Category Theory

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Contents

Remark. **Caution!** A good deal of these notes were copied and pasted from my thesis, typos and nonsequiters abound from the conversion process.

These notes draw on many sources, comingling their observations freely. I hope to provide more comprehensive coverage than any individual source on those aspects of the theory that are especially tricky for me. See inline citations and references.

Category theory provides a unifying language for much of modern math. It provides a "zoomed out" picture of a collection of structures, focusing not on individual objects, but on their relations with one another instead. It also presents a challange to set-theoretic mathematicians: the canonical example of a category is **Set**, the collection of all sets. In **ZFC+FOL**, this category is undefinable.[1]

1 Basics

Definition 1.1. A **category C** consists of the following data:

[1] In particular, its formation is prevented by the axiom of regularity which was included in **ZFC** to avoid the paradoxes of Burali-Forti and Russell. The discovery of said paradoxes motivated Bertrand Russell to invent something he called the "theory of types" (see **??**).

- a collection of **objects**, denoted $\mathrm{Obj}\,\mathbf{C}$,
- for each pair of objects $A, B \in \mathrm{Obj}\,\mathbf{C}$, a collection of **arrows** (or **morphisms**) between them, denoted $\mathrm{Hom}_{\mathbf{C}}(A, B)$,
- for each object $A \in \mathrm{Obj}\,\mathbf{C}$, a distinguished arrow $\mathrm{id}_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$ called the **identity**, and
- for each triple of objects $A, B, C \in \mathrm{Obj}\,\mathbf{C}$, an operation $\circ : \mathrm{Hom}_{\mathbf{C}}(B, C) \times \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{C}}(A, C)$ called **composition**.

These data are subject to the following axioms:

1. composition is associative, and
2. the identity acts as a unit for composition.

When the category in question is clear from context, one writes $f : A \to B$ for $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$.[2]

Definition 1.2. If $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$, then A is the **domain** or **source** of f and B is the **codomain** or **target** of f.

The presentation of the following examples owes a lot to [**?**].

Example 1.3. The following categories are familiar to the student of mathematics:

- **Set**: The category with sets as objects, functions as morphisms, the usual composition of functions, and identity functions.
- **FinSet**: The category with finite sets as objects, functions as morphisms, the usual composition of functions, and identity functions.
- **Grp**: The category of groups with group homomorphisms as morphisms. Note that the identity function of sets is the required identity morphism and that for any homomorphsisms $\ : G \to H$ and $\ : H \to I$, the usual composition of functions defines a homomorphism $\ \circ \ : G \to I$.
- **AbGrp**: The category of abelian groups (this can be considered a **subcategory** of **Grp**).

[2]We use $\to$ to distinguish this from the type of functions $A \to B$.

- F**-Vect**: The category of vector spaces over a field F with linear transformations as morphisms.
- $\mathbf{Rep}_G^F$: The category of (finite-dimensional) representations (V, ρ) of a group G over F with equivariant maps as morphisms.

Example 1.4. Morphisms don't have to be functions:

- For any group $(G, \circ, e)$, there is a corresponding category $\underline{G}$ with a single object (denoted $*$) and $\mathrm{Hom}_{\underline{G}}(*, *) := G$. Composition is given by the group operation.
- For any set or type A, the **discrete category on** A has as objects the members or elements as A, and no other arrows except the requisite identities.
- For any preorder $(A, \leq)$ (a collection with a reflexive, transitive binary relation), there is a category $\mathbf{Pre}(A)$ which has as objects the elements of A and an arrow $a \to b$ just if ab. Conversely, any category with at most one arrow between any two objects defines a preorder.
- There is a category $\mathbf{1}$ with one object and only the identity arrow, called the **unit** or (foreshadowing **??**) **terminal** category.
- For a topological space X, the category PX of paths on X has as objects the points of X and for $x, y \in X$, $\mathrm{Hom}_{PX}(x, y)$ is the set of paths (continuous functions $[0, 1] \to X$) from x to y. Composition of paths p, q is written $p + q$. There is a zero path 0_x $(t \mapsto x)$ which serves as the identity.
- The fundamental groupoid $\Pi(X)$ of a space (X, τ) is the category with points of X as objects and equivalence classes under homotopy (rel endpoints) of paths from $x \in X$ to $y \in X$ as morphisms $\mathrm{Hom}_{\Pi(X)}(x, y)$. Composition is given by path concatenation.

Example 1.5. These preorders are commonly viewed in the context of category theory:

- The category ω is the preorder category $\mathbf{Pre}(\mathbb{N})$, that is, it has as objects the natural numbers, and arrows $n \to \mathsf{succ}\, n$ (plus the requisite composites $n \to m$ where $n < m$).

- For a topological space (X, τ), the category $\mathrm{Open}(X) := \mathbf{Pre}(\tau)$ is the preorder category on the open subsets where $U \leq V$ iff $U \subseteq V$ (this preorder has the rich structure of a Boolean algebra).

Definition 1.6. An **isomorphism** is an arrow $f : A \to B$ such that there exists an arrow $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Such a g is the **inverse** of f.

Just as with functions, inverses (should they exist) are unique.

Definition. A groupoid is a category in which every morphism is an isomorphism.

Example. If the connected components of (X, τ) consist of single points, then

$$\mathrm{Hom}_{\Pi(X)}(x, y) = \begin{cases} \varnothing & \text{if } x \neq y \\ \{\mathrm{id}_x\} & \text{if } x = y \end{cases}$$

A groupoid with this property is called discrete.

For any convex subset X of a normed vector space V, the straight line homotopy gives an equivalence between all paths in $PX(x, y)$. Therefore, $\mathrm{Hom}_{\Pi(X)}(x, y)$ has exactly one object for all $x, y \in X$. A groupoid with this property is called 1-connected and a tree groupoid. If $\Pi(X)$ is a tree groupoid, then X is path-connected. If every path-component of X is 1-connected, then X is simply connected.

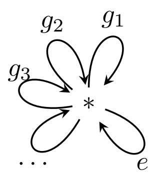

Figure 1: A schematic of the category $\underline{G}$ for a group G with identity e and elements $g_1, g_2, \ldots$. Compositions and inverses not shown.

Definition 1.7. A subcategory D of C is full if for all objects x, y of D, $\mathrm{Hom}_D(x, y) = \mathrm{Hom}_C(x, y)$. A subcategory is called wide if $\mathrm{Obj}(D) = \mathrm{Obj}(C)$.

Example 1.8. Full subcategories of **Top** include Hausdorff spaces, metrizable spaces, and compact spaces. Wide subcategories of **Top** include those with only open maps, only closed maps, or only isomorphisms.

Definition 1.9. A commutative diagram is a way to visualize equations between arrows involving composition. Technically a diagram in $\mathbf{C}$ is a directed graph with vertices labeled by $\operatorname{Obj}\mathbf{C}$ and edges e from A to B labeled by arrows in $\operatorname{Hom}_{\mathbf{C}}(A, B)$. A diagram **commutes**, or is commutative, if the composition of the arrows labeling the edges of any two directed paths with the same endpoints are equal.

Example 1.10. If $f : A \to B$, $g : B \to C$, $h : A \to C$, and $g \circ f = h$, then the following diagram commutes, making a **commutative triangle**:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

If $f : A \to B$, $g : B \to D$, $h : A \to C$, $i : C \to D$, and $g \circ f = i \circ h$, then the following diagram commutes, making a **commutative square**:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"] & B \arrow[d, "g"] \\ C \arrow[r, "i"] & D \end{tikzcd}$$

Definition 1.11. A morphism $f : X \to Y$ is

- a **monomorphism** is left-cancellable, i.e. for every $h, k : W \to X$, $f \circ h = f \circ k$ implies $h = k$, and
- an **epimorphism** is right-cancellable, i.e. for every $h, k : Y \to Z$, $h \circ f = k \circ h$ implies $h = k$.

If $X \xrightarrow{f} Y \xrightarrow{g} X$ are such that $g \circ f = \mathrm{id}_X$, then g is a **retraction** of f and f is a **section** of g. In this case, f is a monomorphism, g is an epimorphism, and they are both called **split**.

Lemma 1.12. ◦ The composition of monics is monic. If $g \circ f$ is monic, so is f.

- ∘ The composition of epis is epic. If $g \circ f$ is epic, so is g.

