EX.0.5.7, Sauer

a. Find the Taylor polynomial of degree 4 for $f(x) = \ln(x)$ about the point $x = 1$.

b. Use the result of (a) to approximate $f(0.9)$ and $f(1.1)$.

c. Use the Taylor remainder to find an error formula for the Taylor polynomial. Give error bounds for each of the two approximations made in part (b). Which of the two approximations in part (b) do you expect to be closer to the correct value?

d. Use a calculator to compare the actual error in each case with your error bound from part (c).

EX.0.5.7, Sauer, solution, Langou

- Only turning the Python code is not a good answer.
- The copy-paste from this PDF to python code does not work great. It is better to copy-paste from colab.
- The Colab Jupyter Notebook is available at: `https://colab.research.google.com/drive/11-RqoQ1hYF1tFyqU_vIXGVTQsVm57Lt1`.
- The Python code and its ouput is at the end of this document.

a. We recall the formula for the Taylor polynomial of degree 4 of a function f at x_0:

$$p_4(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f^{(3)}(x_0)(x - x_0)^3 + \frac{1}{4!}f^{(4)}(x_0)(x - x_0)^4.$$

We want to use this formula for $f(x) = \ln x$ and $x_0 = 1$. First, we compute the first four derivatives of $f(x) = \ln x$:

$$f(x) = \ln x, \quad f'(x) = (-1)^0\frac{0!}{x}, \quad f''(x) = (-1)^1\frac{1!}{x^2}, \quad f^{(3)}(x) = (-1)^2\frac{2!}{x^3}, \quad f^{(4)}(x) = (-1)^3\frac{3!}{x^4}.$$

Then, we evaluate these derivatives at $x_0 = 1$:

$$f(1) = 0, \quad f'(1) = 0!, \quad f''(1) = -1!, \quad f^{(3)}(1) = 2!, \quad f^{(4)}(1) = -3!.$$

Finally, we substitute $f(1)$, $f'(1)$, $f''(1)$, $f^{(3)}(1)$, $f^{(4)}(1)$ in the formula for $p_4(x)$ and obtain:

$$p_4(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4.$$

b. $p_4(0.9) = -0.105358333333\ldots$ and $p_4(1.1) = 0.095308333333\ldots$

c. We have $x_0 = 1$. If x is any real number in $(0, \infty)$, then the function $f(x) = \ln x$ is $k+1$ times continuously differentiable in either the interval $[x, x_0]$ or $[x_0, x]$, whichever makes sense. The assumptions of the Taylor's Theorem with Remainder (Theorem 0.8 page 21) are therefore satisfied.

Therefore the theorem tells us that there exists c in between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f^{(3)}(x_0)(x - x_0)^3 + \frac{1}{4!}f^{(4)}(x_0)(x - x_0)^4 + \frac{1}{5!}f^{(5)}(c)(x - x_0)^5.$$

In other words, using $x_0 = 1$, $p_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$ and the fact that $f^{(5)}(x) = (-1)^4\left(\frac{4!}{x^5}\right)$, we obtain that there exists c in between x and 1 such that:

$$f(x) = p_4(x) + \frac{1}{5c^5}(x-1)^5.$$

This enables to control the error made by approximating $f(x)$ with $p_4(x)$ with the formula:

$$|f(x) - p_4(x)| \leq \frac{1}{5c^5}|x-1|^5, \quad \text{where } c \text{ is in between } x \text{ and } 1.$$

Assume that $x > 1$, (so we use the formula $p_4(x)$ to approximate $f(x)$ when x is on the right of 1,) then since c is in the interval $(1, x)$, we have $1 < c < x$ and so $x^{-5} < c^{-5} < 1$, the left part is of interest:

$$\frac{1}{c^5} < 1$$

and so we obtain a new error bound (without c) as:

$$|f(x) - p_4(x)| \leq \frac{1}{5}(x-1)^5.$$

Assume that $0 < x < 1$, (so we use the formula $p_4(x)$ to approximate $f(x)$ when x is on the left of 1,) then since c is in the interval $(x, 1)$, we have $x < c < 1$ and so $1 < c^{-5} < x^{-5}$, the left part is of interest:

$$\frac{1}{c^5} < \frac{1}{x^5}$$

and so we obtain a new error bound (without c) as:

$$|f(x) - p_4(x)| \leq \frac{1}{5x^5}(1-x)^5.$$

We conclude by looking at the relative error bound:

$$\frac{|f(x) - p_4(x)|}{|f(x)|} \leq \begin{cases} \frac{1}{5x^5|\ln x|}(1-x)^5, & \text{if } 0 < x < 1 \\ \\ \frac{1}{5\ln x}(x-1)^5, & \text{if } 1 < x \end{cases}.$$

For $x = 0.9$, the bound tells us that

$$\text{rel\_err}(0.9) = |f(0.9) - p_4(0.9)|/|f(0.9)| \leq 3.2e-5.$$

For $x = 1.1$, the bound tells us that

$$\text{rel\_err}(1.1) = |f(1.1) - p_4(1.1)|/|f(1.1)| \leq 2.1e-5.$$

We expect $p_4(1.1)$ to be relatively closer to $f(1.1)$ than $p_4(0.9)$ is to $f(0.9)$ (However this does not have to be true.)

d.

$$\text{rel\_err}(0.9) = |f(0.9) - p_4(0.9)|/|f(0.9)| \approx 2.1e-5$$
$$\text{rel\_err}(1.1) = |f(1.1) - p_4(1.1)|/|f(1.1)| \approx 1.9e-5$$

We check that our upper bounds "works". For $x = 0.9$, the true error, 2.1e-5, is less than the error bound 3.2e-5. For $x = 1.1$, the true error, 1.9e-5, is less than the error bound 2.1e-5.

```
from math import log
```

```
p4 = lambda x : (x-1) \
              - (1./2.) * (x-1)**2 \
              + (1./3.) * (x-1)**3 \
              - (1./4.) * (x-1)**4
```

```
print( p4(0.9) )
print( p4(1.1) )
```

```
-0.10535833333333332
0.09530833333333343
```

```
def our_bound_on_the_error(x):
  if(  0 < x < 1 ): y = 1. / 5. / (x**5) / ( - log(x) ) * ( 1. - x )**5
  if(  1 < x ): y = 1. / 5. / ( log(x) )* ( x - 1. )**5
  return y

print( f"{our_bound_on_the_error( 0.9 ):.2e}" )
print( f"{our_bound_on_the_error( 1.1 ):.2e}" )
```

```
3.21e-05
2.10e-05
```

```
print( f"{abs( log(0.9) - p4(0.9) ) / abs( log(0.9) ):.2e}" )
print( f"{abs( log(1.1) - p4(1.1) ) / abs( log(1.1) ):.2e}" )
```

```
2.07e-05
1.94e-05
```