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Zeros of nonlinear systems with input invariances☆

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ABSTRACT

A nonlinear system possesses an invariance with respect to a set of transformations if its output dynamics remain invariant when transforming the input, and adjusting the initial condition accordingly. Most research has focused on invariances with respect to time-independent pointwise transformations like translational-invariance ($u(t) \mapsto u(t)+p$, $p \in \mathbb{R}$) or scale-invariance ($u(t) \mapsto pu(t)$, $p \in \mathbb{R}_{>0}$). In this article, we introduce the concept of s_0-invariances with respect to continuous input transformations exponentially growing/decaying over time. We show that s_0-invariant systems not only encompass linear time-invariant (LTI) systems with transfer functions having an irreducible zero at $s_0 \in \mathbb{R}$, but also that the input/output relationship of nonlinear s_0-invariant systems possesses properties well known from their linear counterparts. Furthermore, we extend the concept of s_0-invariances to second- and higher-order s_0-invariances, corresponding to invariances with respect to transformations of the time-derivatives of the input, and encompassing LTI systems with zeros of multiplicity two or higher. Finally, we show that nth-order 0-invariant systems realize – under mild conditions – nth-order nonlinear differential operators: when excited by an input of a characteristic functional form, the system's output converges to a constant value only depending on the nth (nonlinear) derivative of the input.

1. Introduction

Systems invariant with respect to a set of pointwise input transformations (see e.g. Adler, Mayo, & Alon, 2014; Goentoro, Shoval, Kirschner, & Alon, 2009; Hironaka & Morishita, 2014; Shoval, Alon, & Sontag, 2011; Shoval et al., 2010) show the same output dynamics when applying a transformation to the systems' input, and adjusting the initial conditions appropriately (see Section 3 for precise definitions). For example, linear time-invariant (LTI) systems with a zero at the origin are *translational invariant*, that is, invariant with respect to translations $u(t) \mapsto u(t)+p$ of the input, with $p \in \mathbb{R}$. Similarly, *scale-invariance* (also referred to as fold-change detection Shoval et al., 2010) is defined with respect to geometric scaling $u(t) \mapsto pu(t)$ of the input, with $p \in \mathbb{R}_{>0}$. In the context of invariant systems, the major result in Shoval et al. (2011) is of specific importance, showing that nonlinear systems are invariant with respect to a certain set of input transformations if and only if they are *equivariant* with respect to the same transformations (see Section 3 for details). Different to invariance, equivariance is a "memoryless" structural property only depending on the current state and input of the system; that a given system is equivariant is, thus, typically easier to prove.

Recently, we have shown that – under mild conditions – invariant systems realize first-order nonlinear differential operators (Lang & Sontag, 2016). That is, there exists a set of characteristic inputs for which the output of an equivariant system remains constant (in general nonzero) when initialized appropriately. Importantly, the constant value of the output only depends on the (nonlinear) derivative of the characteristic input, with the functional form of the derivative defined by the invariance itself. For example, translational invariant systems can realize the differential operator $\frac{d}{dt}$ (i.e., $\bar{y}^* = \alpha(\frac{d}{dt}u(t))$, with u a characteristic input, $\bar{y}^*$ the constant output, and α some nonlinear map) in agreement with the known property that the output of Hurwitz LTI systems with a zero at the origin excited by ramps converge to constant values proportional to the slope of the ramp. Similarly, scale-invariant systems can realize the nonlinear differential

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operator $\frac{d}{dt}\log$ (i.e., $\bar{y}^* = \alpha\left(\frac{d}{dt}\log(u(t))\right)$), with the characteristic inputs given by exponential functions (Lang & Sontag, 2016).

In this article, we introduce two mutually compatible generalizations of invariance: (i) for any given $s_0 \in \mathbb{R}$, s_0-invariant systems are invariant with respect to continuous input transformations exponentially growing/decaying over time, and comprise LTI systems with a zero at s_0; and (ii) second-order s_0-invariant systems are invariant with respect to transformations of the time-derivative of the input, and comprise LTI systems possessing zeros with multiplicity two. Additionally, we show how the latter can be generalized to arbitrary-order invariances. For each of the two generalizations of invariance, we derive the corresponding generalized equivariances, that is, provide a possibility to structurally test if a given system possesses a generalized invariance without having to consider state trajectories or past inputs. Finally, based on the definition of a characteristic model of an s_0-invariant system – a concept related to pole-zero cancellation of LTI systems – we extend our previous results on systems realizing differential operators (Lang & Sontag, 2016) by showing that higher-order 0-invariant systems can realize higher-order differential operators.

The rest of this article is organized as follows. In Section 2, we briefly recapitulate dynamic properties of LTI systems with zeros of arbitrary multiplicity. In Section 3, we provide our definitions of first-order s_0-invariance and s_0-equivariance, and prove their equivalence under mild assumptions. In Section 6, we introduce the notion of second-order s_0-equivariance and invariance. Finally, based on the characteristic model of an s_0-invariant system (Section 4), we define in Sections 5 and 7 systems realizing first- and higher-order differential operators, and establish their close relationship to 0-invariant systems.

2. Zeros of linear time-invariant systems

Consider a single-input, single-output LTI system given by the ordinary differential equations (ODEs)

$$\frac{d}{dt}z(t) = Az(t) + bu(t), \qquad z(0) = \bar{z} \tag{1a}$$

$$y(t) = c^T z(t). \tag{1b}$$

with state $z(t) \in \mathbb{R}^n$, piecewise-continuous input $u(t) \in \mathbb{R}$, output $y(t) \in \mathbb{R}$, system matrix $A \in \mathbb{R}^{n\times n}$, and input and output vectors $b, c \in \mathbb{R}^{n\times 1}$.

An LTI system (1) has a zero at $s_0 \in \mathbb{C}$ if $\det(M(s_0)) = 0$, with $M(s) = \begin{bmatrix} A - sI & b \\ -c^T & 0 \end{bmatrix}$ (see e.g. Brockett, 1965). In the following, we assume that the system (1) is controllable and observable. Since, in this article, we are only interested in real zeros, we furthermore assume $s_0 \in \mathbb{R}$. Then, a necessary and sufficient condition for the system to have a zero at s_0 is that the transfer function $G(s) = -c^T(A - sI)^{-1}b$ evaluates to zero at $s = s_0$. The nullspace of $M(s_0)$ is spanned by the vector $(\bar{z}^1_{s_0}, 1)^T$, with $\bar{z}^1_{s_0} = -(A - s_0 I)^{-1}b$. This implies that $\left(pu^1_{s_0}, p\bar{z}^1_{s_0}\right)$, with $u^1_{s_0}(t) = e^{s_0 t}$ and $p \in \mathbb{R}$, zeros the output (see e.g. Isidori, 1995, p. 162ff), that is, when (1) is initialized at $p\bar{z}^1_{s_0}$ and excited by $pu^1_{s_0}$, the output remains zero.

More generally, if the system possesses a zero $s_0 \in \mathbb{R}$ with multiplicity $m_{s_0} \geq 1$, i.e. if $c^T(A - sI)^{-l}b = 0$ for all $l = 1, \ldots, m_{s_0}$, the corresponding inputs and initial conditions zeroing the output are given by (multiples of) $u^l_{s_0}(t) = t^{l-1}e^{s_0 t}$ and $\bar{z}^l_{s_0} = -(l-1)!(A - s_0 I)^{-l}b$. Due to the superposition principle of linear systems, for an LTI system with n_s distinct zeros s^i_0, $i = 1, \ldots, n_s$, with multiplicities $m_{s^i_0}$, the set $S_0 = \{(u^l_{s^i_0}, \bar{z}^l_{s^i_0})\}_{l=1,\ldots,m_{s^i_0}, i=1,\ldots,n_s}$ spans a vector space of inputs and initial conditions zeroing the output.

The superposition principle has another important consequence: Consider an LTI system initialized at any state $\bar{z} \in \mathbb{R}^n$ and excited by any input $u \in \mathcal{U}$, and let $(u_0, \bar{z}_0) \in \operatorname{span}(S_0)$. Then, the output of the system is *invariant* with respect to the mapping $(u, \bar{z}) \mapsto (u+u_0, \bar{z}+\bar{z}_0)$, that is, $c^T\xi(t, \bar{z}, u) = c^T\xi(t, \bar{z}+\bar{z}_0, u+u_0)$, with $\xi(t, \bar{z}, u) = z(t)$ the solution of (1) for the initial condition $\bar{z}$ and the input u. We can reformulate this property as follows: First, let $\pi_p : \mathbb{R} \to \mathbb{R}$, $\pi_p(\bar{u}) = \bar{u} + p$, describe a set of input transformations for $p \in \mathbb{R}$. Assume that the LTI system has a zero at s_0 with multiplicity m_{s_0}. Then, for every $l = 1, \ldots, m_{s^i_0}$, $\bar{z} \in \mathbb{R}$ and $p \in \mathbb{R}$, there exists a $\bar{z}' \in \mathbb{R}$ such that

$$c^T\xi(t, \bar{z}, u) = c^T\xi(t, \bar{z}', t \mapsto \pi_{pt^{l-1}\exp(s_0 t)}(u(t))), \tag{2}$$

for all $u \in U$ and $t \geq 0$. Note, that (2) follows from our previous analysis, with $\bar{z}' = \bar{z} - p(l-1)!(A - s_0 I)^{-l}b$.