Definition. For an object A of a category $\mathbf{C}$, we define the endomorphisms of A to be

$$\mathrm{End}_{\mathbf{C}}(A) \coloneqq \mathrm{Hom}_{\mathbf{C}}(A, A)$$

Definition. For an object A of a category $\mathbf{C}$, we define the automorphisms of A to be

$$\mathrm{Aut}_{\mathbf{C}}(A) \coloneqq \{f \in \mathrm{End}_{\mathbf{C}}(A) | f \text{ is an isomorphism}\}$$

Remark. The endomorphisms of an object A form a monoid under composition where the identity is id_A. The automorphisms of A are a submonoid of the endomorphisms, and they form a group where the inverse of an element is given by the assumption that each element is an isomorphism. Thus, we may refer to the endomorphism monoid or automorphism group of an object.[3]

Example. Just as with categories and functors, endomorphisms and automorphisms generalize common patterns:

- $\mathrm{Aut}_{\mathbf{Set}}(A) = \mathfrak{S}_A$, the symmetric group on A.
- $\mathrm{Aut}_{F\text{-}\mathbf{Vect}}(V) = \mathrm{GL}(V)$, the general linear group.
- In **Top**, the category of topological spaces, the automorphism group is the homeomorphism group.

1.1 Functors

Definition 1.13. A **functor** F between categories $\mathbf{C}$ and $\mathbf{D}$ consists of the following data:

- a map $F_0 : \mathrm{Obj}\,\mathbf{C} \to \mathrm{Obj}\,\mathbf{D}$, and
- for each pair of objects $A, B \in \mathrm{Obj}\,C$, a map $F_1 : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(F_0(A), F_0(B))$.

These data are subject to the following axioms:

[3] Following the above remarks on groups-as-categories, we might even call these the endomorphism or automorphism category, though this description is less useful because it is less precise.

1. functors preserve composition
2. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all $A \in \mathrm{Obj}\, C$.

We generally leave off the subscripts and parentheses when possible, denoting the application by simply FA or Ff. A functor F from $\mathbf{C}$ to $\mathbf{D}$ may be denoted $F : \mathbf{C} \to \mathbf{D}$. We may define functors with or without names using the following notation:

$$\begin{aligned} \mathbf{C} &\longrightarrow \mathbf{D} \\ A &\longmapsto_0 \dots \\ f &\longmapsto_1 \dots \end{aligned}$$

Example 1.14. For any category $\mathbf{C}$, there is an **identity functor** $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ which acts as the identity on objects and morphisms. The composition of functors is associative, and there is a category **Cat** of "small" categories (it doesn't include itself, for instance).

Example 1.15. Since continuous maps preserve paths, homotopy classes, and composition, the fundamental groupoid Π defines a functor $\mathbf{Top} \to \mathbf{Grpd}$.

Definition 1.16. There is a product in **Cat**; the product $\mathbf{C} \times \mathbf{D}$ has as object pairs of objects (A, B) for $A \in \mathbf{C}$ and $B \in \mathbf{D}$ and similarly for arrows.

Example 1.17. For any category, there is a **diagonal functor**

$$\begin{aligned} \Delta : \mathbf{C} &\longrightarrow \mathbf{C} \times \mathbf{C} \\ A &\longmapsto_0 (A, A) \\ f &\longmapsto_1 (f, f) \end{aligned}$$

Example 1.18. For each category of algebraic objects where the morphisms are the corresponding type of homomorphism, there is a **forgetful functor**, generally denoted U, which takes sets with some structure to their underlying sets and homomorphisms to the corresponding maps of sets. For instance, there is a forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$.

More generally, if $\mathbf{C}$ is like $\mathbf{D}$ but with "more structure" (for instance, spaces with a group action, rather than just spaces), then there is a forgetful functor $\mathbf{C} \to \mathbf{D}$. For example, there is a forgetful functor $\mathbf{Rep}_G^F \to F\text{-}\mathbf{Vect}$.

Example. For many of these "algebraic categories", there is a **free functor** from **Set** which assigns each set to the free object on that set. For instance, there is a free group functor $\mathbf{Set} \to \mathbf{Grp}$ and a free F-vector space functor $\mathbf{Set} \to F\text{-}\mathbf{Vect}$.

Just as a group $(G, \circ, e)$ can be seen as a category $\underline{G}$ with a single object, a G-action can be seen as a functor. A permutation representation is a functor $\underline{G} \to \mathbf{Set}$, whereas a linear representation is a functor $\underline{G} \to F\text{-}\mathbf{Vect}$.

Example 1.19. If $\mathbf{C}$ has binary coproducts, then for any fixed $A, B \in \operatorname{Obj} \mathbf{C}$, one can define the following functors:

$$\begin{aligned} \mathbf{C} &\longrightarrow \mathbf{C} \\ X &\longmapsto_0 A + X \\ f &\longmapsto_1 \mathrm{id}_A + f = [i_1, i_2 \circ f] \end{aligned} \qquad\qquad \begin{aligned} \mathbf{C} &\longrightarrow \mathbf{C} \\ Y &\longmapsto_0 Y + B \\ g &\longmapsto_1 g + \mathrm{id}_B = [i_1 \circ g, i_2] \end{aligned}$$

These two functors interact well, meaning in part that they extend to a bifunctor (**??**)

$$\begin{aligned} \mathbf{C} \times \mathbf{C} &\longrightarrow \mathbf{C} \\ (X, Y) &\longmapsto_0 X + Y \\ (f, g) &\longmapsto_1 f + g \end{aligned}$$

where $(f + g) \circ \mathsf{inl} = f$ and $(f + g) \circ \mathsf{inr} = g$.

Definition 1.20. A **bifunctor** is a functor out of a binary product category (see **??**).

Since functors preserve sources, targets, and composition, they preserve commutative diagrams. If f, g, h form a commutative triangle in $\mathbf{C}$, then their images under $F : \mathbf{C} \to \mathbf{D}$ do in $\mathbf{D}$:

$$
\begin{tikzcd}
A \arrow[rr, mapsto, "F"] \arrow[dr, "f"] \arrow[dd, "h"] & & FA \arrow[dr, "Ff"] \arrow[dd, "Fh"] & \\
& B \arrow[rr, mapsto, "F", crossing over] \arrow[dl, "g"] & & FB \arrow[dl, "Fg"] \\
C \arrow[rr, mapsto, "F"] & & FC &
\end{tikzcd}
$$

Note however that it is possible that $FA = FB = FC$, as in a functor to a category with a single object. However, the equalties between composites still hold. One consequence is that functors preserve isomorphisms, sections, and retractions (Emily Riehl calls this "arguably the first lemma of category theory" [**?**, pp. 18]).

Remark. Functors were first defined in the context of algebraic topology. Emmy Noether discovered that Betti numbers (an important invariant of topological spaces) were the ranks of certain free abelian groups associated with those spaces. Modern homology and homotopy theory grew out of the realization that not only did spaces have these isomorphism-invariant groups associated to them, but the continuous maps between such spaces *induced group homomorphisms* between them.

The most useful invariants in topology were those that gave a "picture" of topological spaces *and their morphisms* in better-understood categories. Category theory expresses the fundamental nature of such invariants as functors, and enables the abstract study of their properties.

1.2 Natural transformations

Definition. A **natural transformation** α between two functors $F, G : \mathbf{C} \to \mathbf{D}$ is a collection $\{\alpha_A : FA \to GA\}_{A \in \mathrm{Obj}\,\mathbf{C}}$ of morphisms in $\mathbf{D}$ indexed by the objects of $\mathbf{C}$, called the **components** of α, subject to the requirement that the following diagram commutes for each $A, B \in \mathrm{Obj}\,\mathbf{C}$ and $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$:

$$
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\
GA \arrow[r, "Gf"] & GB
\end{tikzcd}
$$

We denote a natural transformation α from F to G by $\alpha : F \Rightarrow G$. A natural transformation is a **natural isomorphism** if each component is an isomorphism.

Natural transformations are intimidating at first. Luckily, it often suffices to reason about them componentwise (as in the definition of a natural isomorphism).

Since functors preserve diagrams, the image of a functor $F : \mathbf{C} \to \mathbf{D}$ can be seen as a "picture of shape $\mathbf{C}$ in $\mathbf{D}$". A natural transformation, then is a map between such pictures [**?**, pp. 16]. Reusing the same commutative triangle as above,

$$\begin{tikzcd}
FA \arrow[rr, "\alpha_A"] \arrow[dr, "Ff"] \arrow[dd, "Fh"'] & & GA \arrow[dr, "Gf"] \arrow[dd, "Gh"] & \\
& FB \arrow[rr, "\alpha_B"] \arrow[dl, "Fg"] & & GB \arrow[dl, "Gg"] \\
FC \arrow[rr, "\alpha_C"] & & GC &
\end{tikzcd}$$

Definition. The componentwise composition of natural transformations is again a natural transformation. To see this, just paste the two commutative squares together along their border. This composition is associative, and there is an identity natural transformation, and so we can form the **functor category** of functors $\mathbf{C} \to \mathbf{D}$, denoted $\mathbf{D}^{\mathbf{C}}$.

