EX.1.2.30, Sauer3

Assume that Fixed-Point Iteration is applied to a twice continuously differentiable function $g(x)$ and that $g'(r) = 0$ for a fixed point. Show that if FPI converges to r, the error obeys $\lim_{i\to\infty} e_{i+1}/(e_i)^2 = M$, where $M = |g''(r)|/2$.

EX.1.2.30, Sauer3, solution, Langou

Let g be twice continuously differentiable (in a neighborhood of r) such that $g(r) = r$, (i.e., r is a fixed point of g,) and $g'(r) = 0$. We consider fixed point iteration $x_{i+1} = g(x_i)$ with starting point x_0.
First of, we note that, since the assumptions of Theorem 1.6 are satisfied, we know that FPI converges locally. (I.e., no need to say "if FPI converges to r". FPI does converge to r provided that the starting point x_0 is close enough from r.)
By Taylor theorem with remainder (Theorem 0.8) for g centered at r for the point x_i, we know that there exists c_i in between x_i and r such that

$$g(x_i) = g(r) + (x_i - r)g'(r) + \frac{1}{2}g''(c_i)(x_i - r)^2.$$

We use the fact that $g(r) = r$, $g'(r) = 0$ and $x_{i+1} = g(x_i)$ to get

$$x_{i+1} - r = \frac{1}{2}g''(c_i)(x_i - r)^2.$$

We define the sequence of forward error $e_i = |x_i - r|$ for all i, and so we obtain

$$e_{i+1} = \frac{1}{2}|g''(c_i)|e_i^2.$$

We assume that e_i is not zero, and get

$$\frac{e_{i+1}}{e_i^2} = \frac{1}{2}|g''(c_i)|. \qquad (1)$$

(Note that if, for a given iteration i, e_i is zero, well, we are kind of done, so this is not really a problem.)
Now.
Since, when i goes to infinity, x_i converges towards r (see either our initial remark, or the assumption of the problem), and since c_i in between x_i and r, c_i converges towards r.
Since, when i goes to infinity, c_i converges towards r and g'' is continuous (see our initial assumptions on g), $g''(c_i)$ converges towards $g''(r)$.
Therefore we have

$$\lim_{i\mapsto\infty} \frac{1}{2}|g''(c_i)| = \frac{1}{2}|g''(r)|.$$

If we look back at Equation (1), since we proved that the right-hand side converges when i goes to infinity, this implied that the left-hand side converges when i goes to infinity as well. Therefore

$$\lim_{i\mapsto\infty} \frac{e_{i+1}}{e_i^2} = M < \infty.$$

Moreover we have that

$$M = \frac{1}{2}|g''(r)|.$$

Comments: We proved that, for a function g twice continuously differentiable (in a neighborhood of r) such that $g(r) = r$ and $g'(r) = 0$, then the convergence of FPI is *locally superlinear*. If $g''(r) \neq 0$, the convergence is *locally quadratic*. (See Definition 1.10.) If $g''(r) = 0$, the convergence is faster than quadratic.