LTI systems with a zero at the origin with multiplicity m_0 possess another interesting property: when excited by $u(t) = \sum_{l=0}^{m_0} k_l t^l$, there exists an initial condition $\bar{z}^*$ such that the output remains constant, i.e. such that $c^T\xi(t, \bar{z}^*, u) = \bar{y}^* = -m_0! c^T A^{-m_0-1} b k_{m_0}$. Notably, $\bar{y}^*$ only depends on the coefficient of the monomial of u with degree m_0. Thus, we can equivalently write $\bar{y}^* = -c^T A^{-m_0-1} b \frac{d^{m_0}}{dt^{m_0}} u(t)$, i.e. the output of the system excited by u and initialized at $\bar{z}^*$ is proportional to the m_0th time derivative of the input. If, additionally, the system is Hurwitz, the output converges to $\bar{y}^*$ for any initial condition.

3. First-order s_0-invariance and s_0-equivariance

Throughout the rest of this article, we consider nonlinear systems given by ODEs of the form

$$\frac{d}{dt}z(t) = f(z(t), u(t)), \qquad z(0) = \bar{z} \tag{3a}$$

$$y(t) = h(z(t)). \tag{3b}$$

The vector $z(t) \in Z \subseteq \mathbb{R}^n$ represents the state of the system at time $t \in \mathbb{R}_{\geq 0}$, and $u \in \mathcal{U} \subseteq \mathcal{PC}(\mathbb{R}_{\geq 0}, U)$ a piecewise-continuous (external) input, with $u : \mathbb{R}_{\geq 0} \to U \subseteq \mathbb{R}$. The dynamics are given by the vector field $f : Z \times U \to \mathbb{R}^n$, the initial conditions by $\bar{z} \in Z$, and the output by $y(t) \in \mathbb{R}$, with $h : Z \to \mathbb{R}$. We assume that f and h are analytic, and that for each initial condition $\bar{z} \in Z$ and each input $u \in \mathcal{U}$ there exists a unique, piecewise differentiable and continuous solution of Eq. (3), which we denote by $\xi : \mathbb{R}_{\geq 0} \times Z \times \mathcal{U} \to Z$, $\xi(t, \bar{z}, u) = z(t)$.

In the previous section, we have shown that the output of an LTI system is invariant with respect to the input transformations $\pi_p : U \to U$, $p \in \mathbb{R}$, corresponding to translations of the input (2). In the following, we define an equivalent property for nonlinear systems – referred to as s_0-*invariance* – with respect to input transformations not necessarily corresponding to translations. Different to previous work (Shoval et al., 2011), we restrict ourselves to input transformations forming a one-parameter Lie group under function composition $\circ$ as defined in Bluman and Kumei (1989). This implies that we can parametrize the input transformations $\mathcal{P} = \{\pi_p : U \to U\}_p$ by a parameter $p \in P \subseteq \mathbb{R}$ such that π_p is differentiable in U and analytic in P (Bluman & Kumei, 1989, p. 34). Furthermore, by the first fundamental theorem of Lie (Bluman & Kumei, 1989, p. 37), π_p can always be parametrized such that $P = \mathbb{R}$, such that the law of composition becomes additive ($\pi_{p_1} \circ \pi_{p_2} = \pi_{p_1+p_2}$), and such that $p = 0$ corresponds to the identity transformation ($\pi_0(\bar{u}) = \bar{u}$ for all $\bar{u} \in U$). In the following, we assume that every Lie group is parametrized as described above.

Definition 1 (*First-order s_0-invariance*)**.** Consider the system (3) and a one-parameter Lie group of input transformations $\mathcal{P} = \{\pi_p : U \to U\}_{p\in\mathbb{R}}$. Then, the system is *first-order s_0-invariant* with respect to $\mathscr{P}$ (in short, is $\mathscr{P}[s_0]$-invariant), if for all $p \in \mathbb{R}$ and $\bar{z} \in Z$ there exists a $\bar{z}' \in Z$ such that

$$h(\xi(t, \bar{z}, u)) = h(\xi(t, \bar{z}', t \mapsto \pi_{p\exp(s_0 t)}(u(t)))), \tag{4}$$

for all $u \in \mathcal{U}$ and $t \geq 0$.

Remark 2. For $l = 1$, (2) is a special case of (4), implying that LTI systems with an (irreducible) zero at $s_0 \in \mathbb{R}$ are $\mathscr{P}[s_0]$-invariant, with $\mathscr{P} = \{\pi_p(\bar{u}) = \bar{u} + p\}_{p\in\mathbb{R}}$.

Remark 3. When defining a new output $\hat{y}(t) = h(z(t)) - h(\xi(t, \bar{z}, u))$, the tuple $(\pi_{p\exp(s_0 t)}(u(t)), \bar{z}')$ zeros the output $\hat{y}(t)$. For s_0-invariance, we additionally require that $\bar{z}'$ does not depend on $u(t)$, and that $\pi_{p\exp(s_0 t)}(u(t))$ is a zero input for all u and $\bar{z}$.

Remark 4. Invariance as defined in Shoval et al. (2011) closely resembles s_0-invariance for $s_0 = 0$. However, in Shoval et al. (2011) the input transformations are not required to form a Lie group. Furthermore, invariance (Shoval et al., 2011) requires that the system possesses a globally asymptotic stable (GAS) steady-state $\sigma(\bar{u}) \in Z$ for constant inputs $\bar{u} \in U$ (i.e. $\xi(t, z_0, \bar{u}) \to \sigma(\bar{u})$ for all $z_0 \in Z$), and that $\bar{z} = \sigma(\bar{u})$ and $\bar{z}' = \sigma(\pi_p(\bar{u}))$. Thus, for the invariance as defined in Shoval et al. (2011), (4) only has to hold for $\bar{z}$ reachable from a GAS steady-state.

It is typically not trivial to prove that a given system is s_0-invariant by directly applying Definition 1. However, s_0-invariance is closely related to the "memoryless" structural property s_0-equivariance (compare Shoval et al., 2011):

Definition 5 (*First-order s_0-equivariance*)**.** Consider the system (3) and a one-parameter Lie group of input transformations $\mathscr{P} = \{\pi_p : U \to U\}_{p\in\mathbb{R}}$. Then, the system is *first-order s_0-equivariant* with respect to $\mathscr{P}$ (in short, $\mathscr{P}[s_0]$-equivariant), if there exists a one-parameter Lie group of state transformations $\mathscr{R}[s_0] = \{\rho_p : Z \to Z\}_{p\in\mathbb{R}}$ such that

$$f(\rho_p(z), \pi_p(\bar{u})) = (\partial_p \rho)_p(z) p s_0 + (\partial_z \rho)_p(z) f(z, \bar{u})$$
$$h(\rho_p(z)) = h(z)$$

for all $z \in Z, \bar{u} \in U$ and $\pi_p \in \mathscr{P}$, with $(\partial_x y) = \frac{\partial}{\partial x} y$ the Jacobian of y with respect to x.

Remark 6. Equivariance as defined in Shoval et al. (2011) closely resembles 0-equivariance as defined by us, except that in Shoval et al. (2011) the input and state transformations do not have to form Lie groups. To our knowledge, no general method exists to find all input and state transformations with respect to which a system is s_0-equivariant, but they can often be easily "guessed" as described in Shoval et al. (2011).

The following theorem (compare Theorem 1 in Shoval et al., 2011) establishes a close relationship between s_0-equivariance and s_0-invariance. In this theorem, *observability* refers to the property of a system (3) that for every two different initial conditions the output dynamics must be different for some input, i.e. that $(\forall t \geq 0, \forall u \in U : h(\xi(t, \bar{z}_1, u)) = h(\xi(t, \bar{z}_2, u)) \Rightarrow \bar{z}_1 = \bar{z}_2$ (Sussmann, 1977).

Theorem 7. *An analytic and observable system (3) is $\mathscr{P}[s_0]$-invariant, if and only if it is $\mathscr{P}[s_0]$-equivariant.*

Proof. *Sufficiency*: Suppose that $z(t) = \xi(t, \bar{z}, u)$ is the solution of (3) for initial conditions $\bar{z}$ and input u. Let $z_*(t) = \rho_{p\exp(s_0 t)}(z(t))$. Then,

$$\begin{aligned}\frac{d}{dt} z_*(t) &= (\partial_p \rho)_{p\exp(s_0 t)}(z(t)) s_0 p \exp(s_0 t) \\ &\quad + (\partial_z \rho)_{p\exp(s_0 t)}(z(t)) f(z(t), u(t)) \\ &= f(z_*(t), \pi_{p\exp(s_0 t)}(u(t))),\end{aligned}$$

i.e. $z_*(t) = \xi(t, \bar{z}', t \mapsto \pi_{p\exp(s_0 t)}(u(t)))$ with $\bar{z}' = \rho_p(\bar{z})$. Furthermore, $h(\xi(t, \bar{z}', t \mapsto \pi_{p\exp(s_0 t)}(u(t)))) = h(\rho_{p\exp(s_0 t)}(\xi(t, \bar{z}, u))) = h(\xi(t, \bar{z}, u))$.