Example. The category of representations of G can be seen as the functor category $F\text{-}\mathbf{Vect}^{\underline{G}}$. In this context, equivariant maps are exactly natural transformations.

The abelianization of a group is functorial, and the projection of a group onto its abelianization is a natural transformation from the identity functor to $(-)^{\mathbf{ab}}$.

There is a canonical injective linear map from a vector space to its dual. This map extends by composition to the double dual, and so defines a natural transformation from the identity functor to $(-)^{**}$.

1.3 Duality

Consider a preorder $(A, \leq)$ (that is, a set or type A together with a reflexive, transitive relation $\leq$). Its **opposite preorder** is $(A, \leq^{\mathsf{op}})$ where

$$a \leq^{\mathsf{op}} b \Longleftrightarrow b \leq a$$

Consider a group $(G, \,\cdot\,, e)$. Its **opposite group** has the operation

$$g \cdot^{\mathsf{op}} h :\equiv h \cdot g$$

Now, suppose $(A,)$ has all finite meets. Immediately, $(A, ^{\mathsf{op}})$ has all finite joins. In general, if we know some theorem T about $(A,)$ or $(G, \,\cdot\,, e)$, it seems like there should be a purely mechanical process via which one could discover an appropritate "opposite theorem" that applies to $(A, ^{\mathsf{op}})$ or $(G, \,\cdot^{\mathsf{op}}, e)$.

These are instances of the notion of duality in category theory. It plays an economizing role: one *can*, in fact, state "opposite" theorems via a purely mechanical process, and prove them trivially.

Definition 1.21. The opposite category of $\mathbf{C}$, denoted $\mathbf{C}^{\mathsf{op}}$ has

- objects $\mathrm{Obj}\,\mathbf{C}^{\mathsf{op}} :\equiv \mathrm{Obj}\,\mathbf{C}$,
- arrows $f^{\mathsf{op}} \in \mathrm{Hom}_{\mathbf{C}^{\mathsf{op}}}(B, A)$ for each $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$,
- and composition $f^{\mathsf{op}} \circ^{\mathsf{op}} g^{\mathsf{op}} :\equiv (g \circ f)^{\mathsf{op}}$.

One usually identifies $(\mathbf{C}^{\mathsf{op}})^{\mathsf{op}}$ with $\mathbf{C}$.[4] The following might be called "The Fundamental Meta-theorem of Category Theory":

Theorem 1.22. If S is a statement about $\mathbf{C}$ in the first-order language of category theory (i.e. involving only objects, arrows, composition, and identity), then the statement S^{op} holds of $\mathbf{C}^{\mathsf{op}}$, where S^{op} is obtained from S by

1. switching the domain and codomain of all arrows, and
2. reversing composition.

[4] This is a bit more subtle in type theory! What one wants is a *judgmental* equality $(\mathbf{C}^{\mathsf{op}})^{\mathsf{op}} \equiv \mathbf{C}$. To make this possible, one must include a symmetrized version of the associativity axiom.

Usefully, isomorphisms are self-dual. This means objects are isomorphic in $\mathbf{C}$ if and only if they are so in $\mathbf{C}^{\mathsf{op}}$.

If D is a definition in the language of categories, there is a dual definition D^{op}, obtained in the same way as S^{op} above. If D defines a "foo", then the thing that D^{op} defines is usually called a "cofoo". From now on, dual definitions and statements will be introduced in pairs, and typeset like so:

- ∘ Definition
- ∘ Co-definition

1.4 Equivalence

1.5 Diagram chasing

See [**?**, 2.1] and [**?**, 1.6]. For a good example, see **??**.

Picture the following diagram:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[dr, "g"'] & \bullet \arrow[d, "h"] \arrow[dr, "j"] & \\
& \bullet \arrow[r, "k"'] & \bullet
\end{tikzcd}$$

Suppose we know that the two triangles commute, that is

$$h \circ f = g \qquad \text{and} \qquad k \circ h = j$$

How can we tell that the outer square commutes? We **diagram chase**, redrawing paths using established equalities:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] & \bullet \arrow[dr, "j"] & \\
& \bullet & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "f"] & \bullet \arrow[d, "h"] & \\
& \bullet \arrow[r, "k"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[dr, "g"'] & \bullet & \\
& \bullet \arrow[r, "k"'] & \bullet
\end{tikzcd}$$

We can write this argument just as easily with more standard algebraic **equational reasoning**:

$$\begin{aligned}
j \circ f &= (k \circ h) \circ f \\
&= k \circ (h \circ f) \\
&= k \circ g
\end{aligned}$$

Lemma 1.23. If f is an isomorphism, then the commuting of the following diagrams is equivalent:

$$\begin{tikzcd}
\bullet \arrow[d, "f"'] \arrow[dr, "h"] & \\
\bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[dr, "h"] & \\
\bullet \arrow[u, "f^{-1}"] \arrow[r, "g"'] & \bullet
\end{tikzcd}$$

By equational reasoning. (⇒) Precompose with f^{-1}:

$$\begin{aligned}
g &= g \circ \mathrm{id} \\
&= g \circ (f \circ f^{-1}) \\
&= (g \circ f) \circ f^{-1} \\
&= h \circ f^{-1}
\end{aligned}$$

(⇐) Postcompose with f. □

By diagram chase. (⇒) Consider the following diagram:

$$\begin{tikzcd}
\bullet \arrow[r, "f^{-1}"] \arrow[dr, "g"'] & \bullet \arrow[r, "f"] \arrow[d, "h"] & \bullet \arrow[dl, "g"] \\
& \bullet &
\end{tikzcd}$$

We want to show that $h \circ f^{-1} = g$, assuming $g \circ f = h$. Due to the hypothesis and the definition of isomorphism, we know that the following sequence of paths are equal:

$$\begin{tikzcd}
\bullet \arrow[r, "f^{-1}"] & \bullet \arrow[d, "h"] & \bullet \\
& \bullet &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "f^{-1}"] & \bullet \arrow[r, "f"] & \bullet \arrow[dl, "g"] \\
& \bullet &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet & & \bullet \arrow[dl, "g"] \\
& \bullet &
\end{tikzcd}$$

□

Lemma 1.24.

- Composable sequences with common codomain and beginning at an initial object are equal.
- Composable sequences with common domain and ending at a terminal object are equal.

2 Limits and colimits

Many constructions in categories are defined via a *universal property*. Such characterizations abound in everyday mathematics, as the examples illustrate. All of the definitions to follow are examples of limits and colimits. Ultimately, M-types are constructed as limits.

2.1 Instances

Definition 2.1.

- A **terminal object** is an object $\in \mathrm{Obj}\,\mathbf{C}$ such that for all $A \in \mathrm{Obj}\,\mathbf{C}$, there is exactly one arrow $f : A \to \top$.
- An **initial object** is an object $\bot \in \mathrm{Obj}\,\mathbf{C}$ such that for all $A \in \mathrm{Obj}\,\mathbf{C}$, there is exactly one arrow $f : \bot \to A$.

Example 2.2.

- **Set**:
  - Any singleton set $\{*\}$ is terminal.
  - The empty set $\varnothing$ is initial.[5]
- $\mathcal{U}$:
  - Any contractible type is terminal (for the similarity to the case of sets, note that any contractible type is equivalent to the canonical type with one element, $\mathbf{1}$.
  - The elimination rule for the empty type gives an arrow $\mathbf{0} \to X$ for any X. Is it unique? Apply function extensionality, then it suffices to show that it has the same output as any competitor $f : \mathbf{0} \to X$ on some $e : \mathbf{0}$. But using the elimination rule for $\mathbf{0}$ on e, conclude that it does.
- **Cat**:

[5] This is a little odd: in set-theoretic foundations, functions are actually defined as certain kinds of relations ("functional" ones). Accordingly, the unique function out of them empty set is the unique subset of $\varnothing \times X = \varnothing$, namely $\varnothing$ itself.

  - The category with one object and one arrow (its identity) is terminal.
  - The empty category is initial, for the same reasons as in **Set** or $\mathcal{U}$ (depending on which foundational system one is working in).

