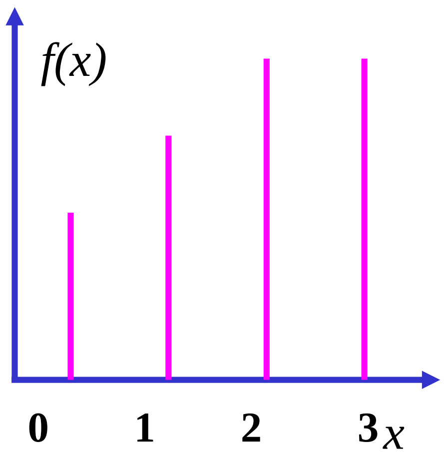
$$\phi(3) = \pi$$

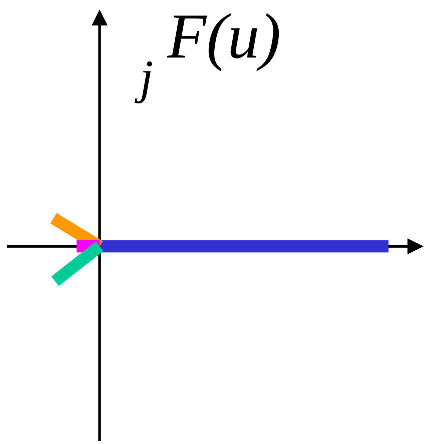
$$\phi(3) = -0.85\pi$$

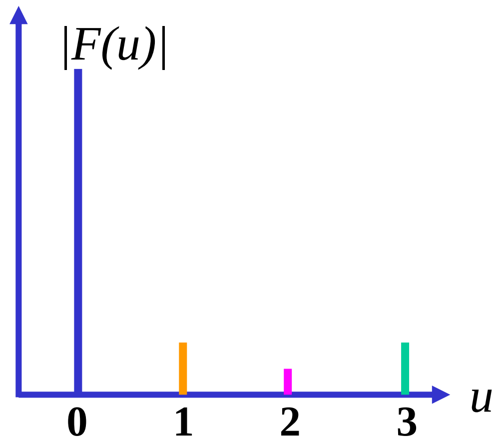
3.2 Fourier Transform and Properties

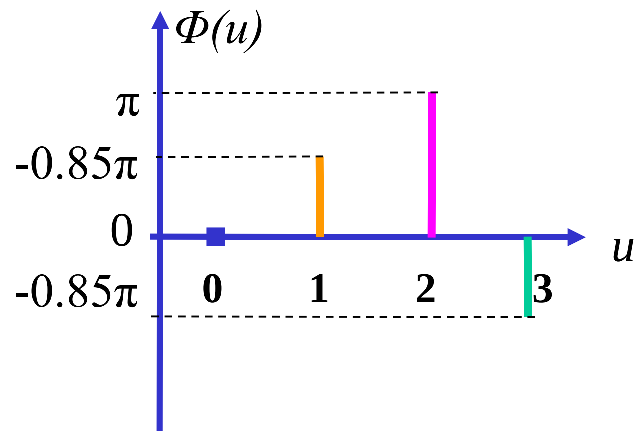
3.2.2 definitions:1-D DFT example

Graphic illustration ：









Fourier Transform and Properties

3.2.2 definitions:1-D DFT example

If a signal is expressed as *f(x)={3,2,3,1,4,5,0,2}*, its DFT are:

$$F(0)=\sum_{x=0}^{7} f(x)\exp(0)=f(0)+f(1)+f(2)+f(3)$$

$$+f(4)+f(5)+f(6)+f(7)=20$$

$$F(1)=\sum_{x=0}^{7} f(x)\exp(-j2\pi x/8)=3e^{0}+2e^{-j\pi/4}+3e^{-j\pi/2}+1e^{-j3\pi/4}+4e^{-j\pi}$$

$$+5e^{-j5\pi/4}+0e^{-j3\pi/2}+2e^{-j7\pi/4}=-2.4142-j0.1716$$

$$F(2)=\sum_{x=0}^{7} f(x)\exp(-j4\pi x/8)=3e^{0}+2e^{-j\pi/2}+3e^{-j\pi}+1e^{-j3\pi/2}+4e^{-j2\pi}$$

$$+5e^{-j5\pi/2}+0e^{-j3\pi}+2e^{-j7\pi/2}=4-j4$$

$$F(3)=\sum_{x=0}^{7} f(x)\exp(-j6\pi x/8)=3e^{0}+2e^{-j3\pi/4}+3e^{-j3\pi/2}+1e^{-j9\pi/4}+4e^{-j3\pi}$$

$$+5e^{-j15\pi/4}+0e^{-j9\pi/2}+2e^{-j21\pi/4}=0.4142+j5.8284$$

$$F(4)=\sum_{x=0}^{7} f(x)\exp(-j8\pi x/8)=3e^{0}+2e^{-j\pi}+3e^{-j2\pi}+1e^{-j3\pi}+4e^{-j4\pi}$$

$$+5e^{-j5\pi}+0e^{-j6\pi}+2e^{-j7\pi}=0$$

$$F(5)=\sum_{x=0}^{7} f(x)\exp(-j10\pi x/8)=3e^{0}+2e^{-j5\pi/4}+3e^{-j5\pi/2}+1e^{-j15\pi/4}+4e^{-j5\pi}$$

$$+5e^{-j25\pi/4}+0e^{-j15\pi/2}+2e^{-j35\pi/4}=0.4142-j5.8284$$

$$F(6)=\sum_{x=0}^{7} f(x)\exp(-j12\pi x/8)=3e^{0}+2e^{-j3\pi/2}+3e^{-j3\pi}+1e^{-j9\pi/2}+4e^{-j6\pi}$$

$$+5e^{-j15\pi/2}+0e^{-j9\pi}+2e^{-j21\pi/2}=4+j4$$

$$F(7)=\sum_{x=0}^{7} f(x)\exp(-j14\pi x/8)=3e^{0}+2e^{-j7\pi/4}+3e^{-j7\pi/2}+1e^{-j21\pi/4}+4e^{-j7\pi}$$

$$+5e^{-j35\pi/4}+0e^{-j21\pi/2}+2e^{-j49\pi/4}=-2.4142+j0.1716$$

3.2 Fourier Transform and Properties

3.2.2 definitions:1-D DFT example

And the Fourier spectra are :

$$|F(0)| = 20 \qquad \phi(0) = 0$$

$$|F(1)| = [(-2.4142)^2 + (-0.1716)^2]^{1/2} = 2.4203 \qquad \phi(1) = -0.9774\pi$$

$$|F(2)| = [4^2 + (-4)^2]^{1/2} = 5.6569 \qquad \phi(2) = -0.25\pi$$

$$|F(3)| = [0.4142^2 + 5.8284^2]^{1/2} = 5.8431 \qquad \phi(3) = 0.4774\pi$$

$$|F(4)| = 0 \qquad \phi(4) = \pi$$

$$|F(5)| = [0.4142^2 + (-5.8284)^2]^{1/2} = 5.8431 \qquad \phi(5) = -0.4774\pi$$

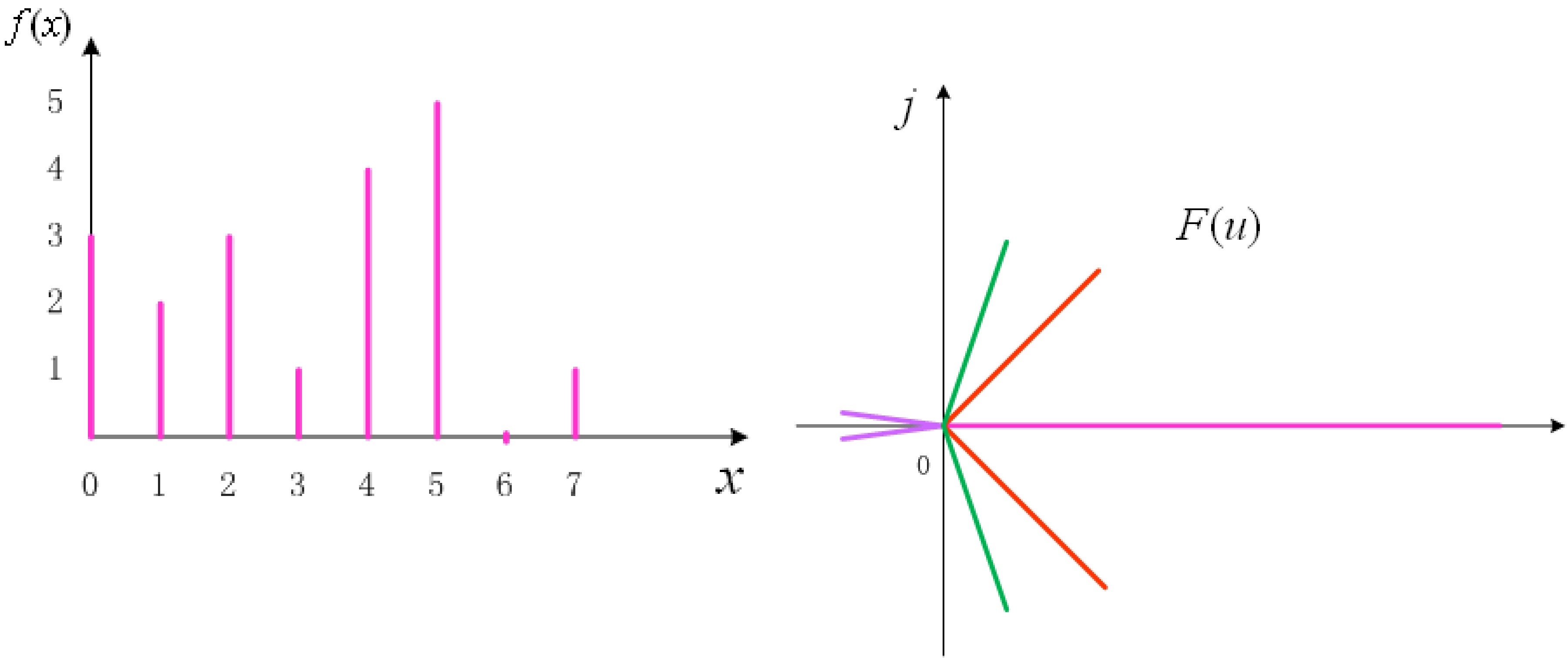
$$|F(6)| = [4^2 + 4^2]^{1/2} = 5.6569 \qquad \phi(6) = 0.25\pi$$

$$|F(7)| = [(-2.4142)^2 + 0.1716^2]^{1/2} = 2.4203 \qquad \phi(7) = 0.9774\pi$$

3.2 Fourier Transform and Properties

3.2.2 definitions:1-D DFT example

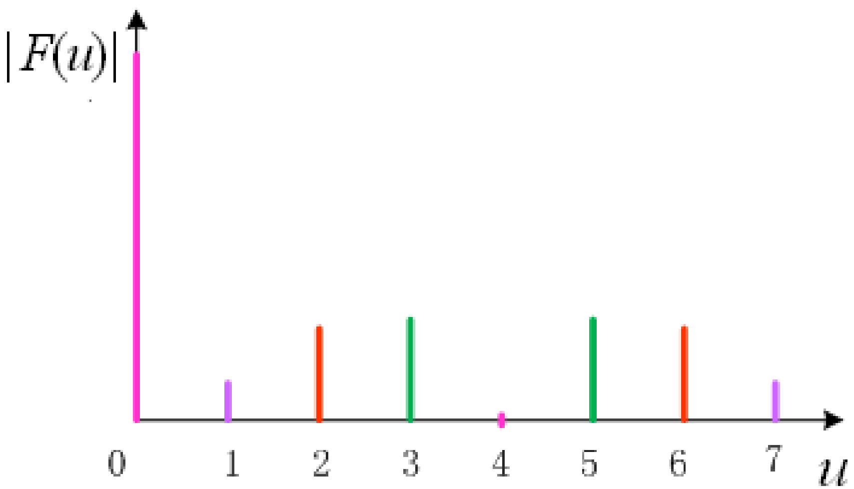
Graphic illustration ：

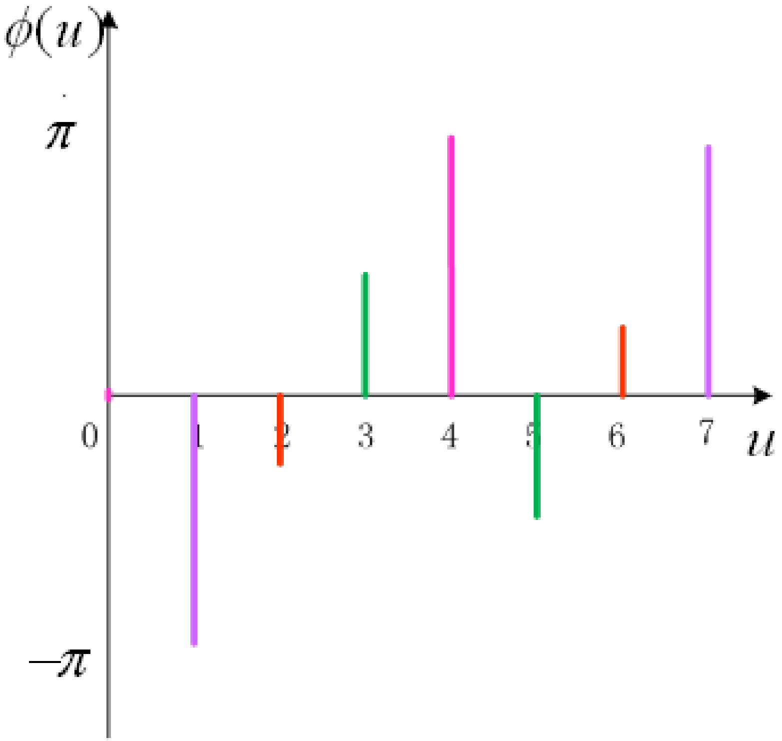


3.2 Fourier Transform and Properties

3.2.2 definitions:1-D DFT example

Graphic illustration ：

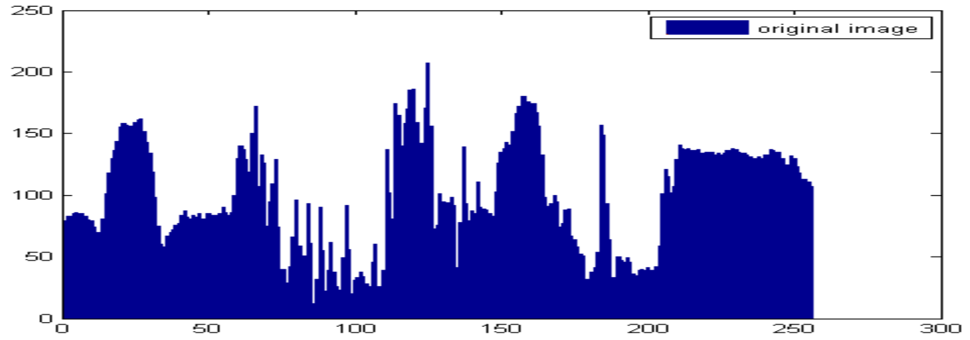




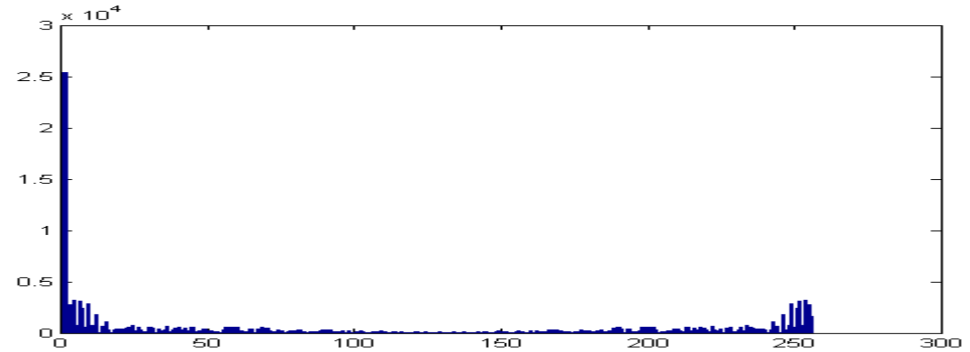
3.2 Fourier Transform and Properties

3.2.2 definitions:1-D DFT example

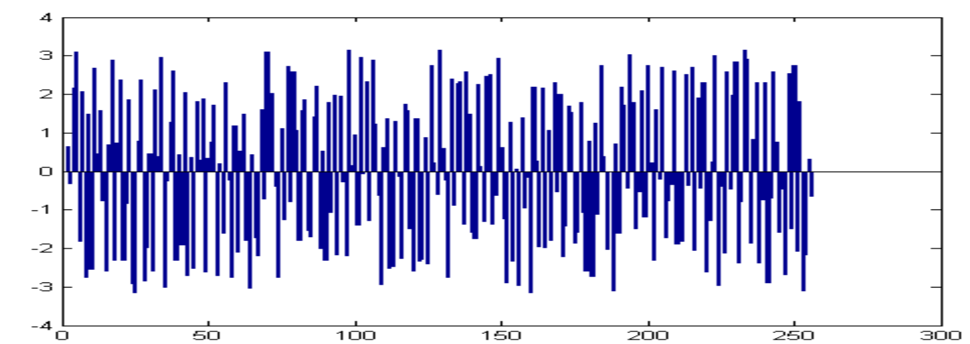
line 128 of Lena



spectrum



phase spectrum



3.2 Fourier Transform and Properties

3.2.2 definitions: 2D-DFT

The Two-Dimensional Discrete Fourier Transform and its Inverse

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} \qquad \begin{matrix} u=0,1,\cdots M-1 \\ v=0,1,\cdots N-1 \end{matrix}$$

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)} \qquad \begin{matrix} x=0,1,\cdots M-1 \\ y=0,1,\cdots N-1 \end{matrix}$$

3.2 Fourier Transform and Properties

3.2.2 definitions: 2D-DFT

Magnitude or spectrum

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

Phase angle or phase spectrum

$$\phi(u,v) = \arctan\left[I(u,v)/R(u,v)\right]$$

Power spectrum
（Spectral density)

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

3.2 Fourier Transform and Properties

3.2.2 definitions ：Display

- Usually the Fourier spectra are displayed as intensity function.
- many image spectra decrease rather rapidly as a function of increasing frequency
- their high-frequency terms have a tendency to become obscured when displayed in image.