Necessity: Let $u(t) = \pi_{q\exp(s_0 t)}(v(t))$ and $p = -q$ in (4). Then, for all $q \in \mathbb{R}$ and $\bar{z}' \in Z$ there exists a $\bar{z} \in Z$ such that $h(\xi(t, \bar{z}, v(t))) = h(\xi(t, \bar{z}', \pi_{q\exp(s_0 t)}(v(t))))$ for all $v \in \mathcal{U}$ and $t \geq 0$, implying that the roles of $\bar{z}$ and $\bar{z}'$ in Definition 1 can be exchanged. Now, consider

$$\frac{d}{dt}\begin{bmatrix}\hat{z}(t)\\ p(t)\end{bmatrix} = \begin{bmatrix}\hat{f}(\hat{z}(t), p(t), u(t))\\ s_0 p(t)\end{bmatrix}, \qquad \begin{bmatrix}\hat{z}(0)\\ p(0)\end{bmatrix} = \begin{bmatrix}\bar{z}'\\ \bar{p}\end{bmatrix} \tag{5a}$$

$$\hat{y}(t) = \hat{h}\left(\begin{bmatrix}\hat{z}(t)\\ p(t)\end{bmatrix}\right) = h(\hat{z}(t)), \tag{5b}$$

with $\hat{f}(\hat{z}(t), p(t), u(t)) = f(\hat{z}(t), \pi_{-p(t)}(u(t)))$, and $\bar{p} \in \mathbb{R}$. Since $p(t) = \bar{p} e^{s_0 t}$,

$$\hat{\xi}\left(t, \begin{bmatrix}\bar{z}'\\ \bar{p}\end{bmatrix}, u\right) = \begin{bmatrix}\xi(t, \bar{z}', t \mapsto \pi_{-\bar{p}\exp(s_0 t)}(u(t)))\\ \bar{p}\exp(s_0 t)\end{bmatrix}$$

$$\begin{aligned}\hat{h}\left(\hat{\xi}\left(t, \begin{bmatrix}\bar{z}'\\ \bar{p}\end{bmatrix}, u\right)\right) &= h(\xi(t, \bar{z}', t \mapsto \pi_{-\bar{p}\exp(s_0 t)}(u(t)))) \\ &= h(\xi(t, \bar{z}, u)),\end{aligned}$$

implying that every state of (5) is indistinguishable from some state in (3). By assumption, (3) is analytic and observable. Since also π_p is analytic in p, it follows that (5) is analytic. Then, by Lemma 7 in Sussmann (1977), there exists a unique homomorphism $\hat{\rho} : Z \times \mathbb{R} \to Z$, such that

$$\hat{\rho}\left(\hat{\xi}\left(t, \begin{bmatrix}\bar{z}'\\ \bar{p}\end{bmatrix}, u\right)\right) = \xi\left(t, \hat{\rho}\left(\begin{bmatrix}\bar{z}'\\ \bar{p}\end{bmatrix}\right), u\right)$$
$$h\left(\hat{\rho}\left(\begin{bmatrix}\bar{z}'\\ \bar{p}\end{bmatrix}\right)\right) = \hat{h}\left(\begin{bmatrix}\bar{z}'\\ \bar{p}\end{bmatrix}\right)$$

for all $u \in \mathcal{U}, \bar{z}' \in Z$ and $\bar{p} \in \mathbb{R}$. Furthermore, by Lemma 5 in Sussmann (1977) (see also the remark below the lemma), the restriction of $\hat{\rho}$ on an orbit of (5) is analytic. Thus,

$$\xi\left(t, \rho_{\bar{p}}(\bar{z}'), u\right) = \rho_{\bar{p}\exp(s_0 t)}\left(\xi\left(t, \bar{z}', t \mapsto \pi_{-\bar{p}\exp(s_0 t)}(u(t))\right)\right)$$
$$h(\bar{z}') = h(\rho_{\bar{p}}(\bar{z}'))$$

with $\rho_{p(t)}(\hat{z}(t)) := \hat{\rho}([\hat{z}(t), p(t)]^T)$. Differentiating by time, we obtain at $t = 0$ that

$$f(\hat{\rho}_{\bar{p}}(\bar{z}'), \pi_{\bar{p}}(\bar{u})) = (\partial_p \rho)_{\bar{p}}\left(\bar{z}'\right) \bar{p} s_0 + (\partial_z \rho)_{\bar{p}}\left(\bar{z}'\right) f(\bar{z}', \bar{u})$$
$$h(\rho_{\bar{p}}(\bar{z}')) = h(\bar{z}')$$

for all $\bar{u} = \pi_{-\bar{p}}(u(0)) \in U, \bar{p} \in \mathbb{R}$ and $\bar{z}' \in Z$. □

Remark 8. In principle, one could generalize s_0-invariance by changing (4) to $h(\xi(t, \bar{z}, u)) = h(\xi(t, \bar{z}', t \mapsto \pi_{p\alpha_{s_0}(t)}(u(t))))$, with $\alpha_{s_0} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ analytic. However, only for $\alpha_{s_0}(t) = e^{s_0 t}$ we obtain a relationship between zeros of LTI systems and s_0-invariance. Furthermore, it is unclear for which other functional forms of α_{s_0} there exists a memoryless equivalence of s_0-equivariance.

3.1. Example 1—LTI system with a zero

Consider the LTI system

$$\frac{d}{dt} x(t) = y(t) - k_1 x(t)$$
$$\frac{d}{dt} y(t) = u(t) - x(t) - k_2 y(t),$$

with parameters $k_1, k_2 \in \mathbb{R}_{>0}$, input $u(t) \in \mathbb{R}$, internal state $x(t) \in \mathbb{R}$, and output $y(t) \in \mathbb{R}$. The transfer function of the system is $G(s) = \frac{s+k_1}{s^2+(k_1+k_2)s+k_1k_2+1}$, with $s \in \mathbb{C}$ the Laplace variable.

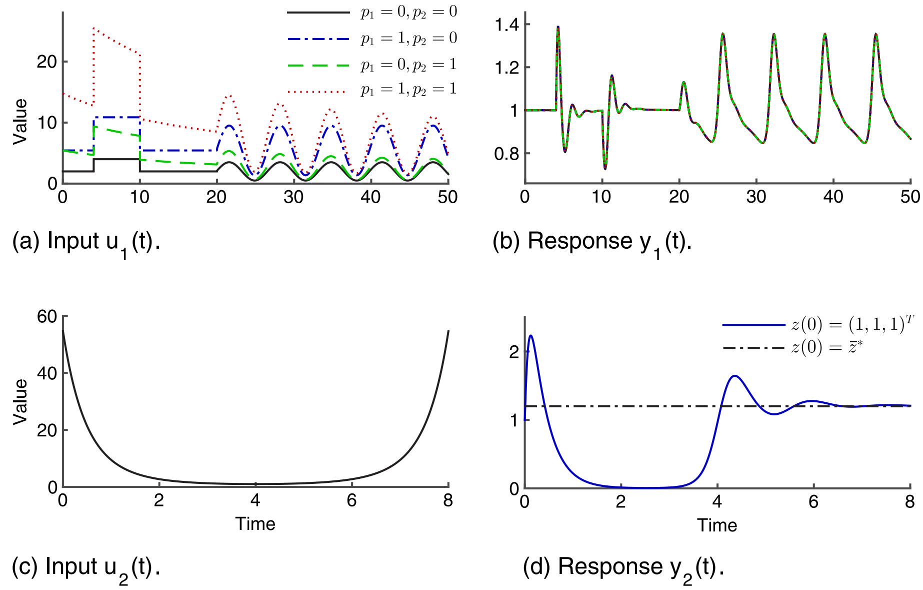


Fig. 1. Input/output dynamics of the multiple-equivariances example (Section 3.2). (a) For $b = 0.04$, the system is excited by the inputs $u_1(t) = \pi_{p_1+p_2\exp(-bt)}(u(t))$, with $p_1 = p_2 = 0$ (black, solid), $p_1 = 1$ and $p_2 = 0$ (blue, dash-dotted), $p_1 = 0$ and $p_2 = 1$ (green, dashed), or $p_1 = p_2 = 1$ (red, dotted), with $u(t)$ an arbitrary reference input. (b) Output dynamics $y_1(t)$ for the inputs depicted in (a), when the system is initialized at $\bar{x}_1 = \sigma_{x_1}(\bar{u}) - \frac{b}{c}p_2, \bar{x}_2 = e^{p_1+p_2}\sigma_{x_2}(\bar{u})$ and $\bar{y} = \sigma_y(\bar{u})$, with $\sigma_{x_1}(\bar{u}) = 0, \sigma_{x_2}(\bar{u}) = \frac{d}{ey_0}\bar{u}$ and $\sigma_y(\bar{u}) = y_0$ the steady-state of the system for the constant input $\bar{u} = 2$. (c) For $b = 0$, the system is excited by $u_2(t) = u_{k_2,k_1,k_0}(t) = \exp(\frac{k_2}{2}t^2 + k_1 t + k_0)$ with $k_2 = 0.5$, $k_1 = -2$ and $k_0 = 4$. (d) Output dynamics $y_2(t)$ for the input depicted in (c), when the system initialized at $(\bar{x}_1^*, \bar{x}_2^*, \bar{y}^*)^T$ (black, dash-dotted), or at $(1, 1, 1)^T$ (blue, solid). The common parameters for (a–d) were set to $a = d = 0.5$, $c = 5$, $e = 3$ and $y_0 = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Since the system has a zero at $s_0 = -k_1$, it is $\mathcal{P}[-k_1]$-equivariant (Theorem 7 and Remark 2), with input and state transformations

$$\mathcal{P}[-k_1] = \{\pi_p(\bar{u}) = \bar{u} + p\}_{p\in\mathbb{R}}$$
$$\mathcal{R}[-k_1] = \left\{\rho_p(x, y) = [x + p, y]^T\right\}_{p\in\mathbb{R}}.$$