- **Grp**, **AbGrp**: The group with one element ("trivial group") is initial *and* terminal. Such an object is, in general, called a **zero object**.
- F-**Vect**: Similarly, the trivial vector space is a zero object (its underlying additive abelian group of vectors is the trivial group!)

Remark 2.3. In particular, for a terminal object , id is the only arrow $\to$.[6]

Lemma 2.4. Terminal objects are unique up to a specified isomorphism.

Proof. Suppose A and B are terminal objects. There are unique arrows $f : A \to B$ and $g : B \to A$. Then $g \circ f : A \to A$ and $f \circ g : B \to B$, but as per **??** $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. $\square$

Definition 2.5. Given two objects $A, B \in \mathrm{Obj}\,\mathbf{C}$,

- a **(binary) product** of A and B consists of an object $C \in \mathrm{Obj}\,\mathbf{C}$ together with arrows $p_1 : C \to A$ and $p_2 : C \to B$ satisfying the following universal property:

  For any other "candidate product" $D \in \mathrm{Obj}\,\mathbf{C}$ with arrows $q_1 : D \to A$ and $q_2 : D \to B$, there is a unique arrow $u : D \to C$ making the following diagram commute:

$$
\begin{tikzcd}
 & D \arrow[dl, "q_1"'] \arrow[d, dashed, "u"] \arrow[dr, "q_2"] & \\
A & C \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

  A product of A and B is denoted by $A \times B$, and the unque arrow u as $\langle f, g \rangle$.

[6] As noted in **??**, this statement holds for initial objects as well. From this point on, I will leave it to the reader to construct the dual of a statement and infer its truth.

- ◦ a **(binary) coproduct** of A and B consists of an object $C \in \operatorname{Obj} \mathbf{C}$ together with arrows $i_1 : A \to C$ and $i_2 : B \to C$ satisfying the following universal property:

  For any other "candidate coproduct" $D \in \operatorname{Obj} \mathbf{C}$ with arrows $j_1 : A \to D$ and $j_2 : B \to D$, there is a unique arrow $u : C \to D$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[dr, "j_1"'] & C \arrow[d, dashed, "u"] & B \arrow[l, "i_2"'] \arrow[dl, "j_2"] \\ & D & \end{tikzcd}$$

  A coproduct A and B is denoted $A + B$, and the unque arrow u as $[f, g]$.

Remark 2.6. The universal properties of the product and coproduct define bijections (there is one and only one function that fills in the diagram)

$$\begin{aligned} \operatorname{Hom}(A, C) \times \operatorname{Hom}(B, C) \cong \operatorname{Hom}(A + B, C) \text{ and} \\ \operatorname{Hom}(C, A) \times \operatorname{Hom}(C, B) \cong \operatorname{Hom}(C, A \times B). \end{aligned}$$

With one additional criterion (called "naturality"), this can be taken as an alternative definition.

Example 2.7.

- **Set**: Let A and B be sets.
  - Their Cartesian product $A \times B :\equiv \{(a, b) : a \in A, b \in B\}$ together with projections $p_1\ (a, b) :\equiv a$ and $p_2\ (a, b) :\equiv b$ is a product.
  - Their disjoint union $A \amalg B :\equiv \{(0, a) : a \in A\}\{(1, b) : b \in B\}$ together with inclusions $i_1\ a :\equiv (0, a)$ and $i_2\ b :\equiv (1, b)$ is a coproduct.[7]

- $\mathcal{U}$: Let $A, B : \mathcal{U}$.
  - The product type $A \times B$ with its projections is a product. (See).

[7] Of course, the identities of the "labels" 0 and 1 are unimportant, so long as they are distinct. One could equally well use "inl" and "inr" so long as inlinr.

- The coproduct type $A + B$ with its injections is a coproduct. The universal property is guaranteed by rec_{A+B}, and can be expressed concisely as an equivalence $(A \to C) \times (B \to C) \simeq A + B \to C$.

- **Cat**:
  - See **??**
  - The coproduct of categories $\mathbf{A}$ and $\mathbf{B}$ is the category $\mathbf{A} + \mathbf{B}$ with objects $\mathrm{Obj}(\mathbf{A} + \mathbf{B}) \coloneqq (\mathrm{Obj}\,\mathbf{A}) + (\mathrm{Obj}\,\mathbf{B})$ and arrows

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{A}+\mathbf{B}}(\mathsf{inl}\,A, \mathsf{inl}\,A') &= \mathrm{Hom}_{\mathbf{A}}(A, A') \\
\mathrm{Hom}_{\mathbf{A}+\mathbf{B}}(\mathsf{inr}\,B, \mathsf{inr}\,B') &= \mathrm{Hom}_{\mathbf{B}}(B, B') \\
\mathrm{Hom}_{\mathbf{A}+\mathbf{B}}(\mathsf{inl}\,A, \mathsf{inr}\,B) &= \mathbf{0} \\
\mathrm{Hom}_{\mathbf{A}+\mathbf{B}}(\mathsf{inr}\,A, \mathsf{inl}\,B) &= \mathbf{0}.
\end{aligned}$$

- **Grp**: Let G, H be groups.
  - The direct product group $G \times H$ is a product.
  - The free product $G*H$ is a coproduct.

- **AbGrp**, R-**Mod**, F-**Vect**: The direct sum is both a product and coproduct.[8]

Lemma 2.8. (Co)products are unique up to a specified isomorphism.

This lemma generalizes: any (co)limits are unique up to specified isomorphism.

Definition 2.9. Given objects $A, B \in \mathrm{Obj}\,\mathbf{C}$ and arrows $f, g : A \to B$,

- an **equalizer** of f and g consists of an object E and an arrow $e : E \to A$ satisfying the following universal property:

  For any other "candidate equalizer" C with arrow $c : C \to A$, there is a unique arrow $u : C \to E$ making the following diagram commute:

[8]In fact, has a stronger property of being a **biproduct**. These three categories share many significant features; one fruitful lens through which to view them is as **abelian categories**.

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
C \arrow[u, dashed, "u"] \arrow[ur, "c"'] & &
\end{tikzcd}$$

- a **coequalizer** of f, g consists of an object Q and an arrow $q : B \to Q$ satisfying the following universal property:

  For any other "candidate coequalizer" C with arrow $c : B \to C$, there is a unique arrow $u : Q \to C$ making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "c"'] & Q \arrow[d, dashed, "u"] \\
& & C
\end{tikzcd}$$

Example 2.10.

- **Set**: Let A, B be sets and $f, g : A \to B$.
  - The equalizer of f and g is the set $E \coloneqq \{a \in A : f\ a = g\ a\}$. The map $E \to A$ is the standard inclusion of a subset (which sends each element of E to itself in A).
  - The functions f and g induce an equivalence relation on B which is the closure of $f\ b g\ b$ for all $b \in B$. The quotient set $B/$ together with the projection $p : B \to B/$ which sends each element of b to its equivalence class is the coequalizer of f and g.
- $\mathcal{U}$: Let $A, B : \mathcal{U}$ and $f, g : A \to B$
  - The equalizer of f and g is the type $E :\equiv \sum_{(a:A)} f\ a = g\ a$. The function $E \to A$ is the first projection.
  - The issue of general quotients of types is actually somewhat intricate. The univalence axiom helps to construct them, but this is outside of the scope of this thesis.
- F-**Vect**: A coequalizer for a linear map $f : A \to B$ is the quotient space $B/\operatorname{im} f$.

Definition 2.11.

- For $A, B, C \in \operatorname{Obj} \mathbf{C}$ with arrows $f : A \to C$ and $g : B \to C$, a **pullback** of f and g consists of an object P together with arrows $p_1 : P \to A$ and $p_2 : P \to B$ satisfying the following universal property:

  For any other "candidate pullback" D with arrows $q_1 : D \to A$ and $q_2 : D \to B$, there is a unique arrow $u : D \to P$ making the following diagram commute:

$$\begin{tikzcd}
D \arrow[drr, bend left, "q_2"] \arrow[dr, dashed, "u"] \arrow[ddr, bend right, "q_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

  The object P is denoted $A \times_C B$, or better $A \times_{\langle f,g \rangle} B$.[9] One says p_2 is the result of *pulling f back along g* and vice versa for p_1.

- For $A, B, C \in \operatorname{Obj} \mathbf{C}$ with arrows $f : C \to A$ and $g : C \to B$, a **pushout** of f and g consists of an object P with arrows $i_1 : A \to P$ and $i_2 : B \to P$ satisfying the following universal property:

  For any other "candidate pushout" D, there is a unique arrow $u : P \to D$ making the following diagram commute:

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\
A \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"'] & P \arrow[dr, dashed, "u"] & \\
& & D
\end{tikzcd}$$

  The object P is denoted $A +_C B$, or better $A +_{[f,g]} B$.

