Probabilistic Graphical Models

Lecture 5: Conditional Random Fields

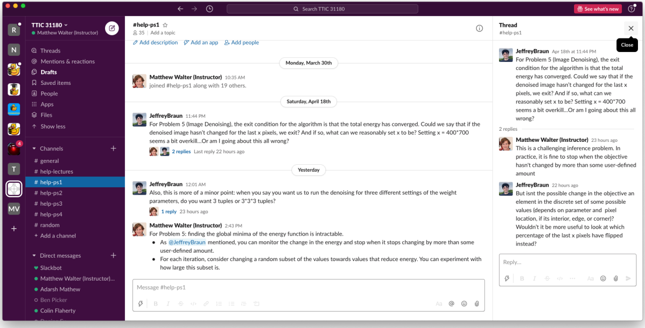
Matthew Walter

TTI-Chicago

April 21, 2020

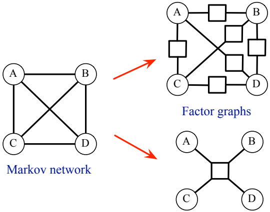
Problem Set 1 and Slack

- See Canvas for updates on Problem Set 1
- Don't forget to join the Slack channel #help-ps1
- See Slack channel for discussion and hints



Factor Graphs (Revisited)

- The Markov network H does not make explicit the structure of the distribution, i.e., maximum cliques vs. complete graph subsets
- A **factor graph** is a bipartite undirected graph with variable nodes (oval) and factor nodes (square). Edges exist only between variable nodes and factor nodes
- Each factor node is associated with a single potential, the scope of which is the variables that are the factor's neighbors



- Distribution is the same as an MRF, just a different data structure

Boltzmann Distribution (Revisited)

- We can rewrite a factor $\phi(\boldsymbol{D}) : \text{Val}(\boldsymbol{D}) \rightarrow \mathbb{R}^+$ as

$$\phi(\boldsymbol{D}) = \exp(-\psi(\boldsymbol{D}))$$

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$$P(X_1, \ldots, X_n) = \frac{1}{Z} \prod_{k=1}^{K} \exp\left(-\psi_k(\boldsymbol{D}_k)\right) = \frac{1}{Z} \exp\left(-\sum_{k=1}^{K} \psi_k(\boldsymbol{D}_k)\right)$$

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- $\sum_{k=1}^{K} \psi_k(\boldsymbol{D}_k)$ is referred to as the "free energy"
- Gives rise to interpretation as energy minimization

$$\arg\max\ P(X_1, \ldots, X_n) = \arg\min\ \sum_{k=1}^{K} \psi_k(\boldsymbol{D}_k)$$

Log-Linear Markov Networks with Features (Revisited)

- A **feature** is a function $f : \text{Val}(\boldsymbol{D}_i) \rightarrow \mathbb{R}$
- A distribution P is a **log-linear model** over a Markov network H if it is associated with
  - A set of features $\boldsymbol{F} = \{f_1(\boldsymbol{D}_1), \ldots, f_K(\boldsymbol{D}_M)\}$ where $\boldsymbol{D}_i$ is a complete subgraph in H
  - A set of weights $\{w_1, \ldots, w_M\}$

  such that

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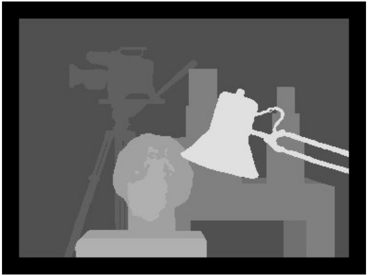
- Multiple features can be defined over the same variables
- Log-linear model can represent tabular potentials, but is more general
- Features and weights can be reused for different factors
- Historically, features designed by hand and weights learned from data

Example: Stereo Vision

Left Image

Right Image

Example: Stereo Vision



Disparity Image

- Dense stereo reconstruction: For every pixel (i_l, j_l) in left image, we are interested in the image-space distance (**disparity**) y_{i_l,j_l} to the corresponding pixel (i_r, j_r) in the right image

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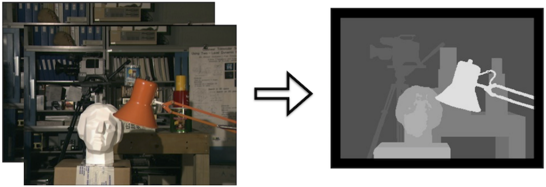

