Summer 2025

MATH 4650-E01
CSCI 4650-E01
MATH 5660-E01

Homework 6

Due on Monday July 21st 2025

Exercises:

1. EX.5.1.2 handwritten+colab
2. EX.5.1.4 handwritten+colab
3. EX.5.1.8 handwritten
4. EX.5.1.10 handwritten
5. EX.5.1.18 handwritten
6. EX.5.1.20 handwritten
7. CP.5.1.2 colab
8. EX.5.2.4.b handwritten
9. EX.5.2.6.b handwritten
10. CP.5.2.1.a colab
11. CP.5.2.2.a colab
12. EX.5.3.2.b handwritten
13. CP.5.3.1.a colab
14. CP.5.4.1.a colab
15. EX.5.5.6.d colab

EX.5.1.2, Sauer3 – Problem rewritten by Julien.
We want to study the three-point centered-difference formula to approximate $f'(x)$ where $f(x) = e^x$ and $x = 0$.

a. (handwritten) For this problem, what is $f'(x)$?

b. (handwritten) What is the **three-point centered-difference formula** in its general form?

c. (python) Use the **three-point centered-difference formula** to approximate $f'(x)$ for (a) $h = 0.1$, (b) $h = 0.01$, and (c) $h = 0.001$? Observe that, as h gets smaller, the approximation gets better. (However if we were to take h too small, we would suffer from cancellation error and start losing accuracy.)

d. (handwritten) What is the **three-point centered-difference** approximation error in its general form? (Hint: There is a "c" in the formula. Specify an interval where c is.)

e. (handwritten) Find lower and upper bounds for the **three-point centered-difference** approximation error (independent of c). We call this an "interval where the approximation error is". We want this interval to be "relevant".

f. (python) Check that the "true approximation error" is within your interval of part (e).

g. (handwritten) In general, we are more interested in guaranteeing that "the error is not larger than". We mean the absolute value of the error. (As opposed to find an interval where the error is.) Based on part (e), find an upper bound for the absolute value of the "true approximation error". We call this an "upper bound on the approximation error".

h. (python) Check that an "upper bound on the approximation error" is an upper bound on the approximation error. We want this upper bound to be "relevant".

EX.5.1.4, Sauer3 – Problem rewritten by Julien.

Carry out the steps of EX.5.1.3 using the three-point centered-difference formula.
We want to study the three-point centered-difference formula to approximate $f'(x)$ where $f(x) = \sin x$ and $x = \frac{\pi}{3}$.

a. (handwritten) For this problem, what is $f'(x)$?

b. (handwritten) What is the **three-point centered-difference formula** in its general form?

c. (python) Use the **three-point centered-difference formula** to approximate $f'(x)$ for (a) $h = 0.1$, (b) $h = 0.01$, and (c) $h = 0.001$? Observe that, as h gets smaller, the approximation gets better. (However if we were to take h too small, we would suffer from cancellation error and start losing accuracy.)

d. (handwritten) What is the **three-point centered-difference** approximation error in its general form? (Hint: There is a "c" in the formula. Specify an interval where c is.)

e. (handwritten) Since we work with small positive h's, let us assume that h is in $(-0.2, 0.2)$, so that $x - h$ and $x + h$ are in $(0, \frac{\pi}{2})$, find lower and upper bounds for the **three-point centered-difference** approximation error (independent of c). We call this an "interval where the approximation error is". We want this interval to be "relevant".

f. (python) Check that the "true approximation error" is within your interval of part (e).

g. (handwritten) In general, we are more interested in guaranteeing that "the error is not larger than". We mean the absolute value of the error. (As opposed to find an interval where the error is.) Based on part (e), find an upper bound for the absolute value of the "true approximation error". We call this an "upper bound on the approximation error".

h. (python) Check that an "upper bound on the approximation error" is an upper bound on the approximation error. We want this upper bound to be "relevant".

EX.5.1.8, Sauer3 Prove the second-order formula for the first derivative

$$f'(x) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} + \mathcal{O}(h^2).$$

EX.5.1.10 Find the error term and order for the approximation formula

$$f'(x) \approx \frac{4f(x+h) - 3f(x) - f(x-2h)}{6h}.$$

EX.5.1.18, Sauer3

Find an upper bound for $E(h)$ the error of the machine approximation of the two-point forward difference formula for the first derivative and then find the h corresponding to the minimum of $E(h)$.