[9] The notation $A \times_C B$ draws an important connection to the product, but omits some the key information: *which* arrows are being pulled back matters, and there might be many choices for the same objects A, B, and C.

Example 2.12.

- **Set**:
  - Let A, B, C be sets, $f : A \to C$, and $g : B \to C$. The pullback of f and g is the set $P :\equiv \{(a, b) \in A \times B : f\, a = g\, b\}$; the maps $P \to A$ and $P \to B$ are just the projections.
  - Let A, B, C be sets, $f : C \to A$, and $g : C \to B$. The pushout of f and g is the disjoint union quotiented by the smallest equivalence relation containing $(0, f\, c)(1, g\, c)$; the maps p_1 and p_2 are the projections to equivalence classes.
- $\mathcal{U}$:
  - Let $A, B, C : \mathcal{U}$, $f : A \to C$, and $g : B \to C$. The pullback of f and g is the type $\sum_{(a:A)} \sum_{(b:B)} f\, a = g\, b$, the functions $P \to A$ and $P \to B$ are just the projections.

Example 2.13. The pullback and pushout are complex; it is less immediately clear where they arise in standard mathematics than, say, the product and coproduct. Three special cases of the pullback in **Set** help motivate these constructions.

1. A pullback where one of A or B is the terminal object in a category defines the notion of a "fiber". Consider the pullback P in the following square (where $\{*\}$ is any one-point set, a terminal object):

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "!"] & B \arrow[d, "g"] \\ \{*\} \arrow[r, "f"'] & C \end{tikzcd}$$

   The arrow f picks out a single element of C, namely $f * \in C$. Call this c. Since the square commutes, $f \circ ! = g \circ p_2$, that is, $(g \circ p_2)\, x = c$ for all $x \in P$. By the explicit construction of **??**,

$$P = \{(*, b) \in \{*\} \times B : f * = g\, b\} \cong \{b \in B : c = g\, b\}.$$

   This says that P is (isomorphic to) the **fiber of** g **above** c.

2. More generally, consider the **inverse image** of a subset SC. Again, let B and C be arbitrary sets and let $f : B \to C$. Let $i : SC$ be the inclusion function. Then the pullback of i and f is

$$\begin{aligned} P &= \{(s, b) \in S \times B : i\, s = f\, b\} \\ &= \{(s, b) \in S \times B : s = f\, b\} \\ &\cong \{b \in B : f\, b \in S\}, \end{aligned}$$

which is the inverse image of S under f, denoted $f^{-1}\, V$.[10] Thus, the inverse image of C under f is the pullback of the inclusion along f.

3. Let S, RC. The pullback of the inclusions $i : S \rightarrowtail C$ and $j : R \rightarrowtail C$ is (isomorphic to) the **intersection** $S \cap R$:

$$\begin{aligned} P &= \{(s, r) \in S \times R : i\, s = j\, r\} \\ &= \{(s, r) \in S \times R : s = r\} \\ &\cong \{c \in C : c \in S \text{ and } c \in R\} \\ &= SR \end{aligned}$$

Example 2.14. In **Mon**, the category of monoids, the kernel of $f : M \to N$ can be represented as a pullback of the unique arrow $! : \{*\} \to N$ along f as in the following diagram:

$$\begin{tikzcd} \ker f \arrow[r, tail] \arrow[d, "!"'] & M \arrow[d, "f"] \\ \{*\} \arrow[r, "!"'] & N \end{tikzcd}$$

Example 2.15. In **Top**, the wedge sum AB is an instance of a pushout. Suppose $a \in A$ and $b \in B$ are the basepoints to be "glued". Define (necessarily continuous) functions

$$\begin{aligned} f : \{*\} &\longrightarrow A \\ * &\longmapsto a \end{aligned} \qquad\qquad \begin{aligned} g : \{*\} &\longrightarrow B \\ * &\longmapsto b \end{aligned}$$

[10] The notation f^{-1} generalizes the notation for the inverse of a function. We can consider f^{-1} to be a function $C \to \mathcal{P}(A)$ into the powerset— the set of all subsets—of A. In the case that f is bijective, it sends each $c \in C$ to a singleton, and defines a function $C \to A$, also denoted f^{-1}.

The equivalence relation only relates $f\,*$ and $g\,*$, that is ab. Thus,

$$A +_{[f,g]} B = A + B/(f\,* g\,*) = A + B/(ab) = AB$$

2.2 General construction

??, **??**, **??**, and **??** are all instances of a general construction of *limits* and *colimits*. Some preliminary constructions are necessary before these definitions; these will help make sense of the terms "candidate _" and "candidate co_" used above.

Definition 2.16. A **graph** G consists of a (not necessarily finite) collection G_0 of vertices, and a map G_1 which assigns to each ordered pair of vertices a collection of directed edges between them.[11]

Definition 2.17. A **diagram** D for a graph G is a collection of objects and arrows "of shape G". Specifically, D consists of a map $D_0 : G_0 \to \operatorname{Obj}\mathbf{C}$ and for each pair (u, v) of vertices in G_0, a map $D_1^{(u,v)} : G_1(u, v) \to \operatorname{Hom}(D_0\,u, D\,_0 v)$. When convenient, we drop the sub- and super-scripts.

More elegantly,

Definition 2.18. A **diagram** in $\mathbf{C}$ is a functor $F : \mathbf{J} \to \mathbf{C}$ from a small category.

Definition 2.19. Let D be a diagram in $\mathbf{C}$ for a graph G.

- A **cone** for D consists of an object $C \in \operatorname{Obj}\mathbf{C}$ (the **apex**) together with **projections** $\{p_v : C \to D\,v\}_{v \in G_0}$ such that the following triangle commutes for all $u, v \in G_0$ and $e \in G_1(u, v)$:

$$\begin{tikzcd} & C \arrow[dl, bend right, "p_u"'] \arrow[dr, bend left, "p_v"] & \\ D\,u \arrow[rr, "D\,e"] & & D\,v \end{tikzcd}$$

[11] This definition may seem obtuse to a classically-trained mathematician, but it translates more elegantly into the type-theoretic setting.

- A **cocone** for D consists of an object $C \in \mathrm{Obj}\,\mathbf{C}$ (the **nadir**) together with **injections** $\{i_v : D\,v \to C\}_{v \in G_0}$ such that the following triangle commutes for all $u, v \in G_0$ and $e \in G_1(u, v)$:

$$\begin{tikzcd} D\,u \arrow[rr, "D\,e"] \arrow[dr, bend right, "i_u"'] & & D\,v \arrow[dl, bend left, "i_v"] \\ & C & \end{tikzcd}$$

Example 2.20. A cone for a diagram of shape $\cdot \quad \cdot$ picks out a "candidate product" in $\mathbf{C}$. The following correspondences illustrate other "candidate" constructions:

- Terminal/Initial: [empty]
- (Co)product: $\cdot \quad \cdot$
- (Co)equalizer: $\cdot \rightrightarrows \cdot$
- Pullback/Pushout: $\cdot \longleftarrow \cdot \longrightarrow \cdot$

Definition 2.21. Given a diagram D for a graph G in $\mathbf{C}$,

- a **limit** of D consists of a cone $(L, \{p_v\})$ satisfying the following universal property:

  For any other cone $(C, \{q_v\})$, there is a unique arrow $u : C \to L$ making the following diagram commute for all $u, v \in G_0$ and $e \in G_1(u, v)$:

$$\begin{tikzcd} & C \arrow[d, dashed, "u"] \arrow[ddl, bend right, "q_u"'] \arrow[ddr, bend left, "q_v"] & \\ & L \arrow[dl, bend right, "p_u"] \arrow[dr, bend left, "p_v"'] & \\ D\,u \arrow[rr, "D\,e"'] & & D\,v \end{tikzcd}$$

- a **colimit** of D consists of a cocone $(L, \{i_v\})$ satisfying the following universal property:

  For any other cone $(C, \{j_v\})$, there is a unique arrow $u : L \to C$ making the following diagram commute:

$$\begin{tikzcd}
D\,u \arrow[rr, "D\,e"] \arrow[dr, "i_u"] \arrow[ddr, bend right, "j_u"'] & & D\,v \arrow[dl, "i_v"'] \arrow[ddl, bend left, "j_v"] \\
& L \arrow[d, dashed, "u"] & \\
& C &
\end{tikzcd}$$

If $\mathbf{C}$ has a (co)limit for a diagram D, its apex (resp. nadir) is denoted $\mathsf{Lim}\,D$ (resp. $\mathsf{Colim}\,D$). The collection of (co)cones over a diagram D with apex (resp. nadir) X is denoted $\mathsf{Cone}\,D\,X$ (resp. $\mathsf{Cocone}\,D\,X$).

