EX.2.7.2.a, Sauer3

Use the Taylor expansion to find the linear approximation $L(x)$ to $F(x)$ near x_0.

$$\text{(b)} \left(\begin{array}{cccc} F: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & \begin{pmatrix} u \\ v \end{pmatrix} & \mapsto & \begin{pmatrix} 1+e^{u+2v} \\ \sin(u+v) \end{pmatrix} \end{array} \right) \qquad \text{at } x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

EX.2.7.2.a, Sauer3, solution, Langou

Colab: `https://colab.research.google.com/drive/18eWOQvhLWWa_V_dDK7pgV8oTgiGXRO9j`

The Taylor expansion of F at x_0 of order 1 is

$$F(x) = F(x_0) + DF(x_0)(x - x_0) + e(x), \quad \text{where } \|e(x)\| = \mathcal{O}(\|x - x_0\|^2).$$

And so, the linear approximation $L(x)$, where $(L : \mathbb{R}^2 \to \mathbb{R}^2)$, to $F(x)$ near x_0 is given by

$$L(x) = F(x_0) + DF(x_0)(x - x_0).$$

We evaluate F at x_0:

$$F(x_0) = F(\begin{pmatrix} 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1+e^0 \\ \sin(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

We compute $\big(DF : \mathbb{R}^2 \to \mathcal{M}_{2\times 2}(\mathbb{R})\big)$, the Jacobian of F.

$$DF(\begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} \frac{\partial}{\partial u}(1+e^{u+2v}) & \frac{\partial}{\partial v}(1+e^{u+2v}) \\ \frac{\partial}{\partial u}(\sin(u+v)) & \frac{\partial}{\partial u}(\sin(u+v)) \end{pmatrix} = \begin{pmatrix} e^{u+2v} & 2e^{u+2v} \\ \cos(u+v) & \cos(u+v) \end{pmatrix}.$$

We evaluate DF at x_0:

$$DF(x_0) = DF(\begin{pmatrix} 0 \\ 0 \end{pmatrix}) = \begin{pmatrix} e^0 & 2e^0 \\ \cos(0) & \cos(0) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

And so

$$L(x) = F(x_0) + DF(x_0)(x - x_0).$$

gives at x_0 and using $x = \begin{pmatrix} u \\ v \end{pmatrix}$

$$L(\begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u-0 \\ v-0 \end{pmatrix}.$$

So that

$$\boxed{L(\begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} 2+u+2v \\ u+v \end{pmatrix}.}$$

Not needed for full credit

```python
import numpy as np
from math import exp
from math import sin
```

```python
x0 = np.array( [ 0., 0. ] )

F = lambda x : np.array( [ 1. + exp( x[0] + 2. * x[1] ) ,
                           sin( x[0] + x[1] ) ] )
L = lambda x : np.array( [ 2. + x[0] + 2. * x[1] ,
                           x[0] + x[1] ] )

for x in [ np.array( [ 0.1,    -0.2    ] ),
           np.array( [ 0.01,   -0.02   ] ),
           np.array( [ 0.001,  -0.002  ] ),
           np.array( [ 0.0001, -0.0002 ] ) ]:

  Fx = F(x)
  Lx = L(x)

  print( " x = [", f"{x[0]:+10.8f}",\
         "]    F(x) = [", f"{Fx[0]:+10.8f}" ,\
         "]    L(x) = [", f"{Lx[0]:+10.8f}" ,\
         "]")
  print( "     [", f"{x[1]:+10.8f}",\
         "]           [", f"{Fx[1]:+10.8f}" ,\
         "]           [", f"{Lx[1]:+10.8f}" ,\
         "]")
  print( " || x - x0 || = ",\
        f"{np.linalg.norm(x-x0,np.infty):7.2e}",\
         " || F(x) - L(x) || = ",\
        f"{np.linalg.norm(Fx-Lx,np.infty):7.2e}")
  print( "\n" )
```

```
 x = [ +0.10000000 ]    F(x) = [ +1.74081822 ]    L(x) = [ +1.70000000 ]
     [ -0.20000000 ]           [ -0.09983342 ]           [ -0.10000000 ]
 || x - x0 || =  2.00e-01  || F(x) - L(x) || =  4.08e-02


 x = [ +0.01000000 ]    F(x) = [ +1.97044553 ]    L(x) = [ +1.97000000 ]
     [ -0.02000000 ]           [ -0.00999983 ]           [ -0.01000000 ]
 || x - x0 || =  2.00e-02  || F(x) - L(x) || =  4.46e-04


 x = [ +0.00100000 ]    F(x) = [ +1.99700450 ]    L(x) = [ +1.99700000 ]
     [ -0.00200000 ]           [ -0.00100000 ]           [ -0.00100000 ]
 || x - x0 || =  2.00e-03  || F(x) - L(x) || =  4.50e-06


 x = [ +0.00010000 ]    F(x) = [ +1.99970004 ]    L(x) = [ +1.99970000 ]
     [ -0.00020000 ]           [ -0.00010000 ]           [ -0.00010000 ]
 || x - x0 || =  2.00e-04  || F(x) - L(x) || =  4.50e-08
```

We see that the norm of the error of approximating $F(x)$ with the linear function $L(x)$, $e(x) = F(x) - L(x)$, is a quadratic function of the distance between x and x_0.

$$\|e(x)\|_\infty = \|F(x) - L(x)\|_\infty \approx \|x - x_0\|_\infty^2.$$

So that for example if $\|x - x_0\|_\infty \approx 1e - 4$ then $\|F(x) - L(x)\|_\infty \approx 1e - 8$.