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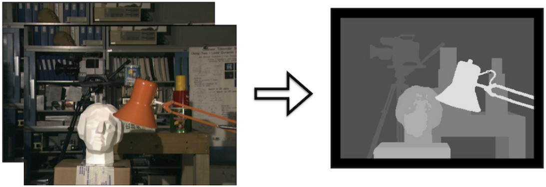

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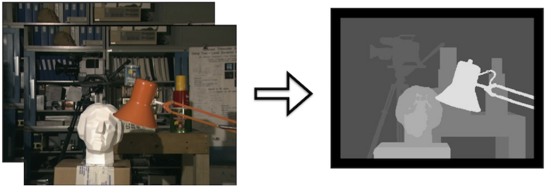

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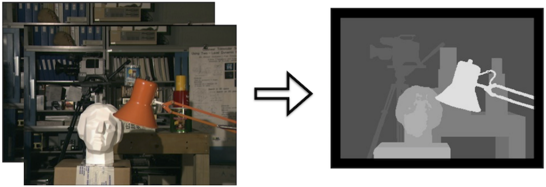

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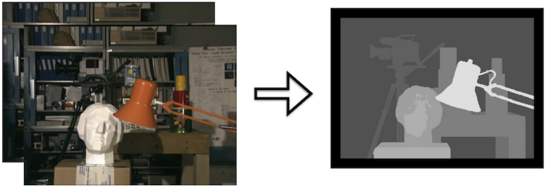

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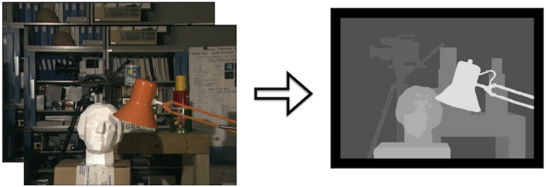

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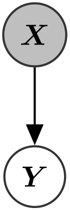
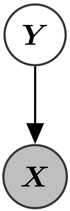
  - Both involve a (conditional) distribution over natural images!!!
- Requires that we choose a parametric form over the pixel values
- Non-local features (e.g., image gradients) are useful, but difficult to capture with a joint distribution

Generative vs. Discriminative Models

- Let $\boldsymbol{X}$ denote the input/observation (e.g., an image) and $\boldsymbol{Y}$ be the output (e.g., disparity, label, etc.)
- Flexibility in the structure and parametrization of the model



Discriminative



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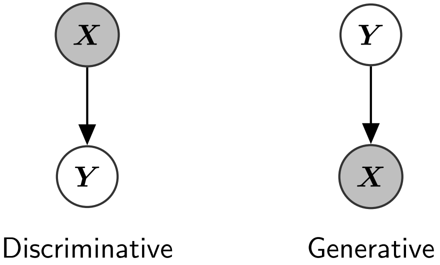

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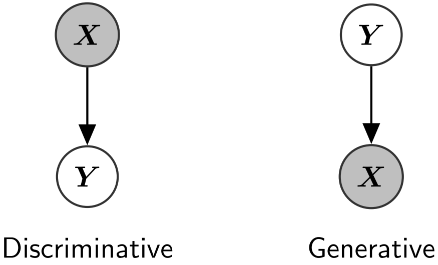
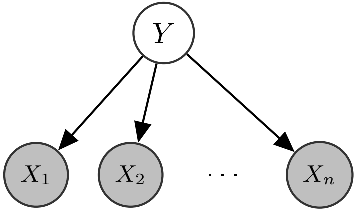


- Discriminative models are concerned with modeling $P(\boldsymbol{Y} \mid \boldsymbol{X} = \boldsymbol{x})$, where $\boldsymbol{X} = \boldsymbol{x}$ is treated as a parameter
- Generative models are interested in the joint distribution $P(\boldsymbol{X}, \boldsymbol{Y}) = P(\boldsymbol{X} \mid \boldsymbol{Y})P(\boldsymbol{Y})$
  - $P(\boldsymbol{X} \mid \boldsymbol{Y})$: Given the target label, generate the input

