Name:____________________________________

MATH 3191 E01 :: Fall 2020 :: Exam 2 (key)

1. For each matrix equality, give the elementary coefficient matrix that enables to go from the right to the left. You can either multiply by the elementary matrix on the left or on the right but not on both sides.

(a) ($L_2 \leftrightarrow L_5$)

$$\begin{pmatrix} -5 & -8 & 8 \\ 4 & 4 & -6 \\ 4 & 0 & -3 \\ -7 & 6 & 4 \\ -3 & 3 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -5 & -8 & 8 \\ -3 & 3 & 8 \\ 4 & 0 & -3 \\ -7 & 6 & 4 \\ 4 & 4 & -6 \end{pmatrix}.$$

(b) ($C_3 \leftarrow C_3 + C_2$)

$$\begin{pmatrix} -5 & -8 & 0 \\ -3 & 3 & 11 \\ 4 & 0 & -3 \\ -7 & 6 & 10 \\ 4 & 4 & -2 \end{pmatrix} = \begin{pmatrix} -5 & -8 & 8 \\ -3 & 3 & 8 \\ 4 & 0 & -3 \\ -7 & 6 & 4 \\ 4 & 4 & -6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) ($L_4 \leftarrow 2L_4$)

$$\begin{pmatrix} -5 & -8 & 8 \\ -3 & 3 & 8 \\ 4 & 0 & -3 \\ -14 & 12 & 8 \\ 4 & 4 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -5 & -8 & 8 \\ -3 & 3 & 8 \\ 4 & 0 & -3 \\ -7 & 6 & 4 \\ 4 & 4 & -6 \end{pmatrix}$$

(d) ($C_1 \leftrightarrow C_5$)

$$\begin{pmatrix} 4 & 6 & -5 & 0 & 2 \\ 0 & 2 & 8 & -7 & 2 \\ -1 & -6 & -9 & 10 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 6 & -5 & 0 & 4 \\ 2 & 2 & 8 & -7 & 0 \\ 7 & -6 & -9 & 10 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(e) ($L_1 \leftarrow L_1 - L_2$)

$$\begin{pmatrix} 0 & 4 & -13 & 7 & 4 \\ 2 & 2 & 8 & -7 & 0 \\ 7 & -6 & -9 & 10 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & -5 & 0 & 4 \\ 2 & 2 & 8 & -7 & 0 \\ 7 & -6 & -9 & 10 & -1 \end{pmatrix}$$

2. Give the inverse of the following matrices. If you determine that the matrix is not invertible, this might be possible. Say it then.

(a)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \text{this matrix is not invertible}$$

(e)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{array}{l} \text{this matrix is not square,} \\ \text{so it is certainly not invertible} \end{array}$$

3. Circle: “makes no sense” or “cannot answer” or answer the question.
• “makes no sense” means the problems makes no sense. More specifically it means that the first sentence that starts with “Let A ...” and ends at its period is not possible. No such matrix A exists.
• “cannot answer” means “Such matrices A exists, but we do not have enough information to answer the question”.
• No justification needed.

(a) Let A be a 10-by-8 matrix such that the dim(Null (A))=2. What is the rank of A?

makes no sense   cannot answer   the rank of A is 6

(b) Let A be a 8-by-7 matrix such that the rank(A) = 7. Are the columns of A linearly dependent?

makes no sense   cannot answer   yes   no

be careful: Are the columns of A linearly **dependent**?

(c) Let A be a 5-by-8 matrix such that the columns are linearly independent. What is the dimension of the row space of A?

makes no sense   cannot answer   the dimension of the row space of A is _______

(d) Let A be a 9-by-7 matrix such that the rank(A) = 5. What is the dimension of the null space of A?

makes no sense   cannot answer   the dimension of the null space of A is 2

(e) Let A be a 4-by-7 matrix. Are the rows of A linearly independent?

makes no sense   cannot answer   yes   no

be careful: Are the **rows** of A linearly independent?

4. Either indicate whether that the matrix-matrix multiplication is not possible, xor, if it is possible, indicate the dimension of the output.

