# Advanced Digital Communications

## Lecture 3: Digital Communication Through Band-Limited Channels

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## Previous Lecture

**Lecture 2: Optimum Receivers for AWGN Channels**

I. Signal Space Representation of Waveforms

II. Waveform and Vector AWGN Channels

III. Optimal Detection and Error Probability for Band-Limited Signaling

IV. Optimal Detection in Presence of Uncertainty: Noncoherent Detection

V. A Comparison of Digital Signaling Methods

# Outline

I. Characterization of Band-Limited Channels

II. Signal Design for Band-limited Channels

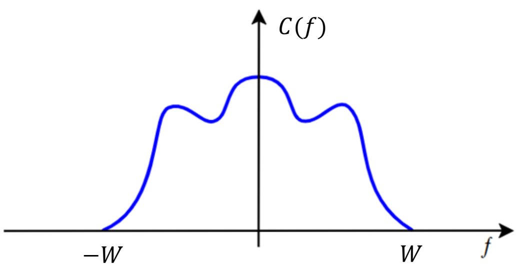
III. Optimum Receiver for Channels with ISI and AWGN

IV. Linear Equalization



# Characterization of Band-Limited Channels

- For a **bandlimited channel** with bandwidth $W$ Hz, such as a telephone channel, it may be **characterized as a linear filter** having an equivalent lowpass frequency-response characteristic $C(f)$ with $C(f) = 0$ for $|f| > W$.



# Characterization of Band-Limited Channels

In a band-limited channel, if the transmitted bandpass signal has the form

$$s(t) = \text{Re}\left[v(t)e^{j2\pi f_c t}\right] \text{ with } v(t) = \sum_n I_n g(t - nT)$$

the equivalent lowpass received signal is

$$r_l(t) = \boxed{\int_{-\infty}^{\infty} v(\tau)c(t-\tau)\,d\tau} + z(t)$$

the convolution $v(t) \star c(t)$

$c(t)$: the impulse response corresponding to a linear filter with frequency-response characteristic $C(f)$

$\{I_n\}$: the discrete information sequence

$g(t)$: the signal pulse shape

$z(t)$: the additive noise

The characteristics of $C(f)$ significantly affect the received signal.



# Characterization of Band-Limited Channels

- The frequency response $C(f)$ could be expressed as

$$C(f) = |C(f)|e^{j\theta(f)}$$

$|C(f)|$: the amplitude-response characteristic $\theta(f)$ : the phase-response characteristic

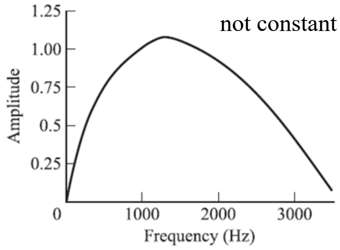
- The envelope delay characteristic is defined as

$$\tau(f) = -\frac{1}{2\pi}\frac{d\theta(f)}{df}$$

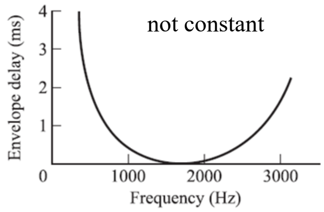
- A band-limited channel is said to be ***nondistorting* or *ideal*** if the amplitude response $|C(f)|$ is constant for all $|f| \le W$ and $\theta(f)$ is a linear function of frequency, i.e., $\tau(f)$ is a constant for all $|f| \le W$.
- If $|C(f)|$ is not constant for all $|f| \le W$, **the channel distorts** the transmitted signal $s(t)$ **in amplitude**. If $\tau(f)$ is not constant for all $|f| \le W$, **the channel distorts** the transmitted signal $s(t)$ **in delay**.

