INDUCED REPRESENTATIONS AND THEIR CHARACTERS

These notes contain an exposition of the Borel-Weil-Bott theorem, which identifies irreducible representations of a compact Lie group G as sections of certain line bundles over the flag variety G/T ("induced" from representations of T). We also include an exposition of the Weyl character formula as a character formula for the induced representation.

All representations will be finite dimensional over $\mathbf{C}$.

1. Lie Theory

We will be switching between three types of objects:

(1) Complex semisimple Lie algebras $\mathfrak{g}_{\mathbf{C}}$.
(2) Complex connected Lie group $G_{\mathbf{C}}$ whose Lie algebra is semisimple.
(3) Real compact connected Lie group G whose Lie algebra is semisimple.

Imposing the condition that objects in (2) and (3) being simply connected, these will be classified by Dynkin diagrams and have essentially the same representation theories. Otherwise, the representation theories can be slightly different (see Remark 1.4). Our focus will be the representation theory of (3), but we need (1) and (2) as convenient models.

Here are some common examples, where $G_{\mathbf{C}}$ is the complexification of G and $\mathfrak{g}_{\mathbf{C}}$ is the Lie algebra of $G_{\mathbf{C}}$ (notice that special orthogonal groups are not simply connected):

G	$G_{\mathbf{C}}$	$\mathfrak{g}_{\mathbf{C}}$	Type
$\mathrm{SU}(n+1)$	$\mathrm{SL}(n+1,\mathbf{C})$	$\mathfrak{sl}(n+1,\mathbf{C})$	$A_n, n\geq 1$
$\mathrm{SO}(2n+1),\mathrm{Spin}(2n+1)$	$\mathrm{SO}(2n+1,\mathbf{C}),\mathrm{Spin}(2n+1,\mathbf{C})$	$\mathfrak{so}(2n+1,\mathbf{C})$	$B_n, n\geq 2$
$\mathrm{Sp}(n)$	$\mathrm{Sp}(2n,\mathbf{C})$	$\mathfrak{sp}(2n,\mathbf{C})$	$C_n, n\geq 3$
$\mathrm{SO}(2n),\mathrm{Spin}(2n)$	$\mathrm{SO}(2n,\mathbf{C}),\mathrm{Spin}(2n,\mathbf{C})$	$\mathfrak{so}(2n,\mathbf{C})$	$D_n, n\geq 4$

We will abbreviate $\mathfrak{g}$ for $\mathfrak{g}_{\mathbf{C}}$ from now on.

Choose $\mathfrak{h}\subseteq\mathfrak{g}$ a Cartan subalgebra (a maximal subalgebra such that the adjoint representation is diagonalizable). This corresponds to choosing a maximal torus $T<G$ or $T_{\mathbf{C}}<G_{\mathbf{C}}$, which is unique up to conjugation. Just like every representation of T splits as a direct sum of 1-dimensional irreducible representations, every representation of $\mathfrak{h}$ also splits as a direct sum of such irreducible representations. In particular, this applies to the adjoint action $\mathfrak{h}\circlearrowright\mathfrak{g}$:

$$\mathrm{ad}:\mathfrak{h}\to\mathfrak{gl}(\mathfrak{g}_{\mathbf{C}}),\quad \mathrm{ad}(h)=[h,-].$$

Under this action, $\mathfrak{g}$ has a Cartan decomposition

$$\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi\subseteq\mathfrak{h}^*}\mathfrak{g}_\alpha.$$

Here $\mathfrak{g}_\alpha$ is the weight space (common eigenvector) of the $\mathfrak{h}$ action with weight (common eigenvalue) α: namely, for $x \in \mathfrak{g}_\alpha$ and $h \in \mathfrak{h}$,

$$[h, x] = \alpha(h)x.$$

Here $\Phi \subseteq \mathfrak{h}^*$ is the set of weights with non-empty weight spaces, called the roots of $\mathfrak{g}$.