(a)

$$\begin{pmatrix} -1 & -8 & -2 & -9 & -2 \\ -1 & -7 & 7 & -2 & 3 \\ 7 & -7 & 6 & 1 & 3 \end{pmatrix} \begin{pmatrix} -4 & -8 \\ -1 & -3 \\ -10 & -6 \\ 10 & 0 \\ -7 & -3 \end{pmatrix}$$

makes no sense $\qquad$ $\boxed{\text{the output is } \underline{3}\text{-by-}\underline{2}}$

(b)

$$\begin{pmatrix} -4 & -8 \\ -1 & -3 \\ -10 & -6 \\ 10 & 0 \\ -7 & -3 \end{pmatrix} \begin{pmatrix} -1 & -8 & -2 & -9 & -2 \\ -1 & -7 & 7 & -2 & 3 \\ 7 & -7 & 6 & 1 & 3 \end{pmatrix}$$

$\boxed{\text{makes no sense}}$ $\qquad$ the output is ________-by-________

(c)

$$\begin{pmatrix} 9 \\ 8 \\ -9 \\ 5 \end{pmatrix} \begin{pmatrix} -5 & -2 & 1 & 9 \end{pmatrix}$$

makes no sense $\qquad$ $\boxed{\text{the output is } \underline{4}\text{-by-}\underline{4}}$

(d)

$$\begin{pmatrix} -5 & -2 & 1 & 9 \end{pmatrix} \begin{pmatrix} 9 \\ 8 \\ -9 \\ 5 \end{pmatrix}$$

makes no sense $\qquad$ $\boxed{\text{the output is } \underline{1}\text{-by-}\underline{1}}$

5. Compute the entry (2,5) of this matrix-matrix multiplication.

$$C = \begin{pmatrix} 4 & 1 & -7 \\ 3 & 8 & 7 \\ -6 & 3 & 3 \\ -7 & -6 & -2 \\ 10 & -3 & -6 \\ -7 & -1 & -1 \\ -9 & 10 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 & -4 & 6 & -2 \\ 2 & 0 & 2 & 0 & 1 \\ -5 & -5 & 0 & 5 & 0 \end{pmatrix}$$

The (2,5) entry of C is $c_{2,5} = \underline{2}$.

$$c_{2,5} = \begin{pmatrix} 3 & 8 & 7 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = (3)(-2) + (8)(1) + (7)(0) = -6 + 8 + 0 = 2.$$

6. Give the solution set for the following system using "inverse of a matrix". So you must compute the inverse of the 2-by-2 matrix and then compute x with the formula $x = A^{-1}b$.

$$\begin{cases} 2x_1 - x_2 = 1 \\ 3x_1 - 5x_2 = 0 \end{cases}$$

Let A, x, and b such that

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -5 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and } \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We want to find x such that $Ax = b$.

We can compute A^{-1} from the inverse of a 2-by-2 matrix formula, we get

$$A^{-1} = \frac{1}{(2)(-5)-(3)(-1)} \begin{pmatrix} -5 & 1 \\ -3 & 2 \end{pmatrix} = \frac{1}{-10+3} \begin{pmatrix} -5 & 1 \\ -3 & 2 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} -5 & 1 \\ -3 & 2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 5 & -1 \\ 3 & -2 \end{pmatrix}..$$

Then we compute $x = A^{-1}b$, we get

$$x = A^{-1}b = \frac{1}{7} \begin{pmatrix} 5 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

$$\boxed{x = \begin{pmatrix} \frac{5}{7} \\ \frac{3}{7} \end{pmatrix}}$$

<u>other acceptable answers:</u>

$$x = \frac{1}{7} \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

or

$$x_1 = \frac{5}{7}, \qquad x_2 = \frac{3}{7}$$

<u>check</u>

$$Ax = \begin{pmatrix} 2 & -1 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} \frac{5}{7} \\ \frac{3}{7} \end{pmatrix} = \begin{pmatrix} \frac{1}{7}((2)(5)+(-1)(3)) \\ \frac{1}{7}((3)(5)+(-5)(3)) \end{pmatrix} = \begin{pmatrix} \frac{1}{7}(7) \\ \frac{1}{7}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = b.$$

7. Let

$$A = \begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We consider $Ax = b$ which is equivalent to the following system of equations

$$\begin{cases} 2x_1 & - & 3x_2 & = & 1, \\ 4x_1 & - & 6x_2 & = & 2. \end{cases}$$

Student claims

"the matrix A = $\begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix}$ does not have an inverse, so there are no solutions $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to the linear system of equations."

