EX.1.4.5, Sauer3

Consider the equation $8x^4 - 12x^3 + 6x^2 - x = 0$. For each of the two solutions $x = 0$ and $x = 1/2$, decide which will converge faster (say, to eight-place accuracy), the Bisection Method or Newton's Method, without running the calculations.

EX.1.4.5, Sauer3, solution, Langou

First of, we check that $x = 0$ and $x = 1/2$ are solutions of $8x^4 - 12x^3 + 6x^2 - x = 0$. Clearly, $f(x)$ is twice continuously differentiable with

$$f'(x) = 32x^3 - 36x^2 + 12x - 1$$
$$f''(x) = 96x^2 - 72x + 12$$

For $x = 0$ we have $f'(0) \neq 0$ such that we satisfy the conditions for Theorem 1.11. Hence, Newton's Method converges quadratically and is faster that the Bisection Method.
For $x = 1/2$, we have $f'(1/2) = 0$ so we cannot use Theorem 1.11. Since $f''(1/2) = 0$ and $f'''(1/2) \neq 0$ we know that $x = 1/2$ is a root of multiplicity three. Using Theorem 1.12, Newton's Method is locally convergent to $r = 1/2$ and the convergence rate is

$$S = \frac{(m-1)}{m} = \frac{2}{3}.$$

Since this is a slower convergence rate than the Bisection Method, which has a convergence rate of $S = 1/2$, the Bisection Method will converge faster to eight-place accuracy.

Note that

$$8x^4 - 12x^3 + 6x^2 - x = 8x(x - \frac{1}{2})^3,$$

which is another way to see that 0 *is a simple root and* 1/2 *is a root of multiciplity 3.*