3.2 Fourier Transform and Properties

3.2.2 definitions :Display

- A useful processing technique which compensates for this difficulty consists of displaying the function

$$D_1(u,v) = \log(1 + |F(u,v)|)$$

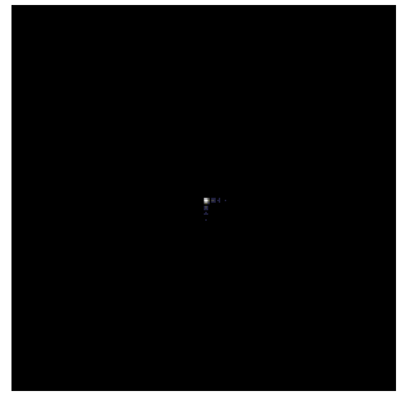
Or

$$D_2(u,v) = \begin{cases} |F(u,v)| + 100 \\ 255 \quad \text{if } |F(u,v)| + 100 > 255 \end{cases}$$

- instead of $|F(u,v)|$

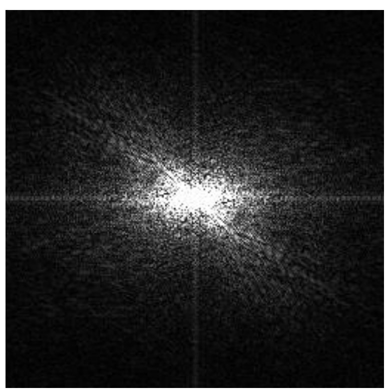
3.2 Fourier Transform and Properties

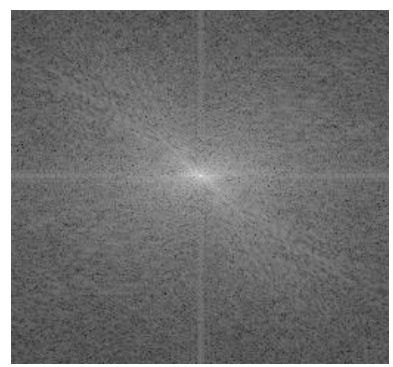
3.2.2 definitions :Display



$|F(u,v)|$

$D_2(u,v)$



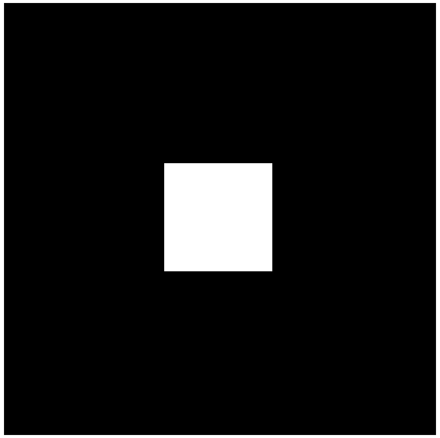


$D_1(u,v)$

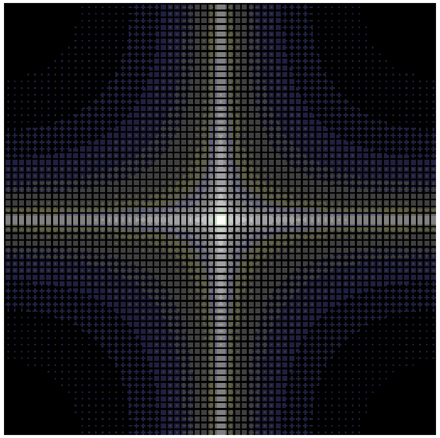
3.2 Fourier Transform and Properties

3.2.2 definitions: 2D-DFT

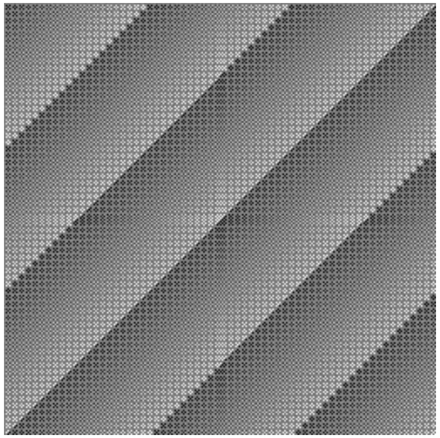
image



Magnitude



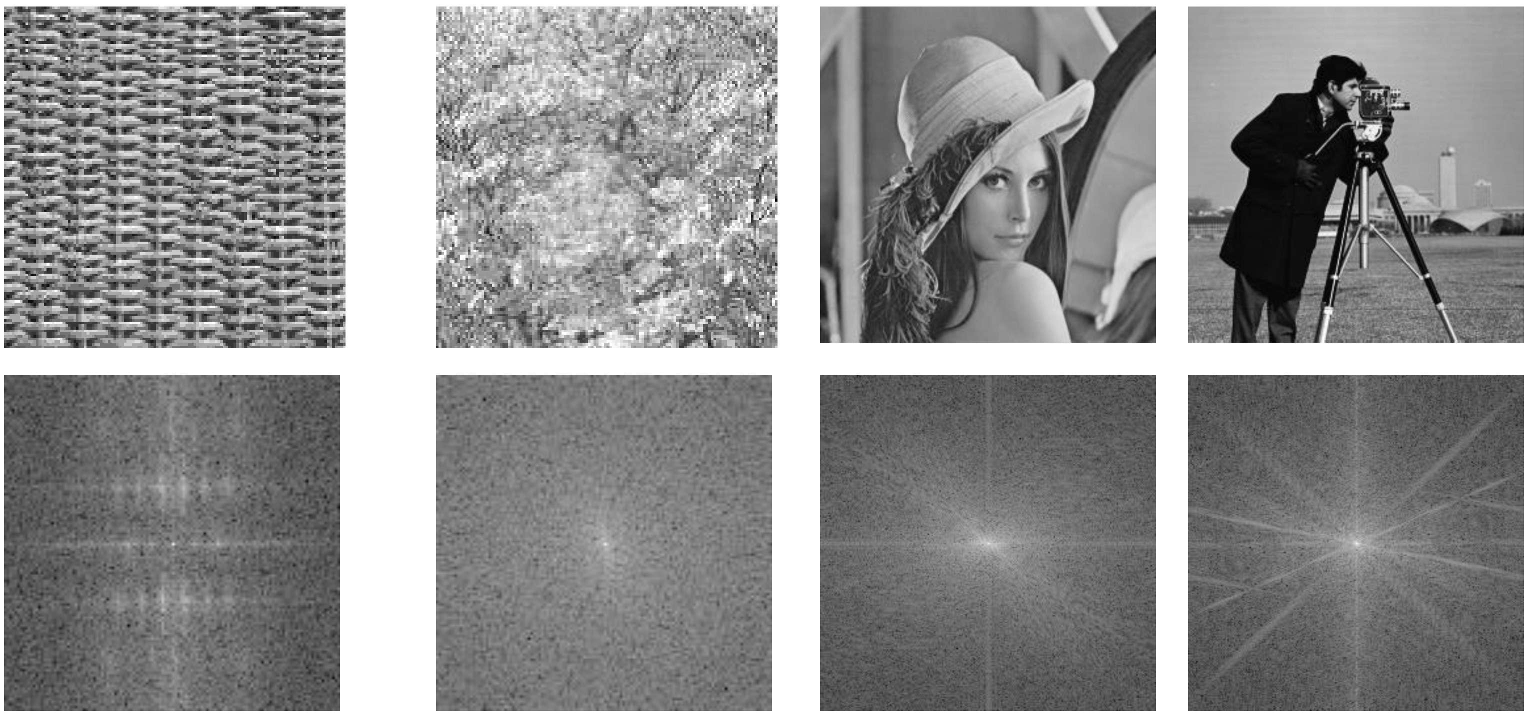
Phase



3.2 Fourier Transform and Properties

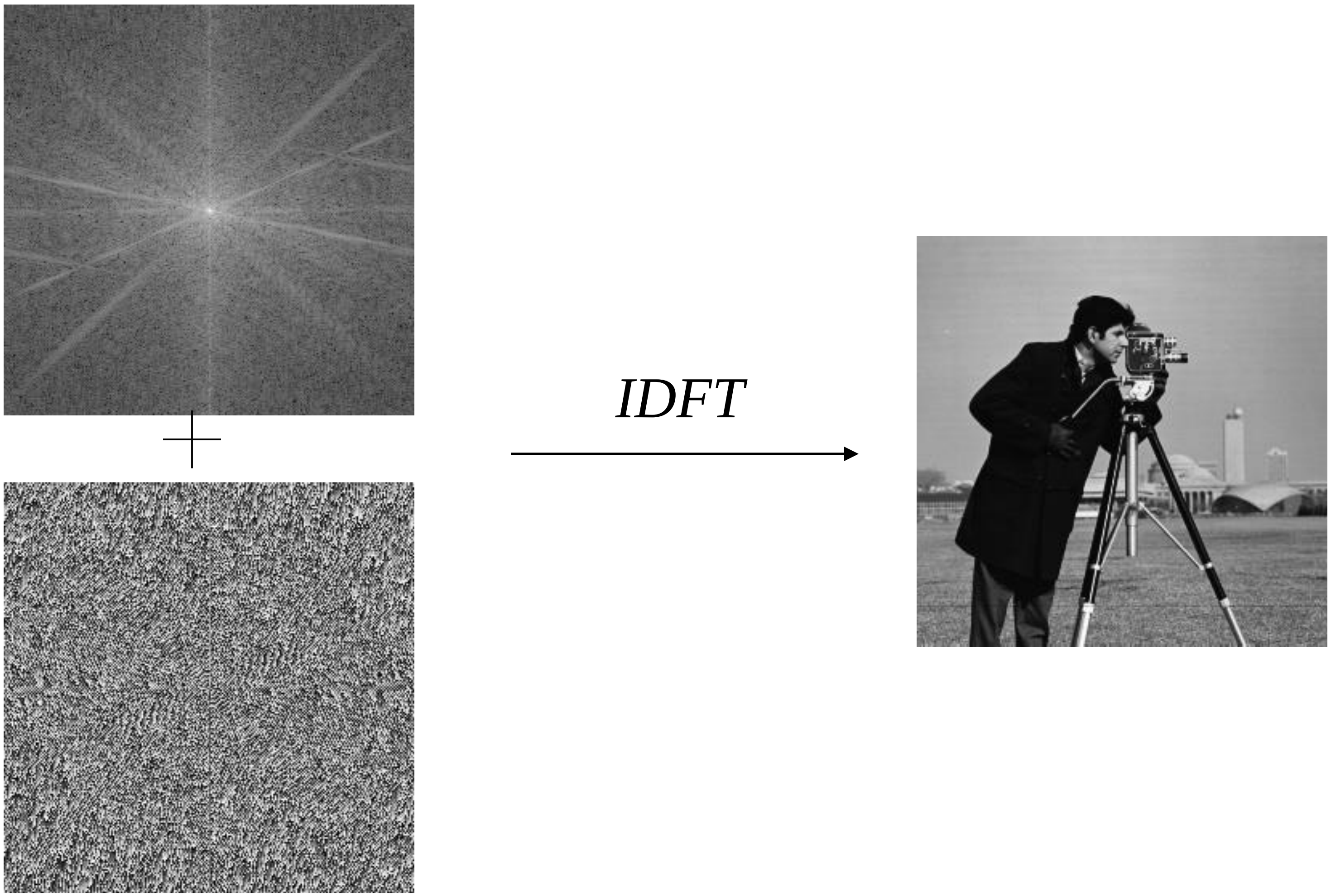
3.2.2 definitions: 2D-DFT

Typical images and their spectra



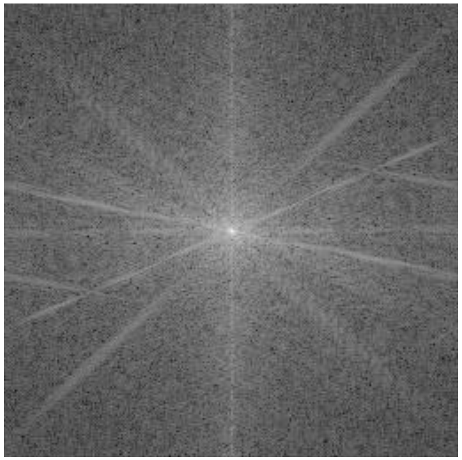
3.2 Fourier Transform and Properties

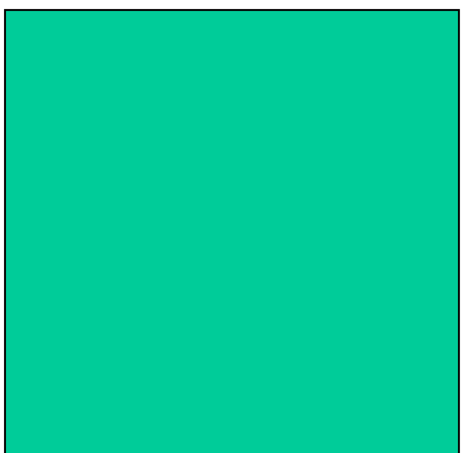
3.2.2 definitions: 2D-IDFT



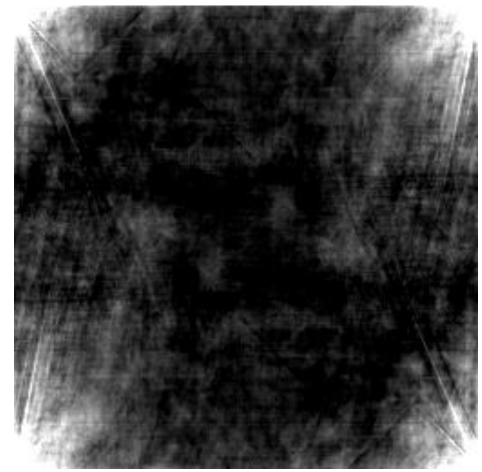
3.2 Fourier Transform and Properties

3.2.2 definitions: 2D-IDFT



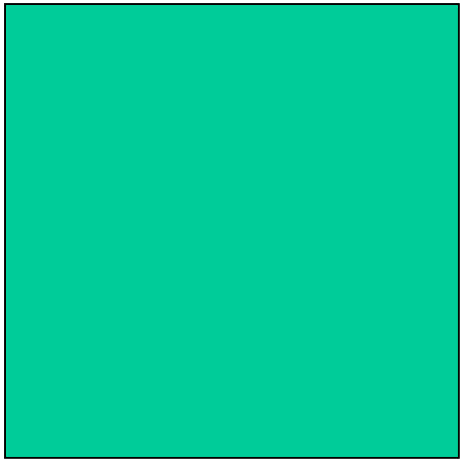


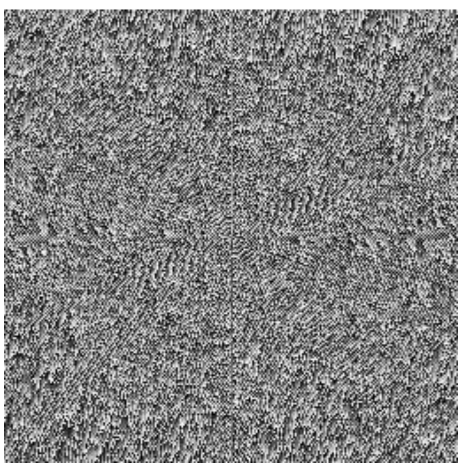
IDFT $\longrightarrow$



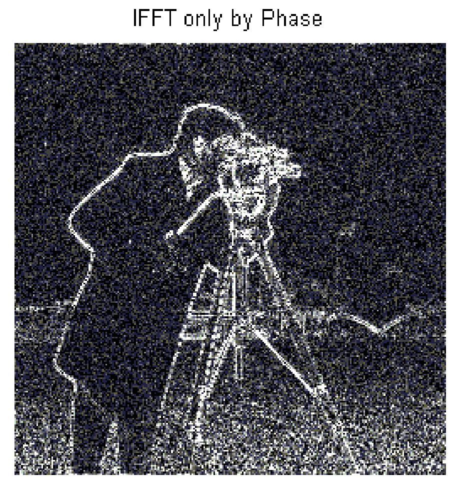
3.2 Fourier Transform and Properties

3.2.2 definitions: 2D-IDFT





IDFT



3.2 Fourier Transform and Properties

3.2.3 Properties: separability

The DFT pair can be expressed in the separable forms:

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\exp\left[\frac{-j2\pi ux}{N}\right]\sum_{y=0}^{N-1}f(x,y)\exp\left[\frac{-j2\pi vy}{N}\right]$$

$$f(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\exp\left[\frac{j2\pi ux}{N}\right]\sum_{v=0}^{N-1}F(u,v)\exp\left[\frac{j2\pi vy}{N}\right]$$

3.2 Fourier Transform and Properties

3.2.3 Properties: separability

The principal of the separability property is that *f(x,y)* or *F(u,v)* can be obtained in two steps by successive applications of the 1-D Fourier transform or its inverse

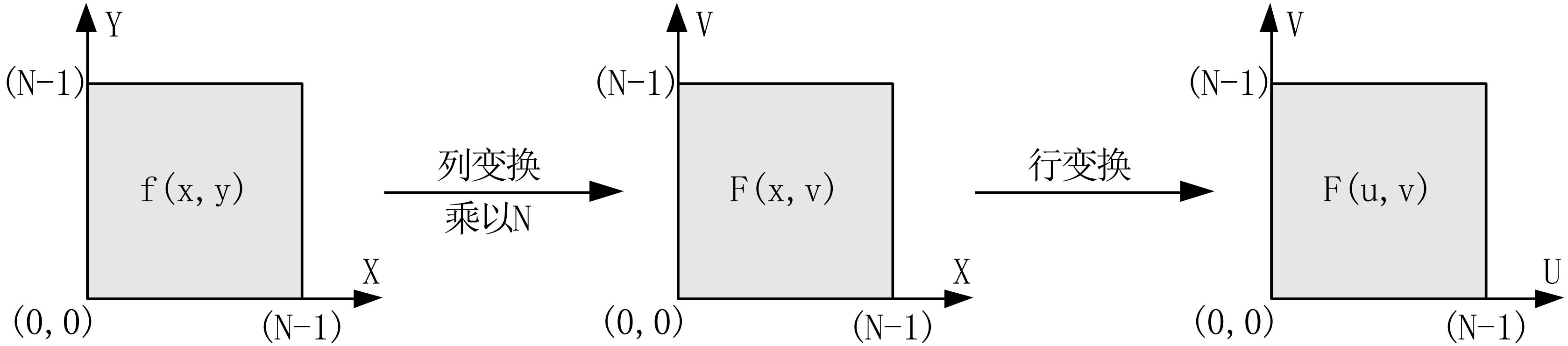
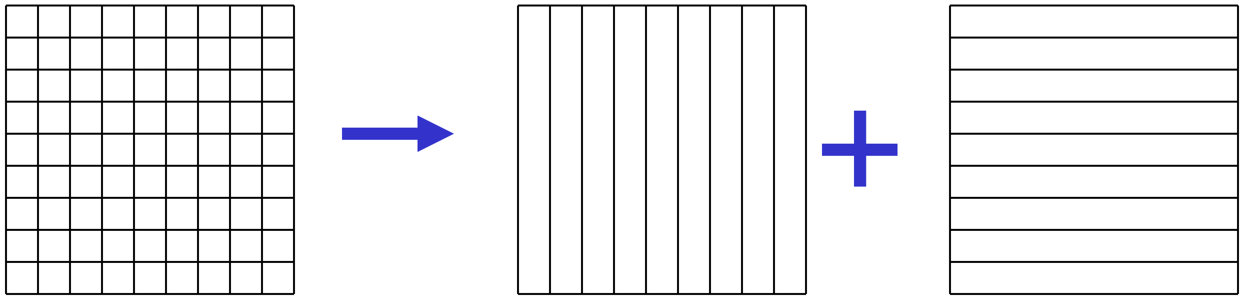


