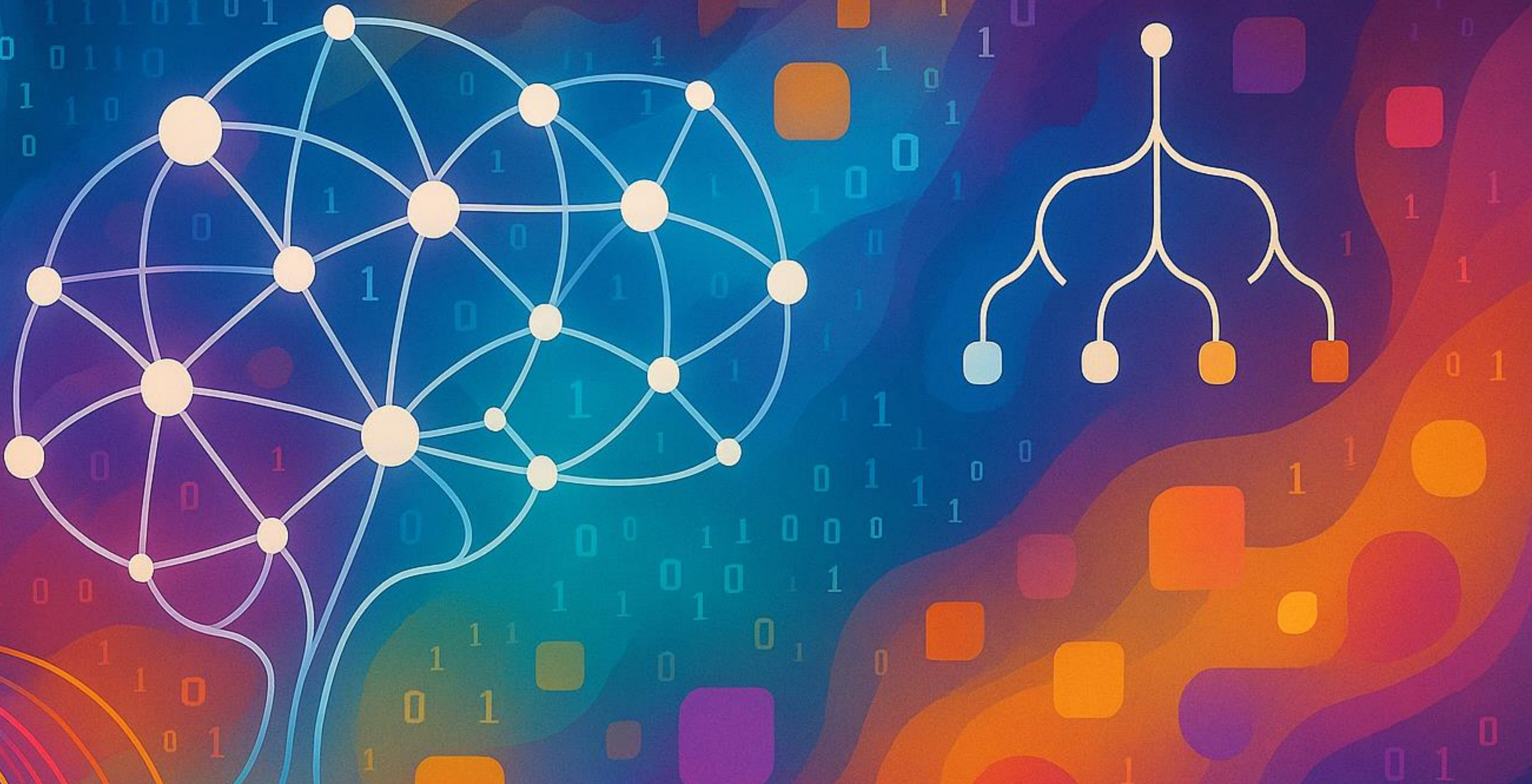

# Tutorial on Language Generation

***Organizers:*** *Moses Charikar, Anay Mehrotra, Charlotte Peale, Chirag Pabbaraju, Grigoris Velegkas*

# Validity–Breadth Trade-Off (Part II)

**This Talk:**

- Lower Bound Proof for "Uniqueness Property"
- Algorithm for Achieving Breadth Infinitely Often

# Lower Bound on Breadth

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*Uniqueness condition*: if a generator satisfies $P$ for some $L$, it cannot satisfy $P$ for some other $L'$

# Angluin's Condition for Identification

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Any collection $\mathcal{L}$ is identifiable in the limit iff every $L \in \mathcal{L}$ has a *finite* tell-tale subset $T_L \subseteq L$, i.e.,

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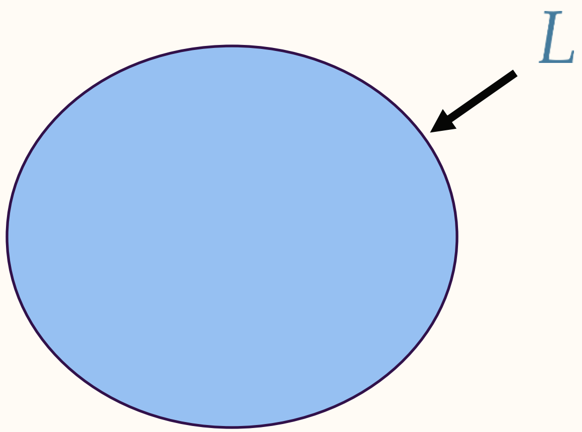

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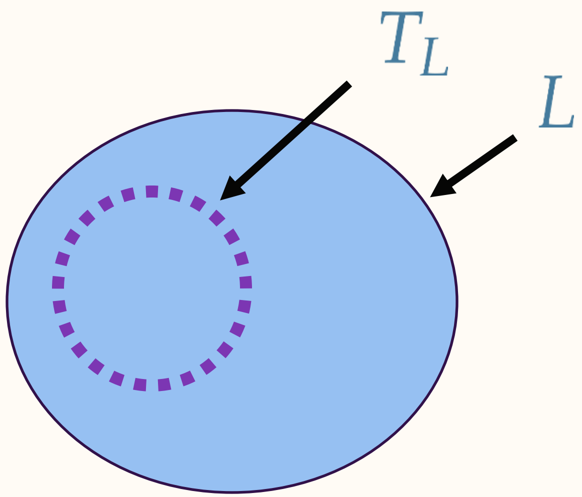

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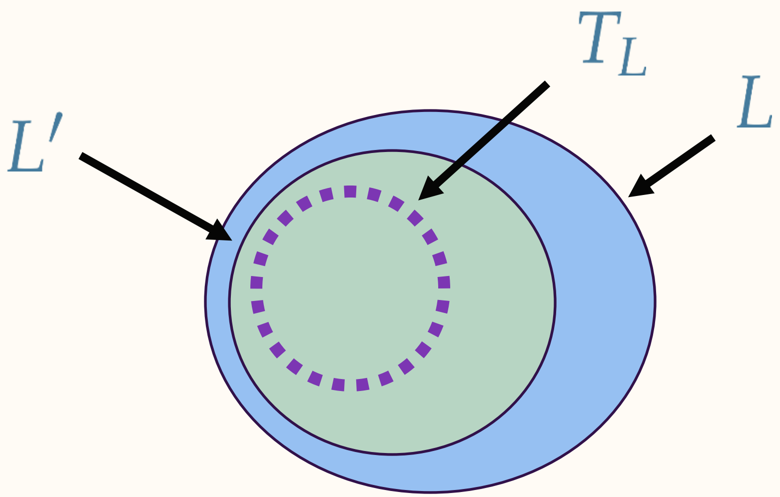