3.2. Example 2—multiple invariances

Consider the system (compare Shoval et al., 2011, Figure 1c)

$$\frac{d}{dt}x_1(t) = a\,(y(t) - y_0) - bx_1(t)$$
$$\frac{d}{dt}x_2(t) = cx_2(t)(x_1(t) + y(t) - y_0)$$
$$\frac{d}{dt}y(t) = d\frac{u(t)}{x_2(t)} - ey(t),$$

with parameters $a, c, d, e \in \mathbb{R}_{>0}$ and $b \in \mathbb{R}_{\geq 0}$, internal states $x_1(t) \in \mathbb{R}$ and $x_2(t) \in \mathbb{R}_{>0}$, output $y(t) \in \mathbb{R}_{\geq 0}$, and input $u(t) \in \mathbb{R}_{>0}$. The system is $\mathcal{P}[0]$-equivariant and, for $b \neq 0$, additionally also $\mathcal{P}[-b]$-equivariant, with input and state transformations given for both cases by

$$\mathcal{P}[s_0] = \{\pi_p(\bar{u}) = e^p\bar{u}\}_{p\in\mathbb{R}}$$
$$\mathcal{R}[s_0] = \left\{\rho_p(x_1, x_2, y) = \left[x_1 + \frac{s_0}{c}p, e^p x_2, y\right]^T\right\}_{p\in\mathbb{R}}.$$

Thus, for $b > 0$, the output dynamics are invariant with respect to geometrically scaling the input by some fixed value (blue curves in Fig. 1(a), (b)), or by an exponentially decaying value (green curves). Furthermore, these input transformations can be combined (red curves). The case $b = 0$ will be further discussed in Section 7.1.

4. The characteristic model

In this section, we derive the *characteristic model* of an s_0-equivariant system important for the generalization of s_0-equivariance and s_0-invariance to higher orders (Section 6). For this, consider a $\mathcal{P}[s_0]$-equivariant system (3) excited by the input $u(t) = \pi_{p(t)}(u^0(t))$, with $u^0(t) \in \mathcal{U}$ an arbitrary external input, and $p : \mathbb{R}_{\geq 0} \to \mathbb{R}$ some yet not specified differentiable function. Setting $\hat{z}(t) = \rho_{-p(t)}(z(t))$, differentiating by time and using the definition of s_0-equivariance applied to $-p(t)$, we obtain

$$\begin{aligned}\frac{d}{dt}\hat{z}(t) &= (\partial_z\rho)_{-p(t)}(z(t))f(z(t), \pi_{p(t)}(u^0(t))) \\ &\quad - (\partial_p\rho)_{-p(t)}(z(t))\frac{d}{dt}p(t) \\ &= f(\hat{z}(t), u^0(t)) - \eta(\hat{z}(t))\left(\frac{d}{dt}p(t) - p(t)s_0\right),\end{aligned}$$

with $\eta(\hat{z}) := (\partial_p\rho)_0(\hat{z})$ the *infinitesimals* of ρ_p (Bluman & Kumei, 1989, p. 37). Furthermore, $h(z(t)) = h(\rho_{p(t)}(\hat{z}(t))) = h(\hat{z}(t))$.

In the following, we refer to

$$\frac{d}{dt}\hat{z}(t) = f(\hat{z}(t), u^0(t)) - \eta(\hat{z})u^1(t), \tag{6a}$$
$$\hat{z}(0) = \rho_{-\bar{p}}(\bar{z}) \tag{6b}$$
$$y(t) = h(\hat{z}(t)) \tag{6c}$$

as the first-order characteristic model of the system (3) with respect to its $\mathcal{P}[s_0]$-equivariance (in short, the $\mathcal{P}[s_0]$-*characteristic model*), with the additional input $u^1(t) = \frac{d}{dt}p(t) - s_0 p(t)$, and $p(0) = \bar{p} \in \mathbb{R}$.

Given u^0 and u^1, the corresponding input u to the $\mathcal{P}[s_0]$-equivariant system (3) is generated by the *input module* of the system with respect to $\mathcal{P}[s_0]$ (in short, the $\mathcal{P}[s_0]$-*input module*) given by (Fig. 2)

$$\frac{d}{dt}p(t) = s_0 p(t) + u^1(t), \qquad p(0) = \bar{p} \tag{7a}$$
$$u(t) = \pi_{p(t)}(u^0(t)). \tag{7b}$$

Lemma 9. *The input/output dynamics of a model composed of the input module of a $\mathcal{P}[s_0]$-equivariant system (3) and the*

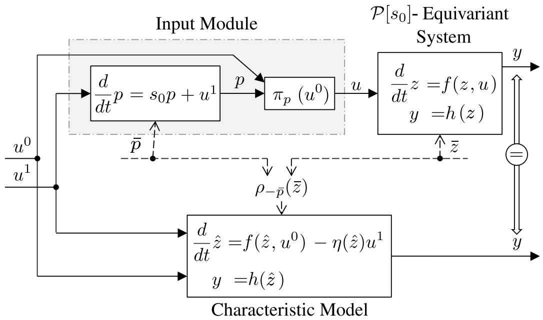


Fig. 2. Relationship between the characteristic model (6) of a $\mathcal{P}[s_0]$-equivariant system, the input module (7), and the system itself (3). Solid arrows depict signal transduction, and dashed arrows initial conditions. The input module receives the external signals u^0 and u^1, and generates the signal u. When using u as the input to the $\mathcal{P}[s_0]$-equivariant system, the system's output equals the output of the characteristic model excited by u^0 and u^1 and initialized at $\rho_{-\bar{p}}(\bar{z})$, with $\bar{z}$ the initial condition of the $\mathcal{P}[s_0]$-equivariant system and $\bar{p}$ the initial condition of the input module.

$\mathcal{P}[s_0]$-equivariant system itself, with the output of the input module feeding into the $\mathcal{P}[s_0]$-equivariant system, are equal to the input/output dynamics of the $\mathcal{P}[s_0]$-characteristic model. The composed model is not observable. If the $\mathcal{P}[s_0]$-equivariant system is observable, its characteristic model is observable, too.

Proof. The equivalence of the input/output dynamics follows directly from the definition of the $\mathcal{P}[s_0]$-characteristic model. The input module maps u^1 and u^0 to $u(t) = \pi_{\bar{p}\exp(s_0 t)}(v(t))$, with $v(t) = \pi_{\int_0^t u^1(t-\tau)e^{s_0\tau}d\tau}\left(u^0(t)\right)$. Thus, the effect of the initial condition $\bar{p}$ corresponds to a transformation of the input $v(t)$ with respect to which the system is invariant. Thus, the composed model is not observable, since (by Definition 1) for every $\bar{p}, \bar{p}' \in \mathbb{R}$ and every $\bar{z} \in Z$ there exists a $\bar{z}' \in Z$ such that the composed model initialized at $(\bar{p}, \bar{z})$ has the same output dynamics for all u^0 and u^1 as when initialized at $(\bar{p}', \bar{z}')$. For $u^1(t) = 0$, the dynamics of the characteristic model (6) are equivalent to the $\mathcal{P}[s_0]$-equivariant system (3). Thus, observability of (6) follows from observability of (3). □

Remark 10. The input model of an LTI system with a zero at $s_0 \in \mathbb{R}$ is linear, and has a pole at s_0. Then, the characteristic model corresponds to the composed model after *pole-zero cancellation*.

4.1. Example 1 (continued)

The $\mathcal{P}[-k_1]$-input module of the linear example system (Section 3.1) is given by

$$\frac{d}{dt}p(t) = u^1(t) - k_1 p(t), \qquad p(0) = \bar{p}$$
$$u(t) = p(t) + u^0(t),$$

with transfer function $G_u(s) = [1, \frac{1}{s+k_1}]$. With $\eta(\hat{z}) = [1, 0]^T$, the $\mathcal{P}[-k_1]$-characteristic model is given by

$$\frac{d}{dt}\hat{x}(t) = \hat{y}(t) - k_1\hat{x}(t) - u^1(t)$$
$$\frac{d}{dt}\hat{y}(t) = u^0(t) - \hat{x}(t) - k\hat{y}(t).$$

The transfer function $\hat{G}(s) = \frac{1}{s^2+(k_1+k_2)s+k_1k_2+1}[s+k_1, 1]$ of the characteristic model equals the product $G(s)G_u(s)$ of the transfer functions of the original system and of the input model after pole-zero cancellation.

