EX.1.2.15, Sauer3

Which of the following three Fixed-Point Iteration converge to $\sqrt{5}$? Rank the ones that converge from fastest to slowest.

$$(a)\quad x \longrightarrow \frac{4}{5}x + \frac{1}{x}; \quad (b)\quad x \longrightarrow \frac{x}{2} + \frac{5}{2x}; \quad (c)\quad x \longrightarrow \frac{x+5}{x+1}.$$

EX.1.2.15, Sauer3, solution, Langou

a. We consider $\left(g_1 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{4}{5}x + \frac{1}{x}\right)$, first we have $g_1(\sqrt{5}) = \frac{4}{5}\sqrt{5} + \frac{1}{\sqrt{5}} = \sqrt{5}$, so $r = \sqrt{5}$ is a fixed point of g_1. We compute $g_1'(x)$ and get $g_1'(x) = \frac{4}{5} - \frac{1}{x^2}$, therefore $g_1'(\sqrt{5}) = \frac{3}{5}$. We conclude that Fixed Point Iteration on g_1 is locally convergent to the fixed point $r = \sqrt{5}$. The convergence is linear with rate $\frac{3}{5}$.

b. We consider $\left(g_2 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{x}{2} + \frac{5}{2x}\right)$, first we have $g_2(\sqrt{5}) = \frac{\sqrt{5}}{2} + \frac{5}{2\sqrt{5}} = \sqrt{5}$, so $r = \sqrt{5}$ is a fixed point of g_2. We compute $g_2'(x)$ and get $g_2'(x) = \frac{1}{2} - \frac{5}{2}\frac{1}{x^2}$, therefore $g_2'(\sqrt{5}) = 0$. We conclude that Fixed Point Iteration on g_2 is locally convergent to the fixed point $r = \sqrt{5}$. The convergence is superlinear. (See pg. 42 #1.2.EX.30 which is Exercise #6 of this homework where we prove that the convergence is at least quadratic.)

c. We consider $\left(g_3 : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{x+5}{x+1}\right)$, first we have $g_2(\sqrt{5}) = \frac{\sqrt{5}+5}{\sqrt{5}+1} = \sqrt{5}$, so $r = \sqrt{5}$ is a fixed point of g_3. We compute $g_3'(x)$ and get $g_3'(x) = -\frac{4}{(x+1)^2}$, therefore $g_3'(\sqrt{5}) = -\frac{4}{(x+1)^2} \approx -0.38$. We conclude that Fixed Point Iteration on g_3 is locally convergent to the fixed point $r = \sqrt{5}$. The convergence is linear with rate ≈ 0.38.

All three Fixed-Point Iteration (g_1, g_2, and g_3) converge to $\sqrt{5}$. The fastest convergence is with (b) g_2 (superlinear convergence), then comes (c) g_3 (linear convergence with rate ≈ 0.38), then comes (a) g_1 (linear convergence with rate 0.6).