Proof of Orthogonality of Matrix Q

Matrix Definition

The entries of the matrix Q are defined as:

$$Q[i+1, j+1] = \cos\left(\frac{\pi(2i+1)j}{2n}\right).$$

Using Euler's formula, $\cos(x) = \mathrm{Re}(e^{ix})$, this can be rewritten as:

$$Q[i+1, j+1] = \mathrm{Re}\left(e^{i\frac{\pi(2i+1)j}{2n}}\right).$$

Inner Product of Two Columns

To check orthogonality, compute the inner product of two columns $Q[:, j]$ and $Q[:, k]$:

$$\mathbf{q}_j^T \mathbf{q}_k = \sum_{i=0}^{n-1} Q[i+1, j+1] Q[i+1, k+1].$$

Substituting $Q[i+1, j+1] = \mathrm{Re}\left(e^{i\frac{\pi(2i+1)j}{2n}}\right)$ and $Q[i+1, k+1] = \mathrm{Re}\left(e^{i\frac{\pi(2i+1)k}{2n}}\right)$, the inner product becomes:

$$\mathbf{q}_j^T \mathbf{q}_k = \sum_{i=0}^{n-1} \mathrm{Re}\left(e^{i\frac{\pi(2i+1)j}{2n}}\right) \mathrm{Re}\left(e^{i\frac{\pi(2i+1)k}{2n}}\right).$$

Using the identity $\mathrm{Re}(a)\mathrm{Re}(b) = \frac{\mathrm{Re}(ab)+\mathrm{Re}(a\bar{b})}{2}$, this becomes:

$$\mathbf{q}_j^T \mathbf{q}_k = \frac{1}{2} \sum_{i=0}^{n-1} \left(\mathrm{Re}\left(e^{i\frac{\pi(2i+1)(j+k)}{2n}}\right) + \mathrm{Re}\left(e^{i\frac{\pi(2i+1)(j-k)}{2n}}\right)\right).$$

Simplification Using Geometric Series

Consider a general term of the form:

$$S = \sum_{i=0}^{n-1} e^{i\frac{\pi(2i+1)m}{2n}},$$

where $m = j + k$ or $m = j - k$. This is a geometric series with common ratio:

$$r = e^{i\frac{\pi m}{2n}}.$$

The sum of a finite geometric series is:

$$S = \frac{1 - r^n}{1 - r},$$

where $r^n = e^{i\frac{\pi mn}{2n}} = e^{i\frac{\pi m}{2}}$.

Case 1: $m \neq 0$ (when $j \neq k$)

For $m \neq 0$, the numerator $1 - r^n$ simplifies to 0 because $e^{i\frac{\pi m}{2}}$ completes a full rotation for integer m. Thus:

$$S = 0.$$

Case 2: $m = 0$ (when $j = k$)

For $m = 0$, $r = 1$, and the series becomes:

$$S = \sum_{i=0}^{n-1} 1 = n.$$

Orthogonality Condition

From the above:

- When $j \neq k$, the inner product $\mathbf{q}_j^T \mathbf{q}_k = 0$ (orthogonal columns).
- When $j = k$, the inner product $\mathbf{q}_j^T \mathbf{q}_j = n$, which means the columns are scaled but consistent.

Normalization

After normalizing each column (dividing by $\sqrt{n}$), the matrix Q becomes orthonormal:

$$Q^T Q = I.$$

Thus, Q is an orthogonal matrix.