EX.5.1.20, Sauer3

Prove the second-order formula for the third derivative

$$f'''(x) = \frac{f(x-3h) - 6f(x-2h) + 12f(x-h) - 10f(x) + 3f(x+h)}{2h^3} + O(h^2).$$

CP.5.1.2, Sauer3 – Problem rewritten by Julien.

We want to study the three-point centered-difference formula to approximate $f'(x)$ where $f(x) = (x+1)^{-1}$ and $x = 1$.

a. (python) Make a table and plot of the error of the three-point centered-difference formula for $f'(x)$, where $f = (x+1)^{-1}$ and $x = 1$, with 10^{-1},..., 10^{-12}, as in the table in Section 5.1.2. Draw a plot of the results.

b. (python) Use the formula

$$h_{\text{opt}} \approx \left(\frac{3\varepsilon_{\text{mach}}}{M}\right)^{\frac{1}{3}},$$

(given below Equation(5.12) in the book,) to compute an approximation to h_{opt}. In this formula M is $|f'''(x)|$, and $\varepsilon_{\text{mach}}$ is the machine precision that you can get in python with `np.finfo(float).eps`.

c. (python) Redo part (a) by adding $E(h)$ from Equation (5.11) to your table and to your plot. You can approximate the quantity $f'''(x)$ by M.

EX.5.2.4.b, Sauer3

Apply the composite Simpson's Rule with $m = 1$, 2 and 4 panels to the following integral. Compute the error by comparing with the exact value from calculus.

$$\text{(b)} \qquad \int_0^1 \frac{1}{1+x^2}\,dx$$

EX.5.2.6.b, Sauer3

Apply the composite Midpoint Rule with $m = 1$, 2 and 4 panels to approximate the following integral.

$$\text{(b)} \qquad \int_0^1 \frac{e^x - 1}{x}\, dx$$

Some comments:

- We note that in zero the function
$$f(x) = \frac{e^x - 1}{x}$$
is not defined. This is why we need an open rule (called in the book: "open Newton-Cotes method" or "midpoint rule"). A closed rule would use $f(0)$ and that would be a problem.

- To repeat, for this integral, Simpson's rule would utterly fail becasue $f(0)$ is not defined, so we need to use a midpoint rule. (To avoid 0.)

- We also note that this is not that bad since
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$
So we could very easily (and legitimally) extend f by continuity at 0, by defining $f(0) = 1$, and then we could use a closed rule. (Be careful when x gets close to 0 though. But this issue will be for "open" and "closed" rules.)

- We note that there is no closed-form formula for this integral and so the only way to obtain a value for this integral is by numerical integration.

CP.5.2.1.a, Sauer3

Use the composite Trapezoid Rule with $m = 16$ and 32 panels to approximate the definite integral. Compare with the correct integral and report the two errors.

$$\text{(a)} \qquad \int_0^4 \frac{x}{\sqrt{x^2+9}} dx$$

CP.5.2.2.a, Sauer3

Use the composite Simpson's Rule with $m = 16$ and 32 panels to approximate the definite integral. Compare with the correct integral and report the two errors.

$$\text{(a)} \qquad \int_0^4 \frac{x}{\sqrt{x^2+9}}dx$$

EX.5.3.2.b, Sauer3

Apply Romberg Integration to find R_{33} for the integral

$$\int_0^1 \frac{1}{1+x^2}\, dx.$$

CP.5.3.1.a, Sauer3

Use Romberg Integration approximation R_{55} to approximate the definite integral below. Compare with the correct integral, and report the error.

$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\, dx.$$

CP.5.4.1.a, Sauer3

Use Adaptive Trapezoid Quadrature to approximate the definite integral within $0.5 \cdot 10^{-8}$. Report the number of subintervals used

$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx.$$

EX.5.5.6.d, Sauer3

Approximate the integral below, using $n = 4$ Gaussian Quadrature.

$$\int_{-3}^{3} e^{-\frac{x^2}{2}}\, dx$$