# Characterization of Band-Limited Channels

*An example*: a telephone channel




amplitude-response characteristic

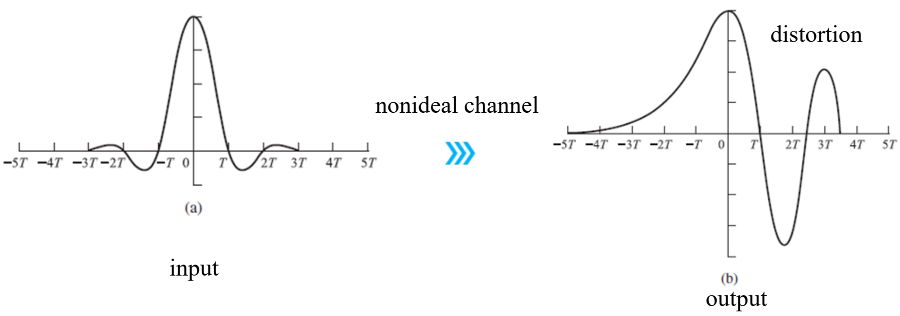



envelope delay characteristic



# Characterization of Band-Limited Channels

- **The nonideal channel frequency-response characteristic causes the amplitude and delay distortion.**




- At the receiver side, zero-crossings are no longer periodically spaced.
- The nonideal channel results in intersymbol interference.

## Outline





## Signal Design for Band-limited Channels

- Suppose that the lowpass transmitted signal has the form

$$v(t) = \sum_{n=0}^{\infty} I_n g(t - nT)$$

the received signal in band-limited channels with impulse response $c(t)$ could be represented as

$$r_l(t) = \sum_{n=0}^{\infty} I_n h(t - nT) + z(t)$$

$$\text{with} \quad h(t) = \int_{-\infty}^{\infty} g(\tau)c(t-\tau)\,d\tau$$

$\{I_n\}$: the discrete information-bearing sequence

$g(t)$: the band-limited signal pulse shape with a frequency-response characteristic $G(f) = 0$ for $|f| > W$

$z(t)$: the additive white Gaussian noise

## Signal Design for Band-limited Channels

For the received signal

$$r_l(t) = \sum_{n=0}^{\infty} I_n h(t - nT) + z(t)$$

the optimum filter from the point of view of signal detection is one matched to the received pulse, and has the frequency response $H^*(f)$.

- The output of the receiving filter is denoted by

$$y(t) = \sum_{n=0}^{\infty} I_n x(t - nT) + v(t)$$

$x(t)$: the pulse representing the response of the receiving filter to the input pulse $h(t)$

$v(t)$: the response of the receiving filter to the noise $z(t)$



## Signal Design for Band-limited Channels

Suppose that $y(t)$ is sampled at times $t = kT + \tau_0$, $k = 0, 1, \ldots$, then we have

$$y(kT + \tau_0) \equiv y_k = \sum_{n=0}^{\infty} I_n x(kT - nT + \tau_0) + v(kT + \tau_0)$$

equivalently

$$y_k = \sum_{n=0}^{\infty} I_n x_{k-n} + v_k$$

equivalently

$$y_k = x_0 \left( I_k + \frac{1}{x_0} \boxed{\sum_{\substack{n=0 \\ n \neq k}}^{\infty} I_n x_{k-n}} \right) + v_k, \qquad k = 0, 1, \ldots$$

$\tau_0$ : the transmission delay through the channel

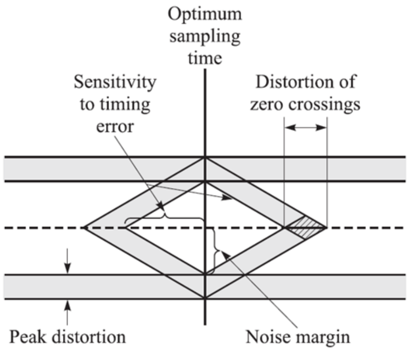
**intersymbol interference (ISI)**

$I_k$: the desired information symbol at the $k$-th sampling instant

an arbitrary scale factor

# Signal Design for Band-limited Channels

- **ISI and Eye Pattern:**
  - The ISI could be viewed on an oscilloscope.
  - The resulting oscilloscope display is called an ***eye pattern*** because of its resemblance to the human eye.
  - The effect of ISI is to cause the eye to close, thereby reducing the margin for additive noise to cause errors.
  - The ISI distorts the position of the zero-crossings. Thus, it causes the system to be more sensitive to a synchronization error.




