EX.1.1.1, Sauer3

Use the Intermediate Value Theorem to find an interval of length one that contains a root of the equation. (a) $x^3 = 9$, (b) $3x^3 + x^2 = x + 5$, (c) $\cos^2(x) + 6 = x$.

EX.1.1.1, Sauer3, solution, Langou

- See
  https://colab.research.google.com/drive/1VJ3_rGh9d07bndBNkzysv7xzU7U_35LN

(a) $x^3 = 9$

The function $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3 - 9$ is continuous, moreover, by direct evaluation, we see that

$$f(2) = -1 \quad \text{ and } f(3) = 18,$$

so

$$f(2) < 0, \quad f(3) > 0,$$

therefore (Intermediate Value Theorem), there exists $x_\star \in (2, 3)$ such that $f(x_\star) = 0$.

Not needed for full credit:
(1) the interval $(2, 3)$ is of length 1.
(2) we proved that there exists $x_\star \in (2, 3)$ such that $f(x_\star) = 0$, it is clear that this $x_\star$ is also such that $x^3 = 9$, and so is a solution of our initial equation.
(3) for this problem it is easy to see that $x_\star = \sqrt[3]{9}$.

```
f = lambda x : ( x ** 3 ) - 9.
print( f(2.), f(3.) )
```

```
-1.0 18.0
```

(b) $3x^3 + x^2 = x + 5$

The function $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 3x^3 + x^2 - x - 5$ is continuous, moreover, by direct evaluation, we see that

$$f(1) = -2 \quad \text{ and } f(2) = 21,$$

so

$$f(1) < 0, \quad f(2) > 0,$$

therefore (Intermediate Value Theorem), there exists $x_\star \in (1, 2)$ such that $f(x_\star) = 0$.

Not needed for full credit:
The closed-form formula for $x_\star$ is

$$x_\star = \frac{1}{9}\left(-1 + \sqrt[3]{593 - 27\sqrt{481}} + \sqrt[3]{593 + 27\sqrt{481}}\right).$$

It is not clear this formula is very useful and it is probably more complicated to compute $x_\star$ through this formula than to run a root solver on $3x^3 + x^2 - x - 5$.
Not needed for full credit:

```
f = lambda x : 3. * ( x ** 3 ) + ( x ** 2 ) - x - 5.
print( f(1.), f(2.) )
```

```
-2.0 21.0
```

(c) $\cos^2(x) + 6 = x$

The function $f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \cos^2(x) + 6 - x$ is continuous, moreover, by direct evaluation, we see that

$$f(1) \approx 0.922 \quad \text{ and } f(2) \approx -0.432,$$

so

$$f(6) > 0, \quad f(7) < 0,$$

therefore (Intermediate Value Theorem), there exists $x_\star \in (6, 7)$ such that $f(x_\star) = 0$.

Not needed for full credit:

```
from math import cos
f = lambda x : ( cos(x) )**2 + 6. - x
print( f(6.), f(7.) )
```

```
0.922 -0.432
```