Language Robot Controller Supplemental

1 Passivity Criteria

A passive system is a system that constrained in such a way that it does not inject excessive energy or instability into the interaction [1]. More formally a system is passive with respect to an input output pair $(u(t), y(t))$ if and only if there exists a positive definite storage function V over the system such that:

$$V(t) - V(0) \leq \int_0^t u(t)^T \cdot y(t) dt \quad \forall t > 0 \tag{1}$$

Theorem 1: Consider the system with dynamics governed by:

$$M\ddot{x} = p^2 K(x_d - x) - \big(pD - \tfrac{\dot{p}}{p} M\big)\dot{x} + p^2(F_{\text{drive}}(x_d) + F_{\text{ext}}). \tag{2}$$

If F_{drive} is differentiable and tangent to the trajectory, M, D and K are positive definite, K is orthogonal, and

$$\exists\, p_{\text{min}} > 0 \text{ such that } p_{\text{min}} < p \quad \forall, t > 0$$

then for any linear trajectory, the system is passive with respect to the the force-velocity $(F_{\text{ext}}, \dot{x})$ input output pair.

Proof: Differentiating (1), one sees it is sufficient to show that there exists positive semi-definite storage function such that:

$$\dot{V} \leq u(t)^T \cdot y(t) \quad \forall t > 0 \tag{3}$$

We dedicate the rest of the section to constructing such a function. We first proceed with some notation. Let T, denote the oriented line drawn out by our trajectory in state space. Furthermore let N denote the unit vector parallel to T.Since $\boldsymbol{F}_{\text{drive}}(x_d)$ is parallel to N we write:

$$F_{\text{drive}}(x_d) = f(x_d)N \tag{4}$$

for some positive definite scalar function f.

For any vector x, let $x_{||}$ denote the component parallel to N and let $x_{\perp}$ denote the component perpendicular to N. Since $x_{||}$ is the projection of x onto the span of N, it is the closest point on T. Therefore :

$$x - x_d = x - x_{||} = x_{\perp} \tag{5}$$

Substituting (4) and (5) into (2) yields:

$$M\ddot{x} = -p^2 K x_\perp - (pD - M\frac{\dot{p}}{p})\dot{x} - p^2 f(x_d) N + p^2 F_{ext} \tag{6}$$

We define our storage function V as follows :

$$V(x, \dot{x}, p) = \tfrac{1}{2}\, x_\perp^T K x_\perp + \phi(x_{||}) + \tfrac{1}{2} p^{-2} \dot{x}^T M \dot{x} \tag{7}$$

where $\phi(x_{||})$ is the line integral of the driving force along the trajectory T to the end point, defined as $\phi(x_{||}) = \int_{x_{||}}^{0} f(x) N dx$. We claim V is positive semi definite and satisfies (3). We first show our function is positive definite. Since K is positive definite:

$$\forall x_\perp \neq 0, \quad x_\perp^T K x_\perp > 0 \tag{8}$$

$$x_\perp = 0 \implies x_\perp^T K x_\perp = 0 \tag{9}$$

Since M is positive definite and $p > p_{\text{min}} > 0$:

$$\forall \dot{x} \neq 0, \quad p^{-2} \dot{x}^T M \dot{x} > 0 \tag{10}$$

$$\dot{x} = 0 \implies p^{-2} \dot{x}^T M \dot{x} = 0 \tag{11}$$

Since $f(x_d)N$ lies parallel to the trajectory, the line integral is positive and therefore :

$$\forall x_{||} \neq 0, \quad \phi(x_{||}) > 0 \tag{12}$$

$$x_{||} = 0 \implies \phi(x_{||}) = 0 \tag{13}$$

Thus each term is non negative and $V = 0$ if and only if $\dot{x} = x = x_{||} + x_\perp = 0$. Therefore V is positive definite. We now show V satisfies (3). Taking the time derivative of V yields:

$$\dot{V} = \dot{x}_\perp^T K x_\perp - f(x_{||}) N \dot{x}_{||}^T - \dot{p} p^{-3} \dot{x}^T M \dot{x} + p^{-2} \dot{x}^T M \ddot{x} \tag{14}$$