Moreover there is a symmetric bilinear form on $\mathfrak{g}$ called the Killing form:

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbf{C}, \quad \kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y)).$$

It turns out that $\kappa|_{\mathfrak{h}}$ is non-degenerate, and this provides a canonical isomorphism $\mathfrak{h}^* \cong \mathfrak{h}$, sending

$$\lambda \mapsto t_\lambda \text{ such that } \kappa(t_\lambda, -) = \lambda.$$

Furthermore, one "transports" the bilinear form κ to $\mathfrak{h}^*$ by defining

$$(\lambda, \mu) := \kappa(t_\lambda, t_\mu).$$

Consider the $\mathbf{Q}$-span of the roots Φ in $\mathfrak{h}^*$, denoted $\mathfrak{h}^*_{\mathbf{Q}}$. It turns out that this is a $\mathbf{Q}$-vector space of dimension $\dim_{\mathbf{C}}(\mathfrak{h}^*)$. Furthermore, the bilinear form $(-,-)$ restricts to $\mathfrak{h}_{\mathbf{Q}}$ is positive definte. The triple $(\mathfrak{h}^*_{\mathbf{Q}}, \Phi, (-,-))$ together consist a root system. Simple Lie algebras can be classified by their root systems, which in turn are classified by Dynkin diagrams.

Example 1.1. We will illustrate this and many other general concepts by the example of $\mathfrak{sl}(3, \mathbf{C})$.

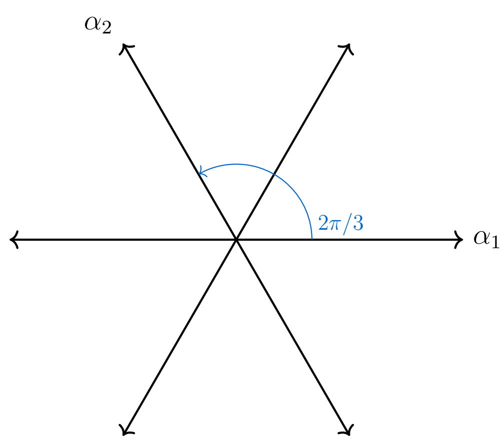

FIGURE 1. Root system of $\mathfrak{sl}(3, \mathbf{C})$, which has type A_2. Two positive roots α_1 and α_2 are chosen.

Identify $\mathfrak{sl}(3, \mathbf{C})$ with traceless 3×3 complex matrices. Choose the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{sl}(3, \mathbf{C})$ to be the diagonal matrices in $\mathfrak{sl}(3, \mathbf{C})$, which has dimension 2. There are 6 roots $\varepsilon_i - \varepsilon_j$ for $1 \leq i, j \leq 3$ and $i \neq j$. The 1-dimensional space $\mathfrak{g}_\alpha$ corresponding to the root α is spanned by the elementary matrix E_{ij}.

Here are some general notions applicable to any root system that can be illustrated in this special case:

(1) Weyl group: The group

$$W = \langle s_\alpha : \alpha \in \Phi \rangle$$

generated by s_α's, where

$$s_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha.$$

is reflection along the hyperplane orthogonal to α.

(2) Simple roots: A collection of roots $\Delta \subset \Phi$ such that Δ is a $\mathbf{Q}$-basis of $\mathfrak{h}^*_{\mathbf{Q}}$ and $\Phi \subset \mathbf{Z}\Delta$. There are many potential choices of Δ, and different choices differ by a Weyl group action. This choice splits $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^+ := \mathbf{N}\Delta \cap \Phi$ is the positive roots.

(3) Weight lattice: For each $\alpha \in \Phi^+$, there exist a unique $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$. Define

$$P := \{\lambda \in \mathfrak{h}^* : \lambda(h_\alpha) \in \mathbf{Z},\ \forall \alpha \in \Phi\} \subset \mathfrak{h}^*_{\mathbf{Q}}.$$

We call the set

$$P^+ := \{\lambda \in \mathfrak{h}^* : \lambda(h_\alpha) \in \mathbf{N},\ \forall \alpha \in \Phi^+\} \subset P$$

the dominant weights, and the set $P^- := -P^+$ the anti-dominant weights.

The representations of semisimple Lie algebras are determined by highest (or lowest) weights.

Theorem 1.2. *We have the following commutative diagram of sets:*

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{finite dimensional}\\ \text{irreducible}\\ \text{representations of } \mathfrak{g}\end{array}\right\} \arrow[r, "V_\rho \mapsto (V_\rho)^*", "\cong"'] \arrow[d, "\text{highest weight}"', "\cong"] & \left\{\begin{array}{c}\text{finite dimensional}\\ \text{irreducible}\\ \text{representations of } \mathfrak{g}\end{array}\right\} \arrow[d, "\cong"', "\text{lowest weight}"] \\
P^+ \arrow[r, "\lambda \mapsto -\lambda", "\cong"'] & P^-
\end{tikzcd}$$

In words, finite dimensional irreducible representations of $\mathfrak{g}$ correspond bijectively to weights in P^+ (resp. P^-) by taking the highest (resp. lowest) weight. Moreover, for any $\lambda \in P^+$, the irreducible representation with lowest weight $-\lambda$ is dual to the irreducible representation with highest weight λ.