Generative vs. Discriminative Classifiers: Naive Bayes

- Classify e-mails as being spam ($Y = 1$) or not spam ($Y = 0$)
  - Let $i \in \{1, \ldots, n\}$ index English words
  - $X_i = 1$ if word i appears in the e-mail
  - E-mails are drawn from the joint distribution $P(Y, X_1, \ldots, X_n)$
- Words are conditionally independent (d-separated) given Y

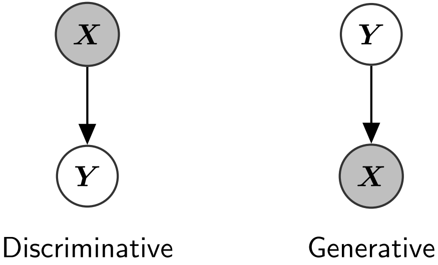


- Prediction follows via Bayes' Rule as

$$P(Y = 1 \mid x_1, \ldots, x_n) = \frac{P(Y = 1) \prod_{i=1}^{n} P(x_i \mid Y = 1)}{\sum_{y=\{0,1\}} P(Y = y) \prod_{i=1}^{n} P(x_i \mid Y = 1)}$$

Generative vs. Discriminative Models

- These are **equivalent** models of $P(\boldsymbol{Y}, \boldsymbol{X})$



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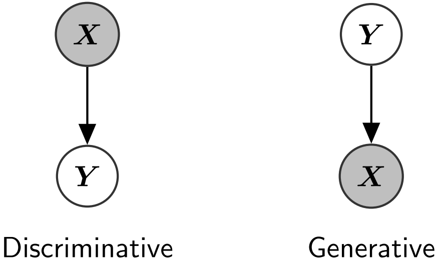

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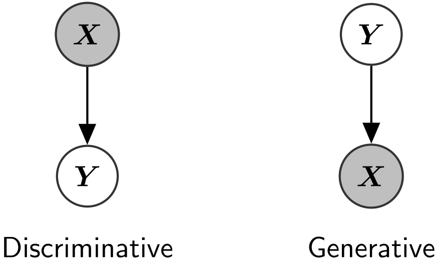

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- A **generative** model requires representing both $P(\boldsymbol{Y})$ and $P(\boldsymbol{X} \mid \boldsymbol{Y})$
  - Generative, since we can *generate* $\boldsymbol{X}$ given $\boldsymbol{Y}$
  - $P(\boldsymbol{Y} \mid \boldsymbol{X})$ is determined via Bayes' Rule

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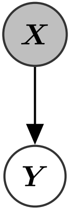

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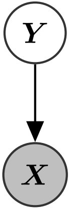

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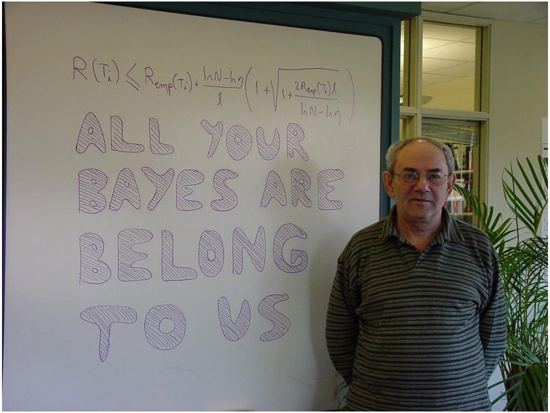
- A **discriminative** model only requires a representation of the conditional distribution $P(\boldsymbol{Y} \mid \boldsymbol{X})$
  - Discriminative, since we can *discriminate* between different $\boldsymbol{Y}$
  - We never need to estimate $P(\boldsymbol{X})$ (which can be hard)

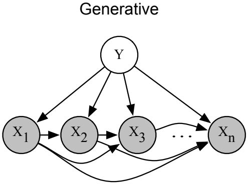
Generative vs. Discriminative Models

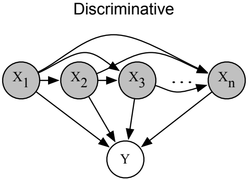


"one should solve the (classification) problem directly and never solve a more general problem as an intermediate step" (Vapnik, 1998)

Generative vs. Discriminative Models

- For the two models to be equivalent, we need:





