EX.2.4.2.a, Sauer3

Find the $PA = LU$ factorization (using partial pivoting) of the following matrix:

$$\text{(a)} \quad \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{pmatrix}.$$

EX.2.4.2.a, Sauer3, solution, Langou

Colab: `https://colab.research.google.com/drive/1sOhfk0aQ4j6b1sVMLZ_YVFi_S-mX84h3`

We perform the PA = LU factorization

$$\begin{array}{cccc}
 & P & L & U \\
 & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{pmatrix} \\
\begin{array}{c} R_0 \leftrightarrow R_1 \\ \rightsquigarrow \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix} \\
\begin{array}{c} R_1 \leftarrow R_1 - \frac{1}{2}R_0 \\ R_2 \leftarrow R_2 + \frac{1}{2}R_0 \\ \rightsquigarrow \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{pmatrix} \\
\begin{array}{c} R_2 \leftrightarrow R_1 \\ \rightsquigarrow \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\
\begin{array}{c} R_2 \leftarrow R_2 - \frac{1}{3}R_1 \\ \rightsquigarrow \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} & \begin{pmatrix} 2 & 1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}
\end{array}$$

We obtain

$$\boxed{P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 2 & 1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}}$$

We can check that

- P is a permutation matrix
- U is upper triangular,
- L is lower unit triangular such that all entries below the diagonal are less or equal to 1,

- P times A is L times U, indeed

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

not needed for full credit.

We often store the matrices L and U in place of A. In the end, the unit diagonal of L is not stored. (Because this is all ones, we do not need to store these ones, we know there are here.) And then the zeros in the upper part of L are not stored, and the the zeros in the lower part of U are not stored. And then P is to store as a permutation of the indexes from 0 to $n-1$.
The algorithm would run as follows:

$$\begin{array}{ccc}
 & P & L\backslash U \\
 & \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{pmatrix} \\[2ex]
\begin{array}{c} R_0 \leftrightarrow R_1 \\ \rightsquigarrow \end{array} & \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix} \\[2ex]
\begin{array}{c} R_1 \leftarrow R_1 - (\frac{1}{2})R_0 \\ R_2 \leftarrow R_2 - (-\frac{1}{2})R_0 \\ \rightsquigarrow \end{array} & \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 & 1 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \end{pmatrix} \\[2ex]
\begin{array}{c} R_1 \leftrightarrow R_2 \\ \rightsquigarrow \end{array} & \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 & -1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\[2ex]
\begin{array}{c} R_2 \leftarrow R_2 - (\frac{1}{3})R_1 \\ \rightsquigarrow \end{array} & \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 & -1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix}
\end{array}$$