Example 1.3. Examples for $\mathfrak{sl}(3, \mathbf{C})$: the adjoint representation and the standard representation. See Figure 2.

Remark 1.4. The finite dimensional irreducible representations of G and $G_{\mathbf{C}}$ are also determined by highest weight theory. However, the classification of those representations can be slightly different.

If G is a real compact Lie group whose complexified Lie algebra is $\mathfrak{g}$ but G is not simply connected, then there is a sub-lattice $L(G) \subset P$ in $\mathfrak{h}^*_{\mathbf{Q}}$ such that $L^+(G) := L(G) \cap P^+$ correspond bijectively to irreducible representations of G. This $L(G)$ corresponds to the dual lattice $\hat{T} = \mathrm{Hom}(T, \mathbb{T})$ of the maximal torus $T < G$ (the analytically integral weights). In fact, $P/L(G) \cong \pi_1(G)$ and $Q \subset L(G) \subset P$ where $Q = \mathbf{Z}\Phi$ is called the root lattice. The same happens for $G_{\mathbf{C}}$.

Example 1.5. The simply connected group $\mathrm{SU}(2)$ is a double cover of the group $\mathrm{SO}(3)$. Both have complexified Lie algebra being $\mathfrak{so}(3, \mathbf{C})$. It turns out that $L(\mathrm{SO}(3)) = 2P$ and $L(\mathrm{SU}(2)) = P$, see Figure 3.