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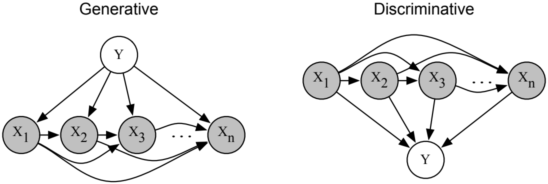

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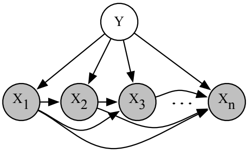
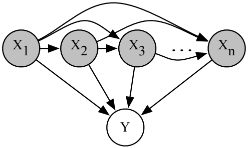


Discriminative



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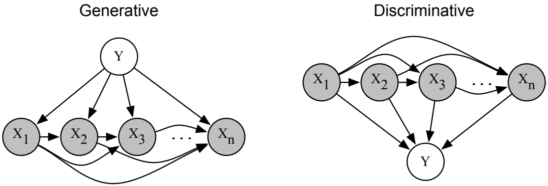

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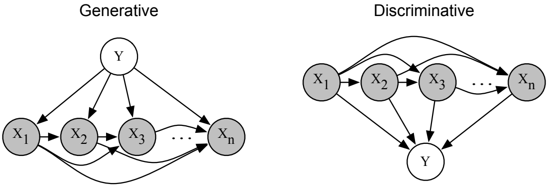

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  - Discriminative model: How do we parametrize $P(\boldsymbol{Y} \mid \boldsymbol{X})$?
    Allows us to ignore encoding distribution over $\boldsymbol{X}$

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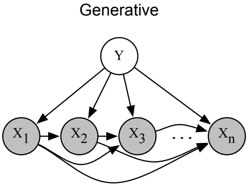

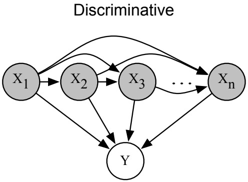

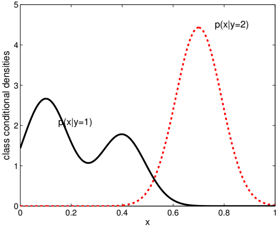

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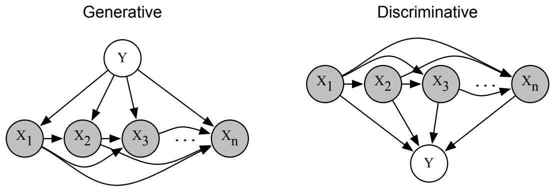

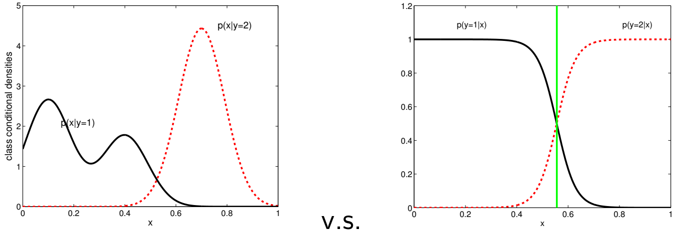

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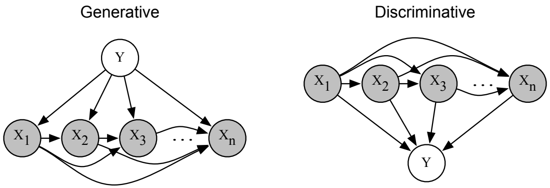
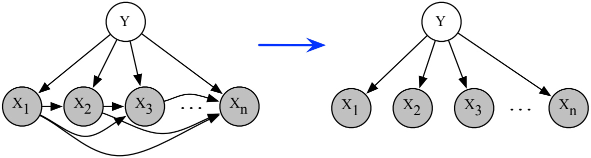


Generative vs. Discriminative Models

- For the generative model, ignore the dependencies: assume that $X_i \perp \boldsymbol{X}_{-i} \mid \boldsymbol{Y}$ (naive Bayes)



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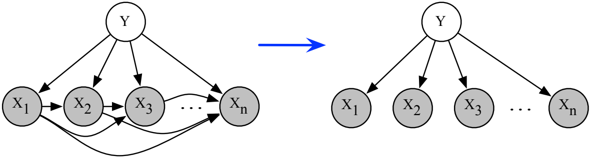


