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CP.1.4.13, Sauer3

For the function

$$f(x) = \left(1 - \frac{3}{4x}\right)^{\frac{1}{3}}$$

a. Find the root of the function.

b. Apply Newton's Method using an initial guess near the root and plot the first 50 iterates. This is another way Newton's Method can fail, by producing a chaotic trajectory.

c. Why are Theorems 1.11 and 1.12 not applicable?

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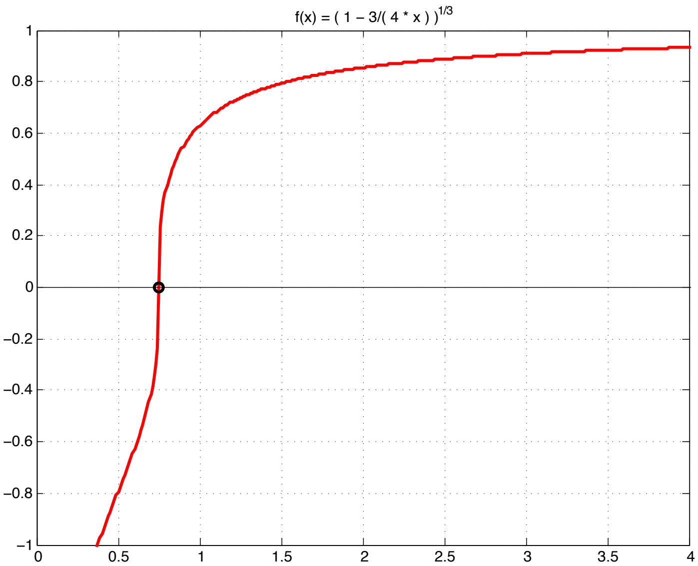
CP.1.4.13, Sauer3, solution, Langou

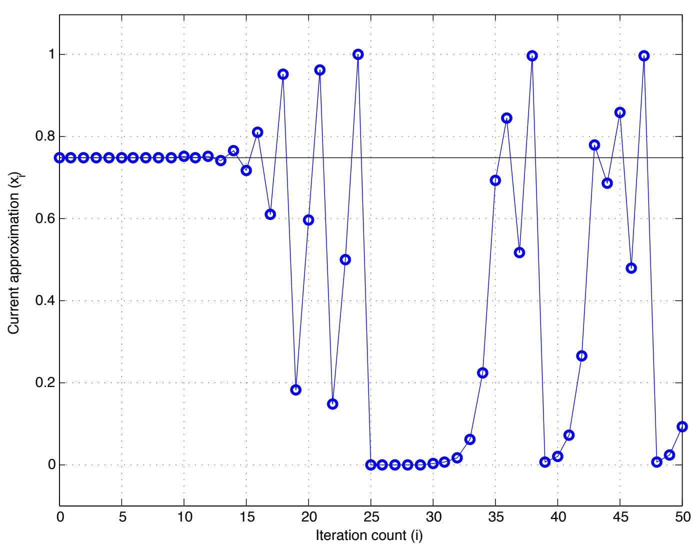
a. To find the root, note that

$$\left(1 - \frac{3}{4x}\right)^{\frac{1}{3}} = \left(\frac{4x-3}{4x}\right)^{\frac{1}{3}}$$

such that the root is $r = \frac{3}{4}$.

b. Newton's Method fails to converge. Left figure is the function f. We see that the slope of the curve is infinite at the root $r = 3/4$. Right figure are the iterates generated by Newton's method. We start at 0.750001, very close from the root $r = 3/4$, and we observe chaotic iterations.





c. Since the derivative

$$f'(x) = \frac{1}{4x^2\left(1 - \frac{3}{4x}\right)^{\frac{2}{3}}}$$

is not defined for $x = 3/4$, the theorems do not apply.

Note: *In this case, Newton's method applied to f reduces to the Fixed Point Iteration scheme $x_{n+1} = g(x_n)$ with $g(x) = 4x(1-x)$, since*

$$x_{n+1} \quad = \quad x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\left(1 - \frac{3}{4x_n}\right)^{\frac{1}{3}}}{\frac{1}{4x_n^2\left(1-\frac{3}{4x_n}\right)^{\frac{2}{3}}}} = 4x_n - 4x_n^2.$$

Now we see that $r = 3/4$ is a fixed point of g (since $g(r) = r$), but $g'(r) = -32$ and so the theorem of convergence for Fixed Point Iteration does not apply.