Effect of ISI on eye opening



# Signal Design for Band-limited Channels

- **Question: How to design the band-limited signals for no intersymbol interference?**
  - The task is to determine the spectral properties of the pulse $x(t)$, hence the transmitted pulse $g(t)$ that results in no intersymbol interference.

# Signal Design for Band-limited Channels

The $k$-th sample value of the receiving filter:

$$y_k = x_0 \left( I_k + \frac{1}{x_0} \sum_{\substack{n=0 \\ n \neq k}}^{\infty} I_n x_{k-n} \right) + v_k, \qquad k = 0, 1, \ldots$$

- **the condition for no intersymbol interference：**

$$x(t = kT) \equiv x_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

***Nyquist pulse-shaping criterion* or *Nyquist condition for zero ISI***



# Nyquist condition for zero ISI

- **Nyquist pulse-shaping criterion or Nyquist condition for zero ISI**

**THEOREM: (NYQUIST).** The necessary and sufficient condition for $x(t)$ to satisfy

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

is that its Fourier transform $X(f)$ satisfy

$$\sum_{m=-\infty}^{\infty} X(f + m/T) = T$$

# Nyquist condition for zero ISI

In general, $x(t)$ is the inverse Fourier transform of $X(f)$. Thus

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, df$$

At the sampling instant $t = nT$, we obtain

$$x(nT) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f nT}\, df$$

Breaking up the integral into integrals covering the finite range of $1/T$, and we have

$$\begin{aligned} x(nT) &= \sum_{m=-\infty}^{\infty} \int_{(2m-1)/2T}^{(2m+1)/2T} X(f) e^{j2\pi f nT} df \\ &= \sum_{m=-\infty}^{\infty} \int_{-1/2T}^{1/2T} X(f + m/T) e^{j2\pi f nT} df \\ &= \int_{-1/2T}^{1/2T} \left[ \sum_{m=-\infty}^{\infty} X(f + m/T) \right] e^{j2\pi f nT} df \end{aligned}$$



# Nyquist condition for zero ISI

Define $B(f) = \sum_{m=-\infty}^{\infty} X(f + m/T)$ ⋙ $x(nT) = \int_{-1/2T}^{1/2T} B(f) e^{j2\pi f nT} df$

$B(f)$ is a periodic function with period $1/T$ and can be expanded by Fourier series coefficients $\{b_n\}$

$$B(f) = \sum_{n=-\infty}^{\infty} b_n e^{j2\pi n f T} \quad \text{with} \quad b_n = T \int_{-1/2T}^{1/2T} B(f) e^{-j2\pi n f T} df$$

Compare the expression of $b_n$ to that of $x(nT)$, we have

$$b_n = T x(-nT)$$

Since $x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ ⋙ $b_n = \begin{cases} T & n = 0 \\ 0 & n \neq 0 \end{cases}$ ⋙ $B(f) = T$

equivalently

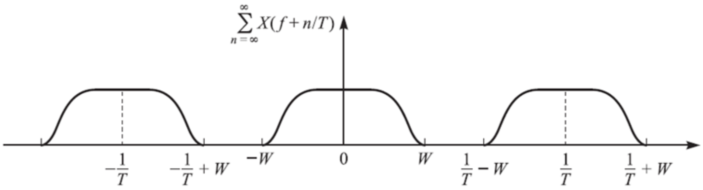
$$\sum_{m=-\infty}^{\infty} X(f + m/T) = T$$

# Nyquist condition for zero ISI

## Some Discussions:

For the band-limited channel with a bandwidth of $W$, we have $C(f) \equiv 0$ for $|f| > W$. Consequently, $X(f) = 0$ for $|f| > W$.

