**EX.4.1.6, Sauer3**

Let $A$ be an $n$-by-$n$ nonsingular matrix. (a) Prove that $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$. (b) Let $b$ be a vector of length $n$, then $Ax = b$ has exactly one solution. We call $x$ this solution. Prove that $x$ is also the unique solution of the normal equations associated with $A$ and $b$.



## EX.4.1.6, Sauer3, solution, Langou

Note that "$A$ is an $n$-by-$n$ nonsingular matrix" is the same as "$A$ is an $n$-by-$n$ invertible matrix".

a. We have that

$$A^{-1}A = I.$$

We transpose this relation, this reads:

$$\left(A^{-1}A\right)^T = I^T.$$

Now, on the left-hand side, we use $(AB)^T = B^T A^T$, on the right-hand side, we use the fact that $I^T = I$.

$$A^T \left(A^{-1}\right)^T = I.$$

From the above relation, we see that (1) $A^T$ is invertible (since there exists a matrix $C$, namely $\left(A^{-1}\right)^T$, such that $A^T C = I$) and (2) $\left(A^{-1}\right)^T$ is the inverse of $A^T$, in other words

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1}.$$

b. Let $A$ be an $n$-by-$n$ nonsingular matrix. Let $b$ be a vector of length $n$. We want to prove that the solution of the least squares problem $x_{\mathrm{LS}} = \arg\min_x \|b - Ax\|_2$ is also the solution of the linear system of equation $Ax_{\mathrm{SV}} = b$, that is $x_{\mathrm{SV}} = A^{-1}b$.

(a) $x_{\mathrm{LS}}$ is given as the solution of the normal equations:

$$x_{\mathrm{LS}} \quad = \quad (A^T A)^{-1} A^T b$$

Since $A$ is square invertible, so is $A^T$. (We just proved this in (a)). We also know that: "If $A$ and $B$ are invertible, then (1) $AB$ is invertible and (2) $(AB)^{-1} = B^{-1}A^{-1}$." Using all this together we can conclude that (1) $A^T A$ is invertible and (2) $\left(A^T A\right)^{-1} = A^{-1}\left(A^T\right)^{-1}$. Applying this in the normal equation gives us

$$x_{\mathrm{LS}} \quad = \quad A^{-1}\left(A^T\right)^{-1} A^T b.$$

This gives

$$x_{\mathrm{LS}} \quad = \quad A^{-1}b.$$

Therefore

$$x_{\mathrm{LS}} \quad = \quad x_{\mathrm{SV}}.$$

(b) Another to go is to go back to the basic. We start form the least squares definition of $x_{\text{LS}}$:

$$x_{\text{LS}} = \arg\min_x \|b - Ax\|_2.$$

Now, on the one hand, a norm is always nonnegative, so

$$0 \leq \|b - Ax_{\text{LS}}\|_2.$$

On the other hand, since $A$ is invertible, there exists $x_{\text{SV}}$ such that $Ax_{\text{SV}} = b$, which we write as

$$\|b - Ax_{\text{SV}}\|_2 = 0,$$

The value of $\|b - Ax_{\text{LS}}\|_2$ should be minimal overall $x$ so, for sure, the value of $\|b - Ax_{\text{LS}}\|_2$ needs to be less than the value of $\|b - Ax_{\text{SV}}\|_2$, which is zero. We therefore get:

$$\|b - Ax_{\text{LS}}\|_2 \leq 0.$$

We find that

$$0 \leq \|b - Ax_{\text{LS}}\|_2 \leq 0.$$

Therefore

$$\|b - Ax_{\text{LS}}\|_2 = 0,$$

therefore

$$Ax_{\text{LS}} = b,$$

therefore

$$x_{\text{LS}} = A^{-1}b,$$

therefore

$$x_{\text{LS}} = x_{\text{SV}}.$$