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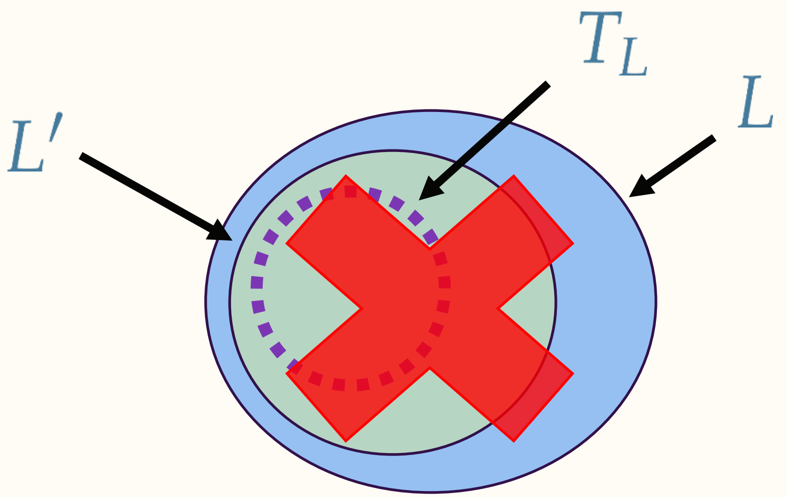
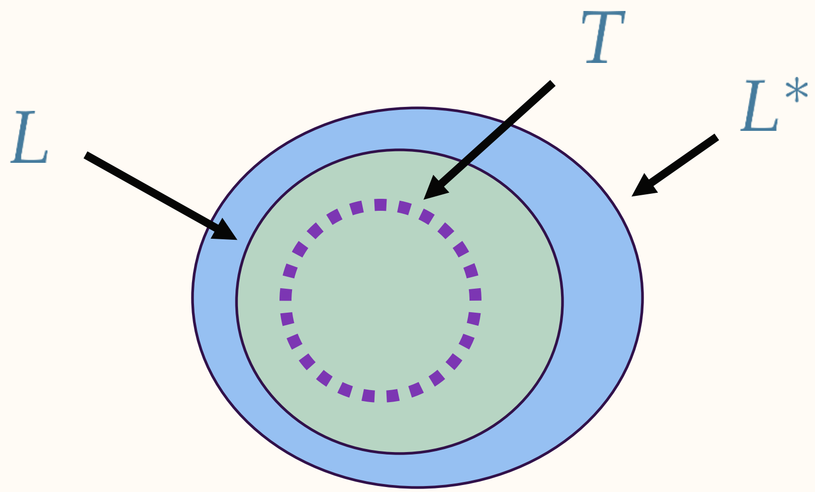


# Angluin's Condition for Identification

- If $L^*$ doesn't have a tell-tale, for every $T$ (finite subset of $L^*$), there is some $L$ such that: i) $T$ is subset of $L$, and ii) $L$ is proper subset of $L^*$



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- Collection isn't identifiable, there's $L^*$ violating Angluin's condition
- $E$, $K$ are chosen via *diagonalization* by applying the (negation of) Angluin's condition on $L^*$ repeatedly

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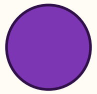
$L^*$



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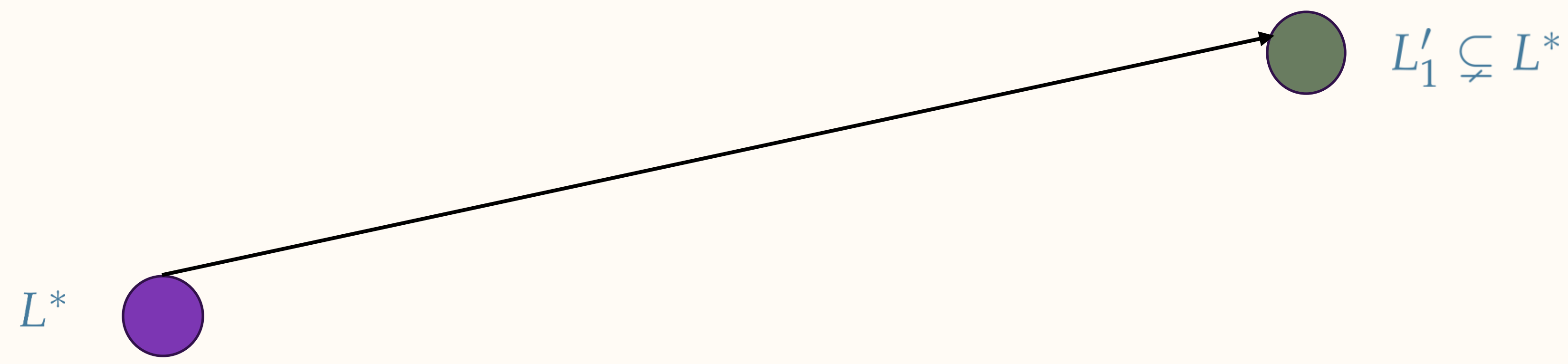

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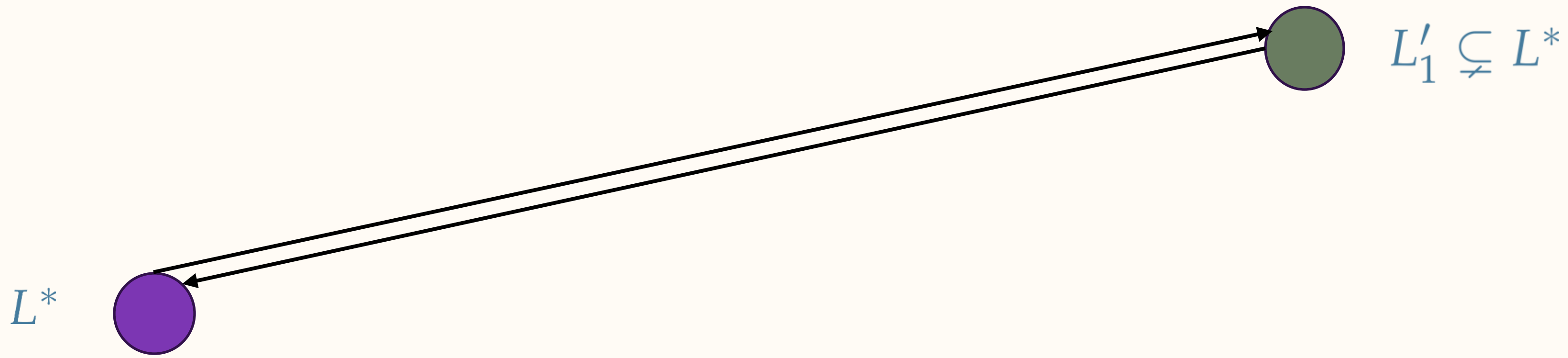

