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**EX.4.3.8.a, Sauer3**

Find the QR factorization and use it to solve the following least squares problem

$$\begin{bmatrix} 1 & 4 \\ -1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \end{bmatrix}.$$

Suggestion: Use the Gram-Schmidt algorithm for getting the QR factorization, and remember that a thin QR factorization is enough for this problem. No need to compute a full QR factorization.

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**EX.4.3.8.a, Sauer3, solution, Langou**

Colab: `https://colab.research.google.com/drive/1cw4J2CTJjaPGahWJBS5ujNKJmtEE4h5M`

We start with performing a QR factorization of $A$ using the Gram-Schmidt algorithm.
Let $a_0$ and $a_1$ be the first and second columns of $A$.
We perform the following computation.

$$\begin{aligned}
w_0 &= a_0, \\
r_{00} &= \|w_0\|_2 = \sqrt{(1)^2 + (-1)^2 + (1)^2 + (1)^2} = 2, \\
q_0 &= \frac{1}{r_{00}} w_0 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \\
r_{01} &= q_0^T a_1 = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \left( (1)(4) + (-1)(1) + (1)(1) + (1)(0) \right) = 2, \\
w_1 &= a_1 - q_0 r_{01} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \end{bmatrix} - 2(\frac{1}{2}) \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \\
r_{11} &= \|w_1\|_2 = \sqrt{(3)^2 + (2)^2 + (0)^2 + (-1)^2} = \sqrt{14}, \\
q_1 &= \frac{1}{r_{11}} w_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix}.
\end{aligned}$$

So we find

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{3}{\sqrt{14}} \\ -\frac{1}{2} & \frac{2}{\sqrt{14}} \\ \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{14}} \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 2 & 2 \\ 0 & \sqrt{14} \end{bmatrix}.$$

We can check that

a. $A = QR$

b. $Q^T Q = I$

c. $R$ is upper triangular

Now we can solve the linear least squares problem using this QR factorization.
Firstly, we form $Q^T b$:

$$Q^T b \quad = \quad , \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{\sqrt{14}} & \frac{2}{\sqrt{14}} & 0 & -\frac{1}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left((1)(3) + (-1)(1) + (1)(1) + (1)(-3)\right) \\ \frac{1}{\sqrt{14}}\left((3)(3) + (2)(1) + (0)(1) + (-1)(-3)\right) \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{14} \end{bmatrix}$$

Secondly, we solve the upper triangular system $Rx = Q^T b$:

$$\begin{bmatrix} 2 & 2 \\ 0 & \sqrt{14} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{14} \end{bmatrix}$$

Using back substitution, we find $x_2 = 1$ and $x_1 = -1$.
The solution to the linear least square is

$$x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We can check that, for this $x$, the associated residual, $b - Ax$ is orthogonal to the columns of $A$. That is

a. $A^T(b - Ax) = 0$.

Indeed

$$\begin{aligned} Ax &= \begin{bmatrix} 1 & 4 \\ -1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}. = \begin{bmatrix} (1)(-1) + (4)(1) \\ (-1)(-1) + (1)(1) \\ (1)(-1) + (1)(1) \\ (1)(-1) + (0)(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix} \\ b - Ax &= \begin{bmatrix} 3 \\ 1 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -2 \end{bmatrix} \\ A^T(b - Ax) &= \begin{bmatrix} 1 & -1 & 1 & 1 \\ 4 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} (1)(0) + (-1)(-1) + (1)(1) + (1)(-2) \\ (4)(0) + (1)(-1) + (1)(1) + (0)(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \checkmark \end{aligned}$$