(a) Does the matrix $A = \begin{pmatrix} 2 & -3 \\ 4 & -6 \end{pmatrix}$ have an inverse? If yes, give the inverse? If no, explain why the inverse does not exist.

Matrix A does not have an inverse. Because, (pick one of these four answers,)

i. The rows of A, $L_1 = \begin{pmatrix} 2 & -3 \end{pmatrix}$ and $L_2 = \begin{pmatrix} 4 & -6 \end{pmatrix}$, are linearly dependent. Indeed $2L_1 - L_2 = 0$.

ii. The columns of A, $C_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $C_2 = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$, are linearly dependent. Indeed $3C_1 - 2C_2 = 0$.

iii. The determinant of A is zero. Indeed $\det(A) = (2)(-6) - (4)(-3) = -12 + 12 = 0$.

iv. If we reduce A to row echelon form, "we lose a row," therefore there is one free variable.

(b) Are there solutions $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to the linear system of equations? If yes, give the solution set? If no, explain why there are no solutions.

Yes, there are solutions to the linear system of equations.

We form the augmented matrix, and row reduce

$$\left(\begin{array}{cc|c} 2 & -3 & 1 \\ 4 & -6 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right)$$

Since the last column (the one corresponding to the right-hand side) is not a pivot column, the linear system of equations is consistent and $\boxed{\text{there exists at least one solution}}$.

Not needed, but not necessarily a bad thing to write what follows. We see that x_1 is leading variable and x_2 is free. So we have one free variable (x_2) and so the linear system of equations has infinitely many solutions. We can set x_2 to parameter t and solve for x_1 and get the solution set as: $\begin{cases} x_1 & = & \frac{1}{2} + \frac{3}{2}t, \\ x_2 & = & t. \end{cases}$ If we want to exhibit one solution, we can take $t = 1$ (for example) and we get that a particular solution is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Not needed, a high level explanation. Remember that, in the square case, if A is not invertible, then it means either there are no solutions or there are infinitely many solutions. Remember that, in the square case, if A is not invertible, then it means that, for some right-hand sides, there is no solution, and, for some other right-hand sides, there are infinitely many solutions. It will depend on the right-hand side. For this problem, we are in the square case, A is not invertible, and the right-hand side is such that there are infinitely many solutions.

Warning. ⚠ A method such as "Cramer's rule" or $x = A^{-1}b$, will NOT "work". These methods only work when there is a unique solution. (So existence and unicity.) So when A is invertible.

8. Compute A^{-1}, the inverse of A.
(If the inverse of A exists. If the inverse of A does not exist, please explain.)

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow L_3 - L_1 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} L_2 \leftrightarrow L_3 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 2 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{matrix} L_2 \leftarrow -L_2 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 3 & 2 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{matrix} L_3 \leftarrow L_3 - 3L_2 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & -2 & 1 & 3 \end{pmatrix}$$

$$\begin{matrix} L_3 \leftarrow L_3/2 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$\begin{matrix} L_1 \leftarrow L_1 - L_3 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 0 & 2 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$\begin{matrix} L_1 \leftarrow L_1 - 2L_2 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$\boxed{A^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -1 \\ -1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix}}$$

Not needed. Check.