图3.2.2　由2步1-D变换计算2-D变换

3.2 Fourier Transform and Properties

3.2.3 Properties: separability

For a N*N image, it can be separated into 2N 1D-DFT



3.2 Fourier Transform and Properties

3.2.3 Properties: periodicity and conjugate symmetry

The discrete Fourier transform and its inverse are *periodic* with Period *N*

$$F(u,v) = F(u+N,v) = F(u,v+N) = F(u+N,v+N)$$

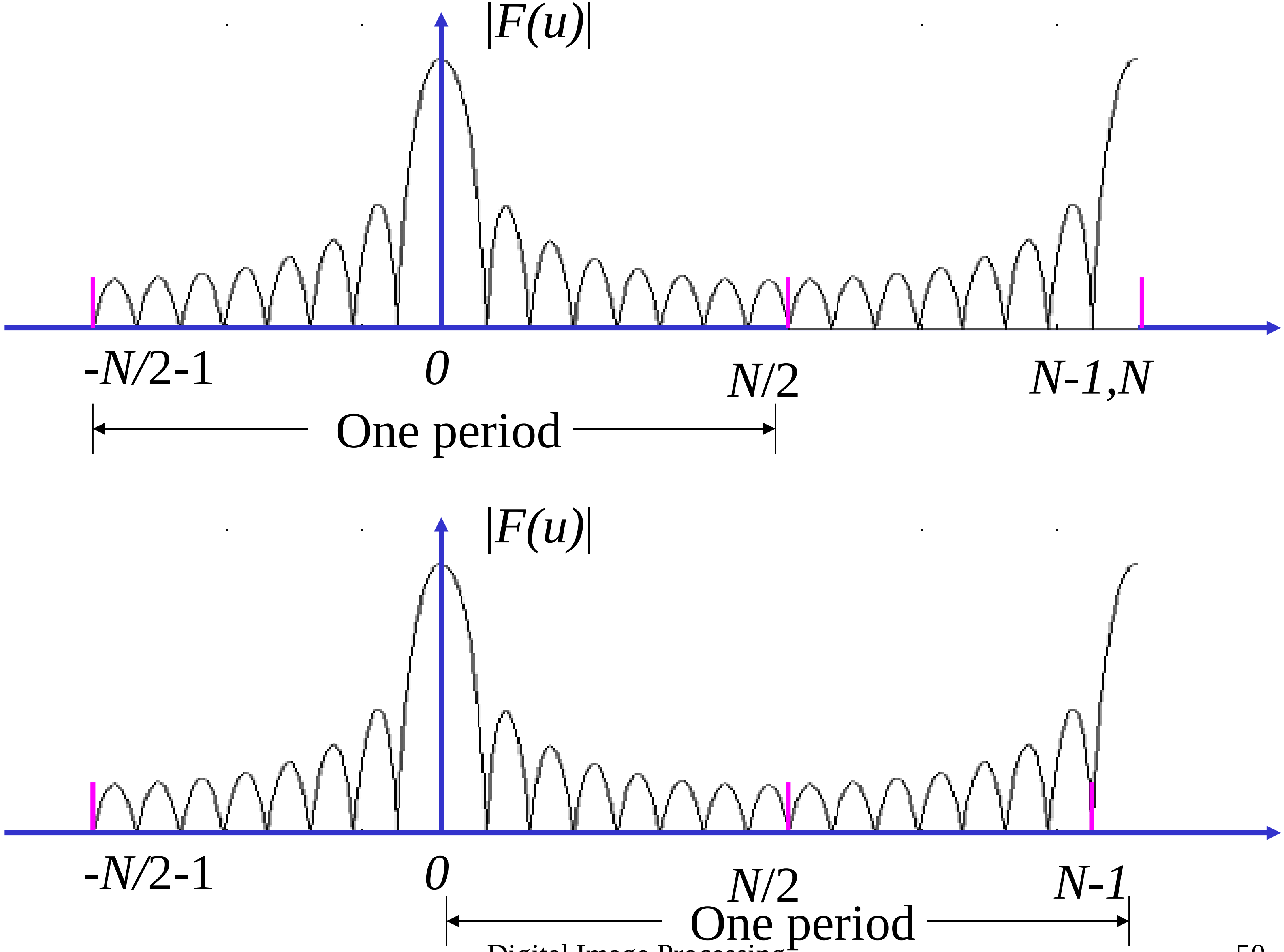
The Fourier transform also exhibits conjugate symmetry since

$$F(u,v) = F^*(-u,-v)$$

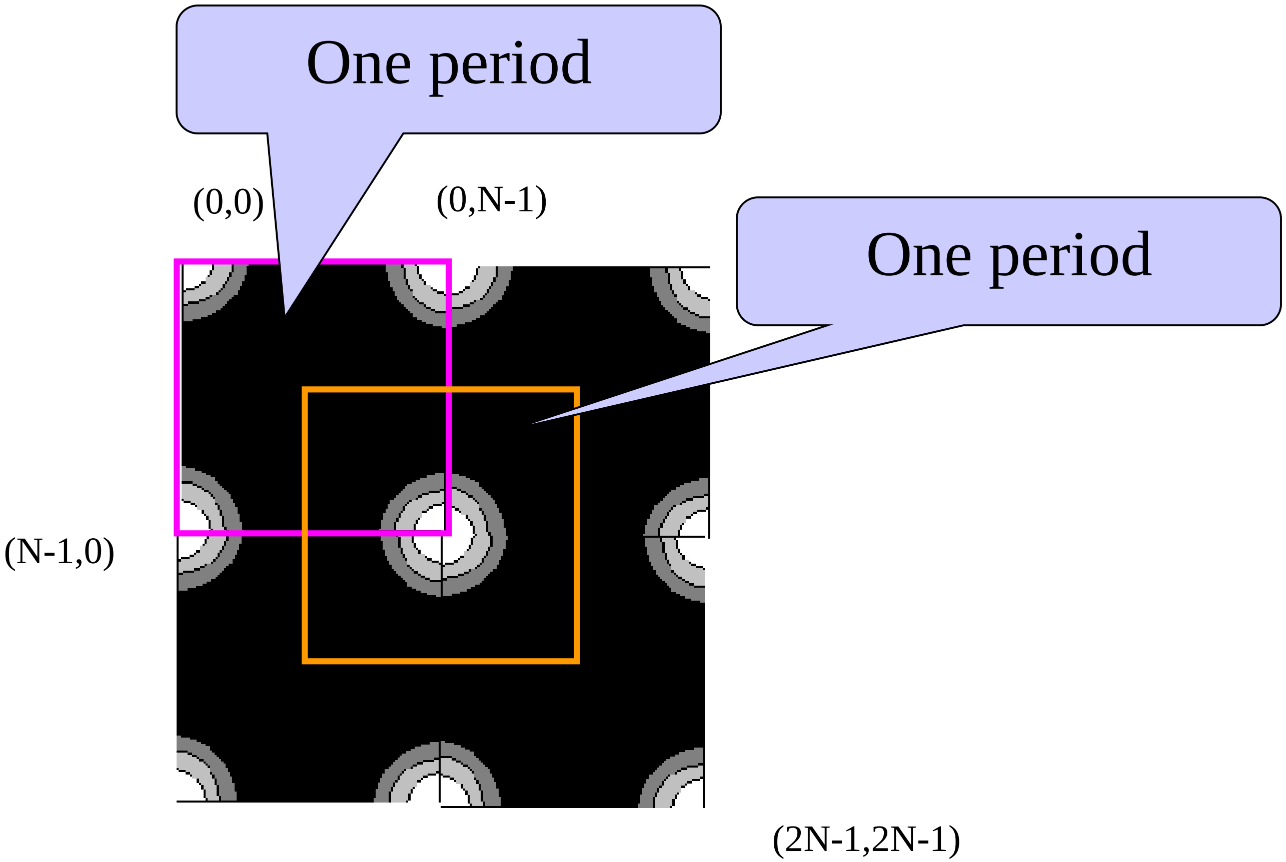
Or more interestingly,

$$\left|F(u,v)\right| = \left|F(-u,-v)\right|$$

1D periodicity and conjugate symmetry

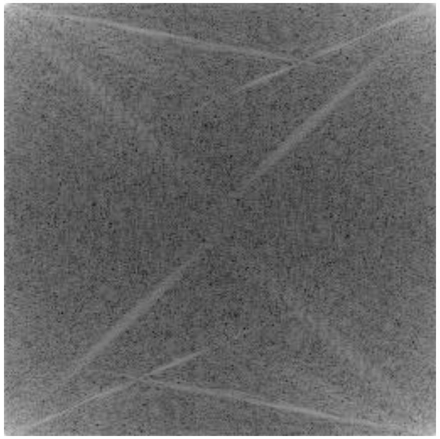


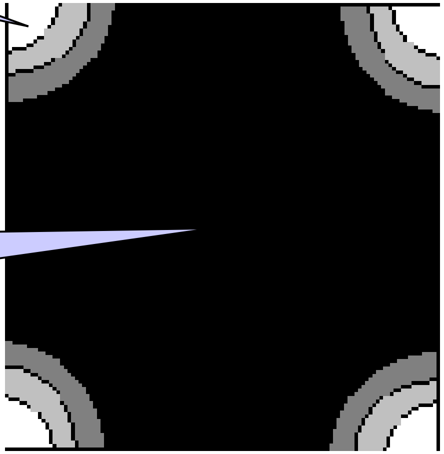
2D periodicity and conjugate symmetry



3.2 Fourier Transform and Properties

3.2.3 Properties: periodicity and conjugate symmetry





3.2 Fourier Transform and Properties

3.2.3 Properties: translation

The translation properties of the Fourier transform pair are given by

$$f(x,y)\exp\left[j2\pi(u_0x+v_0y)/N\right]\Leftrightarrow F(u-u_0,v-v_0)$$

and

$$f(x-x_0,y-y_0)\Leftrightarrow F(u,v)\exp\left[-j2\pi(ux_0+vy_0)/N\right]$$

3.2 Fourier Transform and Properties

3.2.3 Properties: translation

A shift in *f(x,y)* does not affect the magnitude of its Fourier transform since

$$|F(u,v)\exp\left[-j2\pi(ux_0+vy_0)/N\right]|=|F(u,v)|$$

3.2 Fourier Transform and Properties

3.2.3 Properties: translation

Example: in the case $u_0 = v_0 = N/2$, it follows that

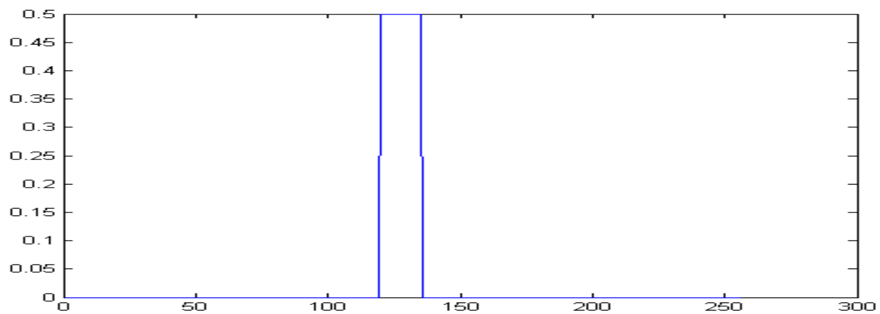
and

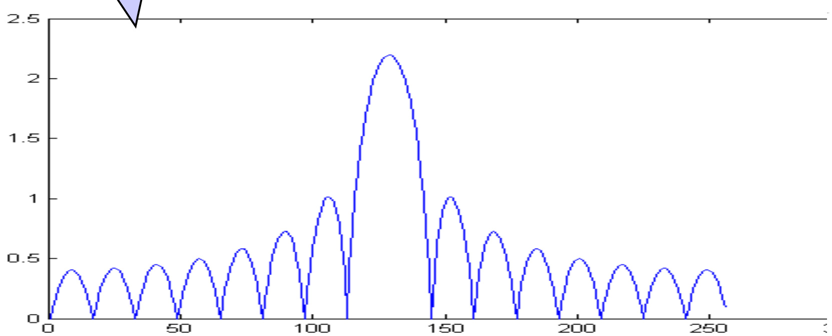
$$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - N/2, v - N/2)$$

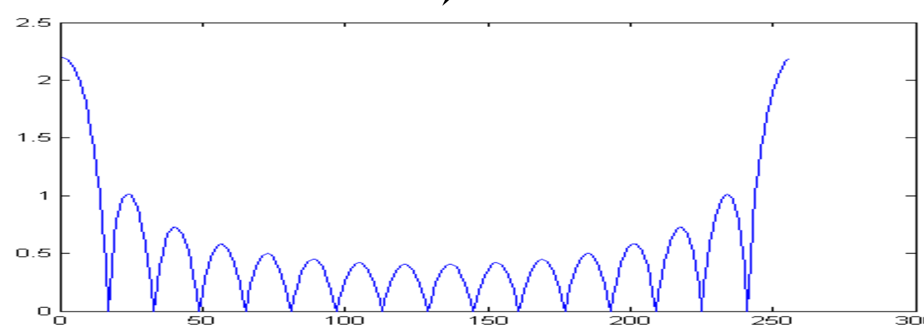
3.2 Fourier Transform and Properties

3.2.3 Properties: translation

Example of One-Dimensional DFT shift





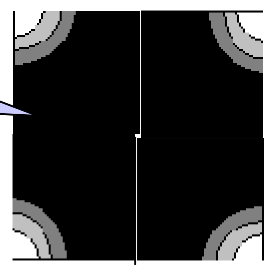


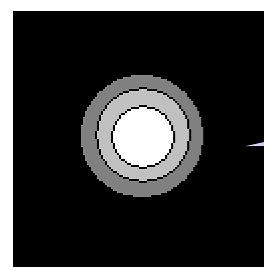
3.2 Fourier Transform and Properties

3.2.3 Properties: translation

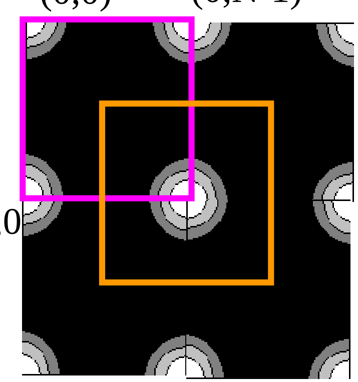
Example of Two-Dimensional DFT shift

F(u,v)
No shift





F(u,v)
shifted

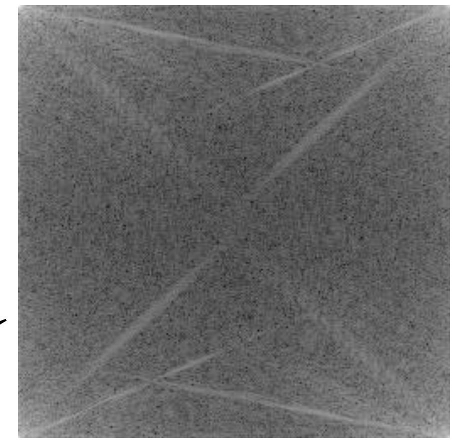


3.2 Fourier Transform and Properties

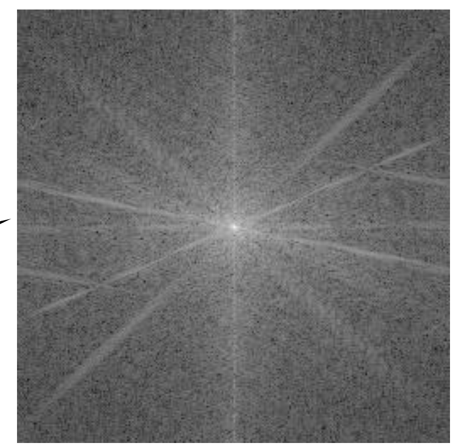
3.2.3 Properties: translation

Example of Two-Dimensional DFT shift

f(x,y)



F(u)
No shift



F(u,v)
shifted

3.2 Fourier Transform and Properties

3.2.3 Properties: rotation

If we introduce the polar coordinates

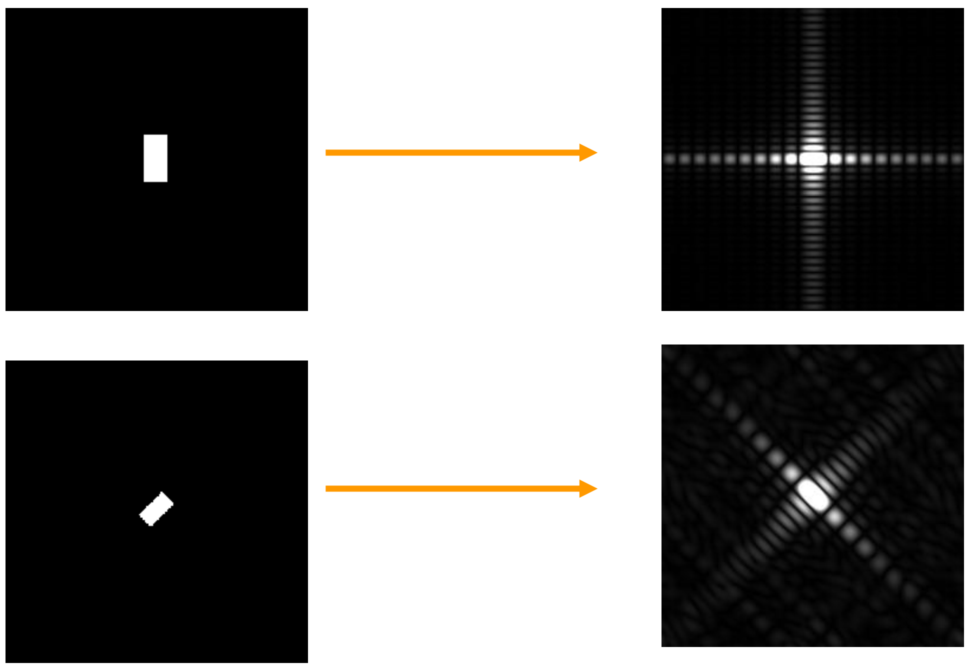
$$x = r\cos(\theta) \quad y = r\sin(\theta) \; u = \omega\cos(\phi) \; v = \omega\sin(\phi)$$

Then f(x,y) and F(u,v)become $f(r,\theta)$ and $F(\omega,\phi)$ respectively

$$f(r,\theta+\theta_0) \Leftrightarrow F(w,\phi+\theta_0)$$

3.2 Fourier Transform and Properties

3.2.3 Properties: rotation



3.2 Fourier Transform and Properties

3.2.3 Properties: Distributivity

It follows directly from the definition of the transform pair that,

$$F\left\{f_1(x,y)+f_2(x,y)\right\}=F\left\{f_1(x,y)\right\}+F\left\{f_2(x,y)\right\}$$

And, in general that,

$$F\left\{f_1(x,y)\cdot f_2(x,y)\right\}\neq F\left\{f_1(x,y)\right\}\cdot\left\{f_2(x,y)\right\}$$

3.2 Fourier Transform and Properties

3.2.3 Properties: scaling

It is also easy to show that for two scalar *a* and *b*

$$af(x, y) \Leftrightarrow aF(u, v)$$

and

$$f(ax, by) \Leftrightarrow \frac{1}{|ab|} F\left(\frac{u}{a}, \frac{v}{b}\right)$$

