EX.3.2.1, Sauer3

a. Find the degree 2 interpolating polynomial $p_2(x)$ through the points $(0, 0)$, $(\frac{\pi}{2}, 1)$, and $(\pi, 0)$.

b. Calculate $p_2(\frac{\pi}{4})$, an approximation for $\sin(\frac{\pi}{4})$.

c. Use Theorem 3.3 to give an error bound for the approximation in part (b).

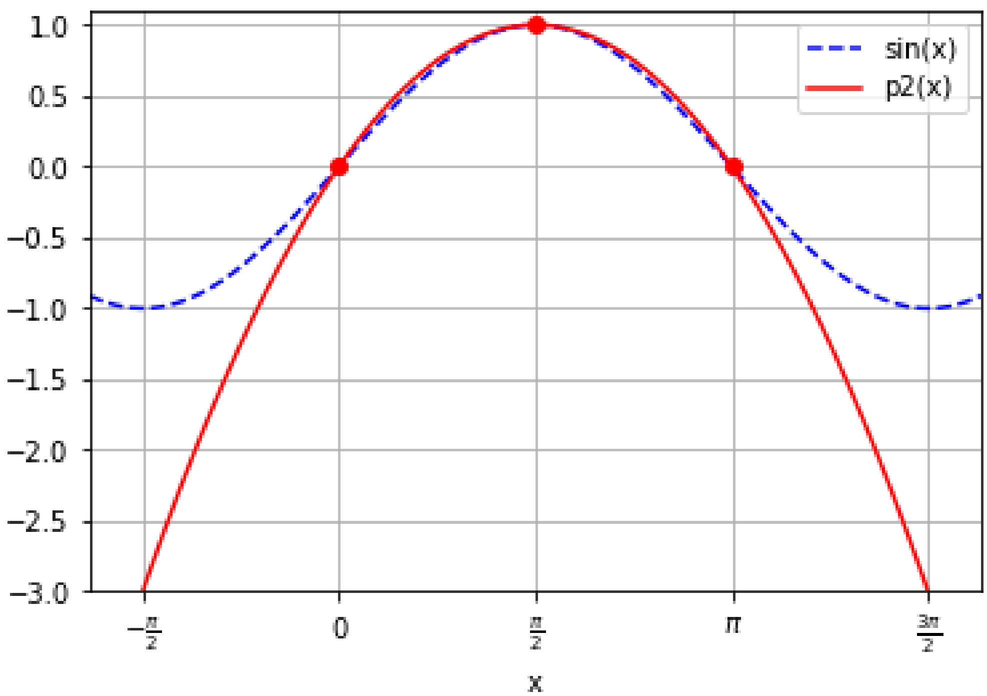
d. Using a Colab Jupyter Notebook, compare the actual error to your error bound.

EX.3.2.1, Sauer3, solution, Langou

Colab Jupyter Notebook: `https://colab.research.google.com/drive/1ZT7KFrkVfylnVCN9uXj-35JVXkNJ8yEL`

(a) Find the degree 2 interpolating polynomial $p_2(x)$ through the points $(0, 0)$, $(\pi/2, 1)$, and $(\pi, 0)$.

The goal is to find the equation of the red parabola (polynomial of at most degree 2) that interpolates sin (the blue curve) at the points $x_0 = 0$, $x_1 = \pi/2$ and $x_2 = \pi$ (the red points).



There are a few method to go about this. We can do Newton's divided difference, or Lagrange interpolation, or "brute force, Vandermonde". Pick the one you want.

Lagrange interpolation

For the points $(0, 0)$, $(\frac{\pi}{2}, 1)$, $(\pi, 0)$, the good news is that we have two functional values out of three that are zeros. So we only need to compute one Lagrange polynomial, and actually, we will have $p_2(x) = \ell(x)$, since the nonzero coefficient is simply 1.

We have

$$\ell(x) = \frac{(x-0)(x-\pi)}{(\frac{\pi}{2}-0)(\frac{\pi}{2}-\pi)} = -\frac{4}{\pi^2}x(x-\pi).$$

And so

$$\boxed{p_2(x) = -\frac{4}{\pi^2}x(x-\pi)}$$

In "standard form", this corresponds to

$$p_2(x) = \frac{4}{\pi}x - \frac{4}{\pi^2}x^2.$$

Newton's divided difference

This is the short, executive version, enough for full credit.