We can perform $A^{-1}A$ check that we get I. We can also check with AA^{-1}. We check $A^{-1}A$ xor AA^{-1}. Whichever. Does not matter. No need to check both, checking only one is enough.

$$
\begin{aligned}
A^{-1}A &= \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -1 \\ -1 & \frac{1}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} (0)(1)+(-\frac{1}{2})(-1)+(\frac{1}{2})(1) & (0)(2)+(-\frac{1}{2})(1)+(\frac{1}{2})(1) & (0)(1)+(-\frac{1}{2})(1)+(\frac{1}{2})(1) \\ (1)(1)+(0)(-1)+(-1)(1) & (1)(2)+(0)(1)+(-1)(1) & (1)(1)+(0)(1)+(-1)(1) \\ (-1)(1)+(\frac{1}{2})(-1)+(\frac{3}{2})(1) & (-1)(2)+(\frac{1}{2})(1)+(\frac{3}{2})(1) & (-1)(1)+(\frac{1}{2})(1)+(\frac{3}{2})(1) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= I \quad \checkmark
\end{aligned}
$$

9. Compute the determinant of the matrix below

$$A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

<u>Method 1:</u>

$$\det(A) = \begin{vmatrix} 1 & 3 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{vmatrix} \overset{L_2 \leftarrow L_2 - L_1}{=} \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{vmatrix} \overset{L_4 \leftarrow L_4 - L_3}{=} \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{vmatrix}$$

$$\overset{L_3 \leftarrow L_3 + L_2}{=} \begin{vmatrix} 1 & 3 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{vmatrix} = (1)(-1)(1)(-2) = 2$$

$$\boxed{\det(A) = 2}$$

<u>Method 2:</u>

$$\det(A) = \begin{vmatrix} 1 & 3 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{vmatrix} \overset{\substack{\text{expand with} \\ \text{1st column}}}{=} (1)\begin{vmatrix} 2 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} - (1)\begin{vmatrix} 3 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

$$\overset{\substack{\text{expand with} \\ \text{2nd column} \\ \text{in both 3x3}}}{=} (1)(-1)(2)\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} - (1)(-1)(1)\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= \; = (1)(-1)(2)\left((1)(-1) - (1)(1)\right) - (1)(-1)(1)\left((1)(-1) - (1)(1)\right) = (-2)(-2) + (-2) = 2$$

$$\boxed{\det(A) = 2}$$

10. Matrix A and the reduced echelon form of A are given below.

$$A = \begin{pmatrix} 8 & 16 & -1 & 5 & -6 & 33 & -8 & -5 & 8 \\ -10 & -20 & -9 & -37 & 5 & -54 & 2 & -25 & -41 \\ -1 & -2 & -6 & -19 & -5 & -16 & 4 & -11 & -16 \\ -2 & -4 & 4 & 10 & 8 & 2 & 1 & 16 & 11 \\ -1 & -2 & -1 & -4 & -5 & -11 & -1 & -16 & -11 \\ 5 & 10 & -7 & -16 & 5 & 23 & 3 & 24 & 9 \\ -4 & -8 & -3 & -13 & -6 & -29 & 3 & -21 & -21 \\ 6 & 12 & 2 & 12 & -4 & 28 & 4 & 20 & 22 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 5 & 0 & 3 & 3 \\ 0 & 0 & 1 & 3 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Clearly indicate with arrows the columns / rows below that makes a basis for $\text{Row}(A)$.

This answer is OK, point to columns or rows with arrows.

A basis for $\text{Row}(A)$ is

$$\begin{pmatrix} 8 & 16 & -1 & 5 & -6 & 33 & -8 & -5 & 8 \\ -10 & -20 & -9 & -37 & 5 & -54 & 2 & -25 & -41 \\ -1 & -2 & -6 & -19 & -5 & -16 & 4 & -11 & -16 \\ -2 & -4 & 4 & 10 & 8 & 2 & 1 & 16 & 11 \\ -1 & -2 & -1 & -4 & -5 & -11 & -1 & -16 & -11 \\ 5 & 10 & -7 & -16 & 5 & 23 & 3 & 24 & 9 \\ -4 & -8 & -3 & -13 & -6 & -29 & 3 & -21 & -21 \\ 6 & 12 & 2 & 12 & -4 & 28 & 4 & 20 & 22 \end{pmatrix} \qquad \begin{pmatrix} \rightarrow & 1 & 2 & 0 & 1 & 0 & 5 & 0 & 3 & 3 & \leftarrow \\ \rightarrow & 0 & 0 & 1 & 3 & 0 & 1 & 0 & 1 & 2 & \leftarrow \\ \rightarrow & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 1 & \leftarrow \\ \rightarrow & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & \leftarrow \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \end{pmatrix}$$

This is a better answers, but it takes a lot writing, so this is not needed.