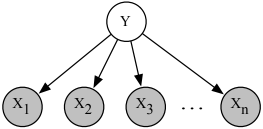
- For the discriminative model, assume that

$$P(Y = 1 \,|\, \boldsymbol{x}; w) = \frac{e^{w_0 + \sum_{i=1}^{n} w_i x_i}}{1 + e^{w_0 + \sum_{i=1}^{n} w_i x_i}} = \frac{1}{1 + e^{-w_0 - \sum_{i=1}^{n} w_i x_i}}$$

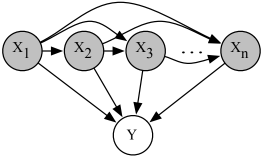
$$= \frac{1}{1 + e^{-z}}$$ **logistic function**

Generative vs. Discriminative Models

Generative (naive Bayes)



Discriminative (logistic regression)



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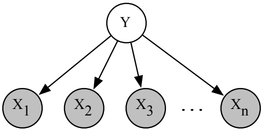
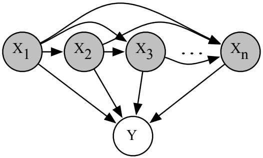
- We can show (problem set) that the first assumption implies the latter
- Every conditional distribution that can be represented via naive Bayes can also be represented using the logistic model

Generative vs. Discriminative Models





- Unlike naive Bayes, logistic models don't assume $X_i \perp \boldsymbol{X}_{-i} \,|\, \boldsymbol{Y}$
- Ignoring dependencies results in double-counting evidence, e.g.,
  - Suppose that $X_i = 1$ ("transaction" in e-mail) and $X_j = 1$ ("account" in e-mail)
  - Irrespective of being spam, these always occur together, i.e., $X_i = X_j$
  - Learning with naive Bayes ($P(X_i \,|\, \boldsymbol{Y}) = P(X_j \,|\, \boldsymbol{Y})$) double-counts evidence
  - Learning with logistic regression sets $w_i = 0$ for one of the words, thereby ignoring it

Generative vs. Discriminative Models

- Discriminative model requires that $\boldsymbol{X}$ be fully observed
  - Generative models allow you to marginalize over unseen variables to compute $P(\boldsymbol{Y} \mid \boldsymbol{X}_o)$
- Maximum likelihood estimation of generative models is more efficient than training discriminative models [Ng and Jordan, 2002][1]
  - Consider number of samples necessary to get close to infinite data case
  - Logistic regression requires $\mathcal{O}(n)$ samples
  - Naive Bayes requires $\mathcal{O}(\log n)$ samples
  - Naive Bayes converges with fewer samples, but not necessarily to better estimates

[1] Ng and Jordan, "On Discriminative vs. Generative Classifiers: A Comparison of Logistic Regression and Naive Bayes," NeurIPS 2002

Generative vs. Discriminative Models

- Ng and Jordan (2002) show that discriminative logistic regression has lower asymptotic error than a generative naive Bayes classifier, but that naive Bayes converges faster

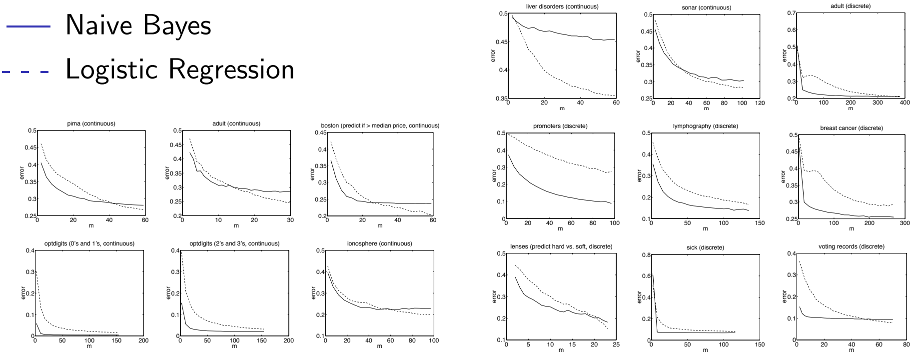
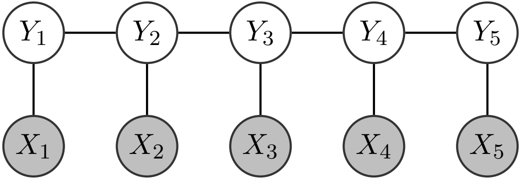


Figure: Error rate vs. number of samples for 15 datasets from the UCI Machine Learning repository. Courtesy: Ng and Jordan, 2002.

