EX.3.1.7, Sauer3

Find $p(0)$, where $p(x)$ is the degree 10 polynomial that is zero at $x = 1, \ldots, 10$ and satisfies $p(12) = 144$.

EX.3.1.7, Sauer3, solution, Langou

Using Lagrange Interpolation, we have

$$\begin{aligned} p(x) &= 0 \times \ell_1(x) + 0 \times \ell_2(x) \ldots + 0 \times \ell_9(x) + 0 \times \ell_{10}(x) + 144 \times \ell_{12}(x) \\ &= 144 \times \ell_{12}(x) \end{aligned}$$

Where $\ell_{12}(x)$ is defined as

$$\begin{aligned} \ell_{12} &= \frac{(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10)}{(12-1)(12-2)(12-3)(12-4)(12-5)(12-6)(12-7)(12-8)(12-9)(12-10)} \\ &= \frac{1}{11 \times 10 \times \ldots \times 2}(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10) \\ &= \frac{1}{11!}(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7)(x-8)(x-9)(x-10) \end{aligned}$$

Note that we do not need $\ell_i(x)$ for $1 \leq i \leq 10$ since the functional values at these interpolation points are zero.

Evaluating $p(x)$ at $x = 0$, we have

$$\begin{aligned} p(0) &= 144 \times \ell_{12}(0) \\ &= 144\frac{1}{11!}(0-1)(0-2)(0-3)(0-4)(0-5)(0-6)(0-7)(0-8)(0-9)(0-10) \\ &= 144\frac{10!}{11!} \\ &= \frac{144}{11} \end{aligned}$$

$$\boxed{p(0) = \frac{144}{11}}$$

This is not a Colab problem but we can quickly check with Colab.

Please be careful though a polynomial of degree 10 will have very high amplitude oscillations and so it gets really tough to evaluate numerically. So do not be surprise if the value returned by Python is a little different from $\frac{144}{11}$. The correct answer is $\frac{144}{11}$. Python gives an approximation using polynomial interpolation. We are not doing any rounding errors. Python is. That being said, always good to check.
Colab: `https://colab.research.google.com/drive/1Uswdel4EIsmdtlilbX06L-6_AVDelJYu?usp=sharing`

```
import numpy as np

x = np.array([  1., 2., 3., 4., 5., 6., 7., 8., 9., 10.,   12. ])
y = np.array([  0., 0., 0., 0., 0., 0., 0., 0., 0.,  0., 144. ])
```

```python
c = np.polynomial.polynomial.polyfit(x,y,10)

print( np.polynomial.polynomial.polyval(0.,c) )
print( 144./11. )
```

```
13.090909081604384
13.090909090909092
```