We use Newton's divided difference. We obtain the following table

$$\begin{array}{c|ccc} 0 & 0 & & \\ \frac{\pi}{2} & 1 & \frac{2}{\pi} & \\ \pi & 0 & -\frac{2}{\pi} & -\frac{4}{\pi^2} \end{array}$$

So that the Newton's form of the polynomial is

$$\boxed{p_2(x) = \frac{2}{\pi}x - \frac{4}{\pi^2}x(x - \frac{\pi}{2}).}$$

This is a more detailed and slower version.

We use Newton's divided difference.
First step, we set up the table with $x_0 = 0$, $x_1 = \frac{\pi}{2}$, $x_2 = \pi$ and $f[x_0] = y_0 = 0$, $f[x_1] = y_2 = 1$, $f[x_2] = y_2 = 0$. We get

$$\begin{array}{c|c} 0 & 0 \\ \frac{\pi}{2} & 1 \\ \pi & 0 \end{array}$$

Second step, we compute

$$\begin{aligned} f[x_0, x_1] &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{1-0}{\frac{\pi}{2} - 0} = \frac{2}{\pi} \\ f[x_1, x_2] &= \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{0-1}{\pi - \frac{\pi}{2}} = -\frac{2}{\pi} \end{aligned}$$

We get

$$\begin{array}{c|cc} 0 & 0 & \\ \frac{\pi}{2} & 1 & \frac{2}{\pi} \\ \pi & 0 & -\frac{2}{\pi} \end{array}$$

Third step, we compute

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{-\frac{2}{\pi} - \frac{2}{\pi}}{\pi - 0} = -\frac{4}{\pi^2}$$

We get

$$\begin{array}{c|ccc} 0 & 0 & & \\ \frac{\pi}{2} & 1 & \frac{2}{\pi} & \\ \pi & 0 & -\frac{2}{\pi} & -\frac{4}{\pi^2} \end{array}$$

We get the coefficients c_0, c_1 and c_2 by reading the diagonal of the table:

$$\begin{array}{c|ccc} 0 & 0 & & \\ \frac{\pi}{2} & 1 & \frac{2}{\pi} & \\ \pi & 0 & -\frac{2}{\pi} & -\frac{4}{\pi^2} \end{array}$$

And so the nested form representation

$$p(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1)$$

gives

$$p(x) = 0 + \frac{2}{\pi}(x-0) - \frac{4}{\pi^2}(x-0)(x-\frac{\pi}{2})$$

Finally:

$$\boxed{p_2(x) = \tfrac{2}{\pi}x - \tfrac{4}{\pi^2}x(x-\tfrac{\pi}{2}).}$$

In "standard form", this corresponds to

$$p_2(x) = \frac{4}{\pi}x - \frac{4}{\pi^2}x^2.$$

```
x = np.array([0., pi/2., pi])
y = np.array([0., 1.,    0.]) # sine values for the three x data points

c = newtdd(x,y)
print( "coefficients of p(x) in nested form:")
for i in range(0,len(c)-1):
    print( f"{c[i]: 20.16f}"+",")
print( f"{c[-1]: 20.16f}")

print( "coefficients of p(x) in nested form (from our handmade computation)):")
print( f"{0.: 20.16f}"+",")
print( f"{2/pi: 20.16f}"+",")
print( f"{-4/pi/pi: 20.16f}")
```

```
coefficients of p(x) in nested form:
  0.0000000000000000,
  0.6366197723675814,
 -0.4052847345693511
coefficients of p(x) in nested form (from our handmade computation)):
  0.0000000000000000,
  0.6366197723675814,
 -0.4052847345693511
```

"brute force, Vandermonde"

We are looking for a, b, and c such that $p_2(x) = a + bx + cx^2$ interpolates sin at $x = 0$, $\frac{\pi}{2}$, and π. We get

at $x=0$	$p_2(0) = \sin(0)$ gives	$a + b0 + c(0)^2 = 0$	so $a = 0$,
at $x=\frac{\pi}{2}$	$p_2(\frac{\pi}{2}) = \sin(\frac{\pi}{2})$ gives	$a + b\frac{\pi}{2} + c(\frac{\pi}{2})^2 = 1$	so, knowing $a=0$, we get $b + \frac{\pi}{2}c = \frac{2}{\pi}$,
at $x=\pi$	$p_2(\pi) = \sin(\pi)$ gives	$a + b\pi + c(\pi)^2 = 0$	so, knowing $a=0$, we get $b + \pi c = 0$.