3.2 Fourier Transform and Properties

3.2.3 Properties: average value

A widely-used definition of the average value of a 2D Discrete function is given by the expression

$$\tilde{f}(x,y) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)$$

Substitution of u-v-0 in definition of 2D DFT yields

$$F(0,0) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y) \longrightarrow \tilde{f}(x,y) = \frac{1}{N}F(0,0)$$

3.2 Fourier Transform and Properties

3.2.3 Properties: convolution

The convolution of two functions *f(x)* and *g(x)*, denoted by *f(x)*g(x)*, is defined by the integral:

$$f(x)*g(x)=\int_{-\infty}^{\infty} f(z)g(x-z)dz$$

Where *z* is a dummy variable of integration

3.2 Fourier Transform and Properties

3.2.3 Properties: convolution

Example1: graphic illustration of convolution *f(x)*g(x)*

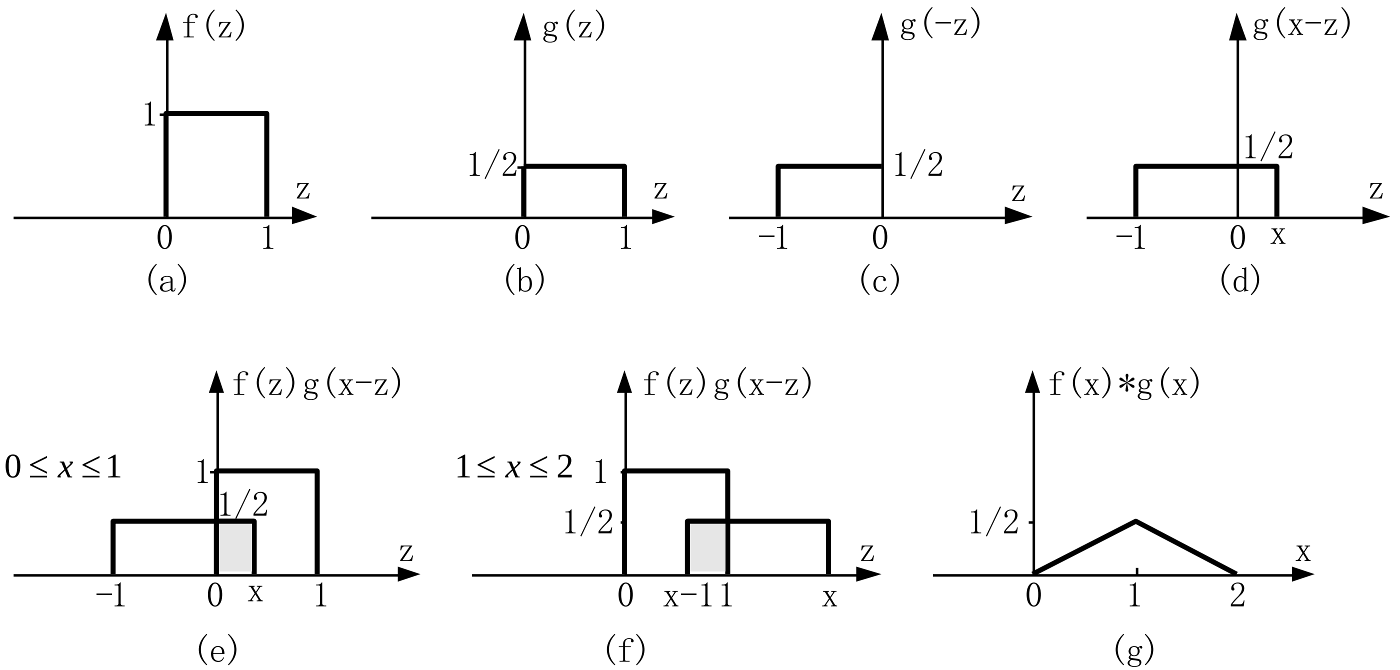
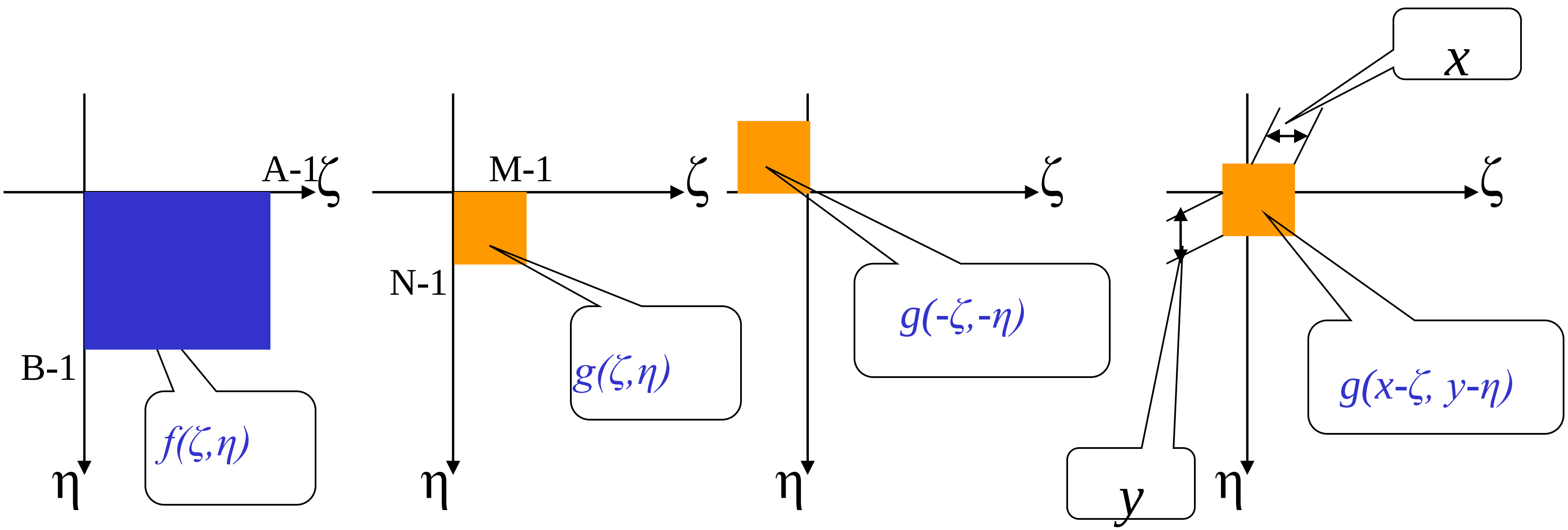


图3.2.3 1-D函数卷积示例

3.2 Fourier Transform and Properties

3.2.3 Properties: convolution

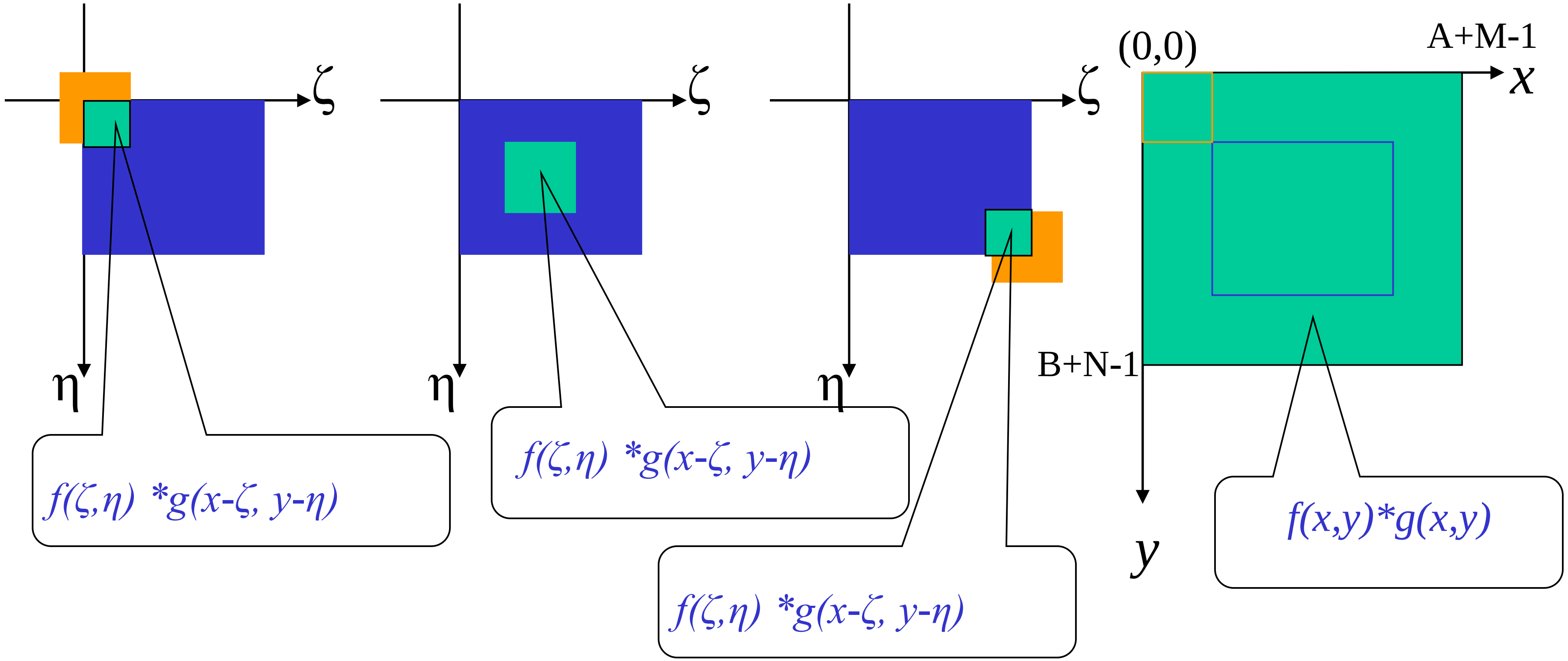
Example2: graphic illustration of convolution $f(x,y)*g(x,y)$



3.2 Fourier Transform and Properties

3.2.3 Properties: convolution

Example2: graphic illustration of convolution *f(x,y)*g(x,y)*



3.2 Fourier Transform and Properties

3.2.3 Properties: convolution

If *f(x,y)* has the Fourier transform *F(u,v)* and *g(x,y)* has the Fourier Transform *G(u,v)*, then *f(x,y)*g(x,y)* has the Fourier transform *F(u,v)G(u,v)*.This result, formally stated as:

$$f(x, y) * g(x, y) \Leftrightarrow F(u, v)G(u, v)$$

And the convolution in *frequency* domain reduces to multiplication In the *spatial*-domain

$$f(x, y)g(x, y) \Leftrightarrow F(u, v) * G(u, v)$$

3.2 Fourier Transform and Properties

3.2.3 Properties: convolution

Definition of 1D-discrete convolution

$$f_e(x) = \begin{cases} f(x) & 0 \le x \le A-1 \\ 0 & A \le x \le M-1 \end{cases}$$

$$g_e(x) = \begin{cases} g(x) & 0 \le x \le B-1 \\ 0 & B \le x \le M-1 \end{cases}$$

$$f_e(x) * g_e(x) = \frac{1}{M}\sum_{m=0}^{M-1} f_e(m) g_e(x-m) \qquad x=0,1,\ldots,M-1$$

3.2 Fourier Transform and Properties

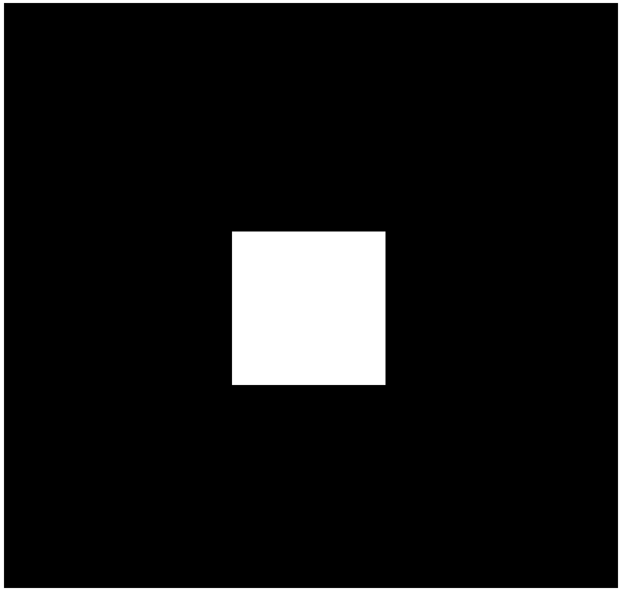
3.2.3 Properties: convolution

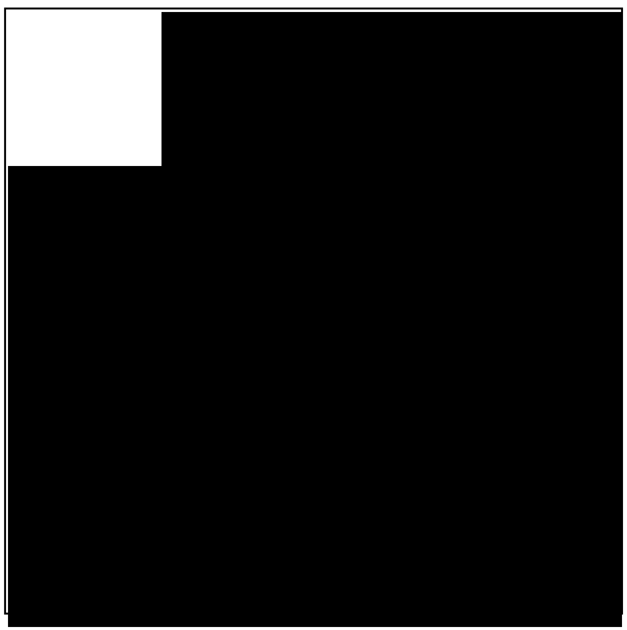
Definition of 2D-discrete convolution

$$f_e(x,y) = \begin{cases} f(x,y) & 0 \le x \le A-1 \text{ and } 0 \le y \le B-1 \\ 0 & A \le x \le M-1 \text{ or } B \le y \le N-1 \end{cases}$$

$$g_e(x,y) = \begin{cases} g(x,y) & 0 \le x \le C-1 \text{ and } 0 \le y \le D-1 \\ 0 & C \le x \le M-1 \text{ or } D \le y \le N-1 \end{cases}$$

$$f_e(x,y) * g_e(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_e(m,n) g_e(x-m, y-n) \qquad \begin{matrix} x = 0,1,\ldots, M-1 \\ y = 0,1,\ldots, N-1 \end{matrix}$$





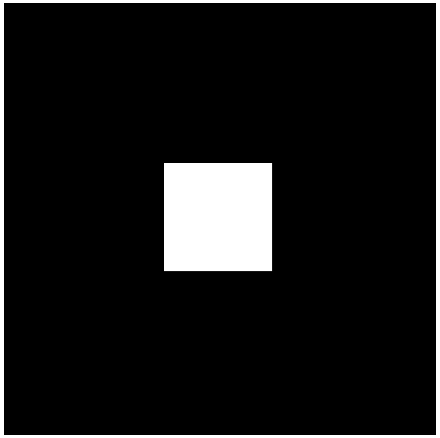
Question:

What is the differences between the amplitude and phase of the two images?

DFT

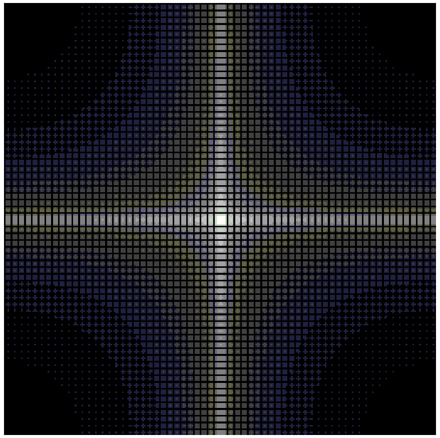
?

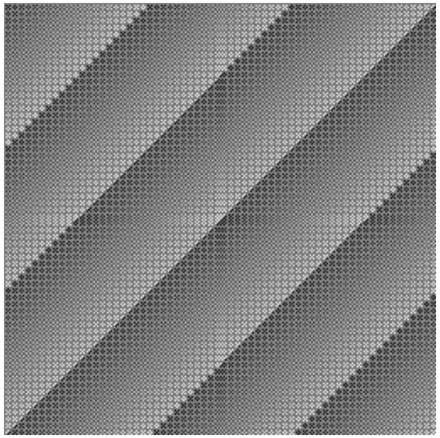
The answer is:



Amplitude

Phase

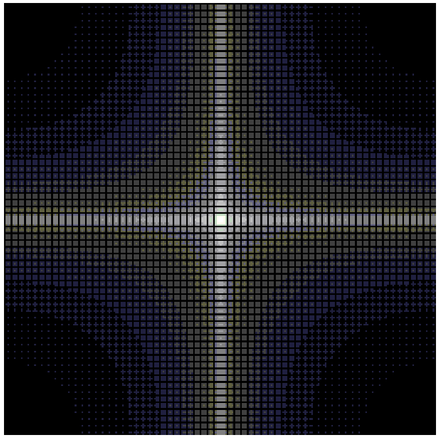


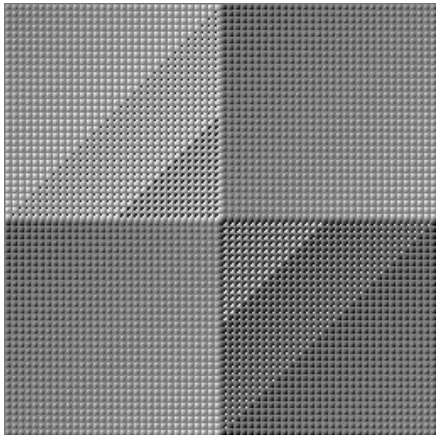


The answer is:

Amplitude

Phase





3.3 Other Separable Transforms

3.3.1 Discrete Cosine Transform (DCT)

The 1-D DCT pair is given by the expression:

$$C(u) = a(u)\sum_{x=0}^{N-1} f(x)\cos\left[\frac{(2x+1)u\pi}{2N}\right] \qquad x=0,1,\ldots N\text{-}1$$

$$f(x) = \sum_{u=0}^{N-1} a(u)C(u)\cos\left[\frac{(2x+1)u\pi}{2N}\right] \qquad u=0,1,\ldots N\text{-}1$$

where

$$a(u) = \begin{cases} \sqrt{1/N} & \text{when } u = 0 \\ \sqrt{2/N} & \text{when } u = 1, 2, \cdots, N-1 \end{cases}$$