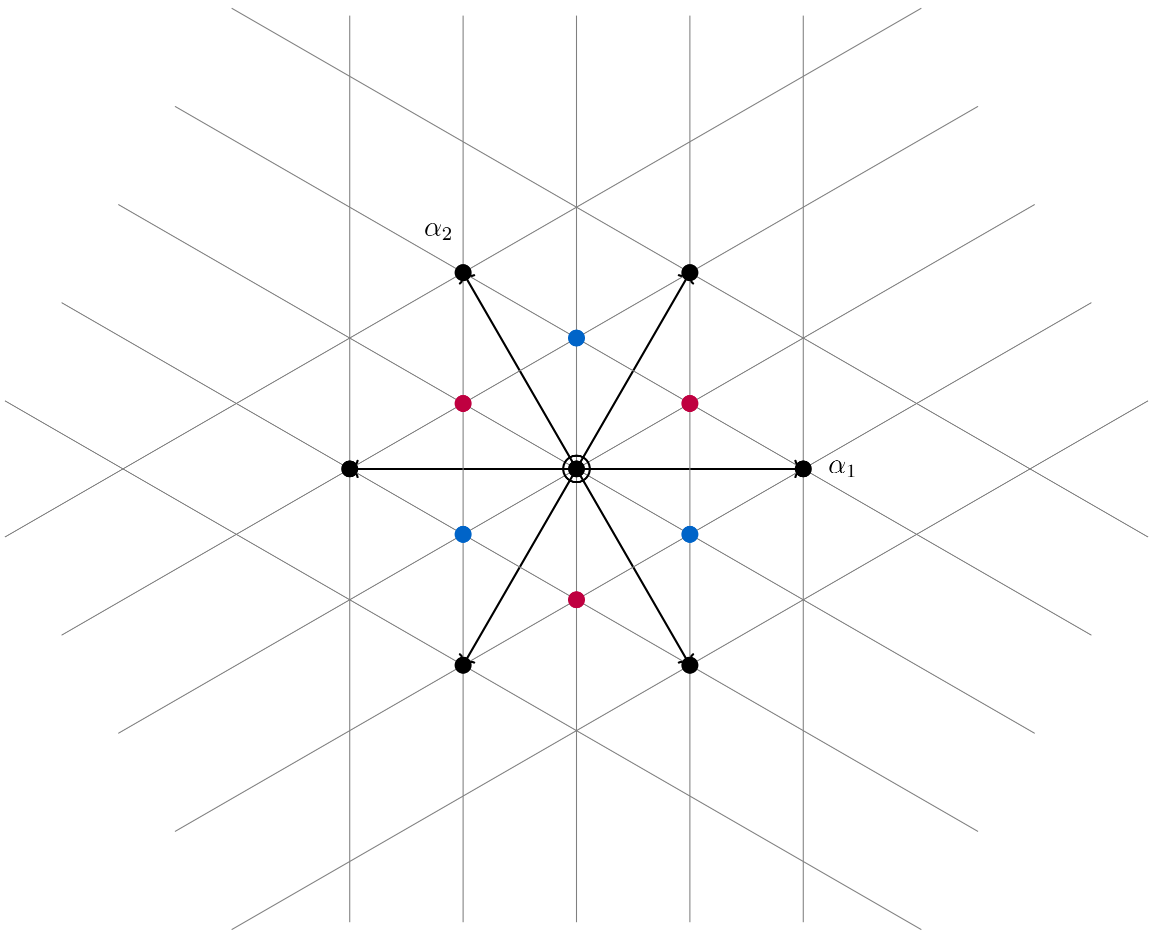

FIGURE 2. Three representations: the standard representation (blue), its dual (red) and the adjoint representation (black). Double dot means the weight space is 2-dimensional.

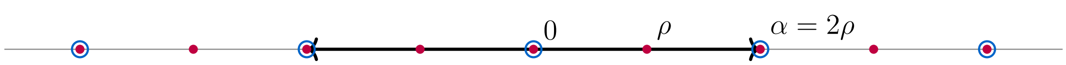

FIGURE 3. Root system of $\mathfrak{sl}(2, \mathbf{C})$. Purple dots are on the weight lattice $L(\mathrm{SU}(2))$, while blue dots are on the weight lattice $L(\mathrm{SO}(3)) = 2P$ of the Lie group $\mathrm{SO}(3)$.

2. Motivating Example: Finite Groups

With the understanding that finite dimensional irreducible representations of G are classified by highest weights, our goals will be:

(1) Find an explicit construction of irreducible representations using data from just the Lie group G.
(2) Find an expression for the character of the representation, using just the highest weight and data from root system.

Both of these goals can be accomplished by constructing representations of G using representations of $T < G$, which is an abelian group and hence significantly easier. Namely, we seek a map

$$\mathrm{Rep}(T) \to \mathrm{Rep}(G).$$

A first attempt would be to mimic the induced representation for finite groups $H < G$

$$\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G).$$

which is a left adjoint to $\mathrm{Res}_G^H : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$. For finite groups, this is accomplished by considering the $\mathbf{C}[G]$-module

$$\mathrm{Ind}_H^G(V_\rho) := \mathbf{C}[G] \otimes_{\mathbf{C}[H]} V_\rho \cong \Gamma(G/H, G \times_H V_\rho)$$

where for $s \in \Gamma(G/H, G \times_H V_\rho)$, G acts via

$$(g.s)([x]) = s(g^{-1}x).$$

Notice that G/H discrete, so there is no need to specify what type of section we are taking.

Moreover, we have the Frobenius formula for the induced representation

$$\chi_{\mathrm{Ind}_H^G(V_\rho)}(g) = \sum_{x \in F} \chi_{V_\rho}(x^{-1}gx) \tag{1}$$

where $F = \{x \in G : x^{-1}gx \in H\}$ is the fixed point set of the action of g on G/H. It will turn out that fixed point considerations for Lie groups will lead to the Weyl character formula.

For Lie groups the situation is more complicated. Given $H < T$ and an H-representation V_ρ, one may still take continuous or C^∞-sections

$$\Gamma(G/H, G \times_H V_\rho),$$

which would be an infinite dimensional representation in general. However, in the case of $H = T$, there is a complex structure (see §3) and one can take the subspace of holomorphic sections,

$$\Gamma_{\mathrm{hol}}(G/T, G \times_T V_\rho) = \mathrm{H}^0(G \times_T V_\rho).$$

This turns out to a finite-dimensional representation of G.

Remark 2.1. The representation $\mathrm{H}^0(G \times_T V_\rho)$ is sometimes also called an induced representation, although it is not a left adjoint to Res_G^T.