We want to solve for b and c

$$\begin{cases} b + \frac{\pi}{2}c = \frac{2}{\pi} \\ b + \pi c = 0 \end{cases}$$

We perform $L_2 \leftarrow L_2 - L_1$ and get $\frac{\pi}{2}c = -\frac{2}{\pi}$, so that $c = \frac{4}{\pi^2}$.
And then, substituing back in L_3, we can find the value of b, we get $b = -\pi c = \frac{4}{\pi}$.
We solve for b and c and get

$$b = \frac{4}{\pi} \quad \text{and} \quad c = -\frac{4}{\pi^2}.$$

Finally

$$p_2(x) = \frac{4}{\pi}x - \frac{4}{\pi^2}x^2.$$

(b) Calculate $p_2(\pi/4)$, an approximation for $\sin(\pi/4)$.

$$p_2(\pi/4) = \frac{4}{\pi}(\pi/4) - \frac{4}{\pi^2}(\pi/4)^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$

```
print( polyval_nested_w_base_points(c,x,pi/4))
```

```
0.75
```

(b) Calculate $p_2(\frac{\pi}{4})$, an approximation for $\sin(\frac{\pi}{4})$.

$$p_2(\frac{\pi}{4}) = \frac{4}{\pi}(\frac{\pi}{4}) - \frac{4}{\pi^2}(\frac{\pi}{4})^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$

```
print( polyval_nested_w_base_points(c,x,pi/4))
```

```
0.75
```

Note: $p_2(\frac{\pi}{4})$ is an "approximation" of $\sin(\frac{\pi}{4})$. We know that

$$\sin(\pi/4) = \frac{\sqrt{2}}{2} \approx 0.7071.$$

So we can see how good $p_2(\frac{\pi}{4})$, which is 0.75, is an approximation of $\sin(\frac{\pi}{4})$, which is 0.7071.

(c) Use Theorem 3.3 to give an error bound for the approximation in part (b).

From Theorem 3.3, we know that, there exists c in $(0, \pi)$ such that

$$f(x) - p_2(x) = \frac{f^{(3)}(c)}{3!}(x)(x - \frac{\pi}{2})(x - \pi).$$

The third derivative of f is $f^{(3)}(x) = -\cos(x)$. We do not know the value of c. But, we know that c is a real number in in $(0, \pi)$. So we cannot know the value of $f^{(3)}(c)$, but we can try to find an upper bound on $|f^{(3)}(c)|$ for all c in $(0, \pi)$. For the function $f(x)$ equal to $\sin(x)$, we know that, for all c in $(0, \pi)$, $|f^{(3)}(c)| \leq 1$. So we will use this to get an upper bound on the approximation error. We now get

$$|f(x) - p_2(x)| \leq \frac{1}{6}|(x)(x - \frac{\pi}{2})(x - \pi)|.$$

So, at $x = \pi/4$, we get

$$|f(x) - p_2(x)| \quad \leq \quad \frac{1}{3!}|(\frac{\pi}{4})(\frac{\pi}{4} - \frac{\pi}{2})(\frac{\pi}{4} - \pi)| = \frac{1}{128}\pi^3$$

So:

$$|f(x) - p_2(x)| \leq \frac{1}{128}\pi^3 \leq 0.242$$

(d) Using a Colab Jupyter Notebook, compare the actual error to your error bound

```
print( abs( sqrt(2)/2 - 0.75 ) )
```

```
0.04289321881345243
```

From Theorem 3.3, we know that an upper bound on the approximation error is 0.242 When we compute the "true" approximation error, we get 0.043. We see that the error bound is above the "true" approximation error. (As is expected by the theory.)