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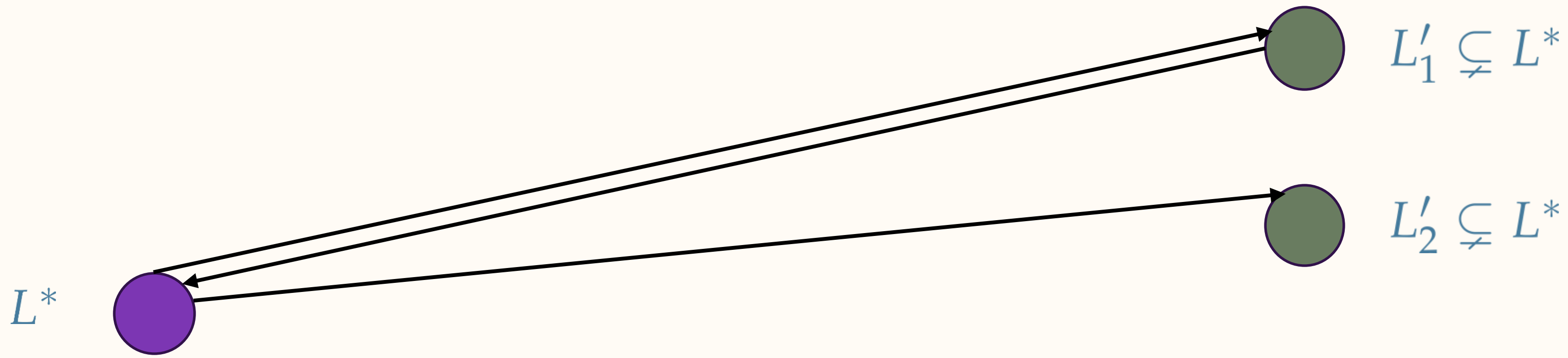

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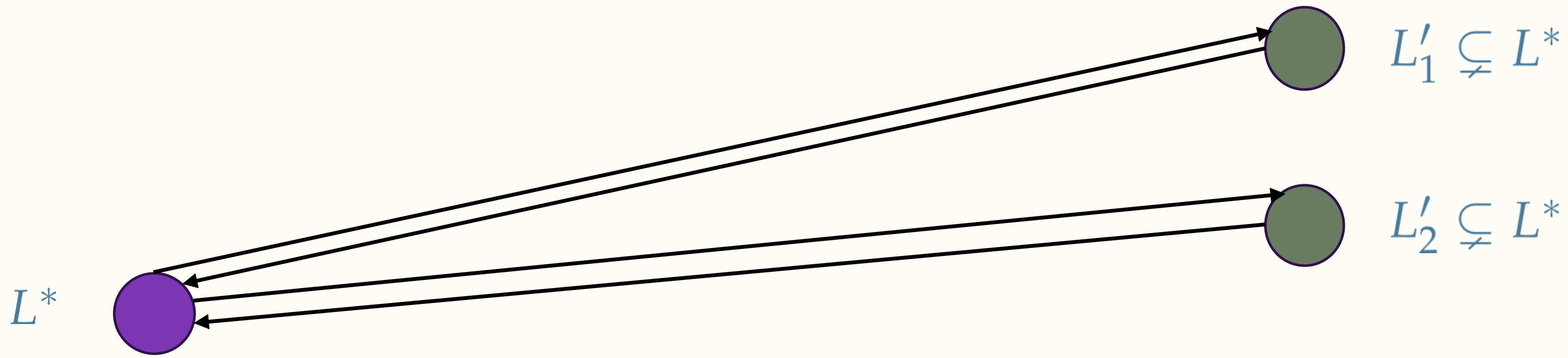

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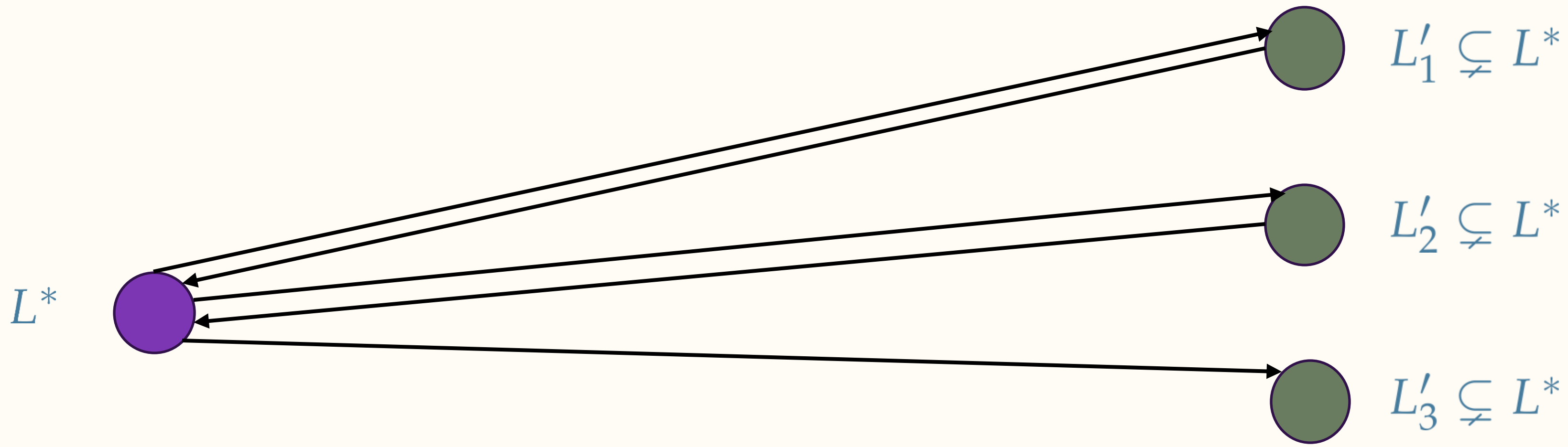