4.2. Example 3—hyperbolic invariances

Consider the system

$$\frac{d}{dt}x_1(t) = (y(t) - y_0)x_2(t)$$
$$\frac{d}{dt}x_2(t) = (y(t) - y_0)x_1(t)$$
$$\frac{d}{dt}y(t) = y_0\left(u(t)x_2(t) - \sqrt{1 + u(t)u(t)}x_1(t)\right) - y(t)$$

with $y_0 \in \mathbb{R}_{>0}$ and $x_1(t), x_2(t), y(t), u(t) \in \mathbb{R}$. The system is $\mathcal{P}[0]$-equivariant, with input and state transformations given by

$$\mathcal{P}[0] = \left\{\pi_p(\bar{u}) = \sinh\left(p + \operatorname{arsinh}(u(t))\right)\right\}_{p\in\mathbb{R}}$$

$$\mathcal{R}[0] = \left\{\rho_p\left(\begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix}\right) = \begin{bmatrix} \cosh(p)x_1 + \sinh(p)x_2, \\ \sinh(p)x_1 + \cosh(p)x_2, \\ y \end{bmatrix}\right\}_{p\in\mathbb{R}}.$$

The $\mathcal{P}[0]$-input module of the system is given by

$$\frac{d}{dt}p(t) = u^1(t), \qquad p(0) = \bar{p}$$
$$u(t) = \sinh\left(p(t) + \operatorname{arsinh}(u^0(t))\right),$$

and, with $\eta(\hat{x}_1, \hat{x}_2, \hat{y}) = (\hat{x}_2, \hat{x}_1, 0)^T$, the $\mathcal{P}[0]$-characteristic model by

$$\frac{d}{dt}\hat{x}_1(t) = (\hat{y}(t) - y_0 - u^1(t))\hat{x}_2(t)$$
$$\frac{d}{dt}\hat{x}_2(t) = (\hat{y}(t) - y_0 - u^1(t))\hat{x}_1(t)$$
$$\frac{d}{dt}\hat{y}(t) = y_0\left(u^0(t)\hat{x}_2(t) - \sqrt{1 + u^0(t)u^0(t)}\hat{x}_1\right) - \hat{y}(t).$$

5. First-order nonlinear differential operators

In Section 2, we have shown that the output of a Hurwitz LTI system with a zero at the origin excited by a ramp input $u_{k_1,k_0}(t) = k_1 t + k_0$ converges to a constant value $y(t) \to \bar{y}^* \propto \frac{d}{dt}u_{k_1,k_0}(t) = k_1$ proportional to the time-derivative (i.e. the slope) of the ramp input. In this section, we shortly summarize our previous results (Lang & Sontag, 2016) generalizing this property to 0-equivariant systems:

Definition 11 (*First-order Differential Operators, Lang & Sontag, 2016*). Consider a nonlinear system (3) and an indexed family of inputs $\mathcal{U}_g = \left\{u_{k_1,k_0} : [0,\infty) \to \mathbb{R} | u_{k_1,k_0}(t) = g(k_1 t + k_0)\right\}_{k_1,k_0\in\mathbb{R}}$ defined by a non-constant piecewise-continuous "prototype" function $g : \mathbb{R} \to \mathbb{R}$. Then, the system *realizes* the (nonlinear) differential operator $D_g : \mathcal{U}_g \to \mathbb{R}$, if there exists a set $K_1 \times K_0 \subseteq \mathbb{R}^2$ with non-empty interior, such that for all inputs $u_{k_1,k_0} \in \mathcal{U}_g$ with $(k_1, k_0) \in K_1 \times K_0$ there exists an initial condition $\bar{z}^* \in Z$ for which the output is constant and independent of k_0, i.e. $\bar{y}^* = h(\xi(t, \bar{z}_u^*, u_{k_1,k_0})) = \alpha_g(k_1) = \alpha_g(D_g u_{k_1,k_0}(t))$ for all $t \geq 0$, with $\alpha_g : K_1 \to \mathbb{R}$ a function which might depend on the specific system. If a system realizes the differential operator D_g, we denote the inputs $\mathcal{U}_g$ as its *characteristic inputs*, and $\bar{\mathcal{U}}_g = \{u_{k_1,k_0} \in \mathcal{U}_g | (k_1, k_0) \in K_1 \times K_0\}$ as its *proper characteristic inputs*. For a given characteristic input $u_{k_1,k_0} \in \bar{\mathcal{U}}_g$, if there exists a neighborhood $\bar{Z} \subseteq Z$ of $\bar{z}^*$ such that the output of the system initialized at every $\bar{z} \in \bar{Z}$ converges to $\bar{y}^*$, we say that the system is *convergent* with respect to the input, and, if $\bar{Z} = Z$, that it is *globally convergent*. If there does not exist a set $K_1 \times K_0$ for which α_g is injective, the system is a *degenerated* realization of D_g.

Consider a $\mathcal{P}[0]$-equivariant system (3) excited by the input $u_{k_1,k_0}(t) = \pi_{k_1 t+k_0}(\bar{u}^0)$, with $k_0, k_1 \in \mathbb{R}$ and $\bar{u}^0 \in U$. Since u_{k_1,k_0} corresponds to the output of the input module (7) initialized at $\bar{p} = k_0$, and excited by $u^0(t) = \bar{u}^0$ and $u^1(t) = k_1$, the output dynamics of the system equal the output dynamics of the characteristic model

$$\frac{d}{dt}\hat{z}(t) = f(\hat{z}(t), \bar{u}^0) - \eta(\hat{z})k_1, \qquad \hat{z}(0) = \rho_{-k_0}(\bar{z}) \tag{8a}$$

$$y(t) = h(\hat{z}(t)). \tag{8b}$$

Note, that the dynamics of this autonomous system of ODEs (8) only depend on k_1, but not on k_0.

Theorem 12 (*First-order Differential Operators, Lang & Sontag, 2016*). *Consider a $\mathcal{P}[0]$-equivariant system (3). If there exists a set $K_1 \times K_0 \subseteq \mathbb{R}^2$ with non-empty interior such that (8) has at least one steady-state $\hat{z}^* \in \rho_{-k_0}(Z)$ for each $(k_1, k_0) \in K_1 \times K_0$, the system realizes the (nonlinear) differential operator $D_{\pi_t(\bar{u}^0)}$ with respect to the characteristic inputs $\mathcal{U}_{\pi_t(\bar{u}^0)}$ defined by the prototype function $\pi_t(\bar{u}^0)$, with $\bar{u}^0 \in U$ and $\pi_p \in \mathcal{P}$. The characteristic inputs $\pi_{k_1 t+k_0}(\bar{u}^0)$, with $(k_1, k_0) \in K_1 \times K_0$ are proper. If for a given $(k_1, k_0) \in K_1 \times K_0$ the steady-state of (8) is (globally) asymptotic stable, the system is (globally) convergent with respect to $\pi_{k_1 t+k_0}(\bar{u}^0)$.*

Proof. The proof is given in Lang and Sontag (2016), Theorem 1. □

Remark 13. If for a constant input $u(t) = \bar{u} \in U$, a $\mathcal{P}[0]$-equivariant system has an exponentially stable steady-state $\bar{z}^* \in \mathrm{int}(Z)$ in the interior of Z, and $\eta(z) = (\partial_p \rho)_0(z)$ is continuously differentiable in a neighborhood of $\bar{z}^*$, we can apply the implicit function theorem to show that the system realizes $D_{\pi_t(\bar{u})}$.

5.1. Example 3 (continued)

For $u^0(t) = 0$ and $u^1(t) = k_1 \in \mathbb{R}$, the characteristic model of the hyperbolic example from Section 4.2 has an infinite number of steady-states $\hat{z}^* = [-1 - \frac{k_1}{y_0}, r, y_0 + k_1]^T$, with $r \in \mathbb{R}$. Thus, the system realizes the differential operator $D_{\sinh} = \frac{d}{dt}\,\mathrm{arsinh}$ for the characteristic inputs $u(t) = \pi_{k_1 t+k_0}(0) = \sinh(k_1 t + k_0)$ (Fig. 3(a), (b)).

When exciting the system by differentiable inputs $u(t)$ not conforming to characteristic inputs, the output of the system is in general different from $D_{\sinh}u(t)$. However, if the input can locally be approximated sufficiently long by characteristic inputs with respect to which the system is convergent, the output still approximately performs the "correct" differential operation, i.e. stays close to $\alpha_{\sinh}(D_{\sinh}u(t)) = y_0 + \frac{d}{dt}\,\mathrm{arsinh}(u(t))$ (Fig. 3(c), (d)).

6. Second-order invariances & equivariances

Recall that LTI systems can not only have several different zeros, but also zeros with multiplicities greater than one, resulting in additional dynamic properties of their input/output relationship (Section 2). In this section, we show that similar holds for second-order s_0-equivariant systems, with the additional dynamic properties described by second-order s_0-invariances. The generalization to arbitrary-order invariances and equivariances is shortly discussed at the end of Section 7.

Intuitively, we define second-order s_0-invariances with respect to continuous transformations of the time-derivative of the input. The mathematical definition, however, is slightly more involved since it also copes with inputs whose time-derivative is not always defined, and is compatible with causal systems:

Definition 14 (*Second-order s_0-invariance*). Consider a nonlinear system (3) and two one-parameter Lie groups of input transformations $\mathcal{P}^1 = \{\pi_p^1 : U \to U\}_{p\in\mathbb{R}}$ and $\mathcal{P}^2 = \{\pi_p^2 : \mathbb{R} \to \mathbb{R}\}_{p\in\mathbb{R}}$. Then, the system is *second-order s_0-invariant* with respect to $\mathcal{P}^1 \times \mathcal{P}^2$ (in short, is $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-invariant), if it is $\mathcal{P}^1[s_0]$-invariant, and if for all $p \in \mathbb{R}$ and $\bar{z} \in Z$ there exists a $\bar{z}' \in Z$ such that

$$\begin{aligned} &h\left(\xi\left(t, \bar{z}, t \mapsto \pi^1_{(T_0 u^1)(t)}\left(u^0(t)\right)\right)\right) \\ &\quad = h\left(\xi\left(t, \bar{z}', t \mapsto \pi^1_{(T_p u^1)(t)}(u^0(t))\right)\right), \end{aligned} \tag{9}$$

for all $t \geq 0$, $u^0 \in \mathcal{U}$, and $u^1 \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R})$, with $(T_p u^1)(t) = \int_0^t e^{s_0(t-\tau)} \pi^2_{p\exp(s_0\tau)}\left(u^1(\tau)\right) d\tau$.

Remark 15. By setting $u^1(t) = 0$, (9) simplifies to $h\left(\xi\left(t, \bar{z}, u^0\right)\right) = h(\xi(t, \bar{z}', t \mapsto \pi^1_{(T_p 0)(t)}(u^0(t))))$.

