EX.2.4.10.a, Sauer3

Assume that $A = (a_{ij})$ is a $n \times n$ matrix with entries $|a_{ij}| \leq 1$ for all $1 \leq i, j \leq n$. Prove that the matrix $U = (u_{ij})$ in its $PA = LU$ factorization satisfies $|u_{ij}| \leq 2^{n-1}$ for all $1 \leq i, j \leq n$.

EX.2.4.10.a, Sauer3, solution, Langou

Assume that $A = (a_{ij})$ is a n-by-n matrix with entries $|a_{ij}| \leq 1$ for all $1 \leq i, j \leq n$.
Note that, at the kth step of the $PA = LU$ factorization, we arrange the rows so that $|a_{kk}^{(k-1)}| \geq |a_{ik}^{(k-1)}|$ for all $i = k+1, \ldots, n$. This is the partial pivoting strategy. Then the elimination step consists in performing $a_{ij}^{(k)} = a_{ij}^{(k-1)} - (a_{ik}^{(k-1)}/a_{kk}^{(k-1)})a_{kj}^{(k-1)}$ for $i > k$ and $j > k$. We assume that the rows of the matrix are already in such a way that no permutation happens. (So in other words, we work with the matrix PA and call it A for simplicity.)
The algorithm starts by setting $A^{(0)}$ as A.
We note that for $1 \leq j \leq n$, $u_{1j} = a_{1j}^{(0)}$. That is the say the first row of U is the first row of $A^{(0)}$. So we have that, for $1 \leq j \leq n$, $|u_{1j}| \leq 1$.
At the first step, $k = 1$, $a_{ij}^{(1)} = a_{ij}^{(0)} - (a_{i1}^{(0)}/a_{11}^{(0)})a_{1j}^{(0)}$ where $i, j > 1$, thus $|a_{ij}^{(1)}| \leq |a_{ij}^{(0)}| + |a_{i1}^{(0)}/a_{11}^{(0)}| \cdot |a_{1j}^{(0)}|$. Now, we have that $|a_{i1}^{(0)}/a_{11}^{(0)}| \leq 1$ (thanks to partial pivoting), and $|a_{ij}^{(0)}| \leq 1$ and $|a_{1j}^{(0)}| \leq 1$, by assumption on the initial matrix. So, we see that, after step 1, $|a_{ij}^{(1)}| \leq 2$ where $i, j > 1$.
We note that for $2 \leq j \leq n$, $u_{2j} = a_{2j}^{(1)}$. That is the say the first row of U is the second row of $A^{(1)}$. So we have that, for $2 \leq j \leq n$, $|u_{2j}| \leq 2$.
At the second step, $k = 2$, $a_{ij}^{(2)} = a_{ij}^{(1)} - (a_{i2}^{(1)}/a_{22}^{(1)})a_{2j}^{(1)}$ where $i, j > 2$, thus $|a_{ij}^{(2)}| \leq |a_{ij}^{(1)}| + |a_{i2}^{(1)}/a_{22}^{(1)}| \cdot |a_{2j}^{(1)}|$. Now, we have that $|a_{i1}^{(1)}/a_{11}^{(1)}| \leq 1$ (thanks to partial pivoting), and $|a_{ij}^{(1)}| \leq 2$ and $|a_{1j}^{(1)}| \leq 2$, by induction. So, we see that, after step 2, $|a_{ij}^{(2)}| \leq 4$ where $i, j > 2$.

$\vdots$

We note that for $n-1 \leq j \leq n$, $u_{n-1,j} = a_{n-1,j}^{(n-2)}$. That is the say the $(n-1)$-th row of U is the $(n-1)$-th row of $A^{(n-2)}$. So we have that, for $n-1 \leq j \leq n$, $|u_{n-1,j}| \leq 2^{n-2}$.
At the $(n-1)$-th step, (last step,) $k = n-1$, there is only one operation to perform $a_{nn}^{(n)} = a_{nn}^{(n-1)} - (a_{n,n-1}^{(n-1)}/a_{n-1,n-1}^{(n-1)})a_{n-1,n}^{(n-1)}$, thus $|a_{nn}^{(n)}| \leq |a_{nn}^{(n-1)}| + |a_{n,n-1}^{(n-1)}/a_{n-1,n-1}^{(n-1)}| \cdot |a_{n-1,n}^{(n-1)}|$. Now, we have that $|a_{n,n-1}^{(n-1)}/a_{n-1,n-1}^{(n-1)}| \leq 1$ (thanks to partial pivoting), and $|a_{nn}^{(n)}| \leq 2^{n-2}$ and $|a_{n-1,n}^{(n-1)}| \leq 2^{n-2}$, by induction. So, we see that, after step n-1, $|a_{nn}^{(n)}| \leq 2^{n-1}$.
We note that, $u_{nn} = a_{nn}^{(n-1)}$. So we have that, $|u_{nn}| \leq 2^{n-1}$.
We proved that the matrix $U = (u_{ij})$ in the $PA = LU$ factorization satisfies $|u_{ij}| \leq 2^{n-1}$ for all $1 \leq i, j \leq n$.