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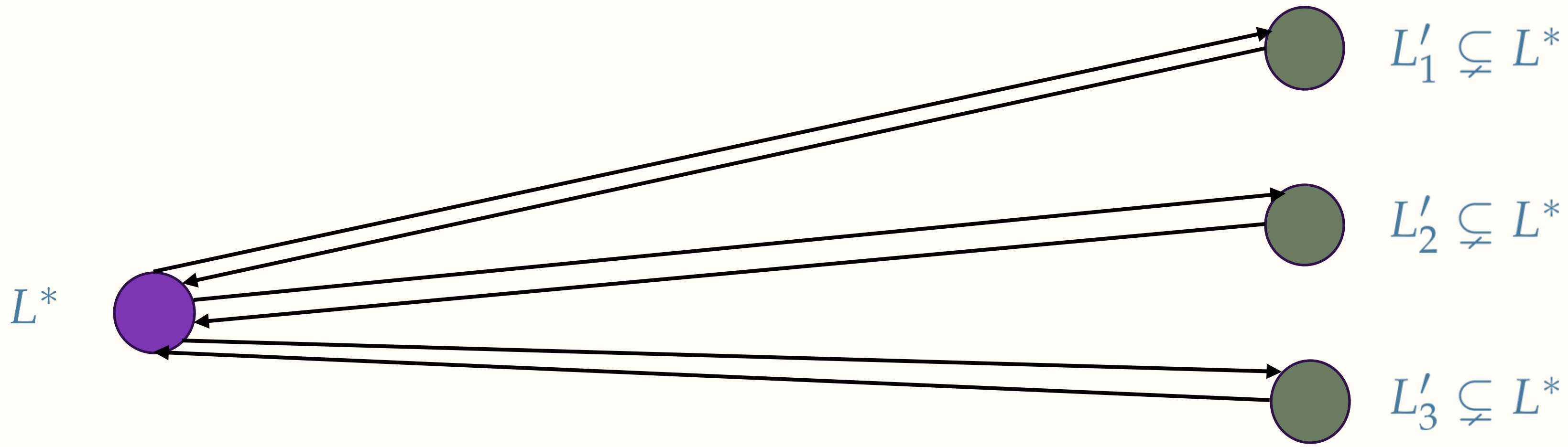

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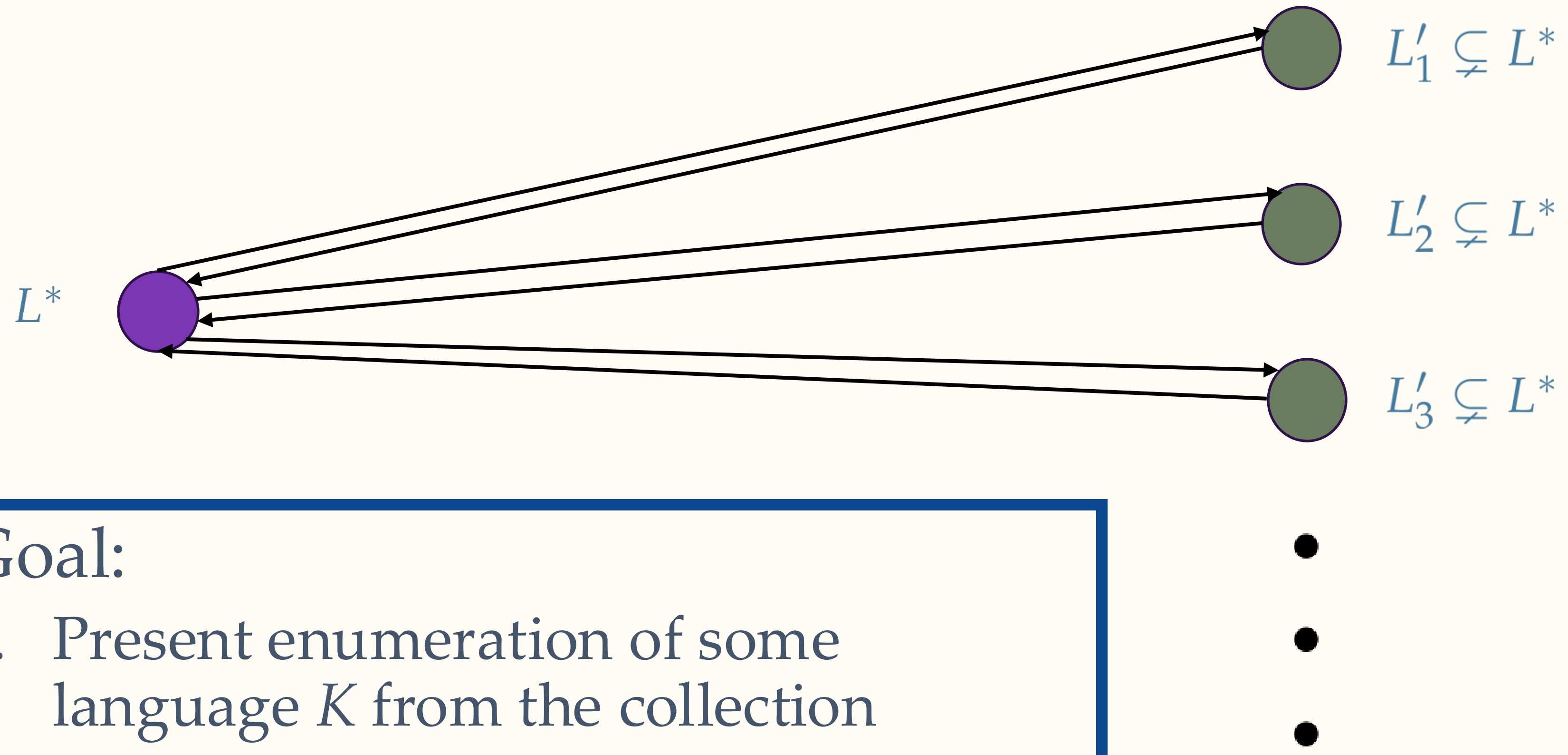


- Goal:
  1. Present enumeration of some language $K$ from the collection
  2. Make the generator miss $P$ infinitely often

**Lower Bound on Generation with Breadth** [CP'25], [KMV'24].
If a notion of breadth $P$ satisfies the *uniqueness condition* and $\mathcal{L}$ isn't identifiable, no generator can achieve $P$.

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- Since $\mathcal{L}$ isn't identifiable there's $L^*$ that violates Angluin's condition

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- Generator achieves $P$ at $t_1$ for $L^*$ ; let $S_{t_1}$ be enumerated elements

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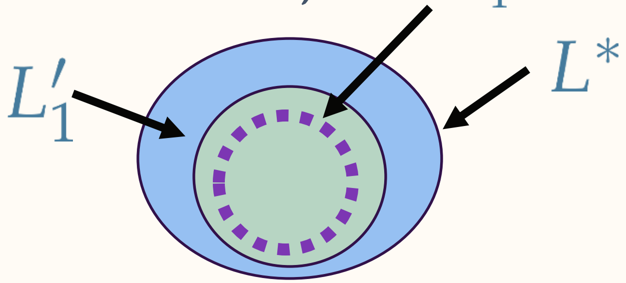

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## At time $t_1$

1. Generator cannot achieve $P$ for $L'_1$ at $t_1$ (uniqueness + strict subset of $L^*$)
2. Adversary switches to enumerating $L'_1$ ("valid" move!)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $S_{t_1+1}$ | $x_1$ | ... | $x_{t_1}$ | $x_{t_1+1}$ | $x_{t_1+2}$ | $x_{t_1+3}$ | $x_{t_1+4}$ | ... |

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2. This is again a valid move