Remark 2.22. If D is a diagram in $\mathbf{C}$, the above universal properties define bijections

$$\mathrm{Hom}(A, \mathsf{Lim}\ D) \cong \mathsf{Cone}\ D\ A \text{ and}$$
$$\mathrm{Hom}(\mathsf{Colim}\ D, A) \cong \mathsf{Cocone}\ D\ A.$$

for any object A. Again, with one additional criterion (called "naturality"), this can be taken as a alternative definitions. (Compare to **??**).

Definition 2.23. A category **C has (finite) (co)limits of shape** G if any (finite) diagram of shape G has a (co)limit in $\mathbf{C}$. A category is **(finitely) (co)complete** or **has (finite) (co)limits** when it has (co)limits of every shape.

Generally, one often says $\mathbf{C}$ "has x" for some object x and collection of arrows defined by a universal property, e.g. "$\mathbf{C}$ has (binary) products" or "$\mathbf{C}$ has internal homs".

3 The Yoneda lemma

Definition 3.1. A **contravariant functor** from $\mathbf{C}$ to $\mathbf{D}$ is a functor $\mathbf{C}^{\mathsf{op}} \to \mathbf{D}$.

The following diagram is due to Riehl [**?**], and demonstrates the differences between co- and contravariant functors:

$$
\begin{tikzcd}
\mathbf{C} \arrow[rr, "F"] & & \mathbf{D} \\
X \arrow[d, "f"] & \mapsto & FX \arrow[d, "Ff"] \\
Y & \mapsto & FY
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}^{\mathsf{op}} \arrow[rr, "F"] & & \mathbf{D} \\
X \arrow[d, "f"] & \mapsto & FX \\
Y & \mapsto & FY \arrow[u, "Ff"']
\end{tikzcd}
$$

Definition 3.2. The covariant and contravariant **functors represented by** X are

$$\mathrm{Hom}(X,-) : \mathbf{C} \to \mathbf{Set} \qquad \mathrm{Hom}(-,X) : \mathbf{C}^{\mathsf{op}} \to \mathbf{Set}$$

given by

$$
\begin{tikzcd}
\mathbf{C} \arrow[rr, "\mathrm{Hom}(X,-)"] & & \mathbf{D} \\
Y \arrow[d, "f"] & \mapsto & \mathrm{Hom}(X,Y) \arrow[d, "f\circ -"] \\
Z & \mapsto & \mathrm{Hom}(X,Z)
\end{tikzcd}
\quad
\begin{tikzcd}
\mathbf{C}^{\mathsf{op}} \arrow[rr, "\mathrm{Hom}(-,X)"] & & \mathbf{D} \\
Y \arrow[d, "f"] & \mapsto & \mathrm{Hom}(Y,X) \\
Z & \mapsto & \mathrm{Hom}(Z,X) \arrow[u, "-\circ f"']
\end{tikzcd}
$$

In fact, these extend to a bifunctor (if $\mathbf{C}$ is locally small),

$$
\begin{aligned}
\mathrm{Hom}(-,-) : C^{\mathsf{op}} \times C &\longrightarrow \mathbf{Set} \\
(X,Y) &\longmapsto_0 \mathrm{Hom}(X,Y) \\
(f,g) &\longmapsto_1 (g \circ - \circ f)
\end{aligned}
$$

Definition 3.3. A (set-valued) **presheaf** on $\mathbf{C}$ is a functor $\mathbf{C}^{\mathsf{op}} \to \mathbf{Set}$ [?].

For objects W, X, Y, Z and morphisms $f : X \to Y$ and $g : Y \to Z$, post-composition by g and pre-composition by f define natural transformations:

$$
\begin{tikzcd}
\mathrm{Hom}(X,Y) \arrow[r, "g\circ -"] \arrow[d, "-\circ f"] & \mathrm{Hom}(X,Z) \arrow[d, "-\circ f"] \\
\mathrm{Hom}(W,Y) \arrow[r, "g\circ -"] & \mathrm{Hom}(W,Z)
\end{tikzcd}
$$

Definition 3.4. A **representation** of a functor F with domain $\mathbf{C}$ is an object $c \in \mathbf{C}$ together with a natural isomorphism between F and the represented functor of the appropriate variance.

3.1 Closure of Cat

The background to fully understand this topic is in **??**.

3.2 The Yoneda embedding

4 Functor algebras

Definition 4.1. An **endofunctor** is a functor with identical domain and codomain.

Endofunctors provide a concise way to encapsulate a **signature**, a way of describing an object of a category together with some arrows into it. Signatures are a basic structure of universal algebra, the zoomed-out study of algebraic structure (as opposed to algebra, the study of specific algebraic structures e.g. monoids, posets, lattices, rings).

Definition 4.2.

- An **algebra** for an endofunctor $F : \mathbf{C} \to \mathbf{C}$ (also called an F**-algebra**) is a pair $(A,)$ of an object $A \in \operatorname{Obj} \mathbf{C}$ and an arrow $: FA \to A$.
- A **coalgebra** for an endofunctor $F : \mathbf{C} \to \mathbf{C}$ (also called an F**-coalgebra**) is a pair $(A,)$ of an object $A \in \operatorname{Obj} \mathbf{C}$ and an arrow $: A \to FA$ [**?**].

There is an unfortunate profusion of terms in mathematics using "algebra" in their name. Functor algebras are unrelated to elementary algebra (the solution of equations with unknowns). Their relation to algebraic structures like groups is a bit more complex.

Remark 4.3. By the universal property of coproducts, an algebra for a functor of the form $A \mapsto (A \times A) + A + 1$ consists of:

- an arrow $A \times A \to A$ (a binary operation on A),

- an arrow $A \to A$ (a unary operation on A),
- and a distinguished element $1 \to A$.

This is exactly the signature of a **group** (the arrow $A \times A \to A$ corresponds the group operation, $A \to A$ to inversion, and $1 \to A$ to the identity).

Not every algebra for this functor will be a group. The functor doesn't encode the *equational laws*, e.g. that the operation $A \times A \to A$ is associative. To represent such relations, one needs the richer structure of a **monad**.

Monads also have algebras, and it is difficult to say whether these generalize F-algebras or the other way around. Particular groups, rings, algebras over a field, etc. are examples of F-algebras (for various choices of F), but the definition of a F-algebra don't encode sufficient information such that *every* F-algebra for the "group signature functor" is indeed a group.

For the remainder of this section when a definition or result is presented only for coalgebras or algebras, it holds dually for the other.

Definition 4.4. A **coalgebra morphism** from (A, α) to (B, β) is an arrow $f : A \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\
FA \arrow[r, "Ff"] & FB
\end{tikzcd}$$

Since F is a functor, the composition of coalgebra morphisms is again a coalgebra morphism. In fact, F-coalgebras have all of the structure of a category, called F-**Coalg**.

Definition 4.5.

- An **initial F-algebra** is an initial object of F-**Alg**.
- An **terminal F-coalgebra** is a terminal object of F-**Coalg** (these are more commonly known as **final coalgebras** in the computer science literature).