The functor $\mathrm{H}^0(-)$ has higher derived functors (equivalently, Dolbeault cohomology groups)

$$\mathrm{H}^i(G \times_T V_\rho) \cong \mathrm{H}^{0,i}(G/T, G \times_T V_\rho).$$

One may collect all the higher derived functors together to get a map

$$\mathrm{Rep}(T) \to \mathrm{Rep}(G), \quad V_\rho \mapsto \sum_i (-1)^i \mathrm{H}^i(G \times_T V_\rho).$$

This alternating sum of sheaf cohomologies turns out to be the pushforward in equivariant K-theory:

$$\mathrm{Rep}(T) \cong \mathrm{K}_T(*) \cong \mathrm{K}_G(G/T) \xrightarrow{\pi_!} \mathrm{K}_G(*) \cong \mathrm{Rep}(G).$$

Indeed, the corresponding complex is $(\Omega^{0,\bullet}(G \times_T V_\rho), \bar\partial)$ and the result follows from equivariant index theorem. We will construct irreducible representations using this pushforward map. We may now rewrite the two goals in terms of equivariant K-theory:

(1) Identify the image of $\mathrm{K}_T(*) \to \mathrm{K}_G(*)$, i.e., the objects $\sum_i (-1)^i \mathrm{H}^i(G \times_T V_\rho)$, in terms of highest weight theory (this is the content of the Borel-Weil-Bott theorem).
(2) Compute the image of V_ρ under $\mathrm{K}_T(*) \to \mathrm{K}_G(*) \to \mathrm{K}_T(*)$, i.e., the character of the image (this is the content of the Weyl character formula).

3. Flag Varieties

We discuss briefly the complex structure on G/T. Let $B^+ < G_{\mathbf{C}}$ be the Borel subgroup satsifying

$$\mathrm{Lie}(B^+) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

One then has a diffeomorphism

$$G/T \cong G_{\mathbf{C}}/B^+$$

This quotient is called the flag variety of G (or $G_{\mathbf{C}}$). This diffeomorphism endows G/T a complex structure coming from that on $G_{\mathbf{C}}/B^+$.

Example 3.1. Take $G = \mathrm{SU}(n)$ and $G_{\mathbf{C}} = \mathrm{SL}(n, \mathbf{C})$. One may choose the positive roots so that $B^+ < G_{\mathbf{C}}$ is the upper-triangular matrices.

Let V be an n-dimensional vector space and choose a basis for V so that we get an isomorphism $V \cong \mathbf{C}^n$. Let G and $G_{\mathbf{C}}$ act on V by matrix multiplication.

Define

$$\mathrm{Fl}(V) := \{(V_i)_{i=0}^n \mid V_i \subset V_{i+1} \subseteq V, \dim(V_i) = i\}.$$

Namely, this is the space of flags in V. The group $G_{\mathbf{C}}$ acts on $\mathrm{Fl}(V)$ transitively, and its stabilizer at $p = (0, \mathbf{C}\{e_1\}, \mathbf{C}\{e_1, e_2\}, \cdots, V)$ is exactly B^+. Thus $G_{\mathbf{C}}/B^+ \cong \mathrm{Fl}(V)$. Similarly, $\mathrm{SU}(n)$ acts on $\mathrm{Fl}(V)$ with stabilizer at p being exactly T, so $G/T \cong \mathrm{Fl}(V)$, and hence we get an isomorphism $G_{\mathbf{C}}/B^+ \cong G/T$.

Remark 3.2. In the literature, people sometimes use $G_{\mathbf{C}}/B^-$ which is also diffeomorphic to G/T. This identification changes the complex structure and leads to a formulation of the Borel-Weil-Bott theorem and Weyl character formula different from the one we give below (using $G_{\mathbf{C}}/B^-$ gives highest weight representations, while our convention $B_{\mathbf{C}}/B^+$ gives lowest weight ones.).

4. The Borel-Weil-Bott Theorem

Let $\lambda : T \to \mathbb{T}$ and denote

$$\mathscr{L}_\lambda := G \times_T \mathbf{C}_\lambda$$

to be the associated line bundle on G/T. Recall that $\mathrm{R}^-(\lambda)$ is the irreducible representation with lowest weight λ.

Theorem 4.1 (Borel-Weil-Bott)**.** *Let $\lambda \in L(G) \subseteq P$ be an integral weight. One has*

(1) If $\lambda \in P_{\mathrm{sing}}$, then $\mathrm{H}^i(G/T, \mathscr{L}_\lambda) = 0$ for all $i \geq 0$.