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# Lower Bound Proof

- If generator doesn't satisfy $P$ for $L^*$ at $t_3 > t_2$ lower bound witnessed

$$\vdots$$

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  - Either infinitely many transitions in the "bipartite graph"
  - Or, if we stop at node, it is never satisfied from some point on for this node
    - Otherwise the adversary would have moved on

# Algorithm for Generation in the Limit

At step t, only consider $L_1, \ldots, L_t$

$L_{n_t}$ : critical language with highest index $n_t \leq t$

Generate a string from $L_{n_t} \setminus S_t$

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**Proof sketch:**

For large enough $t$, target language $K$ is critical and in $L_1, \ldots, L_t$

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**Proof sketch:**

For large enough *t*, target language $K$ is critical and in $L_1, \ldots, L_t$

$L_{n_t} \subseteq K$ , so any string from $L_{n_t} \setminus S_t$ also belongs to $K \setminus S_t$

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- Alternative view: the KM algorithm produces indices $i_1, i_2, \ldots$
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- Idea: The algorithm adds a "backtracking" step in a controlled way

# Chains of Critical Languages

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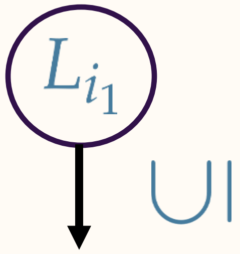
$L_{i_1}$

# Chains of Critical Languages



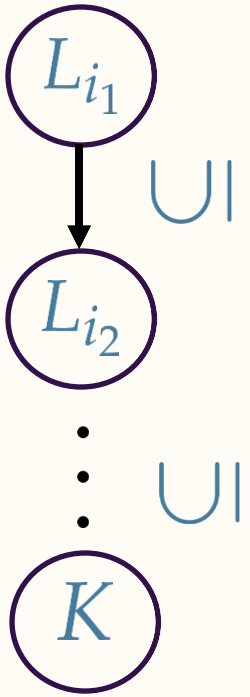
# Chains of Critical Languages

$$
\begin{tikzcd}
L_{i_1} \arrow[d, "\subsetneq"] \\
L_{i_2}
\end{tikzcd}
$$

# Chains of Critical Languages

$$\begin{tikzcd}
L_{i_1} \arrow[d, "\rotatebox{-90}{$\supseteq$}"] \\
L_{i_2} \arrow[d, phantom, "\rotatebox{-90}{$\supseteq$}"] \\
\vdots
\end{tikzcd}$$

# Chains of Critical Languages



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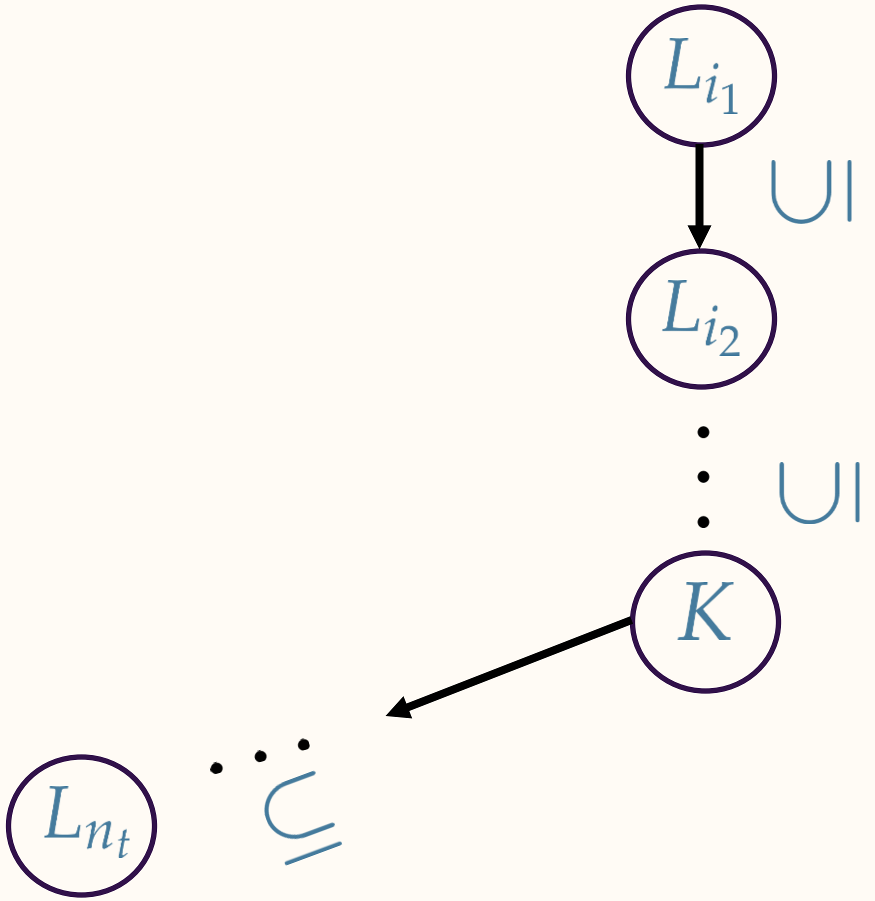

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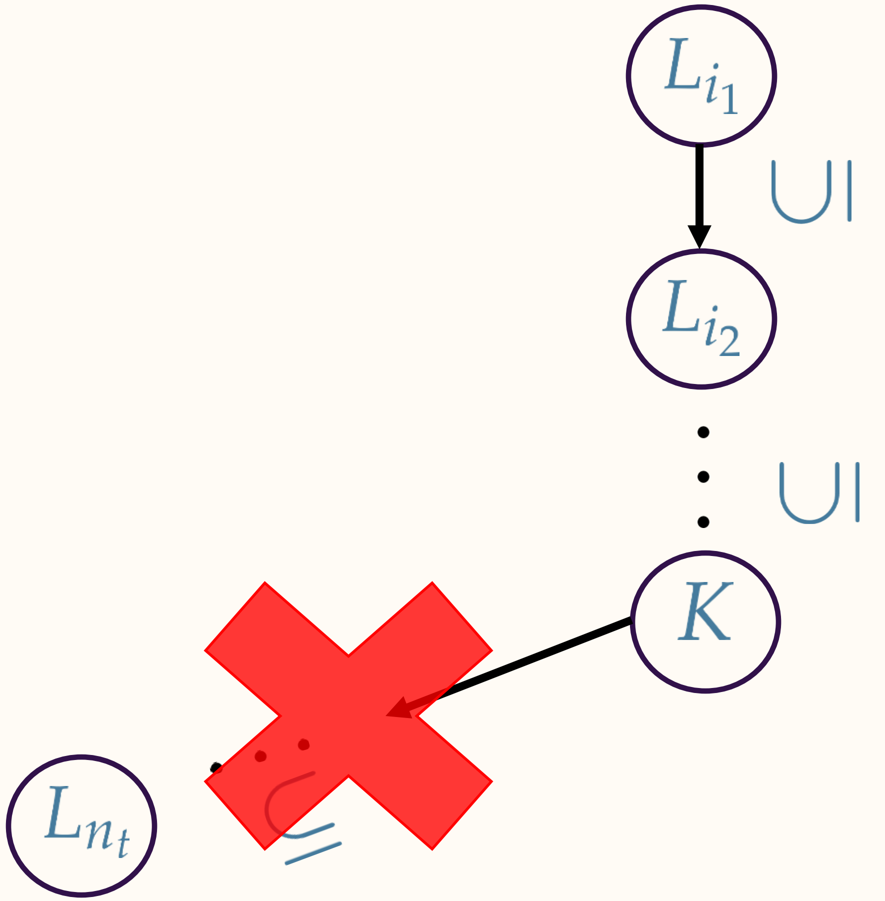

