PRMO-2018

1. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n?

2. In a quadrilateral $ABCD$, it is given that$AB = AD = 13, BC = CD = 20, BD = 24$. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r?

3. Consider all 6-digit numbers of the form abccba where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7.

4. The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is, 1, 6, 5, 8, 9, 0 are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation.

5. Let $ABCD$ be a trapezium in which $AB \parallel CD$ and $AD \perp AB$. Suppose $ABCD$ has an incircle which touches ABat Q and CD at P. Given that $PC = 36$ and $QB = 49$, find PQ.

6. Integers a, b, c satisfy $a + b - c = 1$ and $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$?

7. A point P in the interior of a regular hexagon is at distances $8, 8, 16$ units from three consecutive vertices of the hexagon, respectively. If r is radius of the circumscribed circle of the hexagon, what is the integer closest to r?

8. Let AB be a chord of a circle with centre O. Let C be a point on the circle such that $\angle ABC = 30°$ and O lies inside the triangle ABC. Let D be a point on AB such that $\angle DCO = \angle OCB = 20°$. Find the measure of $\angle CDO$ in degrees.

9. Suppose a, b are integers and $a + b$ is a root of $x^2 + ax + b = 0$.What is the maximum possible value of b^2?

10. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2 + CA^2 + AB^2)/100$.

11. There are several tea cups in the kitchen, some with handles and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?

12. Determine the number of 8-tuples $(\epsilon_1, \epsilon_2, , \epsilon_8)$ such that $\epsilon_1, \epsilon_2, \epsilon_8 \in (1, -1)$and $\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + \ + 8\epsilon_8$ is a multiple of 3.

13. In a triangle ABC, right-angled at$A$, the altitude through A and the internal bisector of A have lengths 3 and 4, respectively. Find the length of the median through A.

14. If $x = \cos 1^\circ \cdot \cos 2^\circ \cdot \cos 3^\circ \cdot \ldots \cdot \cos 89^\circ$ and $y = \cos 2^\circ \cdot \cos 6^\circ \cdot \cos 10^\circ \cdot \ldots \cdot \cos 86^\circ$.We are asked to find the integer nearest to $\frac{2}{7} \log_2 \left(\frac{y}{x}\right)$?

15. Let a and b be natural numbers such that $2a - b$, $a - 2b$ and $a + b$ are all distinct squares. What is the smallest possible value of b?

16. What is the value of $\sum_{1 \le i < j \le 10,\, i+j\ =\text{odd}} (i+j) - \sum_{1 \le i < j \le 10,\, i+j\ =\text{even}} (i+j)$.?

17. Triangles ABC and DEF are such that $\angle A = \angle D, AB = DE = 17, BC = EF = 10$ and $AC - DF = 12$. What is $AC + DF$?

18. If $a, b, c \ge 4$ are integers, not all equal, and $4abc = (a + 3)(b + 3)(c + 3)$, then what is the value of $a + b + c$?

19. Let $N = 6 + 66 + 666 + \ldots\ldots + 666\ldots..66$, where there are hundred $6'$s in the last term in the sum. How many times does the digit 7 occur in the number N?

20. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$.

21. Let ABC be an acute-angled triangle and let H be its orthocentre. Let $G1, G2$ and $G3$ be the centroids of the triangles HBC, HCA and HAB, respectively. If the area of triangle $G_1G_2G_3$is 7 units, what is the area of triangle ABC?

22. A positive integer k is said to be good if there exists a partition of $(1, 2, 3, ..., 20)$ in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many good numbers are there?

23. What is the largest positive integer n such that $\frac{a^2}{\frac{b}{29}+\frac{c}{31}} + \frac{b^2}{\frac{a}{29}+\frac{c}{31}} + \frac{c^2}{\frac{a}{29}+\frac{b}{31}} \ge n(a + b + c)$ holds for all positive real numbers a, b, c.

24. If N is the number of triangles of different shapes (i.e., not similar) whose angles are all integers (in degrees), what is $\frac{N}{100}$?

25. Let T be the smallest positive integer which, when divided by $11, 13, 15$ leaves remainders in the sets $\{7, 8, 9\}$,$\{1, 2, 3\}$,$\{4, 5, 6\}$ respectively. What is the sum of the squares of the digits of T?

26. What is the number of ways in which one can choose 60 unit squares from a 11×11 chessboard such that no two chosen squares have a side in common?

27. What is the number of ways in which one can colour the squares of a 4×4 chessboard with colours red and blue such that each row as well as each column has exactly two red squares and two blue squares?

28. Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N.

29. Let D be an interior point of the side BC of a triangle ABC. Let I_1 and I_2 be the incentres of triangles ABD and ACD respectively. Let AI_1 and AI_2 meet BC in E and F respectively. If $\angle BI_1E = 60°$, what is the measure of $\angle CI_2F$ in degrees ?

30. Let $P(x) = a_0 + a_1x + a_2x^2 + ... + ax^n$ be a polynomial in which ai is a non-negative integer for each $i \in \{0, 1, 2, 3, , n\}$. If $P(1) = 4$ $P(5) = 136$, what is the value of $P(3)$?