Math 5718–001: Exam 1 Name: ______________________________

Wednesday, September 27, 2023

Instructions: Write your solutions in the space provided. This exam is closed book and closed notes.

(1) Let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$. Two true/false problems. If "True", say so and prove it. If "False", say so and give a counterexample and give a counterexample with $E = F = G = \mathbb{R}^2$.

(a) True or False: $\text{Range}(g) \subseteq \text{Range}(g \circ f)$

(b) True or False: $\text{Range}(g \circ f) \subseteq \text{Range}(g)$

(2) Let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$. Two true/false problems. If "True", say so and prove it. If "False", say so and give a counterexample with $E = F = G = \mathbb{R}^2$.

(a) True or False: $\text{Null}(f) \subseteq \text{Null}(g \circ f)$

(b) True or False: $\text{Null}(g \circ f) \subseteq \text{Null}(f)$

(3) Let $v_1, \ldots, v_n$ in V and $T \in \mathcal{L}(V, W)$. Assume that $v_1, \ldots, v_n$ spans V and that T is surjective. Prove that $Tv_1, \ldots, Tv_n$ spans W.

(4) Let $v_1, \ldots, v_n$ in V and $T \in \mathcal{L}(V, W)$. Assume that $v_1, \ldots, v_n$ is linearly independent and that T is injective. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent.

(5) Let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$. Prove that

$$\operatorname{Range}(g \circ f) = \operatorname{Range}(g) \iff \operatorname{Null}(g) + \operatorname{Range}(f) = F.$$

(6) Let V be a finite dimensional vector. Let U and W be two subspaces of V. Show that there exists S, a subspace of V, such that $V = S \oplus U$ and $V = S \oplus W$ if and only if $\dim(U) = \dim(W)$.