3.3 Other Separable Transforms

3.3.1 Discrete Cosine Transform (DCT)

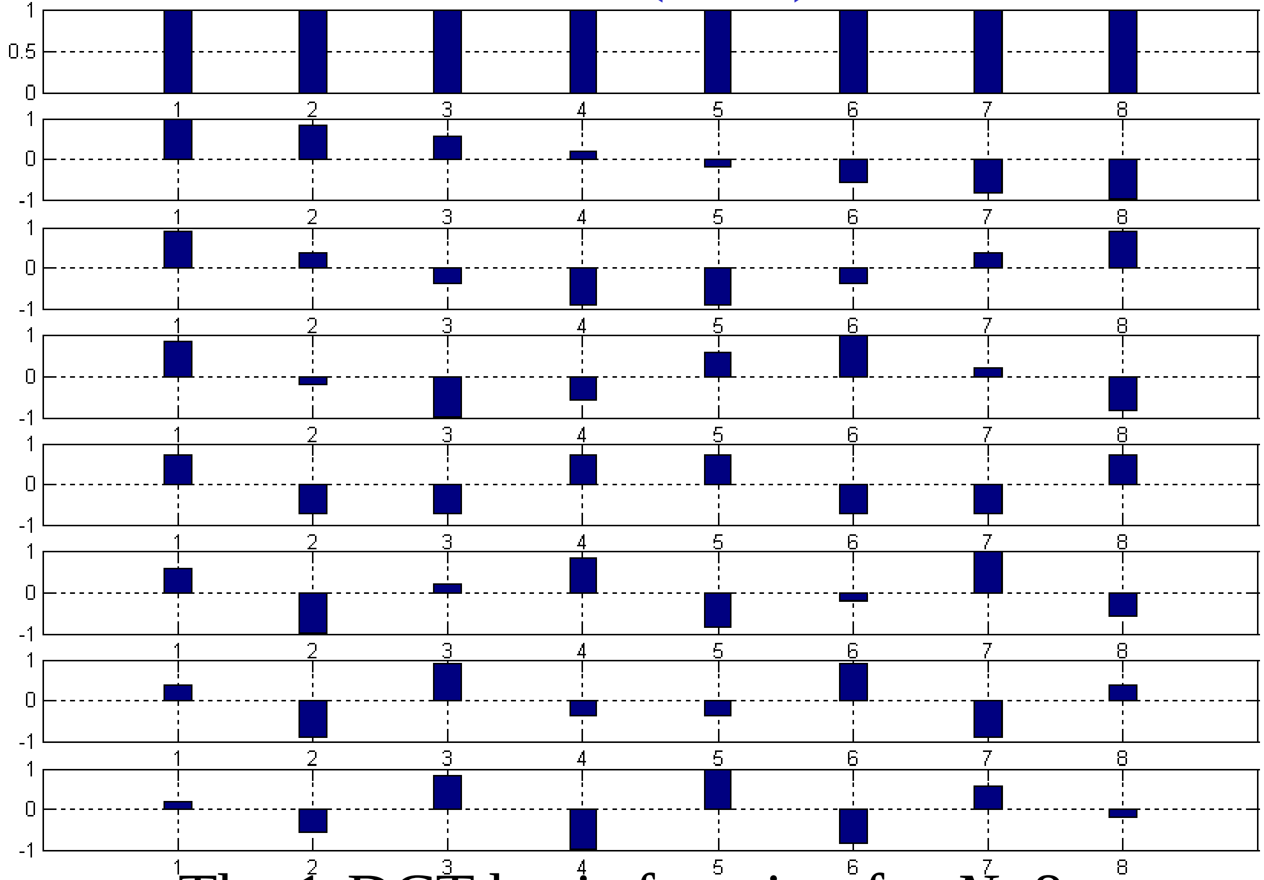
Basis function matrix

$$\begin{array}{ll} & \xrightarrow{\quad x \quad} \\ \downarrow u & \end{array}$$

$$C_N = \sqrt{\frac{2}{N}} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & \cdots & 1/\sqrt{2} \\ \cos \pi/2N & \cos 3\pi/2N & \cdots & \cos(2N-1)\pi/2N \\ \cos 2\pi/2N & \cos 6\pi/2N & \cdots & \cos 2(2N-1)\pi/2N \\ \cos 3\pi/2N & \cos 9\pi/2N & \cdots & \cos 3(2N-1)\pi/2N \\ \cos 4\pi/2N & \cos 12\pi/2N & \cdots & \cos 4(2N-1)\pi/2N \\ \vdots & \vdots & \ddots & \vdots \\ \cos(N-1)\pi/2N & \cos 3(N-1)\pi/2N & \cdots & \cos(2N-1)(N-1)\pi/2N \end{bmatrix}$$

3.3 Other Separable Transforms

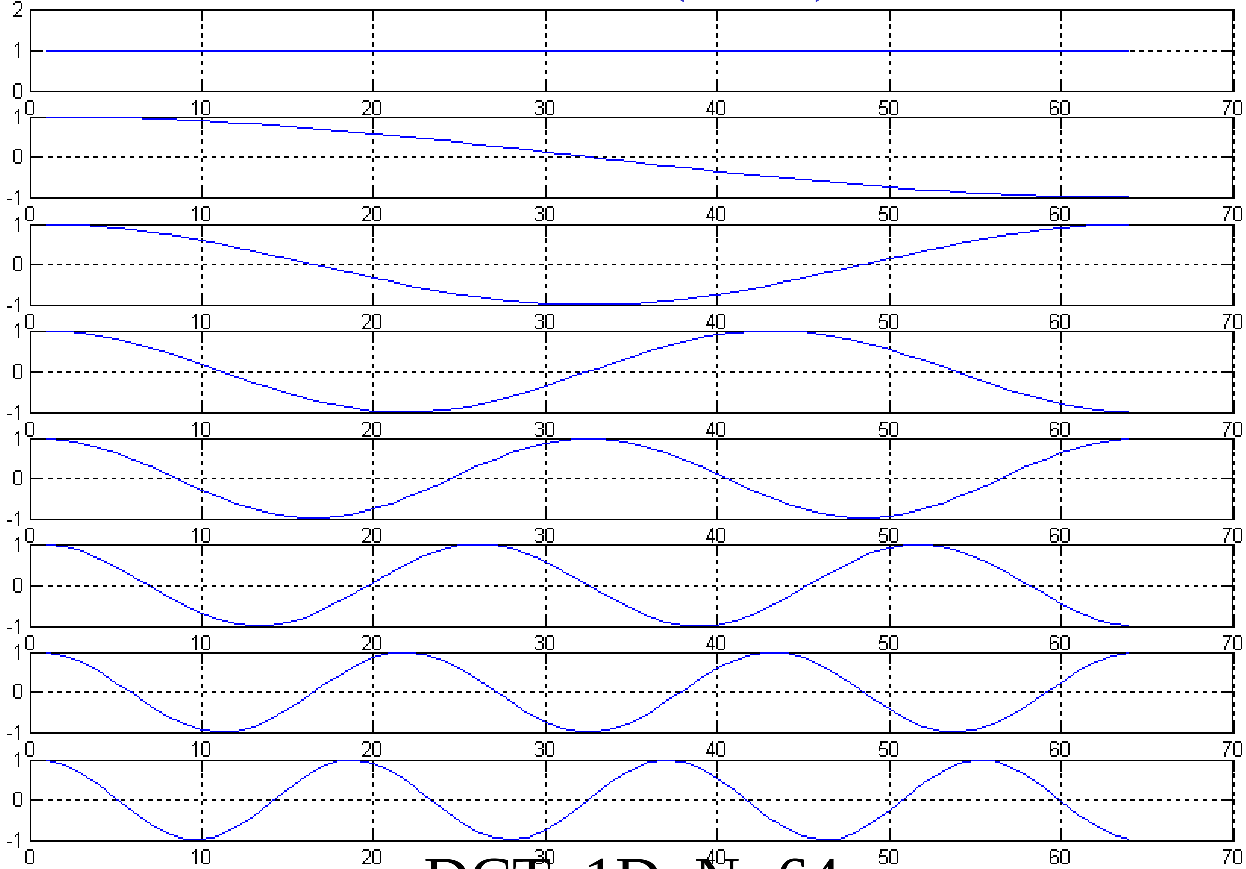
3.3.1 Discrete Cosine Transform (DCT)



The 1-DCT basis function for N=8

3.3 Other Separable Transforms

3.3.1 Discrete Cosine Transform (DCT)



DCT_1D N=64

3.3 Other Separable Transforms

3.3.1 Discrete Cosine Transform (DCT)

The 2-D DCT pair is given by the expression:

$$C(u,v) = a(u)a(v)\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos\left[\frac{(2x+1)u\pi}{2N}\right]\cos\left[\frac{(2y+1)v\pi}{2N}\right]$$

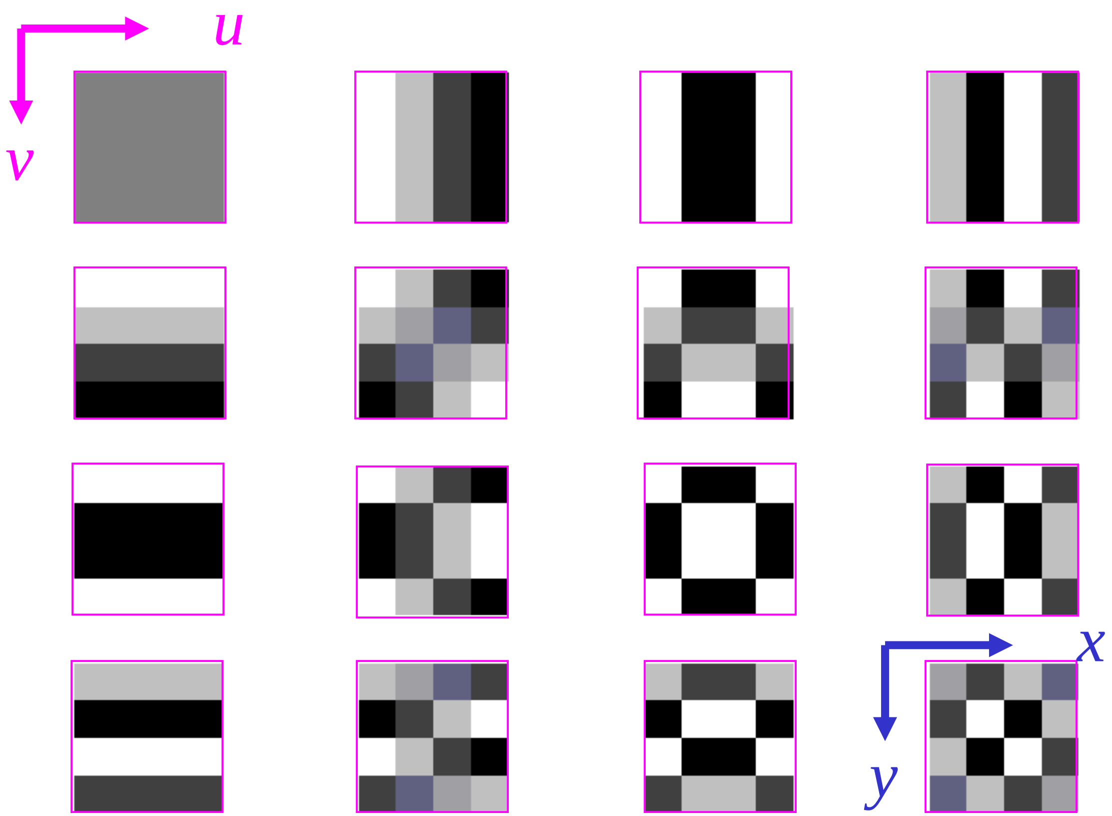
$u,v=0,1,\ldots N\text{-}1$

$$f(x,y) = \sum_{u=0}^{N-1}\sum_{v=0}^{N-1} a(u)a(v)C(u,v)\cos\left[\frac{(2x+1)u\pi}{2N}\right]\cos\left[\frac{(2y+1)v\pi}{2N}\right]$$

$x,y=0,1,\ldots N\text{-}1$

3.3 Other Separable Transforms

3.3.1 Discrete Cosine Transform (DCT)



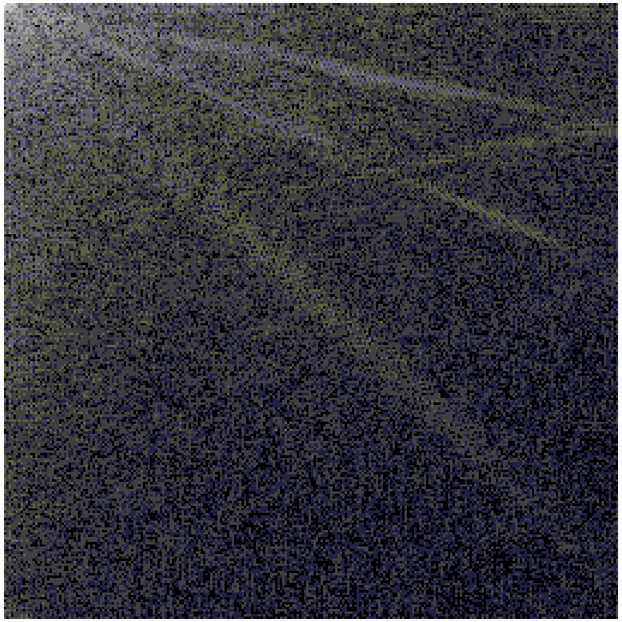
The 2D-DCT basis images for N=4

3.3 Other Separable Transforms

3.3.1 Discrete Cosine Transform (DCT)

An example of 2D-DCT

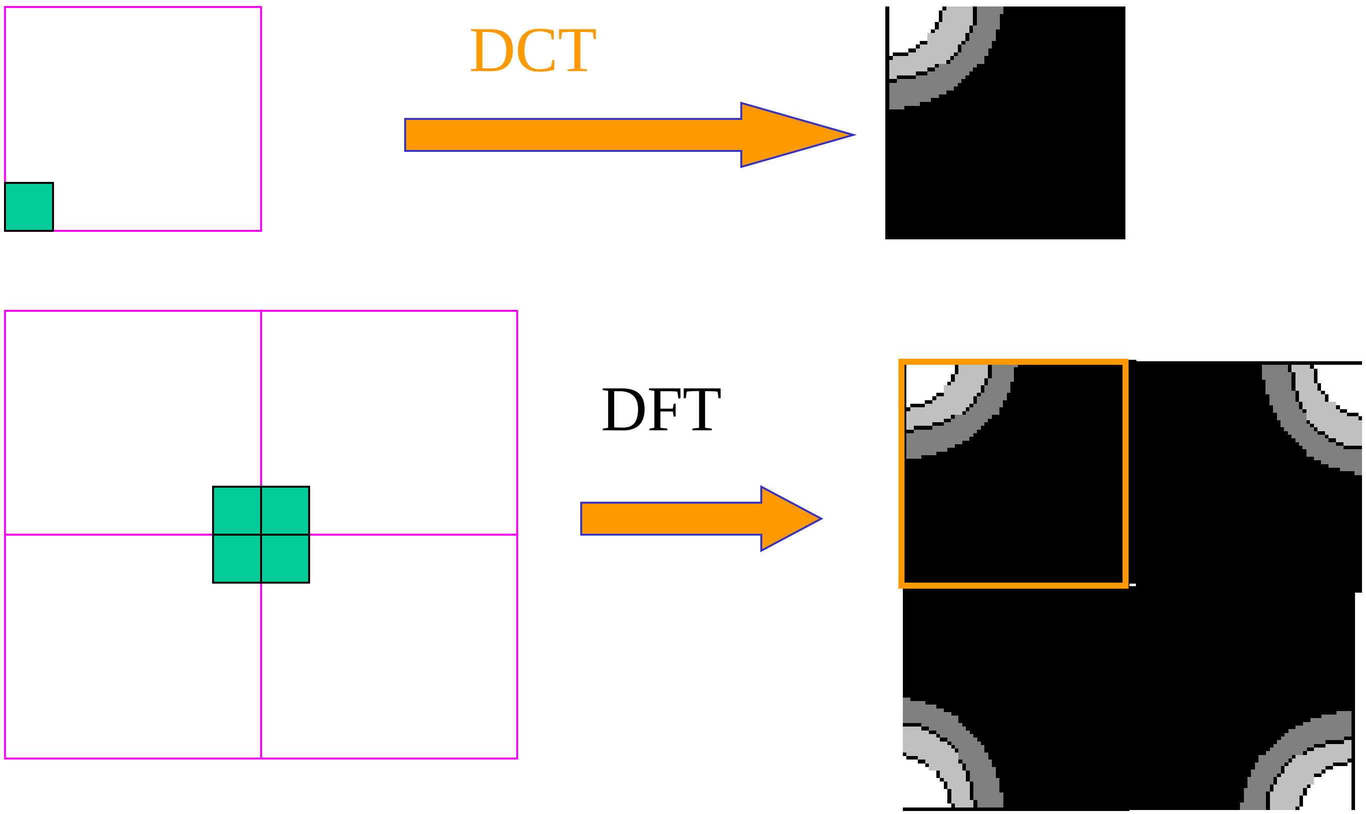
Original Image



3.3 Other Separable Transforms

3.3.1 Discrete Cosine Transform (DCT)

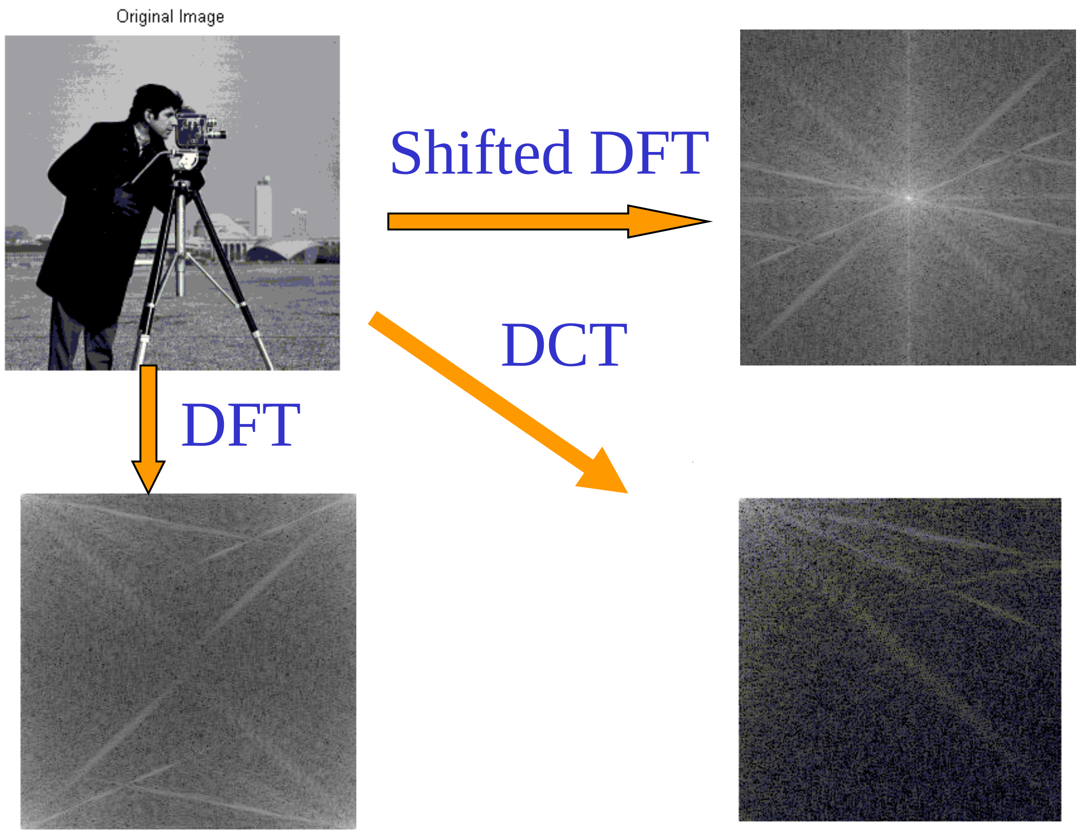
Relationship between DFT and DCT



3.3 Other Separable Transforms

3.3.1 Discrete Cosine Transform (DCT)

Relationship between DFT and DCT



3.3 Other Separable Transforms

3.3.2 Walsh transform

When $N=2^n$, the kernel function is:

$$g(x,u) = \frac{1}{N}\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}$$

the discrete Walsh transform of a function $f(x)$, denote by $W(u)$, is:

$$W(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\prod_{i=0}^{n-1}(-1)^{b_i(x)b_{n-1-i}(u)}$$