Example 4.6. Let $\mathbf{C}$ be a category with distinguished binary coproducts and a terminal object 1. Consider the functor

$$\begin{aligned} F : \mathbf{C} &\longrightarrow \mathbf{C} \\ A &\longmapsto_0 1 + A \\ f &\longmapsto_1 \mathrm{id}_1 + f \end{aligned}$$

where 1 is a terminal object and $+$ is the coproduct bifunctor (**??**) as in **??**. Let's examine what it *means* for some F-algebra (N, η) to be initial. By composing with the coproduct injections, define

$$\begin{aligned} z &: 1 \longrightarrow N & s &: N \longrightarrow N \\ z &= \eta \circ i_1 & s &= \eta \circ i_2 \end{aligned}$$

so that $= [z, s]$:

$$\begin{tikzcd} 1 \arrow[r, "i_1"] \arrow[dr, "z"'] & 1 + N \arrow[d, "\eta"] & N \arrow[l, "i_2"'] \arrow[dl, "s"] \\ & N & \end{tikzcd}$$

Suppose (A, α) is another F-algebra, and define f, g by composition as above so that $\alpha = [f, g]$. By initiality of (N, η), there is a unique arrow u making the following diagram commute:

$$\begin{tikzcd} 1 + N \arrow[r, dashed, "\mathrm{id}_1 + u"] \arrow[d, "{[z,s]}"] & 1 + A \arrow[d, "{[f,g]}"] \\ N \arrow[r, dashed, "u"] & A \end{tikzcd}$$

By functoriality of the coproduct (**??**), we can compose along either the left- or right-hand paths in the above diagram. The above square states $u \circ [z, s] = [f, g] \circ (\mathrm{id}_1 + u)$. Precomposing with $i_1 : 1 \to 1 + N$ yields

$$\begin{aligned} u \circ z = u \circ [z, s] \circ i_1 &= [f, g] \circ (\mathrm{id}_1 + u) \circ i_1 && \text{Above diagram} \\ &= [f, g] \circ [i_1, i_2 \circ u] \circ i_1 && \text{Definition of } + \\ &= [f, g] \circ i_1 \\ &= f \end{aligned}$$

By similar reasoning, precomposing with i_2 yields

$$u \circ s = u \circ [z, s] \circ i_2 = [f, g] \circ (\mathrm{id}_1 + u) \circ i_2 = g \circ u.$$

Combining the above two equations yields the following universal property for $(N,)$. For any object A with arrows $f : 1 \to A$ and $g : A \to A$, there is a unique arrow $u : N \to A$ making the following diagram commute:

$$\begin{tikzcd}
1 \arrow[r, "z"] \arrow[dr, "f"'] & N \arrow[r, "s"] \arrow[d, dashed, "u"] & N \arrow[d, dashed, "u"] \\
 & A \arrow[r, "g"] & A
\end{tikzcd}$$

This is the universal property of a **natural number object** (NNO) [**?**] [**?**]. In **Set**, $\mathbb{N}$ would be an initial algebra. It makes sense to ask if this functor has an initial algebra in any category with a terminal object and binary coproducts. There may be categories without these that have natural number objects, but I'm not aware of any.

Example 4.7.

- An initial algebra for the functor $AB \times A + 1$ has the universal property of a **list object**.
- The initial algebra $A(A \times A) + 1$ has a universal property expressing the induction principle for binary trees.

These examples suggest a new perspective on inductive types: they are the non-equational (or *free*) algebraic structures generated by their constructors (introduction rules).

5 Type theoretic category theory

Type theory is a great foundational system in which to do category theory, particularly because of its hierarchy of universes, which deals elegantly with problems of "size".[12] However, with greater expressiveness, issues of definition

[12] Conversely, certain categories called *topoi* (singular: topos) provide models of type theory. In this case, we can use the logic of type theory to directly prove properties of such categories. In this setting, type theories are called *internal languages* of topoi.

become more prominent; this section explains the relevant considerations.

Unfortunately, terminology varies between the three predominant sources on category theory in univalent type theory [?] [?] [?]. We follow the UniMath library. The following table offers a comparison for those already familiar with [?].

UniMath	**HoTT** book[13]	Obj $\mathbf{C}$	$\mathsf{Hom}_{\mathbf{C}}$	Univalence
Precategory	n/a	Type	Type	No
Category	Precategory	Type	Set	No
Univalent category	Category	Type	Set	Yes
Set category	Strict category	Set	Set	No

For a pioneering development of univalent category theory, see [?] (the formalization of which served as a foundation for UniMath's `CategoryTheory` package).

5.1 Basics

Types in **UTT** are the analogues of sets in **ZFC+FOL**, insofar as they also represent "collections". Therefore, a naïve translation of the definition would define a category to consist of a type $C : \mathcal{U}$ of objects and a family $\mathsf{Hom}_C : C \to C \to \mathcal{U}$ of arrows.

Definition 5.1. A **precategory** consists of the following data:

- a type $\mathbf{C} : \mathcal{U}$ of **objects**,
- a family $\mathsf{Hom}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C} \to \mathcal{U}$ which assigns to each pair of types a **hom-type** which has **morphisms** as its elements,
- a map $\mathrm{id} : \prod_{(c:\mathbf{C})} \mathsf{Hom}_{\mathbf{C}}\, c\, c$ which assigns to each object an **identity morphism**,
- an operation

  $$\circ : \prod_{(a,b,c:\mathbf{C})} \mathsf{Hom}_{\mathbf{C}}\, b\, c \to \mathsf{Hom}_{\mathbf{C}}\, a\, b \to \mathsf{Hom}_{\mathbf{C}}\, a\, c$$

  called **composition**, and

together with maps witnessing[14] the following axioms:

1. left and right identity:

$$\prod_{(a,b:\mathbf{C})} \prod_{(f:\mathsf{Hom}_{\mathbf{C}}\ a\ b)} \mathrm{id}_b \circ f = f$$

$$\prod_{(a,b:\mathbf{C})} \prod_{(f:\mathsf{Hom}_{\mathbf{C}}\ a\ b)} f \circ \mathrm{id}_a = f$$

2. associativity of composition:

$$\prod_{(a,b,c,d:\mathbf{C})} \prod_{(f:\mathsf{Hom}_{\mathbf{C}}\ a\ b)} \prod_{(g:\mathsf{Hom}_{\mathbf{C}}\ b\ c)} \prod_{(h:\mathsf{Hom}_{\mathbf{C}}\ c\ d)} (h \circ g) \circ f = h \circ (g \circ f)$$

The type of all precategories is denoted Precat.

Many concepts from classical category theory can be translated with no further thought into the language of precategories.

Definition 5.2. A **functor** F between $\mathbf{C}, \mathbf{D} : \mathsf{Precat}$ consists of the following data:

- a map $F_0 : \mathbf{C} \to \mathbf{D}$ and
- a map $F_1 : \prod_{(a,b:\mathbf{C})} \mathrm{Hom}_{\mathbf{D}}\ (F_0\ a)\ (F_0\ b)$

together with maps witnessing the following axioms:

1. preservation of identity:

$$\prod_{(a:\mathbf{C})} F_1\ \mathrm{id}_a = \mathrm{id}_{(F_0\ a)}$$

2. preservation of composition:

$$\prod_{(a,b,c:\mathbf{C})} \prod_{(f:\mathsf{Hom}_{\mathbf{C}}\ a\ b)} \prod_{(g:\mathsf{Hom}_{\mathbf{C}}\ b\ c)} F_1\ (g \circ f) = F_1\ g \circ F_1\ f.$$

[14] This is an expressly type-theoretic presentation; one might instead use the more classical "for all objects a, b of $\mathbf{C}$ and arrows $f : A$". However, since such definitions appear in the previous chapter, these are phrased more idiosyncratically to help the reader get used to this more rigid formalism.

The type of functors from $\mathbf{C}$ to $\mathbf{D}$ is denoted $\mathsf{Functor}\,\mathbf{C}\,\mathbf{D}$.

Example 5.3. There is a precategory of precategories, with object type Precat and morphism type $\mathsf{Functor}$;[15] there is also a precategory with object type $\mathsf{Category}$.

Many such "translated" definitions are so near to the originals that we won't treat them here (e.g. commutative diagrams, isomorphisms, and the specific kinds of (co)limits).

However, when proving analogous results about such constructions in precategories, one soon runs into difficulty. The root of the problem is that the hom-types may have "higher homotopical structure", and to account for this structure would require infinitely many additional axioms.[16] For most applications, it suffices to restrict the hom-types to be hom-sets:

Definition 5.4. A precategory $\mathbf{C}$ **has hom-sets** if the type

$$\prod_{(a,b:\mathbf{C})} \mathsf{isSet}\;(\mathsf{Hom}\;a\;b)$$

is inhabited. A precategory with hom-sets is called a **category**.

Example 5.5. The following examples illustrate to readers familiar with category theory the difficulty of working with precategories rather than categories.

- The type $\mathsf{Functor}\,\mathbf{C}\,\mathbf{D}$ doesn't form a precategory with natural transformations as morphisms unless $\mathbf{D}$ has hom-sets.
- Slices (resp. coslices) over (resp. under) an object X don't form precategories unless the homs-types $\mathrm{Hom}\ - X$ (resp. $\mathrm{Hom}\ X\ -$) are all sets.

[15] This is a great example of how universes deal elegantly with problems of size: this precategory isn't a member of itself, because it is really only a precategory of $\mathcal{U}_i$-small precategories for fixed i, and itself is an element of $\mathcal{U}_{\mathsf{succ}\ i}$. A good proof assistant automatically assigns consistent universe indices when possible.

[16] The precise nature of this structure and how it interferes with classical intuitions is outside the scope of this thesis, though certain instructional difficulties will arise in **??** (indeed, that whole chapter may be seen as an attempt to use 1-category theoretic-intuitions outside their domain of applicability).