(2) If $\lambda \in P_{\mathrm{reg}}$, then

$$\mathrm{H}^i(G/T, \mathscr{L}_\lambda) = \begin{cases} \mathrm{R}^-(w_\lambda * \lambda), & i = \ell(w_\lambda), \\ 0, & \text{otherwise.} \end{cases}$$

We explain some notations here. The Weyl vector is the half sum of all positive roots:

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

On the space $\mathfrak{h}^*$ there is a $*$-action of W given by the usual action of W conjugated by a shift:

$$w * \lambda := w(\lambda - \rho) + \rho.$$

For a weight $\lambda \in P$, say λ is singular if there exists a non-identity $w \in W$ such that $w * \lambda = \lambda$. Otherwise there exists a unique $w_\lambda \in W$ such that $w_\lambda * \lambda$ is dominant, and λ is regular. The element w_λ has a length $\ell(w_\lambda)$ (the minimal number of $s_\alpha \in \Delta$ such that their composition is w). Write $P = P_{\mathrm{sing}} \cup P_{\mathrm{reg}}$ as a disjoint union of singular and regular weights. See Figure 4 for the example of $\mathrm{SL}_3(\mathbf{C})$.

The special case where $\lambda \in P^-$ is called the Borel-Weil theorem. In this case, $\ell(w_\lambda) = 0$ and the theorem gives

$$\sum_i (-1)^i \mathrm{H}^i(\mathscr{L}_\lambda) = \Gamma_{\mathrm{hol}}(G/T, \mathscr{L}_\lambda) \cong \mathrm{R}^-(\lambda).$$

In particular we have indeed constructed all the irreducible representations using the pushforward map.

The first step in the proof of the theorem is to analyze $\mathrm{H}^0(\mathscr{L}_\lambda)$. Take a section $s \in V_0$ where V_0 is the space of lowest weight vectors. By construction, $v(1)$ determines its value on the cell U^-B, where U^- is the subgroup corresponding to $\bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha$ in $G_{\mathbf{C}}$. The space U^-B is dense in $G_{\mathbf{C}}$, so V_0 is at most 1-dimensional. It then suffices to prove that $V_0 = 0$ unless $\lambda \notin P^-$. This can be done by reducing to the case of $\mathrm{SL}(2, \mathbf{C})$. See [5] for more details.

5. The Weyl Character Formula

We now turn to the Weyl character formula. There are two "topological" methods to derive this: either from the localization theorem or from the Lefschetz fixed point formula. In both cases, the key is to compute the fixed point $(G/T)^T$ of the left T-action on G/T: they correspond bijectively to elements in $W \cong N(T)/T$.

Theorem 5.1 (Weyl Character Formula). *Fix an integral weight $\lambda \in P^-$. One has*

$$\mathrm{ch}(\mathrm{R}^-(\lambda)) = \sum_{w \in W} w. \left(\frac{e^\lambda}{\prod_{\alpha \in \Delta^+} (1 - e^\alpha)} \right).$$

Here $w \in W$ acts via the usual action (instead of the $*$-action).