Conditional Random Fields



(linear chain CRF)

- Undirected graph with nodes for $\boldsymbol{Y}$ (target variables) and $\boldsymbol{X}$ (observed variables) (alt., partially directed, with $\boldsymbol{X}$ the parent of $\boldsymbol{Y}$)

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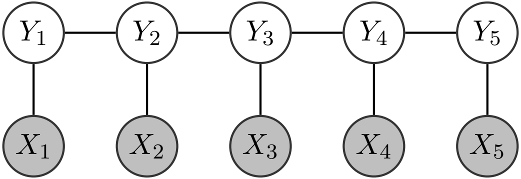

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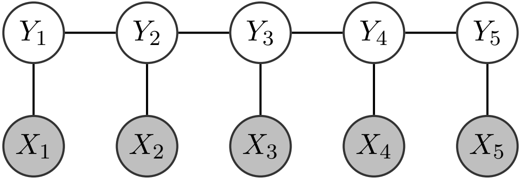

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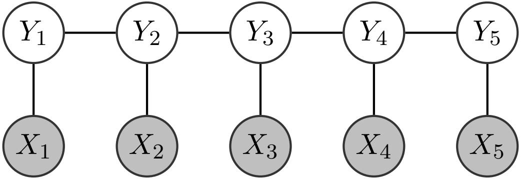

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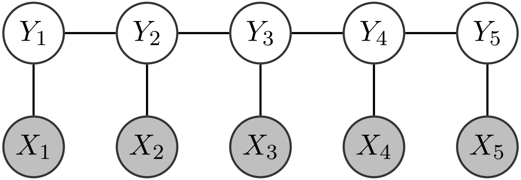
- Represent conditional distribution $P(\boldsymbol{Y} \mid \boldsymbol{X})$ rather than joint distribution $P(\boldsymbol{Y}, \boldsymbol{X})$
- Avoid representing dist. over $\boldsymbol{X} \Rightarrow$ no potentials involving only $\boldsymbol{X}$

Conditional Random Fields



(linear chain CRF)

- A **conditional random field** (CRF) is an undirected graph H over $\boldsymbol{X}$ and $\boldsymbol{Y}$ defined in terms of a set of factors $\phi_1(\boldsymbol{D}_1), \ldots, \phi_m(\boldsymbol{D}_m)$, where $\boldsymbol{D}_i \not\subseteq \boldsymbol{X}$, that encodes the conditional distribution

$$P(\boldsymbol{Y} \mid \boldsymbol{X}) = \frac{1}{Z(\boldsymbol{X})} \prod_{i=1}^{m} \phi_i(\boldsymbol{D}_i)$$

$$Z(\boldsymbol{X}) = \sum_{\boldsymbol{Y}} \prod_{i=1}^{m} \phi_i(\boldsymbol{D}_i)$$

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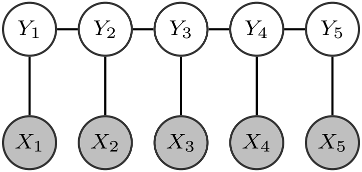
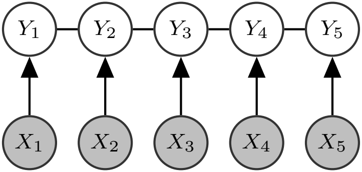
- Two variables are connected by an undirected edge if they appear in the scope of the same factor
- Just like a Markov network, except the partition function depends on (i.e., changes with) the observed variables (input) $\boldsymbol{X}$
- Trained to maximize *conditional* (not joint) probability of the output given the input

Linear-Chain CRFs





- The linear-chain CRF can be represented as an undirected (left) or partially directed graph (right)
- The conditional distribution factorizes as

$$P(\boldsymbol{Y} \mid \boldsymbol{X}) = \frac{1}{Z(\boldsymbol{X})} \prod_{i=1}^{k-1} \phi(Y_i, Y_{i+1}) \prod_{i=1}^{k} \phi(Y_i, X_i)$$

