EX.0.5.5, Sauer

Find the Taylor polynomial of degree 5 about the point $x = 0$ for the following functions:

$$(a) \quad f(x) = e^{x^2} \quad (b) \quad f(x) = \cos(2x) \quad (c) \quad f(x) = \ln(1+x) \quad (d) \quad f(x) = \sin 2x$$

EX.0.5.5, Sauer, solution, Langou

a. $\boxed{\text{Method 1}}$

We know the Taylor series of e^x at 0 by heart:

$$e^x = 1 + x + \frac{1}{2}x^2 + \mathcal{O}(x^3)$$

We plug x^2 and get

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \mathcal{O}(x^6)$$

So we find

$$\boxed{p_5(x) = 1 + x^2 + \tfrac{1}{2}x^4}$$

$\boxed{\text{Method 2}}$

We compute the successive derivatives of e^{x^2}:

$$f(x) = e^{x^2}, \quad f'(x) = 2xe^{x^2}, \quad f''(x) = (4x^2+2)e^{x^2}, \quad f^{(3)}(x) = (8x^3+12x)e^{x^2},$$
$$f^{(4)}(x) = (16x^4+48x^2+12)e^{x^2}, \quad f^{(5)}(x) = (32x^5+160x^3+120x)e^{x^2}.$$

We evaluate them at 0:

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = 2, \quad f^{(3)}(0) = 0, \quad f^{(4)}(0) = 12, \quad f^{(5)}(0) = 0.$$

We write the general formula for the Taylor polynomial of degree 5 of f at 0:

$$p_5(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4 + \frac{1}{5!}f^{(5)}(0)x^5$$

And we plug our previous found values for $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$, $f^{(4)}(0)$, $f^{(5)}(0)$ and we find

$$\boxed{p_5(x) = 1 + x^2 + \tfrac{1}{2}x^4}$$

b. $\boxed{\text{Method 1}}$

We know the Taylor series of $\cos(x)$ at 0 by heart:

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \mathcal{O}(x^6)$$

We plug $2x$ and get

$$\cos(x) = 1 - \frac{2^2}{2}x^2 + \frac{2^4}{4!}x^4 + \mathcal{O}(x^6)$$

So we find

$$\boxed{p_5(x) = 1 - 2x^2 + \frac{2}{3}x^4}$$

$\boxed{\text{Method 2}}$

We compute the successive derivatives of $\cos(2x)$:

$$f(x) = \cos(2x), \quad f'(x) = -2\sin(2x), \quad f''(x) = -4\cos(2x), \quad f^{(3)}(x) = 8\sin(2x),$$
$$f^{(4)}(x) = 16\cos(2x), \quad f^{(5)}(x) = -32\sin(2x).$$

We evaluate them at 0:

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -4, \quad f^{(3)}(0) = 0, \quad f^{(4)}(0) = 16, \quad f^{(5)}(0) = 0.$$

We write the general formula for the Taylor polynomial of degree 5 of f at 0:

$$p_5(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4 + \frac{1}{5!}f^{(5)}(0)x^5$$

And we plug our previous found values for $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$, $f^{(4)}(0)$, $f^{(5)}(0)$ and we find

$$\boxed{p_5(x) = 1 - 2x^2 + \frac{2}{3}x^4}$$

c. $\boxed{\text{Method 1}}$

We know the Taylor series of $\ln(1+x)$ at 0 by heart, so we are done. We get

$$\boxed{p_5(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5}$$

$\boxed{\text{Method 2}}$

We compute the successive derivatives of $\ln(1+x)$:

$$f(x) = \ln(1+x), \quad f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f^{(3)}(x) = \frac{2}{(1+x)^3},$$
$$f^{(4)}(x) = \frac{-3!}{(1+x)^4}, \quad f^{(5)}(x) = \frac{4!}{(1+x)^5}.$$

We evaluate them at 0:

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f^{(3)}(0) = 2, \quad f^{(4)}(0) = -3!, \quad f^{(5)}(0) = 4!.$$

We write the general formula for the Taylor polynomial of degree 5 of f at 0:

$$p_5(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4 + \frac{1}{5!}f^{(5)}(0)x^5$$

And we plug our previous found values for $f(0)$, $f'(0)$, $f''(0)$, $f^{(3)}(0)$, $f^{(4)}(0)$, $f^{(5)}(0)$ and we find

$$\boxed{p_5(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5}$$

d. Since $f(x) = \sin^2(x)$ is an even function, only even power will show up in the Taylor expansion. And since $f(0) = 0$, we only expect to see at most x^2, x^4, etc.

Method 1

We compute $f^{(n)}(0)$ for $n = 0$ to 5:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$	
0	$f^{(0)}(x) = \sin^2(x)$	$f^{(0)}(0) = 0$	0	$= 0$
1	$f^{(1)}(x) = 2\cos(x)\sin(x)$	$f^{(1)}(0) = 0$	0	$= 0$
2	$f^{(2)}(x) = 2(\cos^2(x) - \sin^2(x))$	$f^{(2)}(0) = 2$	$2/2!$	$= 1$
3	$f^{(3)}(x) = -8\cos(x)\sin(x)$	$f^{(3)}(0) = 0$	0	$= 0$
4	$f^{(4)}(x) = -8\cos^2(x) + 8\sin^2(x)$	$f^{(4)}(0) = -8$	$-8/4!$	$= -1/3$
5	$f^{(5)}(x) = 32\cos(x)\sin(x)$	$f^{(5)}(0) = 0$	0	$= 0$

So we get:

$$\boxed{p_5(x) = x^2 - \tfrac{1}{3}x^4}$$

Note: Another way to get the derivatives of f is to remember that $2\cos(x)\sin(x) = \sin(2x)$, so that $f'(x) = \sin(2x)$ and so we would get $f^{(2)}(x) = 2\cos(2x)$, $f^{(3)}(x) = -4\sin(2x)$, $f^{(4)}(x) = -8\cos(2x)$, and $f^{(5)}(x) = 16\sin(2x)$.

Method 2

We know that

$$\sin(x) = x - \frac{1}{3!}x^3 + \mathcal{O}(x^5).$$

We can square the Taylor series of $\sin(x)$ to get the Taylor series of $\sin^2(x)$ and we get

$$\begin{aligned} \sin^2(x) &= \left(x - \frac{1}{3!}x^3 + \mathcal{O}(x^5)\right)\left(x - \frac{1}{3!}x^3 + \mathcal{O}(x^5)\right), \\ &= x^2 - (2\frac{1}{3!})x^4 + \mathcal{O}(x^6), \\ &= x^2 - \frac{1}{3}x^4 + \mathcal{O}(x^6). \end{aligned}$$

So we get:

$$\boxed{p_5(x) = x^2 - \tfrac{1}{3}x^4}$$

Method 3

We know that

$$2\sin^2(x) = 1 - \cos(2x).$$

Since

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \ldots,$$

we have

$$\cos(2x) = 1 - \frac{1}{2!}2^2x^2 + \frac{1}{4!}2^4x^4 - \frac{1}{6!}2^6x^6 \ldots,$$

$$\cos(2x) = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 \ldots,$$

so that

$$1 - \cos(2x) = 2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 \dots,$$

so that

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)) = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 \dots.$$

So we get:

$$\boxed{p_5(x) = x^2 - \tfrac{1}{3}x^4}$$