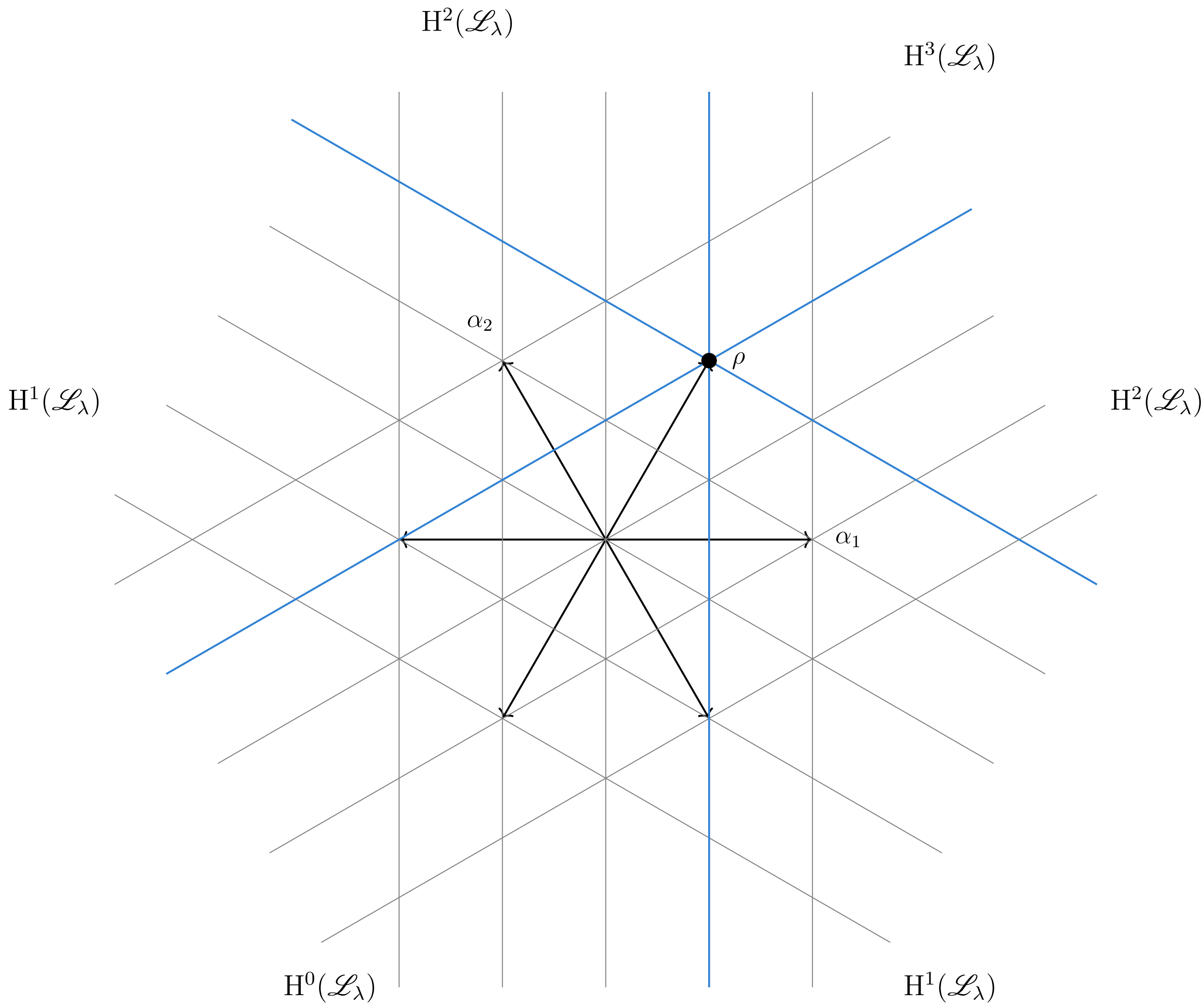

FIGURE 4. The root system of $\mathfrak{sl}(3, \mathbf{C})$ with simple roots α_1 and α_2. Reflections along the blue lines generate the $*$-action of Weyl group. P_{sing} is the intersection of P with the blue lines.

One proof of this is by applying the Lefschetz fixed point formula to the complex $(\Omega^{0,\bullet}(\mathscr{L}_\lambda), \bar{\partial})$. Noticing that the left T-action on G/T has fixed point set given exactly by $N(T)/T \cong W$, one has

$$\begin{aligned}\operatorname{tr}\left(t \mid \sum(-1)^i \mathrm{H}^i(\mathscr{L}_\lambda)\right) &= \operatorname{tr}\left(t \mid \sum(-1)^i \Omega^{0,i}(\mathscr{L}_\lambda)\right) \\ &= \sum_{w \in W} w.\left(\frac{e^\lambda}{\det(1 - d\ell_t)}\right)(t) \\ &= \sum_{w \in W} w.\left(\frac{e^\lambda}{\prod_{\alpha \in \Delta^+}(1 - e^\alpha)}\right)(t).\end{aligned}$$

This is the approach taken in [1]. For a proof using Euler class and localization theorem in K-theory, see [3].

Remark 5.2. This can be generalized to allow any $H < G$ such that G/H admits a complex structure. In the case of finite groups it recovers the Frobenius formula (1).

Remark 5.3. This form of the Weyl character formula is a slight rewriting of

$$\mathrm{ch}(\mathrm{R}^-(\lambda)) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w*\lambda}}{\prod_{\alpha\in\Phi^+}(1-e^{\alpha})}$$

which is more common in the literature. To recover this formula, use that

$$A := e^{-\rho}\prod_{\alpha\in\Phi^+}(1-e^{\alpha}) = \prod_{\alpha\in\Phi^+}(e^{-\alpha/2}-e^{\alpha/2}) \text{ satisfies } w.A = (-1)^{\ell(w)}A.$$

Thus, we obtain

$$\sum_{w\in W} w.\left(\frac{e^{\lambda}}{\prod_{\alpha\in\Delta^+}(1-e^{\alpha})}\right) = \sum_{w\in W} w.\left(\frac{e^{\lambda-\rho}}{A}\right) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda-\rho)}}{A} = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda-\rho)+\rho}}{\prod_{\alpha\in\Phi^+}(1-e^{\alpha})}$$

as desired.