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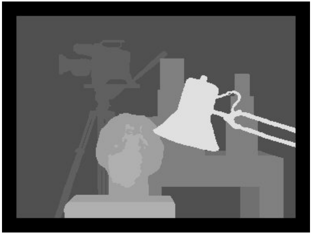
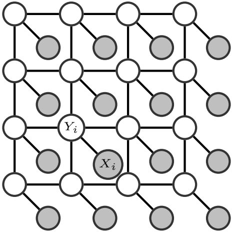
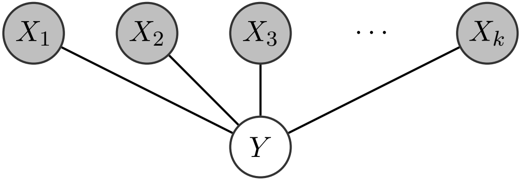




- By not modeling the distribution over $\boldsymbol{X}$, we can consider representations of the data with complex, non-parametric interactions
- We can employ a rich set of features without concern over their joint distribution (e.g., image gradients)

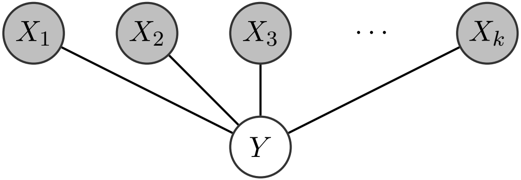
Naive Markov Model



- Let $\boldsymbol{X} = \{X_1.X_2, \ldots, X_k\}$ and Y be binary random variables
- Assume $\boldsymbol{X}$ (observed) and Y are related by the following factors

$$\phi_0(Y) = \exp\{w_0 \mathbb{1}[Y = 1]\}$$
$$\phi_i(X_i, Y) = \exp\{w_i \mathbb{1}[X_i = 1, Y = 1]\}$$

Naive Markov Model



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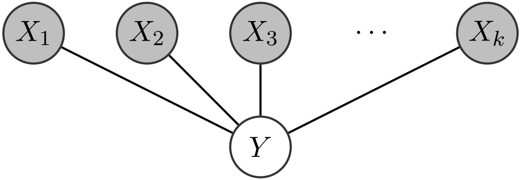
- The conditional distribution becomes

$$\tilde{P}(Y = 1 \,|\, x_1, \ldots, x_k) = \exp\left\{w_0 + \sum_{i=1}^{k} w_i x_i\right\}$$
$$\tilde{P}(Y = 0 \,|\, x_1, \ldots, x_k) = 1$$

Naive Markov Model



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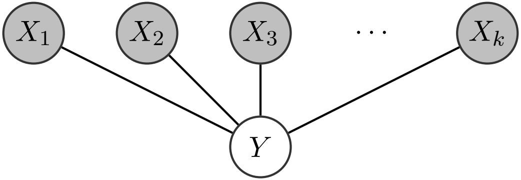

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- Recall earlier discriminative (logistic regression) discussion

CRF Parametrization

- Factors may depend on a large number of variables
- Typically, parametrize factors using log-linear representation

$$\phi_c(\boldsymbol{X}_c, \boldsymbol{Y}_c) = \exp\left(\boldsymbol{w}_c^\top \boldsymbol{f}_c(\boldsymbol{X}_c, \boldsymbol{Y}_c)\right)$$

where

  - $\boldsymbol{f}_c(\boldsymbol{X}_c, \boldsymbol{Y}_c)$ is a feature vector (e.g., local image gradients)
  - $\boldsymbol{w}_c$ is a weight vector that is learned from data

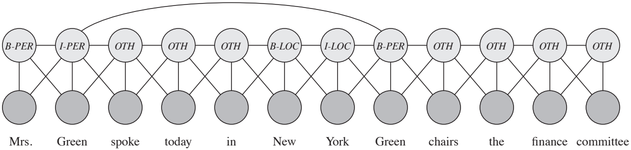
Example: Named Entity Recognition

- Objective: Given a sentence, segment phrases into different locations, people, organizations, etc.