Remark 16. For $u^1(t) = 0$ and $s_0 = 0$, (9) simplifies to $h\left(\xi\left(t, \bar{z}, u^0\right)\right) = h\left(\xi\left(t, \bar{z}', t \mapsto \pi^1_{k_1 t}(u^0(t))\right)\right)$, with $k_1 = \pi_p^2(0)$. Since a $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-invariant system is also $\mathcal{P}^1[s_0]$-invariant, this implies that for all $k_1, k_0 \in \mathbb{R}$ and all $\bar{z} \in Z$ there exists a $\bar{z}'' \in Z$ such that $h\left(\xi\left(t, \bar{z}, u^0\right)\right) = h\Big(\xi\Big(t, \bar{z}'', t \mapsto \pi^1_{k_1 t+k_0}(u^0(t))\Big)\Big)$.

Remark 17. If both input transformations correspond to translations ($\pi_p^1(\bar{u}) = \bar{u}+p^1, \pi_p^2(\bar{u}) = \bar{u}+p^2$), second-order s_0-invariance implies that $h\left(\xi\left(t, \bar{z}, u\right)\right) = h\big(\xi\big(t, \bar{z}', t \mapsto u(t) + p^1 e^{s_0 t} + p^2 t e^{s_0 t}\big)\big)$, in agreement with the zero inputs of LTI systems with a zero at s_0 with multiplicity two (Section 2). Specifically, if $s_0 = 0$, this implies the rejection of ramp inputs.

Definition 18 (*Second-order s_0-equivariance*). Consider the system (3) and two one-parameter Lie groups of input transformations $\mathcal{P}^1 = \{\pi_p^1 : U \to U\}_{p\in\mathbb{R}}$ and $\mathcal{P}^2 = \{\pi_p^2 : \mathbb{R} \to \mathbb{R}\}_{p\in\mathbb{R}}$. Then, the system is *second-order s_0-equivariant* with respect to $\mathcal{P}^1 \times \mathcal{P}^2$ (in short, is $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-equivariant), if there exists two one-parameter Lie groups of state transformations $\mathcal{R}^1 = \{\rho_p^1 : Z \to Z\}_{p\in\mathbb{R}}$ and $\mathcal{R}^2 = \{\rho_p^2 : Z \to Z\}_{p\in\mathbb{R}}$, such that

$$\begin{aligned} &f(\rho_p^1(z), \pi_p^1(\bar{u}^0)) = (\partial_p \rho^1)_p(z) p s_0 + (\partial_z \rho^1)_p(z) f(z, \bar{u}^0) \\ &f(\rho_p^2(z), \bar{u}^0) - \eta^1(\rho_p^2(z))\pi_p^2(\bar{u}^1) \\ &\quad = (\partial_p \rho^2)_p(z) p s_0 + (\partial_z \rho^2)_p(z)\left(f(z, \bar{u}^0) - \eta^1(z)\bar{u}^1\right) \\ &h(\rho_p^1(z)) = h(\rho_p^2(z)) = h(z) \end{aligned}$$

for all $z \in Z$, $\bar{u}^1 \in U$, $\bar{u}^2 \in \mathbb{R}$, and $p \in \mathbb{R}$, with $\eta^1(z) = (\partial_p \rho^1)_0(z)$.

Remark 19. A nonlinear system (3) is $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-equivariant if and only if it is $\mathcal{P}^1[s_0]$-equivariant, and if its $\mathcal{P}^1[s_0]$-characteristic model (6) is $\mathcal{P}^2[s_0]$-equivariant with respect to its input u^1.

Theorem 20. *An analytic and observable nonlinear system (3) is $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-invariant if and only if it is $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-equivariant.*

Proof. *Sufficiency*: Let $z_a(t) = \xi\left(t, \bar{z}, t \mapsto \pi^1_{(T_0 u^1)(t)}\left(u^0(t)\right)\right)$ and $y(t) = h(z_a(t))$. Setting $z_b = \rho^1_{-(T_0 u^1)(t)}(z_a)$, $z_c = \rho^2_{pe^{s_0 t}}(z_b)$ and $z_d = \rho^1_{(T_p u^1)(t)}(z_c)$, and differentiation by time results in

$$\begin{aligned} \frac{d}{dt} z_d(t) &= f\left(z_d(t), \pi^1_{(T_p u^1)(t)}(u^0(t))\right) \\ y(t) &= h(z_d(t)), \end{aligned}$$

with $z_d(0) = \rho_p^2(\bar{z}) =: \bar{z}'$.

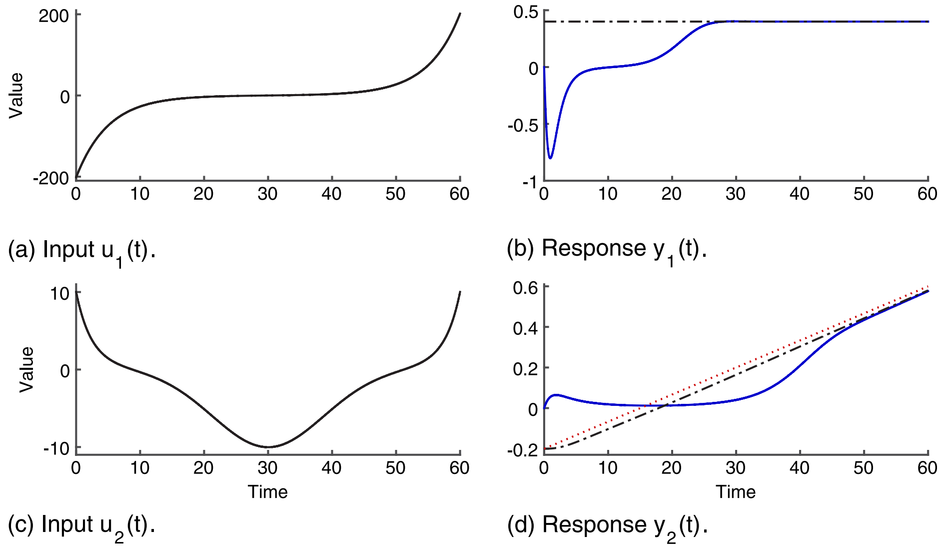


Fig. 3. Dynamics of the hyperbolic invariances example (Section 4.2) realizing the differential operator $\frac{d}{dt}$ arsinh. (a, b) The output y_1 (b) of the system stays constant when excited by the characteristic input $u_1(t) = \sinh(k_1 t + k_0)$ (a), with $k_0 = -6$ and $k_1 = 0.2$, and initialized at $z(0) = \bar{z}^* = \rho_{-k_0}([-1 - \frac{k_1}{y_0}, 2, y_0 + k_1]^T)$ (black, dash-dotted). When initialized at $z(0) = [0, 0.05, 0]^T$ (blue, solid), the output converges to the constant value. (c, d) When initializing the system at $z(0) = \rho_{-k_0}([-1 - \frac{k_1}{y_0}, 2, y_0 + k_1]^T)$ (d, black dotted) or $z(0) = [0, 0.05, 0]^T$ (d, blue solid) and exciting it by $u_2(t) = \sinh(\frac{1}{8}\frac{k_1^2}{k_0}t^2 + k_1 t + k_0)$ (c), with $k_0 = 3$, $k_1 = -0.4$, and $\hat{t} \geq 0$, the output (d) closely follows $\frac{d}{dt}$ arsinh$(u_2(t)) + y_0 = k_1 + \frac{1}{4}\frac{k_1^2}{k_0}t + y_0$ (red, dotted). Note, that the input $u_2(t) = \sinh((k_1 + \frac{1}{4}\frac{k_1^2}{k_0}\hat{t})t + (k_0 - \frac{1}{8}\frac{k_1^2}{k_0}\hat{t}^2) + \frac{1}{8}\frac{k_1^2}{k_0}(t - \hat{t})^2)$ can be well approximated around every $\hat{t} \geq 0$ by a characteristic input if $\left|\frac{k_1^2}{k_0}\right| \ll 1$. (a–d) The parameter y_0 was set to 0.2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Necessity: Let $z_a(t) = \xi\left(t, \bar{z}, t \mapsto \pi^1_{(T_0 w^1)(t)}\left(\bar{u}^0\right)\right)$ and $z_\alpha(t) = \xi\left(t, \bar{z}', t \mapsto \pi^1_{(T_p w^1)(t)}\left(\bar{u}^0\right)\right)$, with $\bar{u}^0 \in U$ and $w \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R})$. By (9), we can choose $\bar{z}$ and $\bar{z}'$ such that $h(z_a(t)) = h(z_\alpha(t))$. By Theorem 7, the system is $\mathcal{P}^1[s_0]$-equivariant with state transformations given by $\mathcal{R}^1[s_0] = \{\rho^1_p : Z \to Z\}_{p \in \mathbb{R}}$. We set $z_b(t) = \rho^1_{-(T_0 w)(t)}(z_a)$ and $z_\beta(t) = \rho^1_{-(T_p w)(t)}(z_a)$. Differentiating by time, the ODEs for z_b and z_β correspond to the $\mathcal{P}^1[s_0]$-characteristic model (6) of the system excited by $u^0(t) = \bar{u}^0$ and $u^1(t) = w(t)$, respectively $u^0(t) = \bar{u}^0$ and $u^1(t) = \pi^2_{pe^{s_0 t}}(w(t))$, and initialized at $\bar{z}$, respectively $\bar{z}'$. Then, $h(z_b(t)) = h(z_\beta(t))$ implies that $h\left(\hat{\xi}\left(t, \bar{z}, \bar{u}^0, w\right)\right) = h\left(\hat{\xi}\left(t, \bar{z}', \bar{u}^0, t \mapsto \pi^2_{pe^{s_0 t}}(w(t))\right)\right)$, with $\hat{\xi}(t, \bar{z}, u^1, u^2) = \hat{z}(t)$ the solution of the $\mathcal{P}^1[s_0]$-characteristic model (6) initialized at $\bar{z}$, and excited by u^1 and u^2. Thus, the $\mathcal{P}^1[s_0]$-characteristic model of the system has to be $\mathcal{P}^2[s_0]$-invariant with respect to its input u^1. The $\mathcal{P}^1[s_0]$-characteristic model is analytic, and – by Lemma 9 – observable. It follows from Theorem 7 that the $\mathcal{P}^1[s_0]$-characteristic model is $\mathcal{P}^2[s_0]$-equivariant. By Remark 19, this implies that (3) is $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-equivariant. □

