EX.4.3.7.a, Sauer3

Use the QR factorization from Exercise EX.4.3.2.a, EX.4.3.4.a, or EX.4.3.6.a to solve the following least squares problem

$$\begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}.$$

EX.4.3.7.a, Sauer3, solution, Langou

Colab: `https://colab.research.google.com/drive/18V9U-OvfbhAnIRA-FP34nMJooaMHC1tk`

In EX.4.3.2.a, we found that a QR factorization of A was

$$Q = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -2 & -2 \\ 1 & -2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix}.$$

Now we can solve the linear least squares problem using this QR factorization.
Firstly, we form Q^Tb:

$$Q^Tb = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}. = \frac{1}{3}\begin{bmatrix} (2)(3) + (-2)(-3) + (1)(6) \\ (-1)(3) + (-2)(-3) + (-2)(6) \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Secondly, we solve the upper triangular system $Rx = Q^Tb$:

$$\begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Using back substitution, the last row reads $3x_2 = -3$, so we get $x_2 = -1$ and, then, since the first row reads $3x_1 + 6x_2 = 6$ and that $x_2 = -1$, we get that $x_1 = 4$.
The solution to the linear least squares is

$$x = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

We can check that, for this x, the associated residual, $b - Ax$ is orthogonal to the columns of A. That is

a. $A^T(b - Ax) = 0$.

Indeed

$$\begin{aligned} Ax &= \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} \\ b - Ax &= \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \\ A^T(b - Ax) &= \begin{bmatrix} 2 & -2 & 1 \\ 3 & -6 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} (2)(-2) + (-2)(-1) + (1)(2) \\ (3)(-2) + (-6)(-1) + (0)(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \checkmark \end{aligned}$$