CP.1.3.2, Sauer3

Let $f(x) = \sin(x^3) - x^3$. (a) Find the multiplicity of the root $r = 0$. (b) Use `scipy.optimize.fsolve` command with initial guess $x = 0.1$ to locate a root. What are the forward and backward errors of `scipy.optimize.fsolve`'s response?

CP.1.3.2, Sauer3, solution, Langou

a. To find the multiplicity of the root $r = 0$, please do not do it like this:

- $f(x) = \sin(x^3) - x^3$, $f(0) = 0$, so $r = 0$ is a root.
- $f'(x) = 3x^2\cos(x^3) - 3x^2$, $f'(0) = 0$, so $r = 0$ is a root of multiplicity at least 2.
- $f''(x) = 6x\cos(x^3) - 9x^4\sin(x^3) - 6x$, $f''(0) = 0$, so $r = 0$ is a root of multiplicity at least 3.
- $f^{(iii)}(x) = (6 - 27x^6)\cos(x^3) - 54x^3\sin(x^3) - 6$, $f^{(iii)}(0) = 0$, so $r = 0$ is a root of multiplicity at least 4.
- $f^{(iv)}(x) = -324x^5\cos(x^3) + (81x^8 - 180x^2)\sin(x^3)$, $f^{(iv)}(0) = 0$, so $r = 0$ is a root of multiplicity at least 5.
- $f^{(v)}(x) = (243x^{10} - 2160x^4)\cos(x^3) + (1620x^7 - 360x)\sin(x^3)$, $f^{(v)}(0) = 0$, so $r = 0$ is a root of multiplicity at least 6.
- $f^{(vi)}(x) = (7290x^9 - 9720x^3)\cos(x^3) + (-729x^{12} + 17820x^6 - 360)\sin(x^3)$, $f^{(vi)}(0) = 0$, so $r = 0$ is a root of multiplicity at least 7.
- $f^{(vii)}(x) = (-2187x^{14} + 119070x^8 - 30240x^2)\cos(x^3) + (-30618x^{11} + 136080x^5)\sin(x^3)$, $f^{(vii)}(0) = 0$, so $r = 0$ is a root of multiplicity at least 8.
- $f^{(viii)}(x) = (-122472x^{13} + 1360800x^7 - 60480x)\cos(x^3) + (6561x^{16} - 694008x^{10} + 771120x^4)\sin(x^3)$, $f^{(viii)}(0) = 0$, so $r = 0$ is a root of multiplicity at least 9.
- $f^{(ix)}(x) = (19683x^{18} - 122472x^{12} + 11838960x^6 - 60480)\cos(x^3) + (472392x^{15} - 1122480x^9 + 3265920x^3)\sin(x^3)$, $f^{(ix)}(0) = -60480 \neq 0$, so $r = 0$ is a root of multiplicity 9.

This is correct but laborious.

We know our Taylor series, so we know that

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

This is an infinite sum. The formula is true for all x. The formula is very accurate with only a few terms when x is close from zero. From the Taylor expansion of sine, we obtain

$$f(x) = \sin(x^3) - x^3 = (x^3 - \frac{1}{3!}x^9 + \frac{1}{5!}x^{15} - \dots) - x^3 = -\frac{1}{3!}x^9 + \frac{1}{5!}x^{15} - \dots$$

The left hand side is nothing else that the Taylor's expansion of f around 0. It has to match the Taylor's formula:

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{3!}f^{(iii)}(0)x^3 + \frac{1}{4!}f^{(iv)}(0)x^4 + \dots$$

Matching term by term the two formulae, we see that

$$f(0) = f'(0) = f''(0) = f^{(iii)}(0) = \ldots = f^{(viii)}(0) = 0,$$

and

$$\frac{1}{9!} f^{(ix)}(0) = -\frac{1}{3!}$$

therefore

$$f^{(ix)}(0) = -\frac{9!}{3!}$$

and so

$$f^{(ix)}(0) = -60480.$$

So we see that the root $r = 0$ has multiplicity 9 since $f(0) = f'(0) = f''(0) = \ldots = f^{(viii)}(0) = 0$ and $f^{(ix)}(0) \neq 0$.