7. Second-order characteristic models & differential operators

In this section, we introduce the notion of second-order characteristic models, corresponding to the characteristic models of first-order characteristic models, as well as systems realizing second-order differential operators. The definitions and results closely follow the ones in Sections 4 and 5, and are mainly stated for completeness. Furthermore, we outline how to extend our theory to arbitrary-order s_0-invariances and s_0-equivariances.

Consider a $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-equivariant system (3) excited by $u(t) = \pi^1_{e^{s_0 t}\bar{p}^1 + \int_0^t e^{s_0(t-\tau)}v(\tau)d\tau}(u^0(t))$, with $v(t) = \pi^2_{e^{s_0 t}\bar{p}^2 + \int_0^t e^{s_0(t-\tau)}u^2(\tau)d\tau}(u^1(t))$, external inputs $u^0(t) \in \mathcal{U}$ and $u^1, u^2 \in \mathcal{PC}(\mathbb{R}_{\geq 0}, \mathbb{R})$, and $\bar{p}^1, \bar{p}^2 \in \mathbb{R}$. Setting $\hat{z}(t) = \rho^1_{-e^{s_0 t}\bar{p}^1 - \int_0^t e^{s_0(t-\tau)}v(\tau)d\tau}(z(t))$ and $\tilde{z}(t) = \rho^2_{-e^{s_0 t}\bar{p}^2 - \int_0^t e^{s_0(t-\tau)}u^2(\tau)d\tau}\left(\hat{z}(t)\right)$ and differentiating by time, we obtain the second-order characteristic model of (3) with respect to its $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-equivariance (in short, the $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-*characteristic model*):

$$\frac{d}{dt}\tilde{z}(t) = f(\tilde{z}(t), u^0(t)) - \eta^1(\tilde{z}(t))u^1(t) - \eta^2(\tilde{z}(t))u^2(t)$$
$$\tilde{z}(0) = \rho^2_{-\bar{p}^2}(\rho^1_{-\bar{p}^1}(\bar{z}))$$
$$y(t) = h(\tilde{z}(t)).$$

with $\eta^2(\tilde{z}) := (\partial_p \rho^2)_0(\hat{z})$.

In our definition of systems realizing second-order differential operators, we restrict ourselves to operators being strictly of a given order, that is, "mixed-order" differential operators like $\frac{d^2}{dt^2}\log + \frac{d}{dt}$ are not considered:

Definition 21 (*Second-order Differential Operators*). Consider the system (3) and an indexed family of inputs

$$\mathcal{U}_{g_2,g_1} = \left\{ u_{k_2,k_1,k_0} : [0,\infty) \to \mathbb{R} \,\middle|\, u_{k_2,k_1,k_0}(t) = g_1\left(\int_0^t g_2(k_2\tau + k_1)d\tau + k_0\right)\right\}_{k_2,k_1,k_0 \in \mathbb{R}}$$

defined by two non-constant piecewise-continuous "prototype" functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$. Then, the system *realizes* the second-order (nonlinear) differential operator $D_{g_2,g_1} = D_{g_2}D_{g_1} : \mathcal{U}_g \to \mathbb{R}$, if there exists a set $K_2 \times K_1 \times K_0 \subseteq \mathbb{R}^3$ with non-empty interior, such that for all inputs $u_{k_2,k_1,k_0} \in \mathcal{U}_{g_2,g_1}$ with $(k_2, k_1, k_0) \in K_2 \times K_1 \times K_0$ there exists an initial condition $\bar{z}^* \in Z$ for which the output is constant and independent of k_1 and k_0, i.e. $\bar{y}^* = h(\xi(t, \bar{z}^*, u_{k_2,k_1,k_0})) = \alpha_{g_2,g_1}(k_2) = \alpha_{g_2,g_1}(D_{g_2,g_1}u_{k_2,k_1,k_0})$ for all $t \geq 0$, with $\alpha_{g_2,g_1} : K_2 \to \mathbb{R}$ a function which might depend on the specific system.

For systems realizing second-order differential operators, (proper) characteristic inputs, (global) convergence and degeneracy can be defined similarly as in Definition 11.

Lemma 22 (*Second-order Differential Operators*). *Consider a $\mathcal{P}^1 \times \mathcal{P}^2[0]$-equivariant system (3). If there exists a set $K_2 \times K_1 \times K_0 \subseteq \mathbb{R}^3$ with non-empty interior such that the $\mathcal{P}^1 \times \mathcal{P}^2[0]$-characteristic model, with $\bar{p}_1 = k_0$ and $\bar{p}_2 = k_1$, excited by $u^0(t) = \bar{u}^0 \in U$, $u^1(t) = \bar{u}^1 \in \mathbb{R}$, and $u^2(t) = k_2$ has at least one steady-state $\tilde{z}^* \in \rho^2_{-k_1}\left(\rho^1_{-k_0}(Z)\right)$ for all $(k_2, k_1, k_0) \in K_2 \times K_1 \times K_0$, the system realizes the (nonlinear) differential operator $D_{\pi^2_t(\bar{u}^1)} D_{\pi^1_t(\bar{u}^0)}$ with respect to the characteristic inputs $\mathcal{U}_{\pi^2_t(\bar{u}^1), \pi^1_t(\bar{u}^0)}$ defined by the prototype functions $\pi^1_t(\bar{u}^0)$ and $\pi^2_t(\bar{u}^1)$.*

Proof. For all $(k_2, k_1, k_0) \in K_2 \times K_1 \times K_0$, the output of the characteristic model initialized at $\tilde{z}^*$ and excited by $u^0(t) = \bar{u}^0 \in U$, $u^1(t) = \bar{u}^1 \in \mathbb{R}$ and $u^2(t) = k_2$ is constant and does not depend on k_0 or k_1. By definition of the characteristic model, these constant output dynamics equal the output dynamics of the original system (3) initialized at $\bar{z}^* = \rho^2_{k_0}(\rho^1_{k_1}(\tilde{z}^*))$ and excited by $u_{k_2,k_1,k_0}(t) = \pi^1_{k_0 + \int_0^t \pi^2_{k_1 + k_2 \tau}(\bar{u}^1) d\tau}(\bar{u}^0)$. $\square$

Finally, let us outline how s_0-equivariance, s_0-invariance and systems realizing differential operators can be generalized to arbitrary order. Recall, that a system is second-order $\mathcal{P}^1 \times \mathcal{P}^2[s_0]$-equivariant if and only if it is first-order $\mathcal{P}^1[s_0]$-equivariant, and if its $\mathcal{P}^1[s_0]$-characteristic model is $\mathcal{P}^2[s_0]$-equivariant with respect to its input u^1 (Remark 19). Then, we can recursively define higher-order equivariances and invariances as follows: A system is $\mathcal{P}^{n+1} \times \cdots \times \mathcal{P}^1[s_0]$-equivariant if and only if it is $\mathcal{P}^n \times \cdots \times \mathcal{P}^1[s_0]$-equivariant and if its $\mathcal{P}^n \times \cdots \times \mathcal{P}^1[s_0]$-characteristic model is $\mathcal{P}^{n+1}[s_0]$-equivariant with respect to the input u^n. Thereby, the $\mathcal{P}^n \times \cdots \times \mathcal{P}^1[s_0]$-characteristic model is recursively defined as the $\mathcal{P}^n[s_0]$-characteristic model of the $\mathcal{P}^{n-1} \times \cdots \times \mathcal{P}^1[s_0]$-characteristic model with respect to the input u^{n-1}. Similarly, we can recursively define $\mathcal{P}^{n+1} \times \cdots \times \mathcal{P}^1[s_0]$-invariance by requiring that the system is $\mathcal{P}^n \times \cdots \times \mathcal{P}^1[s_0]$-invariant, and that it is invariant with respect to transformations of the nth time-derivative of the input in the sense of Definition 14. The relationship between $\mathcal{P}^n \times \cdots \times \mathcal{P}^1[s_0]$-equivariance and -invariance can be established following the methods developed in the proof of Theorem 20. Systems realizing nth-order differential operators can be defined according to Definition 21 using n different prototype functions.

Recall that an observable LTI system with an irreducible zero at s_0 is first-order s_0-equivariant (Remark 2 and Theorem 7), that the transfer function of its first-order s_0-characteristic model corresponds to its transfer function after canceling one occurrence of the zero at s_0 (Remark 10), and that a system is $(n+1)$th-order s_0-equivariant only if its nth-order s_0-characteristic model is s_0-equivariant. Thus, while we only provide detailed examples for second-order invariant systems in the following, we remark that observable LTI systems possessing irreducible zeros with multiplicity $m_{s_0} \geq 2$ are examples for higher-order (m_{s_0}th-order) s_0-invariant systems.

