TTIC 31230 Fundamentals of Deep Learning

Problems for GANs.

Problem 1. Conditional GANs In a conditional GAN we model a conditional distribution $\mathrm{Pop}(y|x)$ defined by a population distribution on pairs $\langle x, y\rangle$. For conditional GANs we consider the probability distribution over triples $\langle x, y, i\rangle$ defined by

$$\begin{aligned}\tilde{P}_\Phi(i=1) &= 1/2 \\ \tilde{P}_\Phi(y|x, i=1) &= \mathrm{pop}(y|x) \\ \tilde{P}_\Phi(y|x, i=-1) &= p_\Phi(y|x)\end{aligned}$$

(a) Write the conditional GAN adversarial objective function for this problem in terms of $\tilde{P}(x, y, i)$, $P_\Phi(y|x)$ and $P_\Psi(i|y, x)$.

Solution:

$$\Phi^* = \underset{\Phi}{\mathrm{argmax}}\ \min_{\Psi}\ E_{x,y,i\sim\tilde{P}(x,y,i)}\ \ -\ln P_\Psi(i|x, y)$$

Problem 2. GAN instability

Consider the following adversarial objective where x and y are scalars (real numbers).

$$\max_x\ \min_y\ xy$$

(a) Write the differential equation for gradient flow of this adversarial objective.

Solution:

ascent $\frac{\partial L}{\partial y} = dy = x\,dt$

$L = xy = x(t)\,y(t) \Rightarrow \frac{\partial L}{\partial x} = dx = y\,dt$

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x\end{aligned}$$

gradient descent

(b) Give a general solution to your differential equation. (Hint: It goes in a circle). You solution should have parameters allowing for any given initial value of x and y.

Solution:

$$\begin{aligned} x &= r_0 \sin(t + \Theta_0) \\ y &= r_0 \cos(t + \Theta_0) \end{aligned}$$

Problem 3. Contrastive GANs.

A GAN can be built with a "contrastive" discriminator. Rather than estimate the probability that y is from the population, the discriminator must select which of $y_1, \ldots, y_N$ is from the population.

More formally, for $N \geq 2$ let $\tilde{P}_\Phi^{(N)}$ be the distribution on tuples $\langle i, y_1, \ldots, y_N \rangle$ defined by drawing one "positive" from Pop and $N - 1$ IID negatives from P_Φ; then inserting the positive at a random position among the negatives; and returning $(i, y_1, \ldots, y_N)$ where i is the index of the positive.

$$\Phi^* = \operatorname*{argmax}_\Phi \min_\Psi \; E_{(i, y_1, \ldots, y_{N+1}) \sim \tilde{P}_\Phi^{(N)}} \; - \ln p_\Psi(i | y_1, \ldots, y_{N+1}) \quad (1)$$

Restate the above definition of $\tilde{P}_\Phi^{(N)}$ and the GAN adversarial objective for the case of conditional constrastive GANs.

Solution:

For $N \geq 2$ let $\tilde{P}_\Phi^{(N)}$ be the distribution on tuples $\langle i, y_1, \ldots, y_N, x \rangle$ defined by drawing a pair $\langle x, y \rangle$ from Pop, where y is a positive sample of $\mathrm{Pop}(y|x)$, drawing IID negatives from $P_\Phi(y|x)$; then inserting the positive at a random position among the negatives; and returning $(i, y_1, \ldots, y_N, x)$ where i is the index of the positive.

$$\Phi^* = \operatorname*{argmin}_\Phi \max_\Psi E_{x \sim \mathrm{pop}} \; E_{(i, y_1, \ldots, y_{N+1}) \sim (\mathrm{pop}(y|x) \hookrightarrow p_\Phi(y|x)^k)} \ln p_\Psi(i | y_1, \ldots, y_{N+1}, x)$$