Substituting (6) for $M\ddot{x}$:

$$\dot{V} = \dot{x}_\perp^T K x_\perp - f(x_{||}) N \dot{x}_{||}^T - \dot{p} p^{-3} \dot{x}^T M \dot{x}$$

$$+ p^{-2} \dot{x}^T (-p^2 K x_\perp - (pD - M\frac{\dot{p}}{p})\dot{x} + p^2 f(x_d) N + p^2 F_{\text{ext}}) \tag{15}$$

Distributing the $p^{-2}\dot{x}^T$:

$$\dot{V} = \dot{x}_\perp^T K x_\perp - f(x_{||}) N \dot{x}_{||}^T - \dot{p} p^{-3} \dot{x}^T M \dot{x} - \dot{x}^T K x_\perp - p^{-1} \dot{x}^T D \dot{x}$$

$$+ \dot{p} p^{-3} \dot{x}^T M \dot{x} + \dot{x}^T f(x_{||}) N + \dot{x}^T F_{\text{ext}} \tag{16}$$

Rearranging terms :

$$\dot{V} = \dot{x}_\perp^T K x_\perp - \dot{x}^T K x_\perp + \dot{p} p^{-3} \dot{x}^T M \dot{x} - \dot{p} p^{-3} \dot{x}^T M \dot{x} +$$

$$-f(x_{||})N\dot{x}_{||}^T + \dot{x}^T f(x_{||})N - p^{-1}\dot{x}^T D\dot{x} + \dot{x}^T F_{\text{ext}} \tag{17}$$

Cancelling terms and rewriting $\dot{x} = \dot{x}_{||} + \dot{x}_{\perp}$:

$$\dot{V} = \dot{x}_{\perp}^T K x_{\perp} - (\dot{x}_{||} + \dot{x}_{\perp})^T K x_{\perp} - f(x_{||})N\dot{x}_{||}^T +$$

$$(\dot{x}_{||} + \dot{x}_{\perp})^T f(x_{||})N - p^{-1}\dot{x}^T D\dot{x} + \dot{x}^T F_{\text{ext}} \tag{18}$$

Since K is orthogonal and $x_{||} \cdot x_{\perp} = 0$, $N \cdot x_{\perp} = 0$ we simplify to:

$$\dot{V} = \dot{x}_{\perp}^T K x_{\perp} - \dot{x}_{\perp}^T K x_{\perp} - f(x_{||})N\dot{x}_{||}^T + f(x_{||})N\dot{x}_{||}^T - p^{-1}\dot{x}^T D\dot{x} + \dot{x}^T F_{\text{ext}} \tag{19}$$

Cancelling terms:

$$\dot{V} = -p^{-1}\dot{x}^T D\dot{x} + \dot{x}^T F_{\text{ext}} \tag{20}$$

Since D is positive definite and $p \geq p_{\text{min}} \geq 0$:

$$-p^{-1}\dot{x}^T D_0\dot{x} \leq 0 \tag{21}$$

Therefore

$$\dot{V} = -p^{-1}\dot{x}^T D_0\dot{x} + \dot{x}^T F_{\text{ext}} \leq \dot{x}^T F_{\text{ext}}. \tag{22}$$

Thus for any linear trajectory our system is passive with respect to the force-velocity, $(F_{\text{ext}}, \dot{x}^T)$, input output pair. Furthermore, reasonably well-behaved trajectories can be effectively approximated as a series of linear trajectories, and as more lines are added, the approximation becomes arbitrarily faithful. This robust heuristic suggests stability even for more complex nonlinear paths, demonstrating the practicality and reliability of our language controller.

References

[1] J. Wyatt, L. Chua, J. Gannett, I. Goknar, and D. Green, "Energy concepts in the state-space theory of nonlinear n-ports: Part i-passivity," *IEEE transactions on Circuits and Systems*, vol. 28, no. 1, pp. 48–61, 1981.