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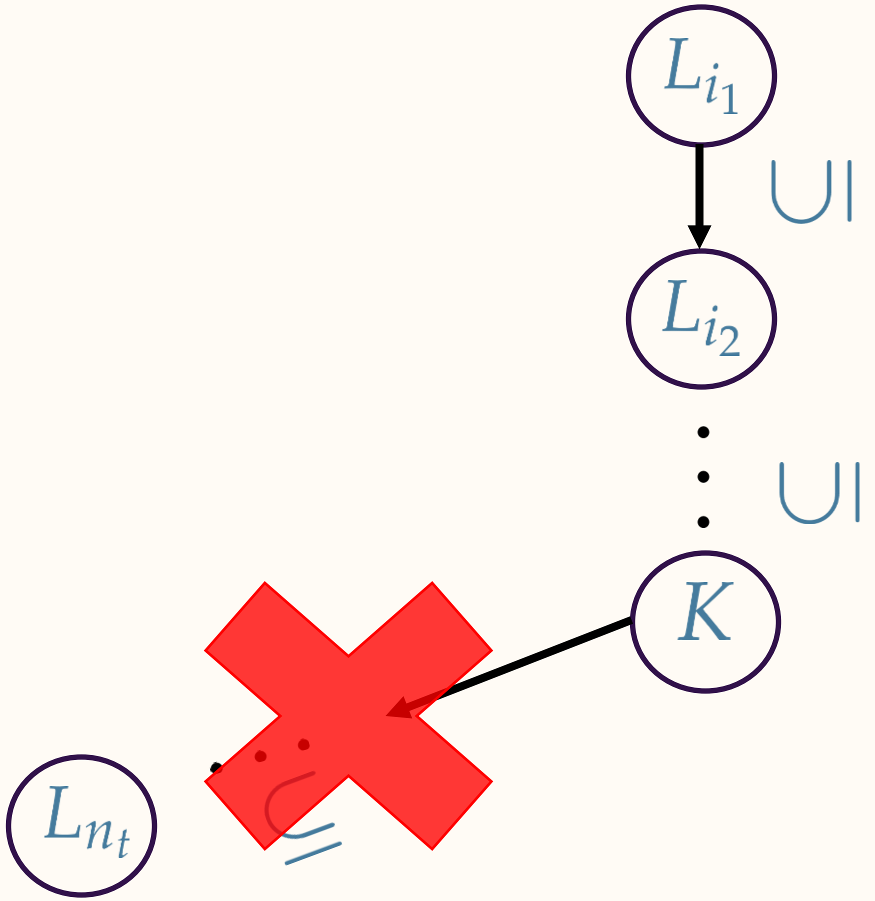

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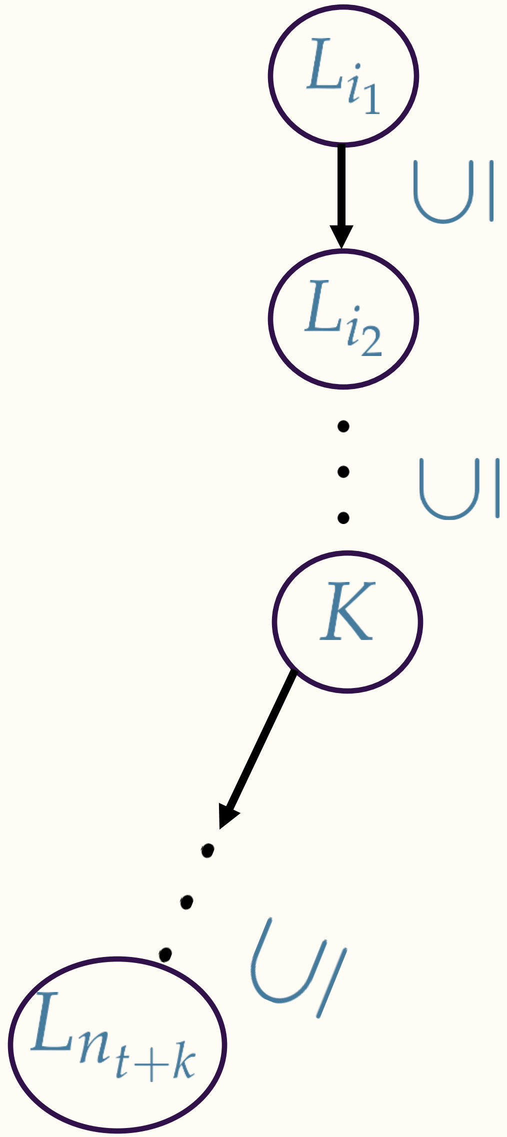

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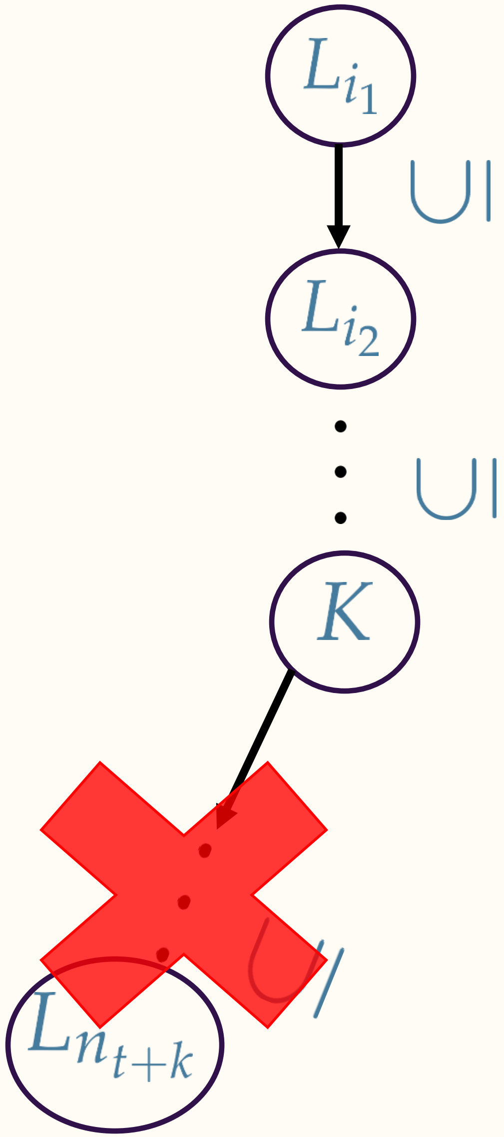