5.2 Products in $\mathcal{U}$

We are frequently concerned with a precategory that is emphatically *not* a category, namely $\mathcal{U}$. The next few lemmas explore binary products.

Lemma 5.6. The precategory of types has binary products.[17]

While there are more direct proofs, the following is based on the observation made in **??**, and will carry over to the case of general limits. We begin a lemma:

Lemma 5.7. Let $\mathbf{C}$ be a precategory, let $a, b, c : \mathbf{C}$, and let $p_1 : c \to a$, $p_2 : c \to b$. Define a function

$$\begin{aligned} \mathsf{postcomp} : \prod_{(d:\mathbf{C})} (d \to c) &\to (d \to a) \times (d \to b) \\ \mathsf{postcomp}\ f &:\equiv (p_1 \circ f, p_2 \circ f) \end{aligned}$$

If $\mathsf{postcomp}$ is an equivalence, c is a product of a and b.

Sketch. Why is this true? If $\mathsf{postcomp}$ is an equivalence, then its fibers are contractible. In particular, for each pair of arrows $f : d \to a$ and $g : d \to b$, there is exactly one arrow $u : d \to c$ in the fiber over them. This means when u is composed projections p_1 and p_2, it gives back f and g (up to propositional equality). This is just a backwards rephrasing of the universal property of the product.

Proof. Fix d, q_1, and q_2 as in the type of $\mathsf{postcomp}$. For (c, p_1, p_2) to be a product of a and b, the following type must be contractible:

$$\sum_{(f:d \to c)} (p_1 \circ f = q_1) \times (p_2 \circ f = q_2)$$

By **??**, it suffices to show that the above type is equivalent to a contractible type. By **??**, the fibers of $\mathsf{postcomp}$ are contractible, so it suffices to show that the above type is equivalent to the fiber over some point in $(d \to a) \times (d \to b)$.

[17] Again, where these are defined by rote translation from **??**.

This point is (q_1, q_2):

$$\begin{aligned}
\mathsf{fiber}\ (q_1, q_2) &\equiv \sum_{(f:d\to c)} \mathsf{postcomp}\ f = (q_1, q_2) && \text{Definition} \\
&\equiv \sum_{(f:d\to c)} (p_1 \circ f, p_2 \circ f) = (q_1, q_2) && \text{Definition} \\
&\simeq \sum_{(f:d\to c)} (p_1 \circ f = q_1) \times (p_2 \circ f = q_2) && \textbf{??}
\end{aligned}$$

□

Lemma 5.8. There is an equivalence of types

$$(C \to A) \times (C \to B)(C \to A \times B)$$

for all $A, B, C : \mathcal{U}$.[18]

Proof. Define

$$\begin{aligned}
g &: (C \to A) \times (C \to B) \to (C \to A \times B) \\
g\ (h, i) &:\equiv \lambda c.\ (h\ c, i\ c)
\end{aligned}$$

and let $\mathsf{postcomp}_{\mathcal{U}}$ be the specialization of $\mathsf{postcomp}$ to the precategory of types. Then

$$\begin{aligned}
(g \circ \mathsf{postcomp}_{\mathcal{U}})\ h &\equiv g\ (\mathsf{pr}_1 \circ h\ , \mathsf{pr}_2 \circ h\ ) \\
&\equiv \lambda c.\ ((\mathsf{pr}_1 \circ h\ )\ c, (\mathsf{pr}_2 \circ h\ )\ c) \\
&= h
\end{aligned}$$

where the final equality is due to function extensionality and the -rule for the product type. Further,

$$\begin{aligned}
(\mathsf{postcomp}_{\mathcal{U}} \circ g)\ (h, i) &\equiv \mathsf{postcomp}_{\mathcal{U}}\ (\lambda c.\ (h\ c, i\ c)) \\
&\equiv (\mathsf{pr}_1 \circ \lambda c.\ (h\ c, i\ c), \mathsf{pr}_2 \circ \lambda c.\ (h\ c, i\ c)) \\
&= (h, i)
\end{aligned}$$

[18] This result originally appeared in Voevodsky's "Foundations" library, a precursor to UniMath.

where the final equality is again due to function extensionality and the -rule for the function type. $\square$

Proof of **??**. **??** proves that $\mathsf{postcomp}$ is an equivalence in $\mathcal{U}$, so by **??**, $\mathcal{U}$ has binary products. $\square$

6 Topos theory

Lemma 6.1. Given a rectangle

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

Then the outer rectangle is a pullback square if TODO

7 Monoidal categories

Definition 7.1. A **monoidal category** consists of a category $\mathbf{C}$ together with

- a bifunctor (**??**) $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$ called the **tensor product** or **monoidal product**,
- an object $I \in \operatorname{Obj} \mathbf{C}$ called the **unit**,
- for all objects $A, B, C \in \operatorname{Obj} \mathbf{C}$, an isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ called the **associator**, and
- for all objects $A \in \operatorname{Obj} \mathbf{C}$, isomorphisms $\lambda_A : I \otimes A \cong A$ and $\rho_A : A \otimes I \cong A$, called the **left unitor** and **right unitor**, respectively.

These isomorphisms are subject to additional "coherence conditions", which won't play a crucial role here. A monoidal category is

- **symmetric** if there are additional isomorphisms $s_{A,B} : A \otimes B \to B \otimes A$ satisfying other coherence conditions; and is

- **cartesian** if the monoidal product $\otimes$ coincides with the categorical product $\times$.

Remark 7.2. For any categorical product there is a specified isomorphism $A \times B \cong B \times A$, so any cartesian monoidal category is also symmetric monoidal.

Example 7.3.

- **Set**, $\mathcal{U}$, **Cat**, **Grp**, and others are cartesian (so necessarily symmetric) monoidal under their categorical products.
- F-**Vect** is symmetric monoidal under $\otimes$; the trivial vector space is a unit.

Definition 7.4. An **internal hom** for a monoidal category $(\mathbf{C}, \otimes)$ consists of

- for each pair of objects $A, B \in \operatorname{Obj} \mathbf{C}$, an object $A \Rightarrow B$, and
- an arrow $\mathsf{eval} : (A \Rightarrow B) \times A \to B$

satisfying the following universal property: for any object $C \in \operatorname{Obj} \mathbf{C}$ and arrow $f : C \otimes A \to B$, there is an arrow $\lambda f : C \to (A \Rightarrow B)$ making the following diagram commute:

$$
\begin{tikzcd}
C \times A \arrow[d, "f \times \mathrm{id}_A"'] \arrow[dr, "f"] & \\
(A \Rightarrow B) \times A \arrow[r, "\mathsf{eval}"'] & B
\end{tikzcd}
$$

When $\mathbf{C}$ is cartesian, $A \Rightarrow B$ is called an **exponential**, and is denoted B^A.

Definition 7.5. A monoidal category $(\mathbf{C}, \otimes)$ is **closed** when it has internal homs.

Remark 7.6. (Compare to **??**) In a closed monoidal category, for all $A, B, C \in \operatorname{Obj} \mathbf{C}$, λ and eval define a bijection

$$\operatorname{Hom}(A \otimes B, C) \cong \operatorname{Hom}(A, B \Rightarrow C).$$

This bijection can be taken as their definition with an additional "naturalality" condition. This bijection is often called **currying**.

Example. **Set** and $\mathcal{U}$ are closed monoidal. Define $B \Rightarrow A :\equiv \mathrm{Hom}(B, A)$. Then this is an element of **Set**/$\mathcal{U}$. Mixing set- and type-theoretical notation, the bijection is given by the mutually inverse functions

$$\begin{aligned}
\mathrm{Hom}(A \times B, C) &\longrightarrow \mathrm{Hom}(A, \mathrm{Hom}(B, C)) \\
f &\longmapsto \lambda a.\ \lambda b.\ f\ a\ b \\
\mathrm{Hom}(A, \mathrm{Hom}(B, C)) &\longrightarrow \mathrm{Hom}(A \times B, C) \\
g &\longmapsto \lambda p.\ g\ (\mathsf{pr}_1\ a)\ (\mathsf{pr}_2\ b)
\end{aligned}$$

Definition 7.7. A category is **Cartesian closed** when it is closed, cartesian monoidal, and has a terminal object. Equivalently, it is cartesian closed when it has

- a terminal object (**??**),
- binary products (**??**), and
- exponentials (**??**).

"CCC" abbreviates "Cartesian closed category".

7.1 Symmetry

7.2 Closure