Where $b_k(z)$ is the kth bit in the binary representation of z.
Eg: $n=3$, $z=6$ (110 in binary), we have that

$b_0(z)=0$, $b_1(z)=1$, and $b_2(z)=1$

3.3 Other Separable Transforms

3.3.2 Walsh transform: 1-D transform

The values of *g(x,u)* are list in below

*u**x*	0	1	2	3	4	5	6	7
0	+	+	+	+	+	+	+	+
1	+	+	+	+	-	-	-	-
2	+	+	-	-	+	+	-	-
3	+	+	-	-	-	-	+	+
4	+	-	+	-	+	-	+	-
5	+	-	+	-	-	+	-	+
6	+	-	-	+	+	-	-	+
7	+	-	-	+	-	+	+	-

3.3 Other Separable Transforms

3.3.2 Walsh transform： 1-D transform

Inverse kernel and transform:

$$h(x,u) = \prod_{i=0}^{n-1} (-1)^{b_i(x) b_{n-1-i}(u)}$$

$$f(x) = \sum_{u=0}^{N-1} W(u) \prod_{i=0}^{n-1} (-1)^{b_i(x) b_{n-1-i}(u)}$$

3.3 Other Separable Transforms

3.3.2 Walsh transform： 1-D matrix expression N=8

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

3.3 Other Separable Transforms

3.3.2 Walsh transform: 1-D transform

The values of *g(x,u)* are list in below for *N*=8

*u**x*	0	1	2	3	4	5	6	7	Sequency of the row
0	+	+	+	+	+	+	+	+	0
1	+	+	+	+	-	-	-	-	1
2	+	+	-	-	+	+	-	-	3
3	+	+	-	-	-	-	+	+	2
4	+	-	+	-	+	-	+	-	7
5	+	-	+	-	-	+	-	+	6
6	+	-	-	+	+	-	-	+	4
7	+	-	-	+	-	+	+	-	5

3.3 Other Separable Transforms

3.3.2 Walsh transform： 1-D ordered Walsh transform

The values of $g(x,u)$ are list in below for N=8

$u \backslash x$	0	1	2	3	4	5	6	7	Sequency of the row
0	+	+	+	+	+	+	+	+	0
1	+	+	+	+	-	-	-	-	1
2	+	+	-	-	-	-	+	+	2
3	+	+	-	-	+	+	-	-	3
4	+	-	-	+	+	-	_	+	4
5	+	-	-	+	-	+	+	-	5
6	+	-	+	-	-	+	-	+	6
7	+	-	+	-	-	+	+	-	7

3.3 Other Separable Transforms

3.3.2 Walsh transform: 2-D transform

The direct and inverse kernel functions are expressed as:

$$g(x,y,u,v)=\frac{1}{N}\prod_{i=0}^{n-1}(-1)^{[b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)]}$$

$$h(x,y,u,v)=\frac{1}{N}\prod_{i=0}^{n-1}(-1)^{[b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)]}$$

And the direct and inverse transforms are:

$$W(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\prod_{i=0}^{n-1}(-1)^{[b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)]}$$

$$f(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}W(u,v)\prod_{i=0}^{n-1}(-1)^{[b_i(x)b_{n-1-i}(u)+b_i(y)b_{n-1-i}(v)]}$$

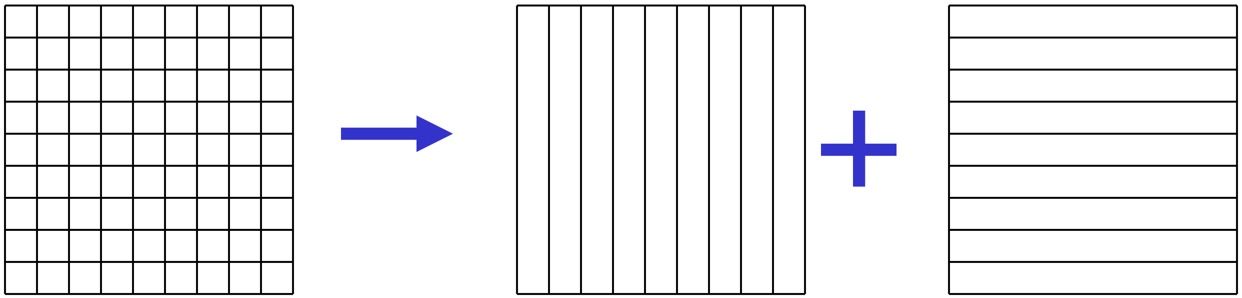
3.3 Other Separable Transforms

3.3.2 Walsh transform： 2-D transform

The direct and inverse kernel functions are *Separable* and *Symmetric*

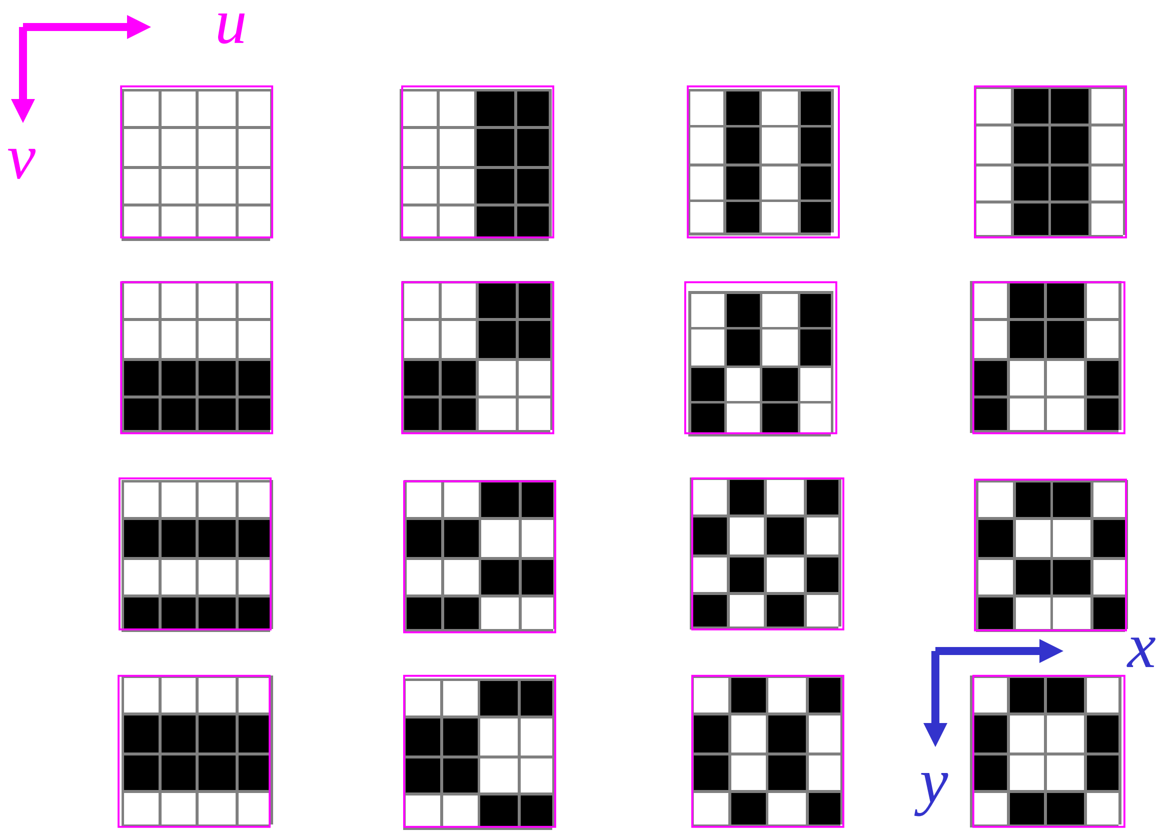
$$g(x, y, u, v) = g_1(x, u) g_1(y, v) = h_1(x, u) h_1(y, v)$$

So it can be implemented in two steps



3.3 Other Separable Transforms

3.3.2 Walsh transform： 2-D transform



The Walsh transform basis images for N=4

3.3 Other Separable Transforms

3.3.3Hadamard transform: 1-D transform

When $N=2^n$, the kernel function is:

$$g(x,u) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1} b_i(x) b_i(u)}$$

Where the summation in the exponent is performed in modulo 2

1-D Hadamard transform of a function *f(x)*, denote by H(*u*), is:

$$H(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)(-1)^{\sum_{i=0}^{n-1} b_i(x) b_i(u)}$$

3.3 Other Separable Transforms

3.3.3 Hadamard transform: 1-D inverse transform

Inverse kernel and transform:

$$h(x,u) = (-1)^{\sum_{i=0}^{n-1} b_i(x) b_i(u)}$$

$$f(x) = \sum_{u=0}^{N-1} H(u)(-1)^{\sum_{i=0}^{n-1} b_i(x) b_i(u)}$$

3.3 Other Separable Transforms

3.3.3 hadamard transform： 1-D matrix expression N=8

$$
\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1
\end{bmatrix}
\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}
$$

$$
\begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1
\end{bmatrix}
\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}
$$

3.3 Other Separable Transforms

3.3.3 Hadamard transform: 1-D transform

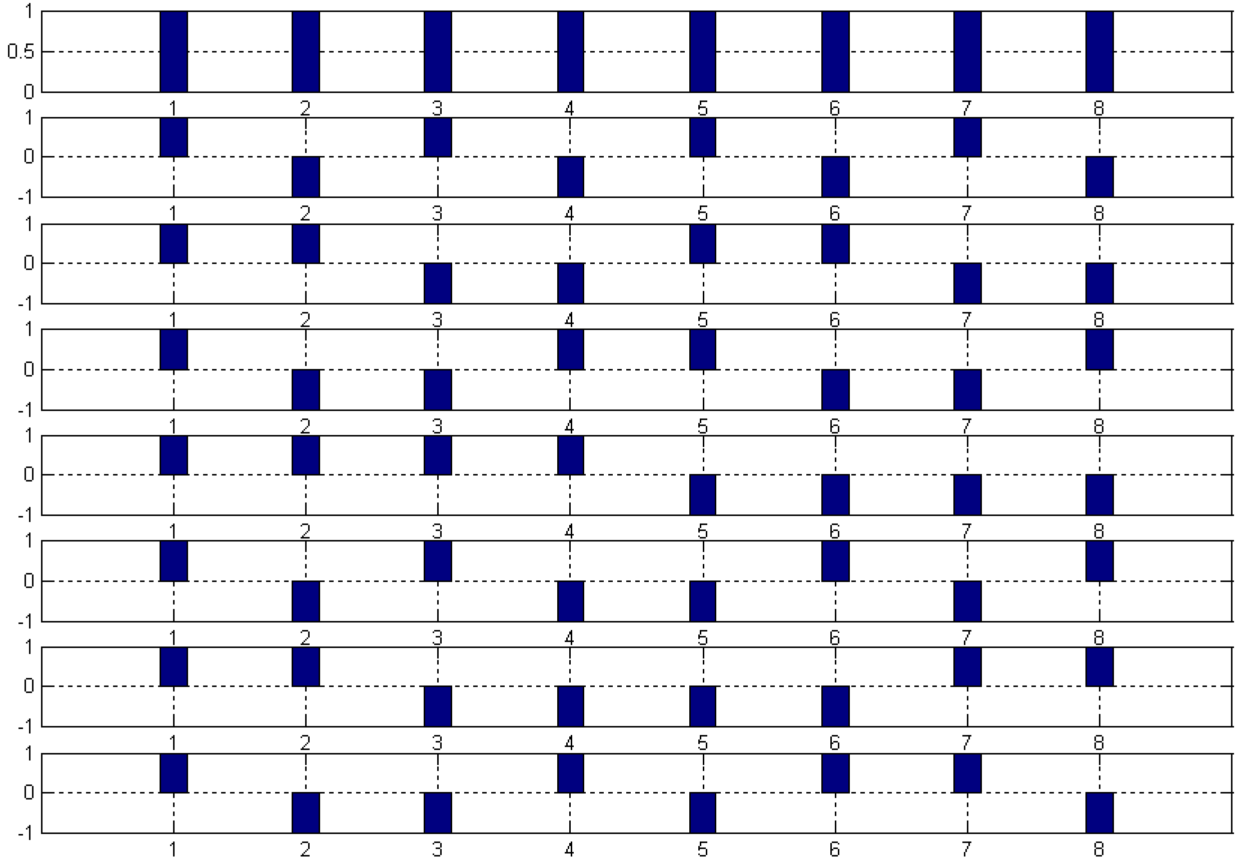
The values of $g(x,u)$ are list in below for N=8

x →, u ↓

$u \backslash x$	0	1	2	3	4	5	6	7	Sequency of the row
0	+	+	+	+	+	+	+	+	0
1	+	-	+	-	+	-	+	-	7
2	+	+	-	-	+	+	-	-	3
3	+	-	-	+	+	-	-	+	4
4	+	+	+	+	-	-	-	-	1
5	+	-	+	-	-	+	-	+	6
6	+	+	-	-	-	-	+	+	2
7	+	-	-	+	-	+	+	-	5

3.3 Other Separable Transforms

3.3.3 Hadamard transform： 1-D transform



The 1-D Hadamard transform basis function for N=8

3.3 Other Separable Transforms

3.3.3 Hadamard transform： 1-D transform

Another way for generating kernel matrix

For the two-by-two case, the kernel matrix is

$$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

And for successively larger N, these can be generated from The block matrix from

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix} = H_2 \otimes H_N$$

3.3 Other Separable Transforms

3.3.3 Hadamard transform： 1-D transform

Another way for generating kernel matrix

For examples

$$H_4 = \frac{1}{\sqrt{4}}\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right]$$

$$H_8 = \frac{1}{\sqrt{8}}\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ \hline 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array}\right]$$

3.3 Other Separable Transforms

3.3.3 Hadamard transform: 1-D ordered Hadamard transform

The values of $g(x,u)$ are list in below for N=8

Same with the Ordered Walsh transform

$u \backslash x$	0	1	2	3	4	5	6	7	Sequency of the row
0	+	+	+	+	+	+	+	+	0
1	+	+	+	+	-	-	-	-	1
2	+	+	-	-	-	-	+	+	2
3	+	+	-	-	+	+	-	-	3
4	+	-	-	+	+	-	-	+	4
5	+	-	-	+	-	+	+	-	5
6	+	-	+	-	-	+	-	+	6
7	+	-	+	-	-	+	+	-	7

3.3 Other Separable Transforms

3.3.3 Hadamard transform： 1-D ordered Hadamard transform

Then the ordered Hadamard transform pair is

$$H(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)(-1)^{\sum_{i=0}^{n-1} b_i(x)p_i(u)}$$

$$f(x) = \sum_{u=0}^{N-1} H(u)(-1)^{\sum_{i=0}^{n-1} b_i(x)p_i(u)}$$

3.3 Other Separable Transforms

3.3.3 Hadamard transform： 2-D transform

The kernel functions of 2-D Hadamard are:

$$h(x,y,u,v) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

$$g(x,y,u,v) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

Both the direct and inverse kernel function are *Separable* and *Symmetric,* because

$$g(x,y,u,v) = g_1(x,u)g_1(y,v) = h_1(x,u)h_1(y,v)$$

3.3 Other Separable Transforms

3.3.3 Hadamard transform： 2-D transform

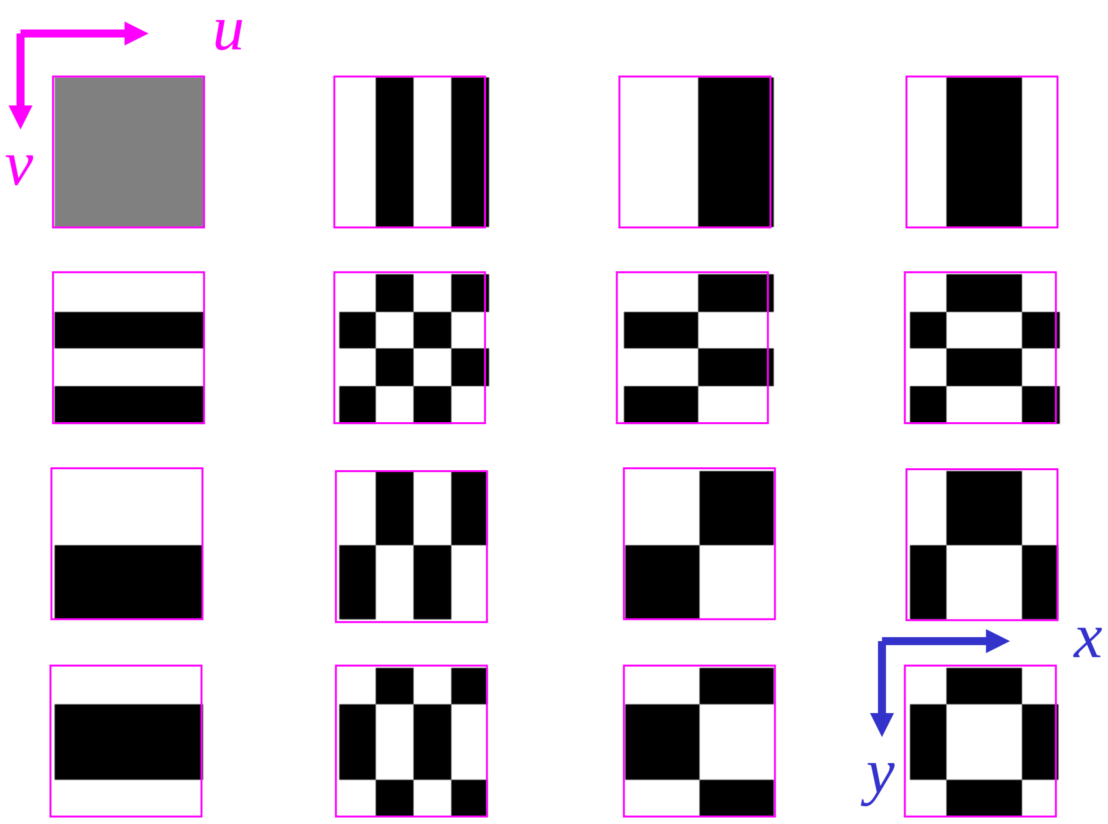
And the 2-D Hadamard transform is defined as:

$$H(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

$$f(x,y)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}H(u,v)(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u)+b_i(y)b_i(v)]}$$

3.3 Other Separable Transforms

3.3.3 Hadamard transform： 2-D transform



The Hadamard transform basis images for N=4

3.3 Other Separable Transforms

3.3.3 Hadamard transform： 2-D ordered transform

The direct and inverse kernel functions of 2-D ordered Hadamard are same as:

$$g(x,y,u,v)=h(x,y,u,v)=\frac{1}{N}(-1)^{\sum_{i=0}^{n-1}[b_i(x)p_i(u)+b_i(y)p_i(v)]}$$

