EX.1.5.6, Sauer3

If the Secant Method converges to r, $f'(r) \neq 0$, and $f''(r) \neq 0$, then the approximate error relationship $e_{i+1} \approx |f''(r)/(2f'(r))|e_i e_{i-1}$ can be shown to hold. Prove that if in addition $\lim_{i\to\infty} e_{i+1}/e_i^\alpha$ exists and is nonzero for some $\alpha > 0$, then $\alpha = (1+\sqrt{5})/2$ and $e_{i+1} \approx |(f''(r)/2f'(r))|^{\alpha-1} e_i^\alpha$.

EX.1.5.6, Sauer3, solution, Langou

We are given that

$$e_{i+1} \approx \left|\frac{f''(r)}{2f'(r)}\right| e_i e_{i-1}$$

and

$$e_{i+1} \approx M e_i^\alpha$$

where M is some nonzero constant. Thus

$$\begin{aligned}
Me_i^\alpha &\approx \left|\frac{f''(r)}{2f'(r)}\right| e_i e_{i-1} \\
Me_i^{\alpha-1} &\approx \left|\frac{f''(r)}{2f'(r)}\right| e_{i-1} \\
e_i &\approx M^{\frac{-1}{\alpha-1}} \left|\frac{f''(r)}{2f'(r)}\right|^{\frac{1}{\alpha-1}} e_{i-1}^{\frac{1}{\alpha-1}}
\end{aligned}$$

For large enough i, we have that $e_i \approx M e_{i-1}^\alpha$ such that

$$\alpha = \frac{1}{\alpha - 1}$$

which has a positive solution of $\alpha = (1+\sqrt{5})/2$. Moreover, we have the condition on M such that

$$\begin{aligned}
M &= M^{\frac{-1}{\alpha-1}} \left|\frac{f''(r)}{2f'(r)}\right|^{\frac{1}{\alpha-1}} \\
M^{\alpha-1} &= M^{-1} \left|\frac{f''(r)}{2f'(r)}\right| \\
M^\alpha &= \left|\frac{f''(r)}{2f'(r)}\right| \\
M &= \left|\frac{f''(r)}{2f'(r)}\right|^{\frac{1}{\alpha}}
\end{aligned}$$

Since $\alpha = 1/(\alpha - 1)$ then

$$M = \left|\frac{f''(r)}{2f'(r)}\right|^{\alpha-1}.$$