EX.1.1.3, Sauer3

Consider the equations (a) $x^3 = 9$, (b) $3x^3 + x^2 = x + 5$, (c) $\cos^2(x) + 6 = x$.
Starting from an interval of length one that contains a root of the equation, apply two steps of the Bisection Method to find an approximate root within 1/8 of the true root.

EX.1.1.3, Sauer3, solution, Langou

`https://colab.research.google.com/drive/19pmIPmjKy23ryyoVU5c0K3zeZdgm96pI`

(a) $x^3 = 9$

We work with $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3 - 9$ and start with $a = 2$ and $b = 3$. (See EX.1.1.1.)

$$
\begin{array}{lrclrclrclr}
\text{Step 0} & a_0 & = & 2.000 & & & & b_0 & = & 3.000 & \\
 & f(a_0) & = & -1.000 & & & & f(b_0) & = & 18.000 & \text{(two function evaluations)} \\
 & & & & x_0 & = & 2.500 & & & & |x_0 - x_\star| \leq 0.500 = \frac{1}{2} \\
\hline\hline
\text{Step 1} & & & & f(x_0) & = & 6.625 & & & & \text{(one function evaluation)} \\
 & a_1 & = & 2.000 & & & & b_1 & = & 1.000 & \\
 & f(a_1) & = & -1.000 & & & & f(b_1) & = & 6.625 & \\
 & & & & x_1 & = & 2.250 & & & & |x_1 - x_\star| \leq 0.250 = \frac{1}{4} \\
\hline
\text{Step 2} & & & & f(x_1) & = & 2.391 & & & & \text{(one function evaluation)} \\
 & a_2 & = & 2.000 & & & & b_1 & = & 2.250 & \\
 & f(a_1) & = & -1.000 & & & & f(b_1) & = & 1.000 & \\
 & & & & x_2 & = & 2.125 & & & & |x_1 - x_\star| \leq 0.125 = \frac{1}{8}
\end{array}
$$

The approximate solution $x_2 = 2.125$ is within 1/8 of the true root.

<u>*Comment:*</u> Starting with an interval of length one, we needed two steps of the bisection method and four function evaluations to guarantee a forward error of less then 1/8. This makes sense.

(b) $3x^3 + x^2 = x + 5$

We work with $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 3x^3 + x^2 - x - 5$ and start with $a = 1$ and $b = 2$. (See EX.1.1.1.)

$$
\begin{array}{lrclrclrclr}
\text{Step 0} & a_0 & = & 1.000 & & & & b_0 & = & 2.000 & \\
 & f(a_0) & = & -2.000 & & & & f(b_0) & = & 21.000 & \text{(two function evaluations)} \\
 & & & & x_0 & = & 1.500 & & & & |x_0 - x_\star| \leq 0.500 = \frac{1}{2} \\
\hline
\text{Step 1} & & & & f(x_0) & = & 5.875 & & & & \text{(one function evaluation)} \\
 & a_1 & = & 1.000 & & & & b_1 & = & 1.500 & \\
 & f(a_1) & = & -2.000 & & & & f(b_1) & = & 6.625 & \\
 & & & & x_1 & = & 1.250 & & & & |x_1 - x_\star| \leq 0.250 = \frac{1}{4} \\
\hline
\text{Step 2} & & & & f(x_1) & = & 1.172 & & & & \text{(one function evaluation)} \\
 & a_2 & = & 1.000 & & & & b_1 & = & 1.250 & \\
 & f(a_1) & = & -2.000 & & & & f(b_1) & = & 1.172 & \\
 & & & & x_2 & = & 1.125 & & & & |x_1 - x_\star| \leq 0.125 = \frac{1}{8}
\end{array}
$$

The approximate solution $x_2 = 2.125$ is within $1/8$ of the true root.

(c) $3x^3 + x^2 = x + 5$

We work with $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \cos^2(x) + 6 - x$ and start with $a = 6$ and $b = 7$. (See EX.1.1.1.)

Step 0	$a_0 = 6.000$		$b_0 = 7.000$	
	$f(a_0) = 0.922$		$f(b_0) = -0.432$	(two function evaluations)
		$x_0 = 6.500$		$\lvert x_0 - x_\star \rvert \leq 0.500 = \frac{1}{2}$
Step 1		$f(x_0) = 0.454$		(one function evaluation)
	$a_1 = 6.500$		$b_1 = 7.000$	
	$f(a_1) = 0.454$		$f(b_1) = -0.432$	
		$x_1 = 6.750$		$\lvert x_1 - x_\star \rvert \leq 0.250 = \frac{1}{4}$
Step 2		$f(x_1) = 0.047$		(one function evaluation)
	$a_2 = 6.750$		$b_1 = 7.000$	
	$f(a_1) = 0.047$		$f(b_1) = -0.432$	
		$x_2 = 6.875$		$\lvert x_1 - x_\star \rvert \leq 0.125 = \frac{1}{8}$

The approximate solution $x_2 = 6.875$ is within $1/8$ of the true root.