A basis for $\text{Row}(A)$ is made of ℓ_1, ℓ_2, ℓ_3, and ℓ_4 such that

$$\begin{aligned} \ell_1 &= \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 5 & 0 & 3 & 3 \end{pmatrix}, \\ \ell_2 &= \begin{pmatrix} 0 & 0 & 1 & 3 & 0 & 1 & 0 & 1 & 2 \end{pmatrix}, \\ \ell_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 1 \end{pmatrix}, \\ \ell_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}. \end{aligned}$$

(b) Clearly indicate with arrows the columns / rows below that makes a basis for $\mathrm{Col}(A)$.

This answer is OK, point to columns or rows with arrows.

A basis for $\mathrm{Col}(A)$ is

$$\begin{array}{ccccccccc} \downarrow & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

$$\begin{pmatrix} 8 & 16 & -1 & 5 & -6 & 33 & -8 & -5 & 8 \\ -10 & -20 & -9 & -37 & 5 & -54 & 2 & -25 & -41 \\ -1 & -2 & -6 & -19 & -5 & -16 & 4 & -11 & -16 \\ -2 & -4 & 4 & 10 & 8 & 2 & 1 & 16 & 11 \\ -1 & -2 & -1 & -4 & -5 & -11 & -1 & -16 & -11 \\ 5 & 10 & -7 & -16 & 5 & 23 & 3 & 24 & 9 \\ -4 & -8 & -3 & -13 & -6 & -29 & 3 & -21 & -21 \\ 6 & 12 & 2 & 12 & -4 & 28 & 4 & 20 & 22 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 5 & 0 & 3 & 3 \\ 0 & 0 & 1 & 3 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccccccccc} \uparrow & & \uparrow & & \uparrow & & \uparrow & & \end{array}$$

This is a better answers, but it takes a lot writing, so this is not needed.

A basis for $\mathrm{Col}(A)$ is made of v_1, v_2, v_3, and v_4 such that

$$v_1 = \begin{pmatrix} 8 \\ -10 \\ -1 \\ -2 \\ -1 \\ 5 \\ -4 \\ 6 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ -9 \\ -6 \\ 4 \\ -1 \\ -7 \\ -3 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -6 \\ 5 \\ -5 \\ 8 \\ -5 \\ 5 \\ -6 \\ -4 \end{pmatrix}, v_4 = \begin{pmatrix} -8 \\ 2 \\ 4 \\ 1 \\ -1 \\ 3 \\ 3 \\ 4 \end{pmatrix}.$$

11. Let A such that

$$A = \begin{pmatrix} 4 & 8 & 8 & 4 \\ -8 & -16 & -9 & -15 \\ 1 & 2 & -3 & -9 \\ 1 & 3 & -7 & 5 \end{pmatrix}$$

All the computation below are correct. The question is: what is the determinant of A?

It is OK to give the answer as a sign ($+$ or $-$) followed by a product of positive numbers. So for example: answers like $(-3 \times 9 \times 7 \times 5 \times 9)$ or $(+2 \times 8 \times 7 \times 7 \times 5)$ would be acceptable.

The determinant of A is __________________.

$$\begin{pmatrix} 4 & 8 & 8 & 4 \\ -8 & -16 & -9 & -15 \\ 1 & 2 & -3 & -9 \\ 1 & 3 & -7 & 5 \end{pmatrix} \quad \begin{matrix} L_1 \to L_1/4 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 2 & 1 \\ -8 & -16 & -9 & -15 \\ 1 & 2 & -3 & -9 \\ 1 & 3 & -7 & 5 \end{pmatrix}$$

$$\begin{matrix} L_2 \leftarrow L_2 + 8L_1 \\ L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 - L_1 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 7 & -7 \\ 0 & 0 & -5 & -10 \\ 0 & 1 & -9 & 4 \end{pmatrix}$$

$$\begin{matrix} L_4 \leftrightarrow L_2 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -9 & 4 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & 7 & -7 \end{pmatrix}$$

$$\begin{matrix} L_3 \leftarrow L_3/5 \\ L_4 \leftarrow L_4/7 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -9 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{matrix} L_4 \leftarrow L_4 + L_3 \\ \rightsquigarrow \end{matrix} \quad \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -9 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

(a) step 1. The operation $L_1 \to L_1/4$ has divided the determinant by 4, so we will need to multiply the final determinant by 4.

