EX.1.2.7, Sauer3

Use Theorem 1.6 to determine whether Fixed-Point Iteration of $g(x)$ is locally convergent to the given fixed point r. (a) $g(x) = (2x-1)^{\frac{1}{3}}, \quad r = 1$, (b) $g(x) = \frac{1}{2}\left(x^3+1\right), \quad r = 1$, (c) $g(x) = \sin x + x, \quad r = 0$.

EX.1.2.7, Sauer3, solution, Langou

a. (a) The function $\left(g : (\frac{1}{2}, \infty) \to \mathbb{R}, \quad x \mapsto (2x-1)^{\frac{1}{3}}\right)$ is continuously differentiable in $(\frac{1}{2}, \infty)$,
   (b) by direct evaluation, we see that $g(r) = r$, (i.e., g(1) = 1,)
   (c) we compute $g'(x)$ and get $g'(x) = \frac{2}{3}(2x-1)^{-\frac{2}{3}}$, evaluate at $r = 1$, and get $g'(r) = \frac{2}{3}$, and see that $|g'(r)| < 1$.

   We conclude (using Theorem 1.6) that Fixed Point Iteration is locally convergent to the fixed point $r = 1$.

b. (a) The function $\left(g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{2}\left(x^3+1\right)\right)$ is continuously differentiable in $\mathbb{R}$,
   (b) by direct evaluation, we see that $g(r) = r$, (i.e., $g(1) = 1$,)
   (c) we compute $g'(x)$ and get $g'(x) = \frac{3}{2}x^2$, evaluate at $r = 1$, and get $g'(r) = \frac{3}{2}$, and see that $|g'(r)| \geq 1$.

   The assumption of Theorem 1.6 are not satisfied. We cannot conclude anything.

c. (a) The function $g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sin x + x$ is continuously differentiable in $\mathbb{R}$,
   (b) by direct evaluation, we see that $g(r) = r$, (i.e., g(0) = 0,)
   (c) we compute $g'(x)$ and get $g'(x) = \cos x + 1$, evaluate at $r = 0$, and get $g'(r) = 2$, and see that $|g'(r)| \geq 1$.

   The assumption of Theorem 1.6 are not satisfied. We cannot conclude anything.