7.1. Example 2 (continued)

For $b > 0$, we have already shown that the example system in Section 3.2 is $\mathcal{P}[0]$-equivariant and $\mathcal{P}[-b]$-equivariant. For $b \to 0$, the two equivariances collapse, such that for $b = 0$ the system possesses only one (first-order) 0-equivariance. However, for $b = 0$ the system is second-order $\mathcal{P}^1 \times \mathcal{P}^2[0]$-equivariant, with $\mathcal{P}^1$ and $\mathcal{R}^1$ as given in Section 3.2, and $\mathcal{P}^2 = \{\pi_p(\bar{u}) = \bar{u} + p\}_{p \in \mathbb{R}}$ and $\mathcal{R}^2 = \{\rho_p(x_1, x_2, y) = [x_1 + \frac{p}{c}, x_2, y]^T\}_{p \in \mathbb{R}}$. The second-order characteristic model is given by

$$\frac{d}{dt}\tilde{x}_1(t) = a\left(\tilde{y}(t) - y_0\right) - \frac{u^2(t)}{c}$$
$$\frac{d}{dt}\tilde{x}_2(t) = c\tilde{x}_2(t)\left(\tilde{x}_1(t) - \frac{u^1(t)}{c} + \tilde{y}(t) - y_0\right)$$
$$\frac{d}{dt}\tilde{y}(t) = d\frac{u^0(t)}{\tilde{x}_2(t)} - e\tilde{y}(t).$$

For $u^0(t) = 1$, $u^1(t) = 0$ and $u^2(t) = k_2 > K^2_L = -acy_0$, the characteristic model has a steady-state at $[\tilde{x}^*_1, \tilde{x}^*_2, \tilde{y}^*]^T = [-\frac{k_2}{ac}, \frac{acd}{ek_2 + acey_0}, y_0 + \frac{k_2}{ac}]^T$. Thus, for $k_2 > K^2_L$, the system realizes the second-order differential operator $\frac{d^2}{dt^2}\log$ (Fig. 1(c), (d)), with the characteristic inputs given by $u_{k_2,k_1,k_0}(t) = e^{\frac{1}{2}k_2 t^2 + k_1 t + k_0}$.

7.2. Example 4—realizing $\frac{d}{dt}\log(\frac{1}{1+v_0}(\frac{d}{dt}(.) + v_0))$

Consider the system

$$\frac{d}{dt}x_1(t) = u(t) - x_1(t)$$
$$\frac{d}{dt}x_2(t) = -ax_2 + b\left(u(t) - x_1(t) + v_0\right)$$
$$\frac{d}{dt}y(t) = c\frac{u(t) - x_1(t) + v_0}{x_2(t)} - dy(t).$$

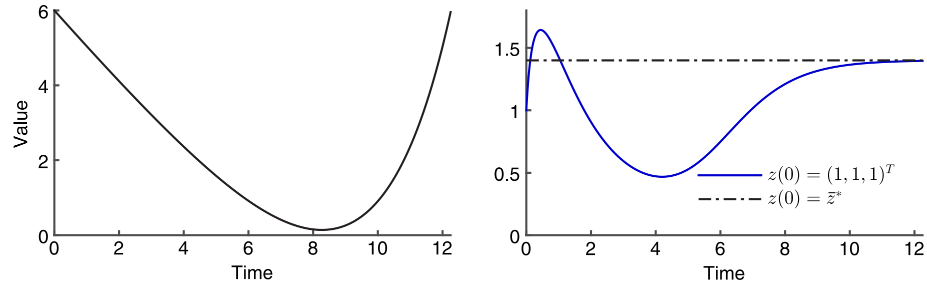
The system is second-order 0-equivariant, with $\pi^1_p(\bar{u}) = \bar{u} + p$, $\rho^1_p(z) = [x_1 + p, x_2, y]^T$ and $\eta^1(z) = [1, 0, 0]^T$, and $\pi^2_p(v) = e^p(v + v_0) - v_0$, $\rho^2_p(\hat{z}) = [e^p(\hat{x}_1 - v_0) + v_0, e^p\hat{x}_2, \hat{y}]^T$ and $\eta^2(\hat{z}) = [\hat{x}_1 - v_0, \hat{x}_2, 0]^T$. The second-order 0-characteristic model is given by

$$\frac{d}{dt}\tilde{x}_1(t) = u^0(t) - \tilde{x}_1(t) - u^1(t) - (\tilde{x}_1(t) - v_0)u^2(t)$$
$$\frac{d}{dt}\tilde{x}_2(t) = -a\tilde{x}_2(t) + b\left(u^0(t) - \tilde{x}_1(t) + v_0\right) - \tilde{x}_2(t)u^2(t)$$
$$\frac{d}{dt}\tilde{y}(t) = c\frac{u^0(t) - \tilde{x}_1(t) + v_0}{x_2(t)} - d\tilde{y}(t).$$

For $u^0(t) = 0$, $u^1(t) = 1$ and $u^2(t) = k_2 > K^2_L = -\min(a, 1)$, the characteristic model has a steady-state at $[\tilde{x}^*_1, \tilde{x}^*_2, \tilde{y}^*]^T = [-\frac{1 - v_0 k_2}{1 + k_2}, \frac{b}{a + k_2}\frac{v_0 + 1}{1 + k_2}, \frac{c(a + k_2)}{bd}]^T$. Thus, for $k_2 > K^2_L$, the system realizes the second-order differential operator $\frac{d}{dt}\log(\frac{1}{1+v_0}(\frac{d}{dt}(.) + v_0))$ (Fig. 4), with the characteristic inputs given by $u_{k_2,k_1,k_0}(t) = \frac{1+v_0}{k_2}(\exp(k_2 t + k_1) - \exp(k_1)) - v_0 t + k_0$.

8. Discussion

Systems invariant with respect to sets of pointwise and time-invariant input transformations have been widely studied in the literature (Adler et al., 2014; Goentoro et al., 2009; Hironaka & Morishita, 2014; Shoval et al., 2011, 2010) with arguably the most prominent examples being scale-invariant systems ($u(t) \mapsto e^p u(t)$, with $p \in \mathbb{R}_{>0}$) and translational-invariant systems ($u(t) \mapsto p + u(t)$, with $p \in \mathbb{R}$). In this article, we introduced the concept of s_0-invariant systems, corresponding to invariances with respect to input transformations exponentially growing/decaying over time. We have shown the close relationship between s_0-invariance and s_0-equivariance, with the latter being a memoryless property only



(a) Input u(t). (b) Response y(t).

Fig. 4. (a, b) Input/output dynamics of the system realizing the differential operator $\frac{d}{dt}\log(\frac{1}{1+v_0}(\frac{d}{dt}(.) + v_0))$ (Section 7.2) when excited by the characteristic input $u_{k_2,k_1,k_0}(t) = \frac{1+v_0}{k_2}(\exp(k_2 t + k_1) - \exp(k_1)) - v_0 t + k_0$ (a) with $k_2 = 0.4$, $k_1 = -4$ and $k_0 = 6$. (b) The black curve represents the output dynamic when the system is initialized at $(\bar{x}_1^*, \bar{x}_2^*, \bar{y}^*)^T$, while the blue curve corresponds to the initial condition $(1, 1, 1)^T$. The parameters were set to $a = b = c = d = v_0 = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

depending on the current state and input of the system. We then extended our framework to second-order s_0-invariant systems, that is, systems invariant with respect to transformations of the time-derivatives of the input. Since this extension is based on a so called characteristic model of an s_0-equivariant system, and since it is possible to recursively define characteristic models of characteristic models, our theory is easily extendable to arbitrary-order s_0-invariant systems. Finally, we introduced the concept of systems realizing first-and higher-order differential operators, that is, systems whose output remains at a constant value only depending on the (first- or higher-order) time-derivative of the input when excited by a characteristic input and initialized appropriately.

First- and higher-order s_0-equivariant systems not only encompass LTI systems possessing zeros at $s_0 \in \mathbb{R}$ with single or higher multiplicities, but their input/output dynamics also show properties generalizing those of LTI systems with transfer function zeros. We expect that in future research, our theoretical framework might be extended to also govern complex conjugated pairs of zeros, e.g. systems invariant with respect to exponentially decaying or increasing oscillatory input-transformations. It might also be possible to generalize certain linear controller or observer design techniques to s_0-equivariant systems.

Practically, we expect our framework to prove valuable in trajectory tracking applications and in the analysis of natural or engineered (biological) systems. For example, the tracking error of type-N servomechanisms converges to zero for polynomial inputs of order $N - 1$. If such a servomechanism would additionally be Nth order 0-invariant, its transient error dynamics would become independent of the current position for $N \geq 1$, of the current velocity ($N \geq 2$), and of the current acceleration ($N \geq 3$). On the other hand, several naturally evolved biomolecular networks are known to possess 0-invariances. For example, bacterial chemotaxis was not only shown to be scale-invariant, but also to realize the differential operator $\frac{d}{dt}\log$ already more than 30 years ago (Block, Segall, & Berg, 1983). Interestingly, while 0-invariance naturally lends itself as a mean to detect changes in otherwise (approximately) constant signals, s_0-invariance, with $s_0 < 0$, lends itself to detect changes in otherwise decaying signals, with exponential decay due to dilution or degradation being a prevalent feature in biological systems.

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