EX.1.5.7, Sauer3

Consider the following four methods for calculating $2^{1/4}$, the fourth root of 2.

(i) Bisection Method applied to $f(x) = x^4 - 2$

(ii) Secant Method applied to $f(x) = x^4 - 2$

(iii) Fixed Point Iteration applied to $g(x) = \frac{x}{2} + \frac{1}{x^3}$

(iv) Fixed Point Iteration applied to $g(x) = \frac{x}{3} + \frac{1}{3x^3}$

For each of these methods,

a. Determine the speed of convergence

b. Are there any methods that will converge faster than all the above methods? If so, name it.

EX.1.5.7, Sauer3, solution, Langou

a. The speed of convergence is:

(i) The Bisection Method will have $S = 1/2$ for the rate of convergence since it cuts the interval in half at every step.

(ii) It is clear that $f(2^{1/4}) = 0$. Moreover, $f'(x) = 4x^3$ and $f'(2^{1/4}) \neq 0$. Thus the Secant Method will obtain superlinear convergence.

(iii) We first check that $x = 2^{1/4}$ is indeed a fixed point for $g(x)$.

$$g(2^{1/4}) = \frac{2^{1/4}}{2} + \frac{1}{(2^{1/4})^3}$$
$$g(2^{1/4}) = 2^{-3/4} + \frac{1}{2^{3/4}}$$
$$g(2^{1/4}) = 2^{1-\frac{3}{4}}$$
$$g(2^{1/4}) = 2^{\frac{1}{4}}$$

Note that $g'(x) = \frac{1}{2} - \frac{3}{x^4}$ and $g'(2^{1/4}) = -1$. Thus Fixed Point Iteration on this function may not converge. (The convergence theorem does not apply.)

(iv) We first check that $x = 2^{1/4}$ is indeed a fixed point for $g(x)$.

$$g(2^{1/4}) = \frac{2^{1/4}}{3} + \frac{1}{3(2^{1/4})^3}$$
$$g(2^{1/4}) = \frac{2}{3(2^{3/4})} + \frac{1}{3(2^{3/4})}$$
$$g(2^{1/4}) = \frac{3}{3(2^{3/4})}$$
$$g(2^{1/4}) = 2^{-\frac{3}{4}}$$

Thus this is not a fixed point for $g(x)$ and the Fixed Point Iteration Method is not applicable.

The fastest convergence will be obtained via the Secant Method followed by the Bisection Method. The first Fixed Point Iteration Methods may fail to converge. The second Fixed Point Iteration Methods does not converge to $2^{1/4}$.

b. Let $f(x) = x^4 - 2$, then note that $f'(x) = 4x^3$ and $f'(2^{1/4}) \neq 0$. Therefore we can apply Theorem 1.11 with Newton's Method and obtain quadratic convergence.