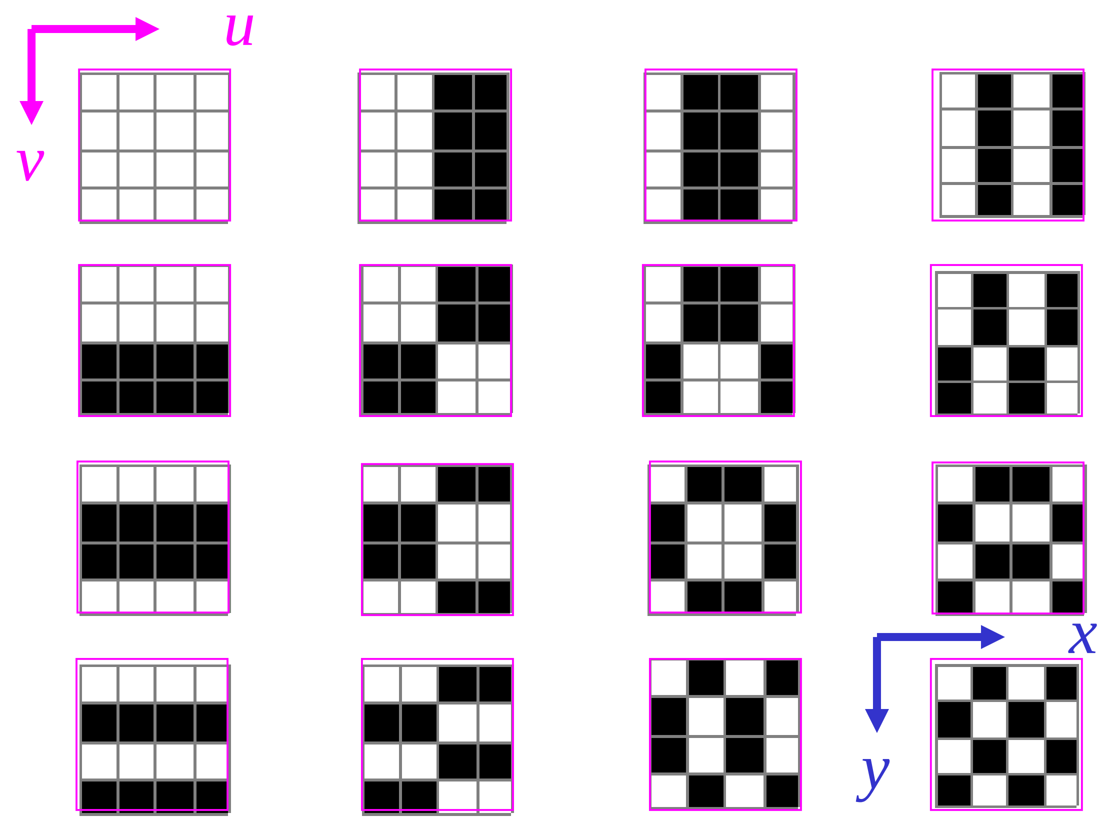
And the 2-D ordered Hadamard transform is defined as:

$$H(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)(-1)^{\sum_{i=0}^{n-1}[b_i(x)p_i(u)+b_i(y)p_i(v)]}$$

$$f(x,y)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}H(u,v)(-1)^{\sum_{i=0}^{n-1}[b_i(x)p_i(u)+b_i(y)p_i(v)]}$$

3.3 Other Separable Transforms

3.3.3 Hadamard transform： 2-D ordered transform



Basis image of 2-D ordered Hadamard transform

3.3 Other Separable Transforms

3.3.4 Haar transform： definitions

1. Haar function

The Haar functions $h_k(z)$ are defined on the interval [0,1], and $k=0,1 \ldots N\text{-}1$, $N=2^n$. Let theInteger $0 \le k \le N-1$ be specified (uniquely) by two other integers, p and q, as

$$k = 2^p + q - 1$$

Where 2^p is the largest power of 2 such that $2^p \le k$ and q-1 is The remainder， except k=0.

For example,

$k=1=2^0+1\text{-}1,$ $\Rightarrow$ $p=0,\ q=1$

$k=23=2^4+8\text{-}1,$ $\Rightarrow$ $p=4,\ q=8$

$k=0,\quad p=0,\ q=0$

$k=100=2^6+37\text{-}1,$ $\Rightarrow$ $p=6,\ q=37$

3.3 Other Separable Transforms

3.3.4 Haar transform： definitions

1. Haar function

The Haar functions are defined by

$$h_0(z) = h_{00}(z) = 1/\sqrt{N} \qquad z \in [0,1]$$

$$h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & \frac{q-1}{2^p} \le z < \frac{q-1/2}{2^p} \\ -2^{p/2} & \frac{q-1/2}{2^p} \le z < \frac{q}{2^p} \\ 0 & \text{others} \end{cases}$$

3.3 Other Separable Transforms

3.3.4 Haar transform： definitions

2. Haar transform

$$H(u) = \sum_{x=0}^{N-1} f(x) h_u\left(\frac{x}{N}\right)$$

$$H = \begin{bmatrix} h_0(0/N) & h_0(1/N) & \cdots & h_0(N-1/N) \\ h_1(0/N) & h_1(1/N) & \cdots & h_1(N-1/N) \\ \vdots & \vdots & \ddots & \vdots \\ h_{N-1}(0/N) & h_{N-1}(1/N) & \cdots & h_{N-1}(N-1/N) \end{bmatrix}$$

3.3 Other Separable Transforms

3.3.4 Haar transform： definitions

2. Haar transform matrix

For example: N=2

$$H_2 = \begin{bmatrix} h_0(0/2) & h_0(1/2) \\ h_1(0/2) & h_1(1/2) \end{bmatrix}$$

$$h_0(0/2) = h_{00}(0/2) = \frac{1}{\sqrt{2}}$$

$$h_0(1/2) = h_{00}(1/2) = \frac{1}{\sqrt{2}}$$

$$h_1(0/2) = h_{01}(0/2) = \frac{1}{\sqrt{2}}$$

$$h_1(1/2) = h_{01}(1/2) = -\frac{1}{\sqrt{2}}$$

k=0 *p=0, q=0*

k=1 *p=0, q=1*

$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

3.3 Other Separable Transforms

3.3.4 Haar transform： definitions

2. Haar transform matrix

For example: N=4

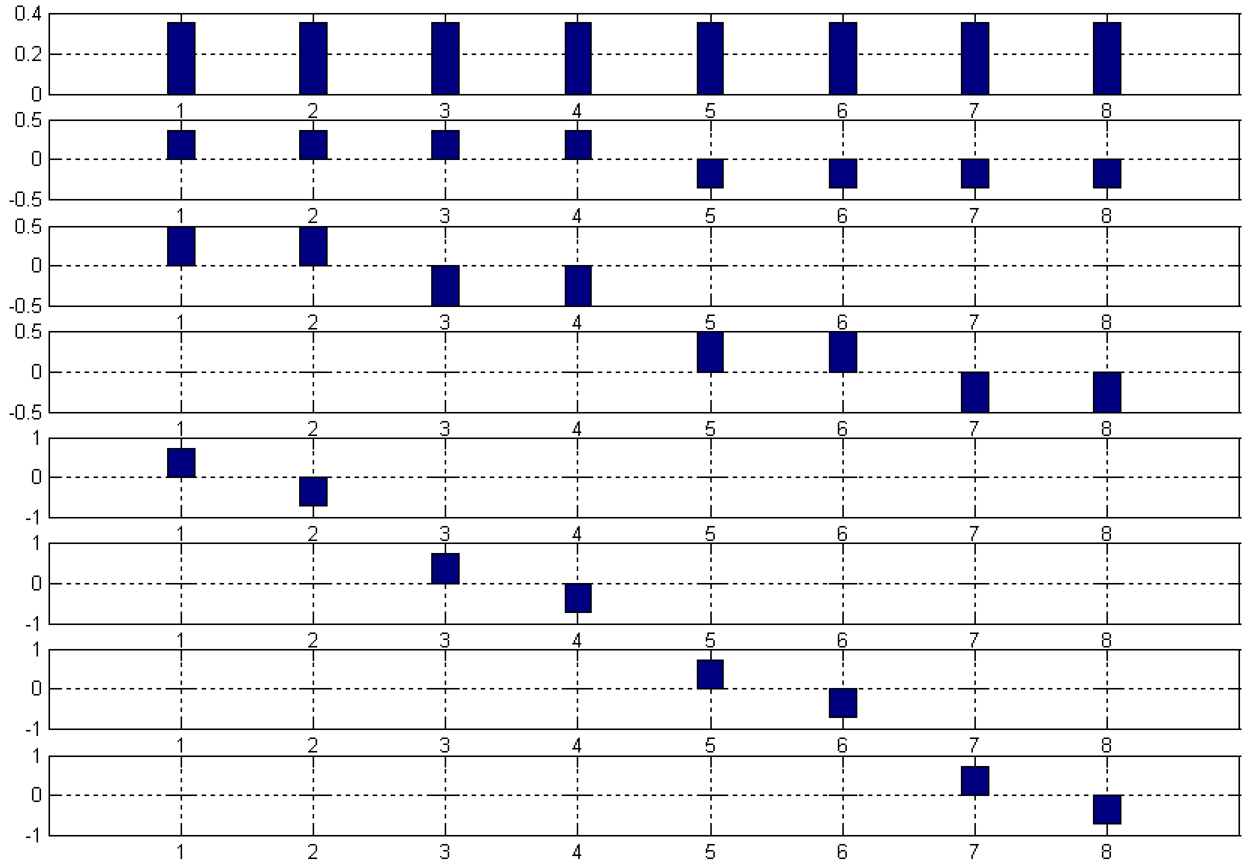
$$H_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$H_8 = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

3.3 Other Separable Transforms

3.3.4 Haar transform： definitions

2. Haar transform matrix



The 1-D Haar transform basis function for N=8

3.3 Other Separable Transforms

3.3.4 Haar transform: definitions

3. Haar transform

Direct transform: $G = HF$

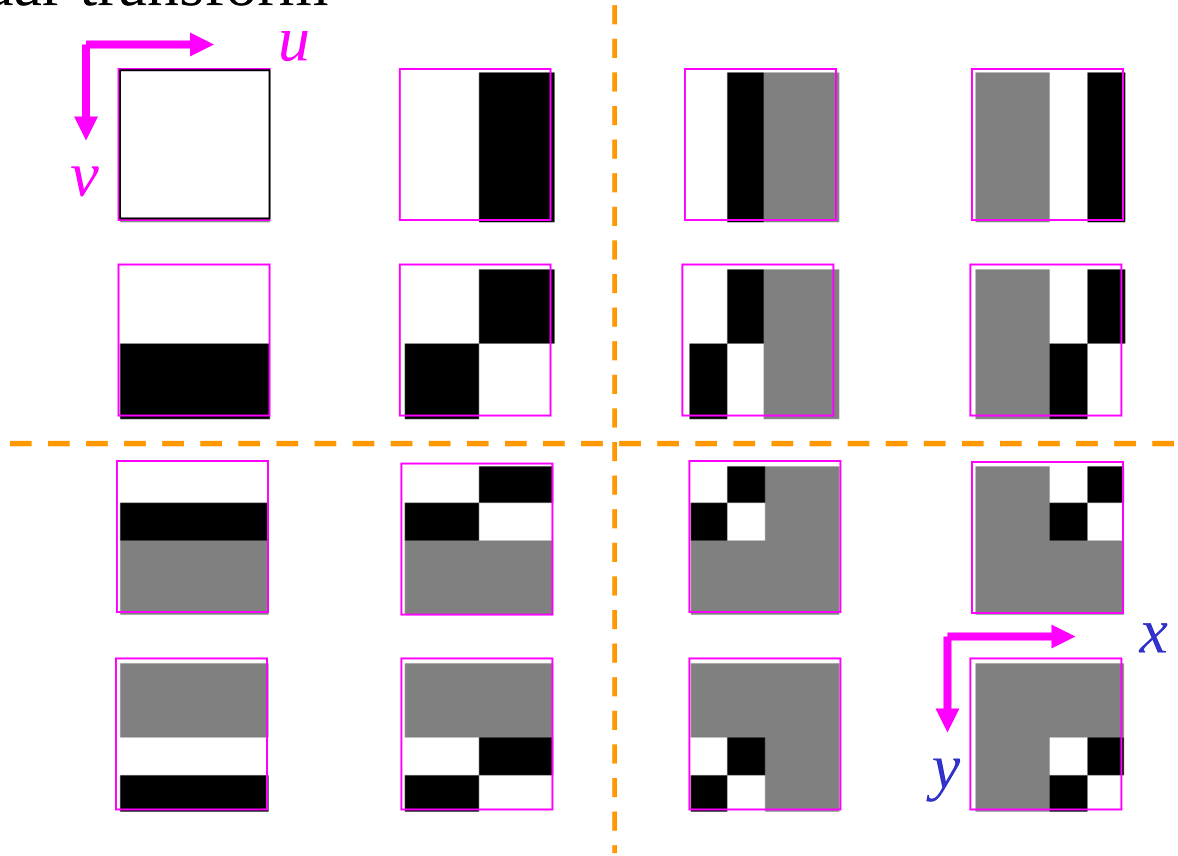
Inverse transform: $F = H^{-1}G$

$$\begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \\ g_6 \\ g_7 \end{bmatrix} = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

3.3 Other Separable Transforms

3.3.4 Haar transform： properties

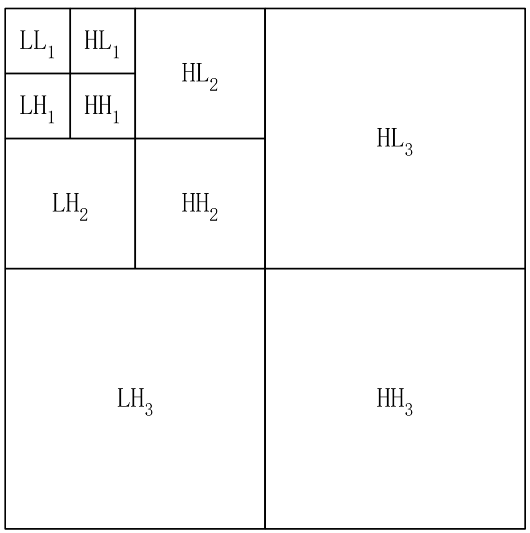
3. Haar transform

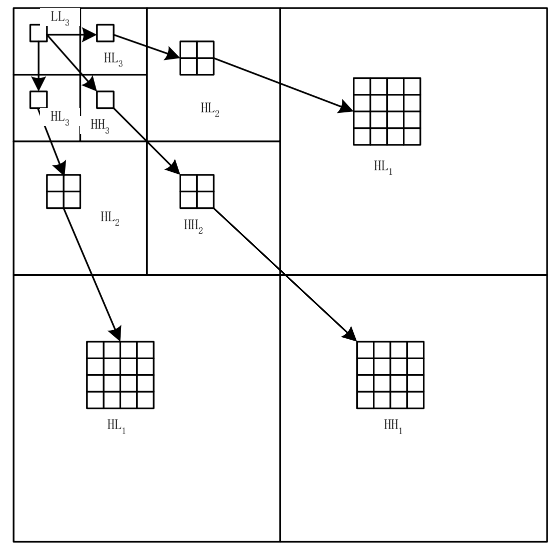


The Haar transform basis images for N=4

3.3 Other Separable Transforms

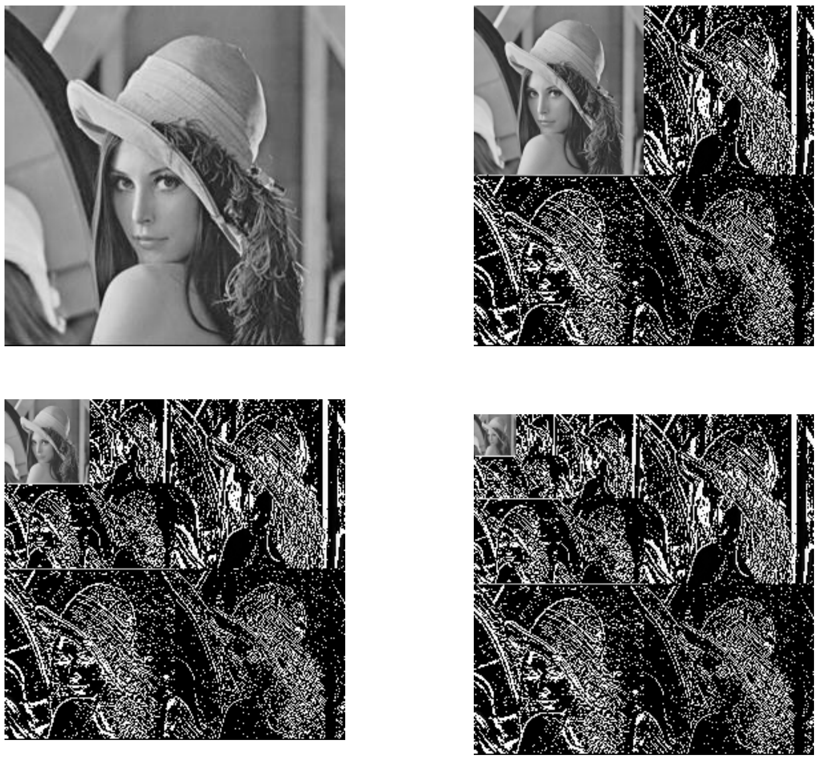
3.3.4 Haar transform： mallat algorithms





3.3 Other Separable Transforms

3.3.4 Haar transform： mallat algorithms



3.4 Hotelling Transforms

3.3.5 Hotelling transform： definitions

KLT: Karhunen-Loeve Transfrom
PCA:Principal Components Analysis

The *kth* image in a image set can be expressed as a vector:

…

$$x_k = \begin{bmatrix} x_k^0 & x_k^1 & \cdots & x_k^{N-1} \end{bmatrix}^T \qquad k=0,1,\ \ldots M-1$$

The covariance matrix of the **x** vector is defined as

$$C_x = E\{(x - m_x)(x - m_x)^T\}$$

where

$$m_x = E\{x\}$$

is the mean vector, E is the expected value

3.4 Hotelling Transforms

3.3.5 Hotelling transform： definitions

They can be approximated from the samples by using the relations

$$m_x = \frac{1}{M}\sum_{k=0}^{M-1} x_k$$

and

$$C_x = \frac{1}{M}\sum_{k=0}^{M-1}(x_i - m_x)(x_i - m_x)^T$$
$$= \frac{1}{M}\sum_{k=0}^{M-1} x_k x_k^T - m_x m_x^T$$