(b) step 2. The operations $L_2 \leftarrow L_2 + 8L_1$, $L_3 \leftarrow L_3 - L_1$, and $L_4 \leftarrow L_4 - L_1$ do not change the determinant.

(c) step 3. The operation $L_4 \leftrightarrow L_2$ has changed the sign of the determinant, so we will need to change the sign of the final determinant.

(d) step 4. The operations $L_3 \leftarrow L_3/5$ $L_4 \leftarrow L_4/7$ have divided the determinant by 5 and 7 respectively, so we will need to multiply the final determinant by 5 and 7.

(e) step 5. The operation $L_4 \leftarrow L_4 + L_3$ does not change the determinant.

(f) step 6. The final matrix

$$\begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & -9 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

is triangular so its determinant is the product of its diagonal entries. So it is (1)(1)(-1)(-3) = 3.

$$\boxed{\det(A) = -(3) \times (4) \times (5) \times (7).}$$

<u>Not needed.</u>

I am sure you can do the computation by hand and, if you do so, you would get

$$\boxed{\det(A) = -420.}$$

12. We want to solve the following linear system for x_4 using Cramer's rule. Please provide the formula for x_4 using Cramer's rule. Please give the formula with two four-by-four determinants. Do not attempt to compute anything, just write the formula. Assume a computer can compute the formula for you if the computer sees your formula. (Which, would computer be able to see, be pretty much true.)

$$\left\{\begin{array}{rcrcrcrcrcr} 4x_1 & + & x_2 & + & 9x_3 & + & 6x_4 & - & 4x_5 & = & 0 \\ 3x_1 & + & 3x_2 & + & x_3 & - & 2x_4 & - & x_5 & = & 9 \\ -9x_1 & - & 2x_2 & - & 3x_3 & - & 8x_4 & + & x_5 & = & 3 \\ -9x_1 & + & 6x_2 & - & 8x_3 & - & 5x_4 & - & x_5 & = & 9 \\ -4x_1 & + & 4x_2 & + & 2x_3 & - & 7x_4 & + & 8x_5 & = & -5 \end{array}\right.$$

$$x_4 = \frac{\begin{vmatrix} 4 & 1 & 9 & 0 & -4 \\ 3 & 3 & 1 & 9 & -1 \\ -9 & -2 & -3 & 3 & 1 \\ -9 & 6 & -8 & 9 & -1 \\ -4 & 4 & 2 & -5 & 8 \end{vmatrix}}{\begin{vmatrix} 4 & 1 & 9 & 6 & -4 \\ 3 & 3 & 1 & -2 & -1 \\ -9 & -2 & -3 & -8 & 1 \\ -9 & 6 & -8 & -5 & -1 \\ -4 & 4 & 2 & -7 & 8 \end{vmatrix}}$$

Irrelevant facts: It turns out that

$$\begin{vmatrix} 4 & 1 & 9 & 0 & -4 \\ 3 & 3 & 1 & 9 & -1 \\ -9 & -2 & -3 & 3 & 1 \\ -9 & 6 & -8 & 9 & -1 \\ -4 & 4 & 2 & -5 & 8 \end{vmatrix} = -55764 \quad \text{and} \quad \begin{vmatrix} 4 & 1 & 9 & 6 & -4 \\ 3 & 3 & 1 & -2 & -1 \\ -9 & -2 & -3 & -8 & 1 \\ -9 & 6 & -8 & -5 & -1 \\ -4 & 4 & 2 & -7 & 8 \end{vmatrix} = 36684$$

and so

$$x_4 = -\frac{1549}{1019}.$$

Name:____________________________________

MATH 3191 E01 :: Fall 2020 :: Exam 2

1	2	3	4	5	6

7	8	9	10	11	12