6. Example of SU(2)

Take $G = \mathrm{SU}(2)$ and $G_{\mathbf{C}} = \mathrm{SL}(2,\mathbf{C})$. Choose the positive root to be $\alpha = \varepsilon_1 - \varepsilon_2$ so that $\rho = \alpha/2$ (see Figure 5). Notice that in this case $\mathrm{R}^+(\lambda)$ coincides with $\mathrm{R}^-(\lambda)$. Also, $L(\mathrm{SU}(2)) = P = \mathbf{Z}\rho$.

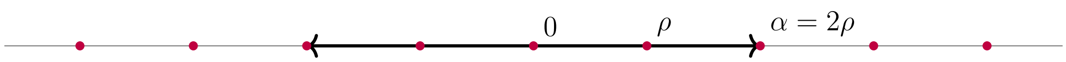

Figure 5. Root system and weight latice of $\mathfrak{sl}(2,\mathbf{C})$.

One has

$$G/T \cong G_{\mathbf{C}}/B^+ \cong \mathbf{CP}^1.$$

Indeed, the flag variety for a 2-dimensional vector space is the same as the collection of 1-dimensional subspaces, i.e., $\mathbf{CP}^1$. Line bundles over $\mathbf{CP}^1$ are the twisted sheaves $\mathscr{O}_{\mathbf{CP}^1}(n)$ for $n \in \mathbf{Z}$. In fact, we have

$$\mathscr{L}_{n\rho} \cong \mathscr{O}_{\mathbf{CP}^1}(-n)$$

as line bundles. Thus,

$$\mathrm{H}^0(\mathscr{L}_{-n\rho}) = \mathrm{H}^0(\mathscr{O}_{\mathbf{CP}^1}(n)) = \begin{cases} \mathbf{C}\{x^j y^{n-j} \mid 0 \le j \le n\}, & n \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

The action of SU(2) (or SL(2)) is by regarding $f(x,y) := x^j y^{n-j}$ as a function in (x,y) and $g.f(x,y) = f(g^{-1}.(x,y))$. The Borel-Weil-Bott theorem indicates that $\mathrm{H}^0(\mathscr{L}_{-n\rho})$ is the irreducible representation $\mathrm{R}^-(-n\rho) \cong \mathrm{R}^+(n\rho)$ of SU(2) (or $\mathrm{SL}(2,\mathbf{C})$).

The maximal torus of SU(2) is the collection of matrices

$$T \cong \mathrm{U}(1) = \left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}\right\}.$$

One may explicitly calculate the character of this representation using the eigenvectors $\{x^j y^{n-j}\}$:

$$\mathrm{ch}(\mathrm{R}^-(-n\rho))\left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) = e^{-in\theta} + e^{-i(n-2)\theta} \cdots + e^{i(n-2)\theta} + e^{in\theta} = \frac{\sin(n+1)\theta}{\sin\theta}.$$

On the other hand, denote

$$h := \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix}.$$

Notice that

$$\alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = 2,$$

and hence the Weyl character formula yields

$$\mathrm{ch}(\mathrm{R}^-(-n\rho))(h) = \frac{e^{-n\rho(h)} - e^{(n+2)\rho(h)}}{1 - e^{2\rho(h)}} = \frac{e^{-in\theta} - e^{i(n+2)\theta}}{1 - e^{i2\theta}} = \frac{\sin(n+1)\theta}{\sin\theta}$$

which recovers the previous computation.

References

[1] Atiyah, M. and Bott, R. *A Lefschetz fixed point formula for elliptic complexes: II. Applications.*
[2] Bott, R. *On induced representations.*
[3] Ganter, N. *The elliptic Weyl character formula.*
[4] Kirillov, A. and Kirillov Jr, A. *Compact groups and their representations.*
[5] Lurie, J. *A proof of the Borel-Weil-Bott theorem.*
[6] Segal, G. *The representation ring of a compact Lie group.*