3.4 Hotelling Transforms

3.3.5 Hotelling transform： definitions

let $\left| C_x - \lambda I \right| = 0$ → Calculated N eigenvalues and arranged as $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{N-1}$

let $\left[C_x - \lambda_i I \right] T_i = 0$ → Calculated N eigenvectors T_i and arranged as

$$A = \begin{bmatrix} T_0^T \\ T_1^T \\ \vdots \\ T_{N-1}^T \end{bmatrix}$$

→ Hotelling transform $\boxed{Y = A(X - m_x)}$

Inverse Hotelling transform $\boxed{X = A^T Y + m_x}$

3.4 Hotelling Transforms

3.3.5 Hotelling transform: properties

Relationship between the eigenvaluse and eigenvectors:

$$\left[C_x - \lambda_i I\right] T_i = 0 \rightarrow C_x T_i = \lambda_i T_i$$

$$A = \begin{bmatrix} T_0^T \\ T_1^T \\ \vdots \\ T_{N-1}^T \end{bmatrix} \rightarrow C_x A^T = A^T \wedge$$

where $\wedge = \begin{bmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_{N-1} \end{bmatrix}$

3.4 Hotelling Transforms

3.3.5 Hotelling transform: properties

mean vector of $\boldsymbol{y}$ $m_y = E\{y\} = E\{(Ax - Am_x)\} = AE\{x\} - Am_x$

$$\boxed{m_y = 0}$$

The covariance matrix of the ***Y*** vector is given by

$$C_y = E\{(Y - m_y)(Y - m_y)^T\}$$

$$= E\left\{(AX - Am_x)(AX - Am_x)^T\right\}$$

$$= E\left\{A(X - m_x)(X - m_x)^T A^T\right\}$$

$$= AE\left\{(X - m_x)(X - m_x)^T\right\} A^T$$

3.4 Hotelling Transforms

3.3.5 Hotelling transform： properties

$$AA^T = I \quad \Rightarrow \quad \boxed{\begin{aligned} C_y &= AC_xA^T \\ &= AA^T \wedge \\ &= \wedge \end{aligned}}$$

C_y is a diagonal matrix with elements equal to the eigenvalues of C_x, that is

$$C_y = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{bmatrix}$$

That's means elements of Y are *uncorrelated*

3.4 Hotelling Transforms

3.3.5 Hotelling transform: inverse transform

Since Hotelling transform is *orthogonal* , so

$$A^{-1} = A^T \qquad \text{and} \qquad X = A^T Y + m_x$$

If we form A from K eigenvectors corresponding to the largest eigenvalues as A_K, the recovered vector will be

$$\hat{X} = A_K^T Y + m_x$$

It can be shown that the mean square error, e_{ms}, between X and $\hat{X}$ is given by the expression

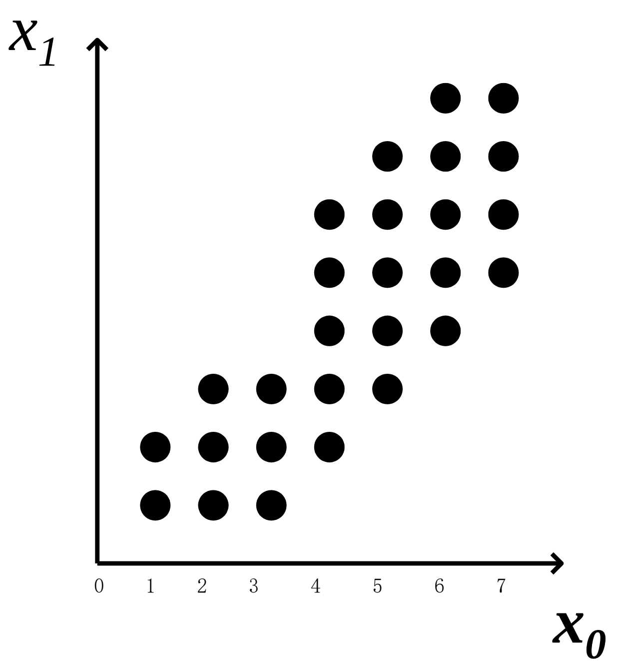
$$e_{ms} = \sum_{j=0}^{N-1} \lambda_j - \sum_{j=0}^{k-1} \lambda_j = \sum_{j=K}^{N-1} \lambda_j$$

3.4 Hotelling Transforms

3.3.5 Hotelling transform: example

Given the samples of 2-dimension vectors shown as below, calculate its Hotelling transform. $N=2$, $M=27$

(1,1) (1,2) (2,1) (2,2) (2,3)
(3,1) (3,2) (3,3) (4,2) (4,3)
(4,4) (4,5) (4,6) (5,3) (5,4)
(5,5) (5,6) (5,7) (6,4) (6,5)
(6,6) (6,7) (6.8) (7,5) (7,6)
(7,7) (7,8)



3.4 Hotelling Transforms

3.3.5 Hotelling transform： example

Let $\boxed{x^k = \begin{bmatrix} x_0^k & x_1^k \end{bmatrix}^T}$ $k=0,1,\ ...26$

$$\boxed{m_x = \frac{1}{27}\sum_{k=0}^{26} x_k = \begin{bmatrix} 4.444 & 4.2963 \end{bmatrix}}$$

$$\boxed{\begin{aligned} C_x &= \frac{1}{27}\sum_{k=0}^{26} (x_i - m_x)(x_i - m_x)^T \\ &= \begin{bmatrix} 3.4103 & 3.2479 \\ 3.2479 & 4.7550 \end{bmatrix} \end{aligned}}$$

3.4 Hotelling Transforms

3.3.5 Hotelling transform： example

let $\left| C_x - \lambda I \right| = 0$ $\rightarrow$ $\begin{vmatrix} 3.4103-\lambda & 3.2479 \\ 3.2479 & 4.7550-\lambda \end{vmatrix} = 0$

$\lambda_0 = 7.3993$ $\lambda_1 = 0.7659$

let $\left[C_x - \lambda_i I \right] T_i = 0$ $\rightarrow$ $T_0 = \begin{bmatrix} 0.6314 \\ 0.7755 \end{bmatrix}$ $T_1 = \begin{bmatrix} -0.7755 \\ 0.6314 \end{bmatrix}$

$$A = \begin{bmatrix} 0.6314 & 0.7755 \\ -0.7755 & 0.6314 \end{bmatrix}$$

$$\boxed{y = A(x - m_x)}$$

3.4 Hotelling Transforms

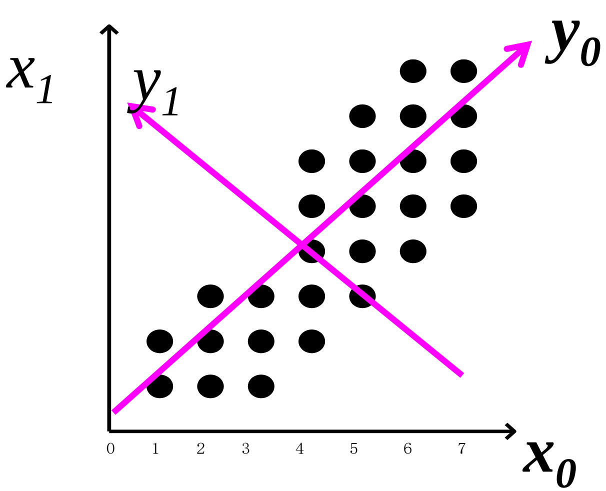
3.3.5 Hotelling transform： example

y =(-4.7309 0.5899) (-3.9555 1.2212) (-4.0996 -0.1856) (-3.3241 0.4458)
(-2.5486 1.0771) (-3.4682 -0.9611) (-2.6927 -0.3297) (-1.9172 0.3017)
(-2.0613 -1.1052) (-1.2859 -0.4738) (-0.5104 0.1576) (0.2651 0.7890)
(1.0406 1.4203) (-0.6545 -1.2493) (0.1210 -0.6179) (0.8965 0.0135)
(1.6719 0.6449) (2.4474 1.2762) (0.7524 -1.3934) (1.5279 -0.7620
(2.3033 -0.1306) (3.0788 0.5008) (3.8543 1.1322) (2.1592 -1.5375)
(2.9347 -0.9061) (3.7102 -0.2747) (4.4857 0.3567)

my = 1.0e-015 *
-0.6908 -0.1069

cy =

7.3993	0.0000
0.0000	0.7659

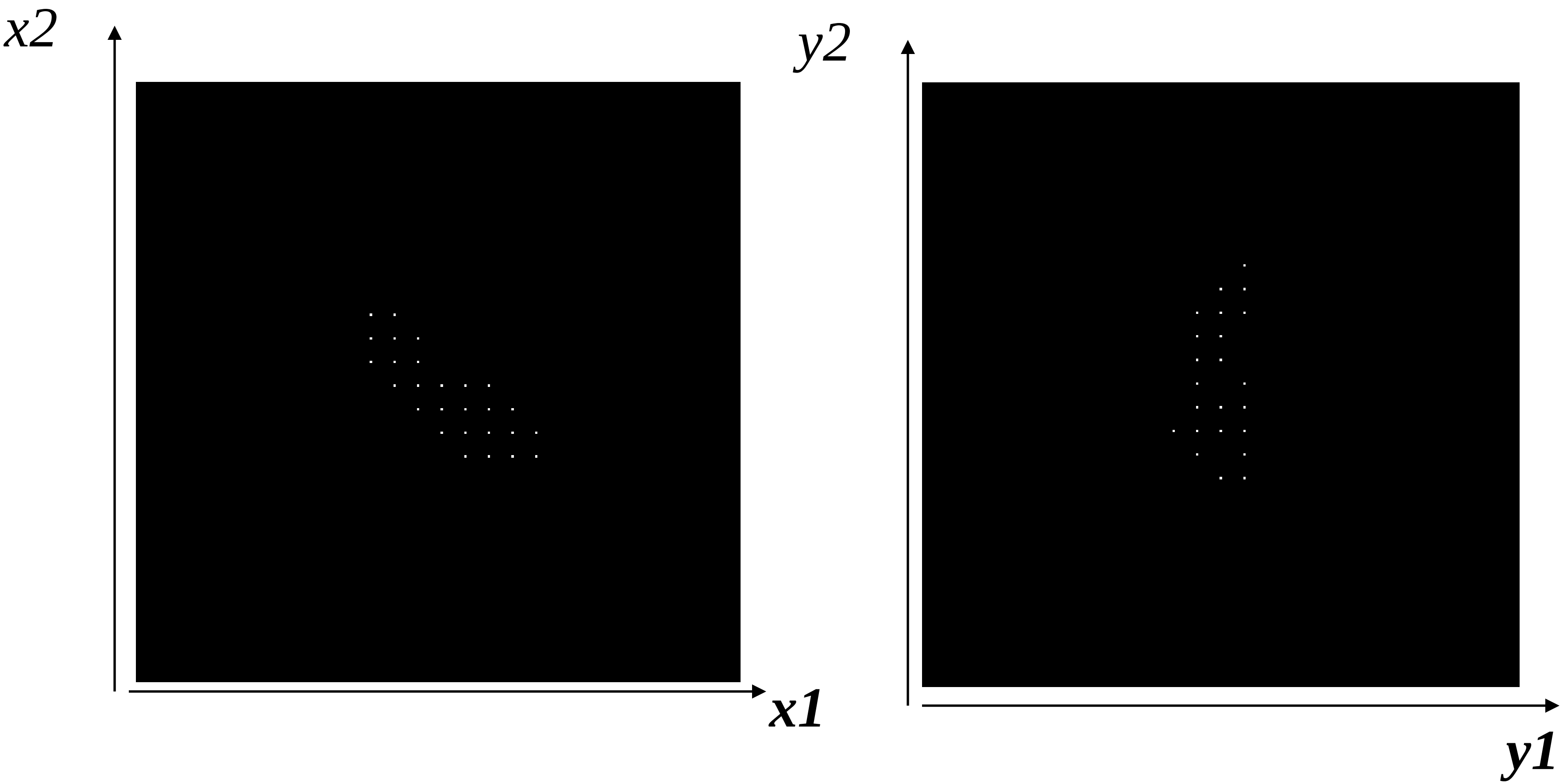


corry =

1.0000	0.0000
0.0000	1.0000

3.4 Hotelling Transforms

3.3.5 Hotelling transform: example



Summary

Basic function

Sinusoidal transforms

(a) Discrete Fourier Transform $e^{i\theta} = \cos\theta + i\sin\theta$

(b) Discrete Cosine Transform $\cos\theta$

(c) Discrete Sine Transform $\sin\theta$

(d) Hartly Transform $\cos\theta + \sin\theta$

Summary

Rectangular wave transforms

(a) Hadamard Transform

(b) Walsh Transform

(c) Slant Transform

(d) Haar Transform

Summary

Eigenvector-based transforms

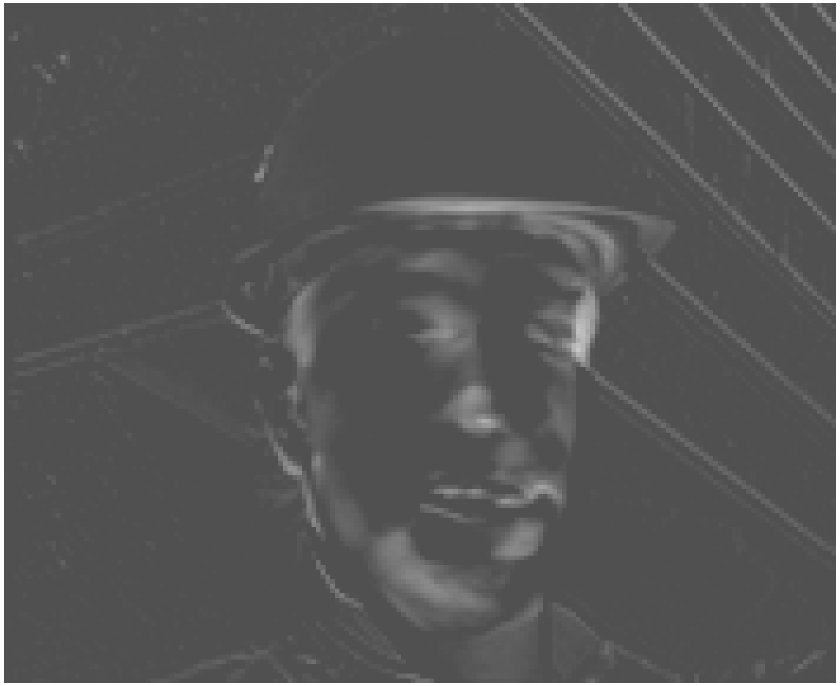
Hotelling Transform (K-LT)

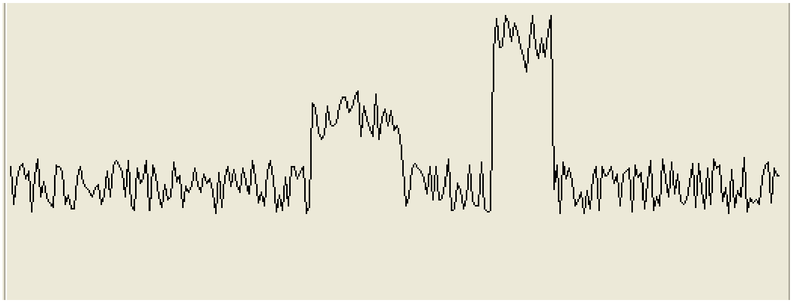
Summary

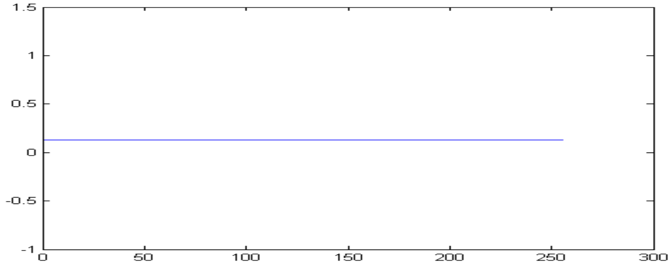
How to Use these Transforms?

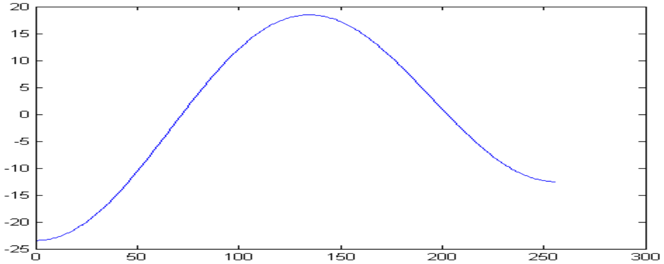
Summary

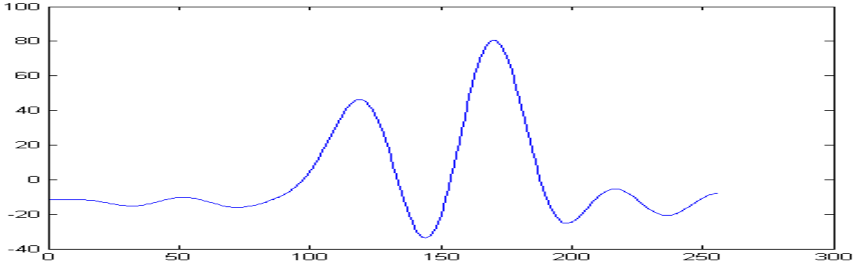
- Future work

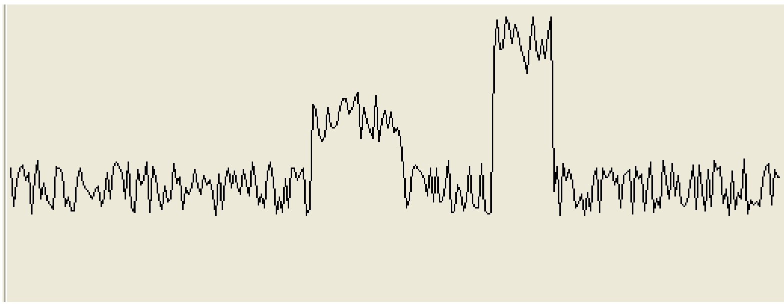


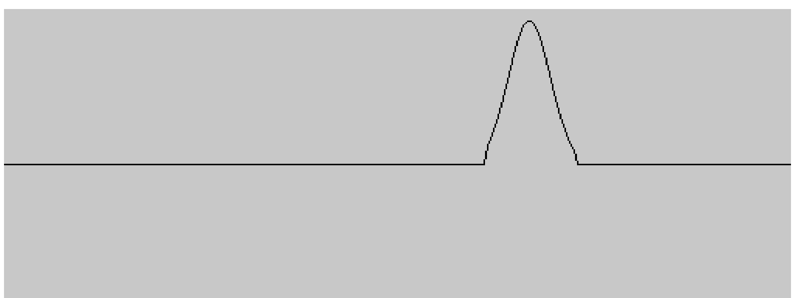



1.5
1
0.5
0
-0.5
-1
0
50
100
150
200
250
300


20
15
10
5
0
-5
-10
-15
-20
-25
0
50
100
150
200
250
300


100
80
60
40
20
0
-20
-40
0
50
100
150
200
250
300





Homework

(1) 计算下图的 DFT, DCT, Hadamard 变换和Haar变换

$$\boxed{\begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}}$$

(2) Page 71(章毓晋) 3.21: 设有一组64*64的图像,它们的协方差矩阵式单位矩阵.如果只使用一半的原始特征值计算重建图像,那么原始图像和重建图像间的均方误差是多少?

(3) 编程实现lena.bmp的离散Fourier变换和离散余弦变换，并显示频谱图像。

The End