

EX.4.3.8.a, Sauer3

Find the QR factorization and use it to solve the following least squares problem

$$\begin{bmatrix} 1 & 4 \\ -1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \end{bmatrix}.$$

Suggestion: Use the Gram-Schmidt algorithm for getting the QR factorization, and remember that a thin QR factorization is enough for this problem. No need to compute a full QR factorization.

EX.4.3.8.a, Sauer3, solution, Langou

Colab: <https://colab.research.google.com/drive/1cw4J2CTJjaPGahWJBS5ujNKJmtEE4h5M>

We start with performing a QR factorization of A using the Gram-Schmidt algorithm.

Let a_0 and a_1 be the first and second columns of A .

We perform the following computation.

$$\begin{aligned} w_0 &= a_0, \\ r_{00} &= \|w_0\|_2 = \sqrt{(1)^2 + (-1)^2 + (1)^2 + (1)^2} = 2, \\ q_0 &= \frac{1}{r_{00}} w_0 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \\ r_{01} &= q_0^T a_1 = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} ((1)(4) + (-1)(1) + (1)(1) + (1)(0)) = 2, \\ w_1 &= a_1 - q_0 r_{01} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \end{bmatrix} - 2 \left(\frac{1}{2}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \\ r_{11} &= \|w_1\|_2 = \sqrt{(3)^2 + (2)^2 + (0)^2 + (-1)^2} = \sqrt{14}, \\ q_1 &= \frac{1}{r_{11}} w_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}. \end{aligned}$$

So we find

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{3}{\sqrt{14}} \\ -\frac{1}{2} & \frac{2}{\sqrt{14}} \\ \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{14}} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 2 & 2 \\ 0 & \sqrt{14} \end{bmatrix}.$$

We can check that

a. $A = QR$

b. $Q^T Q = I$

c. R is upper triangular

Now we can solve the linear least squares problem using this QR factorization.

Firstly, we form $Q^T b$:

$$Q^T b = , \left[\begin{array}{cccc} \frac{1}{\sqrt{14}} & -\frac{1}{\sqrt{14}} & \frac{1}{2} & -\frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} & \frac{2}{\sqrt{14}} & 0 & -\frac{1}{\sqrt{14}} \end{array} \right] \left[\begin{array}{c} 3 \\ 1 \\ 1 \\ -3 \end{array} \right] = \left[\begin{array}{c} \frac{1}{2} ((1)(3) + (-1)(1) + (1)(1) + (1)(-3)) \\ \frac{1}{\sqrt{14}} ((3)(3) + (2)(1) + (0)(1) + (-1)(-3)) \end{array} \right] = \left[\begin{array}{c} 0 \\ \sqrt{14} \end{array} \right]$$

Secondly, we solve the upper triangular system $Rx = Q^T b$:

$$\left[\begin{array}{cc} 2 & 2 \\ 0 & \sqrt{14} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] = \left[\begin{array}{c} 0 \\ \sqrt{14} \end{array} \right]$$

Using back substitution, we find $x_2 = 1$ and $x_1 = -1$.

The solution to the linear least square is

$$x = \left[\begin{array}{c} -1 \\ 1 \end{array} \right].$$

We can check that, for this x , the associated residual, $b - Ax$ is orthogonal to the columns of A . That is

a. $A^T(b - Ax) = 0$.