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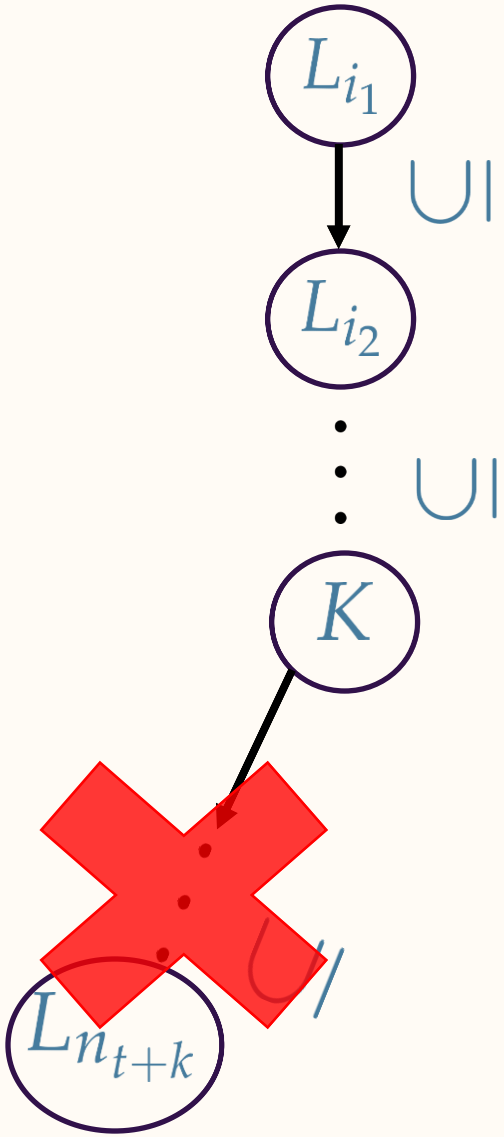

# Algorithm of [Kleinberg and Wei, 2025]

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After large enough $t$, the prefix of the chain of critical languages up to $K$ stops changing

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After large enough $t$, the prefix of the chain of critical languages up to $K$ stops changing

After large enough $t$, if a critical language gets contradicted, then we know it appears after $K$, hence in the next round we can "backtrack" to the "previous" critical language!

# Algorithm of [Kleinberg and Wei, 2025]

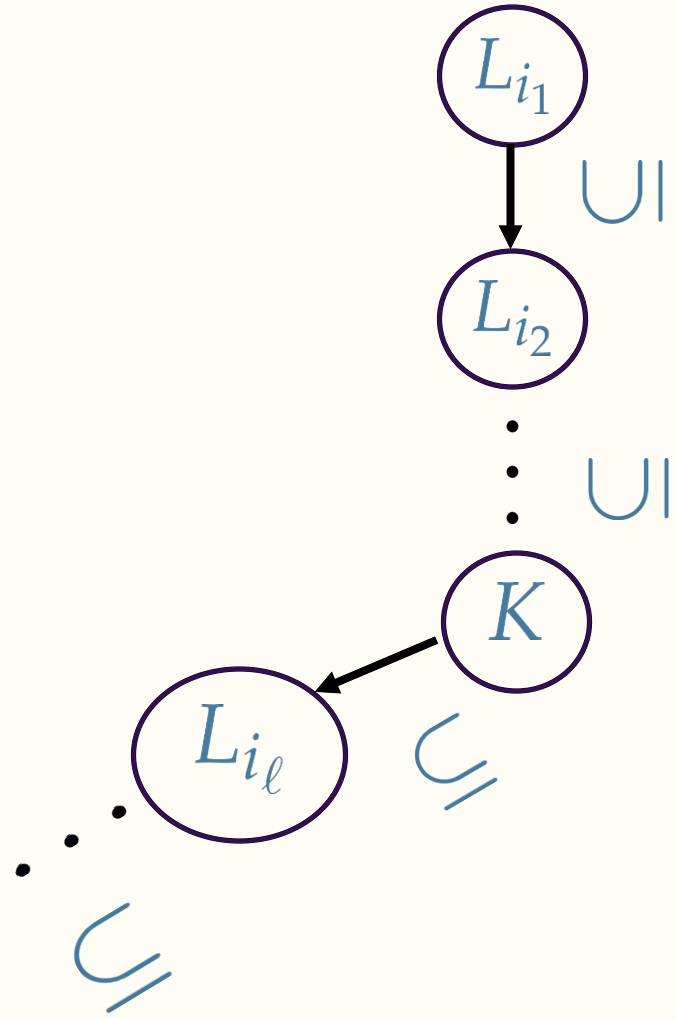
At step t, only consider $L_1, \ldots, L_t$

$L_{n_t}$ : critical language with highest index $n_t \leq t$

If all critical languages from $t-1$ remain critical, output $n_t$

Otherwise output the index of the critical language that appears before the first contradicted language

# Chains of Critical Languages



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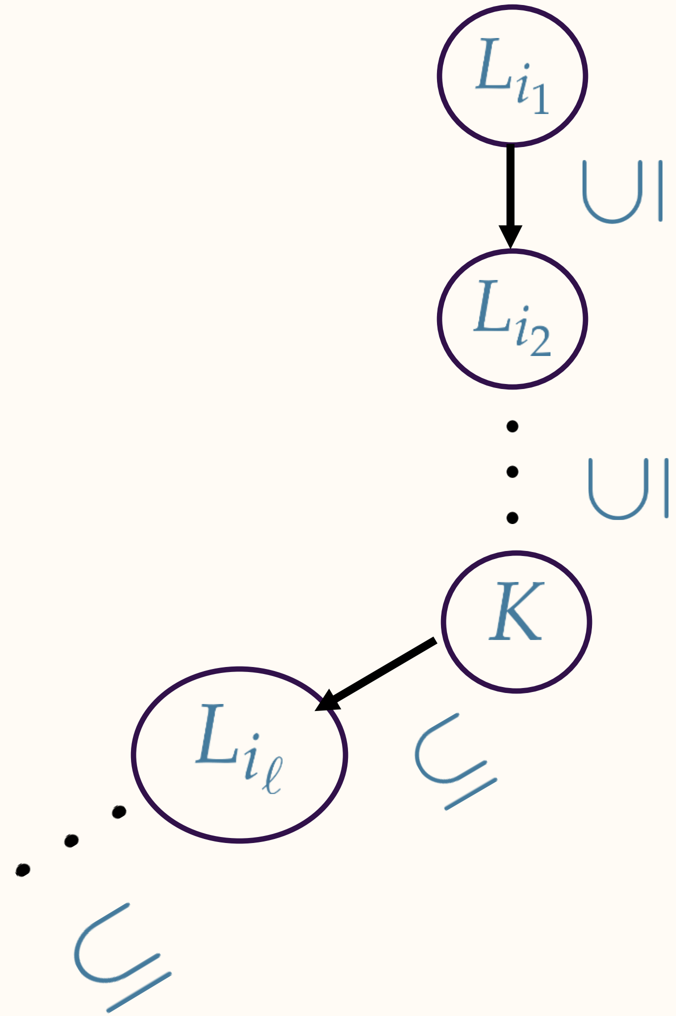