  "Mrs. Green spoke today in New York Green chairs the finance committee"

- Entries often span multiple words and label isn't obvious without considering context
- Define a random variable Y_i for each word X_i that expresses its entity type ("BIO notation")
  - B-PER/B-LOC: Beginning of a person/location
  - I-PER/I-LOC: Inside or end of named entity phrase for person/location
  - OTH: Word is not part of an entity

Example: Named Entity Recognition



Interested in $P(\boldsymbol{Y} \mid \boldsymbol{X}) \Rightarrow$ Model as a CRF w/ three types of factors

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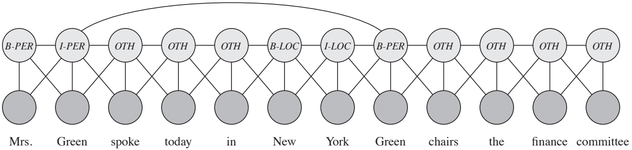

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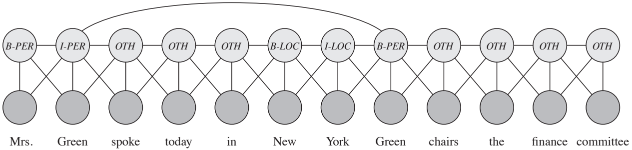

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    - O(1000) features over current word X_t (e.g., capitalized), neighboring words, and entire sequence (e.g., number of sports-related words)

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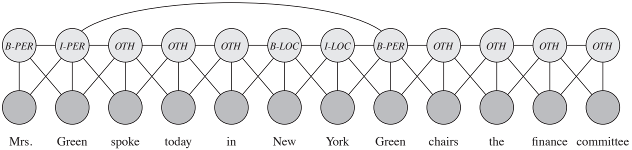

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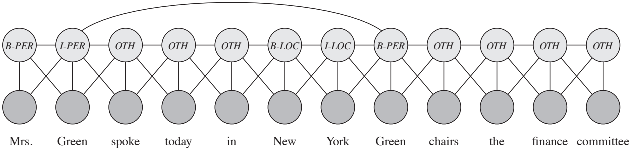

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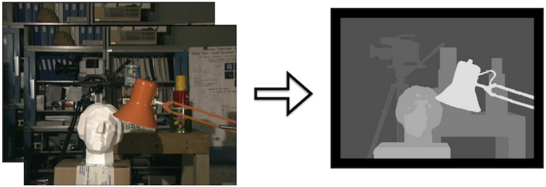
Graph structure changes depending on sentence

Example: Stereo Vision



- Given two images $\boldsymbol{X}_l$ and $\boldsymbol{X}_r$, estimate the disparity $y_{i_l,j_l} \in \boldsymbol{Y}$ between a pixel at (i_l, j_l) in the left image and the corresponding pixel $(i_r, j_r) = (i_l + y_{i_l,j_l}, j_l)$ in the right image (assume 1D)

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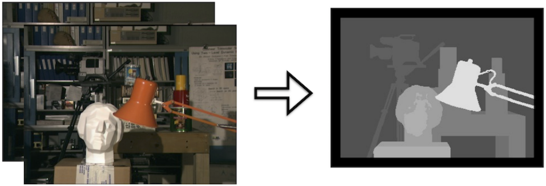

- Define local node potential as

$$\phi_{i_l,j_l}(y_{i_l,j_l}, \boldsymbol{X}) \propto \exp\left(-\frac{1}{2\sigma^2}\left(x_l(i_l, j_l) - x_r(i_l + y_{i_l,j_l}, j_l)\right)^2\right)$$

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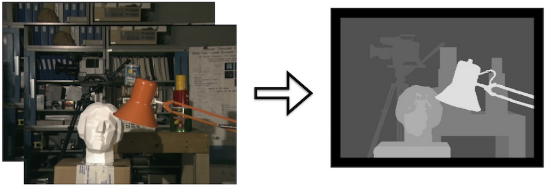

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- Define smoothness potential over neighboring pixels s and t

$$\phi_{s,t}(y_s, y_t) \propto \left(-\frac{1}{2\gamma^2}(y_s - y_t)^2\right)$$

Can be defined only for pairs s and t that don't cross boundaries

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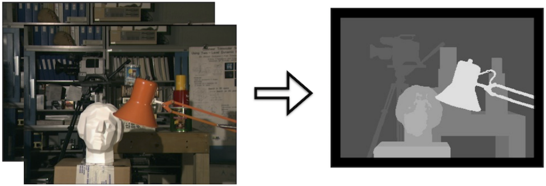

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    - In named entity recognition, we can tie repeated words together

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– CRFs require labeled training data and are slower to train