EX.2.7.2.b, Sauer3

Use the Taylor expansion to find the linear approximation $L(x)$ to $F(x)$ near x_0.

$$\text{(b)} \left(\begin{array}{cccc} F: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & \begin{pmatrix} u \\ v \end{pmatrix} & \mapsto & \begin{pmatrix} u + e^{u-v} \\ 2u+v \end{pmatrix} \end{array} \right) \qquad \text{at } x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

EX.2.7.2.b, Sauer3, solution, Langou

Colab: `https://colab.research.google.com/drive/1RK6y5e_1cRT1euI_-X6j3l3sfNGS-Lff`

The Taylor expansion of F at x_0 of order 1 is

$$F(x) = F(x_0) + DF(x_0)(x - x_0) + e(x), \quad \text{where } \|e(x)\| = \mathcal{O}(\|x - x_0\|^2).$$

And so, the linear approximation $L(x)$, where $(L : \mathbb{R}^2 \to \mathbb{R}^2)$, to $F(x)$ near x_0 is given by

$$L(x) = F(x_0) + DF(x_0)(x - x_0).$$

We evaluate F at x_0:

$$F(x_0) = F(\begin{pmatrix} 1 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 + e^0 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

We compute $\big(DF : \mathbb{R}^2 \to \mathcal{M}_{2\times 2}(\mathbb{R})\big)$, the Jacobian of F.

$$DF(\begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} \frac{\partial}{\partial u}(u + e^{u-v}) & \frac{\partial}{\partial v}(u + e^{u-v}) \\ \frac{\partial}{\partial u}(2u+v) & \frac{\partial}{\partial v}(2u+v) \end{pmatrix} = \begin{pmatrix} 1 + e^{u-v} & -e^{u-v} \\ 2 & 1 \end{pmatrix}.$$

We evaluate DF at x_0:

$$DF(x_0) = DF(\begin{pmatrix} 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 + e^0 & -e^0 \\ 2 & 1 \end{pmatrix}. = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}.$$

And so

$$L(x) = F(x_0) + DF(x_0)(x - x_0).$$

gives at x_0 and using $x = \begin{pmatrix} u \\ v \end{pmatrix}$

$$L(\begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} u-1 \\ v-1 \end{pmatrix} = \begin{pmatrix} 2 + 2(u-1) - (v-1) \\ 3 + 2(u-1) + (v-1) \end{pmatrix} = \begin{pmatrix} 1 + 2u - v \\ 2u+v \end{pmatrix}.$$

So that

$$\boxed{L(\begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} 1 + 2u - v \\ 2u+v \end{pmatrix}.}$$

Not needed for full credit

```python
import numpy as np
from math import exp
from math import sin
```

```python
x0 = np.array( [ 1., 1. ] )

F = lambda x : np.array( [ x[0] + exp( x[0] - x[1] ) ,
                            2. * x[0] + x[1] ] )
L = lambda x : np.array( [ 1. + 2 * x[0]  - x[1] ,
                            2. * x[0] + x[1] ] )

for x in [ np.array( [ 1.1,     0.8    ] ),
           np.array( [ 1.01,    0.98   ] ),
           np.array( [ 1.001,   0.998  ] ),
           np.array( [ 1.0001,  0.9998 ] ) ]:
  Fx = F(x)
  Lx = L(x)

  print( " x = [", f"{x[0]:+10.8f}",\
         "]    F(x) = [", f"{Fx[0]:+10.8f}" ,\
         "]    L(x) = [", f"{Lx[0]:+10.8f}" ,\
         "]")
  print( "      [", f"{x[1]:+10.8f}",\
         "]           [", f"{Fx[1]:+10.8f}" ,\
         "]           [", f"{Lx[1]:+10.8f}" ,\
         "]")
  print( " || x - x0 || = ",\
        f"{np.linalg.norm(x-x0,np.infty):7.2e}",\
         " || F(x) - L(x) || = ",\
        f"{np.linalg.norm(Fx-Lx,np.infty):7.2e}")
  print( "\n" )
```

```
 x = [ +1.10000000 ]    F(x) = [ +2.44985881 ]    L(x) = [ +2.40000000 ]
     [ +0.80000000 ]           [ +3.00000000 ]           [ +3.00000000 ]
 || x - x0 || =  2.00e-01  || F(x) - L(x) || =  4.99e-02


 x = [ +1.01000000 ]    F(x) = [ +2.04045453 ]    L(x) = [ +2.04000000 ]
     [ +0.98000000 ]           [ +3.00000000 ]           [ +3.00000000 ]
 || x - x0 || =  2.00e-02  || F(x) - L(x) || =  4.55e-04


 x = [ +1.00100000 ]    F(x) = [ +2.00400450 ]    L(x) = [ +2.00400000 ]
     [ +0.99800000 ]           [ +3.00000000 ]           [ +3.00000000 ]
 || x - x0 || =  2.00e-03  || F(x) - L(x) || =  4.50e-06


 x = [ +1.00010000 ]    F(x) = [ +2.00040005 ]    L(x) = [ +2.00040000 ]
     [ +0.99980000 ]           [ +3.00000000 ]           [ +3.00000000 ]
 || x - x0 || =  2.00e-04  || F(x) - L(x) || =  4.50e-08
```

We see that the norm of the error of approximating $F(x)$ with the linear function $L(x)$, $e(x) = F(x) - L(x)$, is a quadratic function of the distance between x and x_0.

$$\|e(x)\|_\infty = \|F(x) - L(x)\|_\infty \approx \|x - x_0\|_\infty^2.$$

So that for example if $\|x - x_0\|_\infty \approx 1e - 4$ then $\|F(x) - L(x)\|_\infty^2 \approx 1e - 8$.