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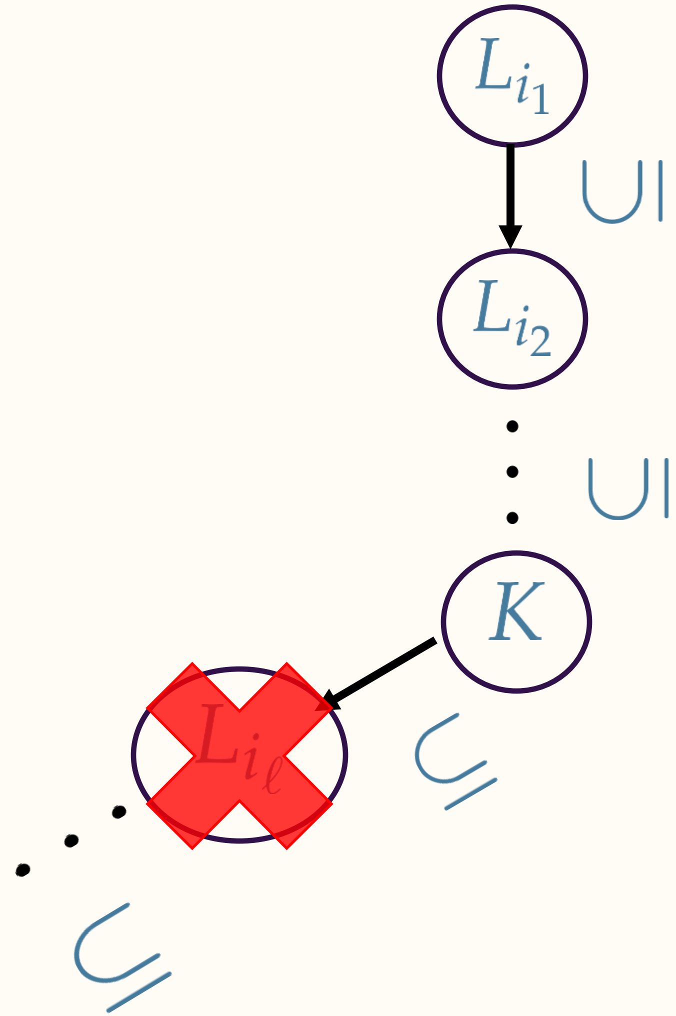

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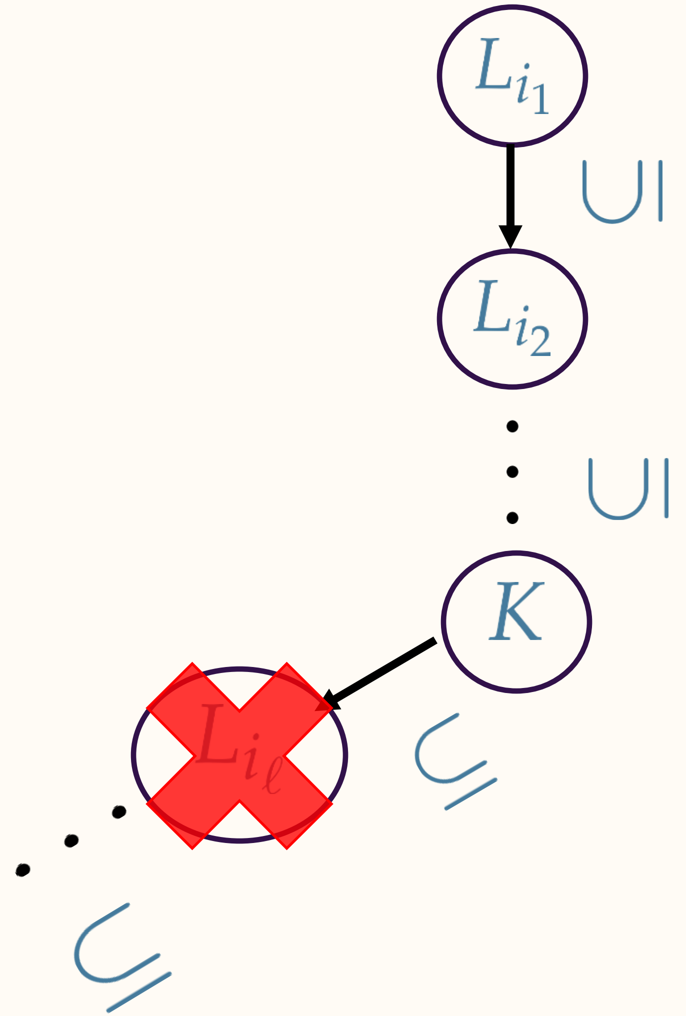

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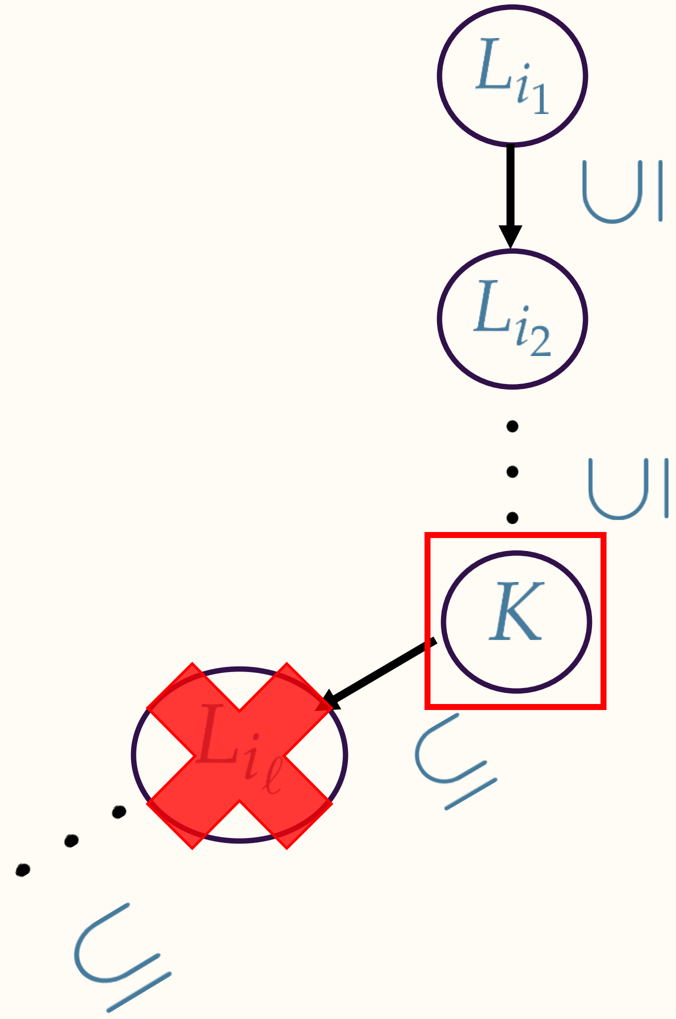

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*Thank you!*