- When $T < 1/2W$, i.e., $1/T > 2W$, since $B(f) = \sum_{n=-\infty}^{+\infty} X(f + n/T)$ consists of non-overlapping replicas of $X(f)$ separated by $1/T$, there is no choice of $X(f)$ to ensure $B(f) \equiv T$ in this case and thus we cannot design a system with no ISI.





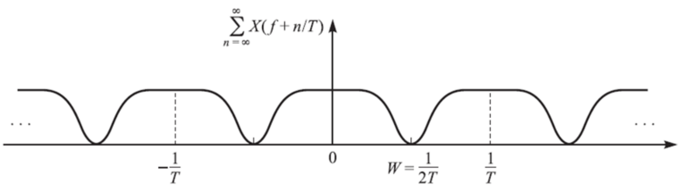

# Nyquist condition for zero ISI

- When $T = 1/2W$, i.e., $1/T = 2W$ (the Nyquist rate), the replications of $X(f)$ separated by $1/T$. It is clear that in this case there exists only one $X(f)$ that results in $B(f) = T$, namely,

$$X(f) = \begin{cases} T & |f| < W \\ 0 & \text{otherwise} \end{cases}$$

which corresponds to the pulse

$$x(t) = \frac{\sin(\pi t/T)}{\pi t/T} \equiv \text{sinc}\left(\frac{\pi t}{T}\right)$$



# Nyquist condition for zero ISI

The pulse $x(t) = \frac{\sin(\pi t/T)}{\pi t/T} \equiv \text{sinc}\left(\frac{\pi t}{T}\right)$

- $x(t)$ is noncausal and nonrealizable

  To make it realizable, usually a delayed version of $x(t)$, i.e., $sinc[\pi(t-t_0)/T]$ is used and $t_0$ is chosen such that for $t < 0$, we have $sinc[\pi(t-t_0)/T] \approx 0$. The sampling time must also be shifted to $mT + t_0$.

- The rate of convergence to zero is slow

  The tails of $x(t)$ decay as $1/t$; consequently, a small mistiming error in sampling the output of the matched filter at the demodulator results in an infinite series of ISI components. Moreover, such a series is not absolutely summable, hence the sum of the resulting ISI does not converge.

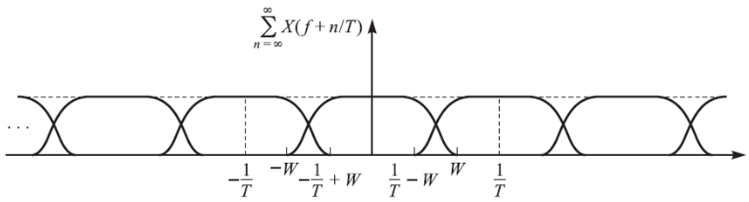


# Nyquist condition for zero ISI

- When $T > 1/2W$, i.e., $\frac{1}{T} < 2W$, $B(f)$ consists of overlapping replications of $X(f)$ separated by $1/T$.

  In this case, there exist numerous choices for $X(f)$ such that $B(f) \equiv T$.

  - A particular pulse spectrum that has desirable spectral properties and has been widely used in practice is the **raised cosine spectrum**.

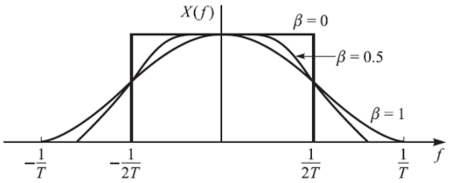


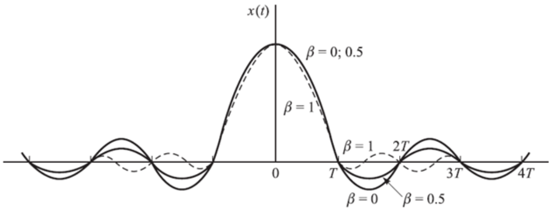
# Raised Cosine Pulse

- **The raised cosine pulse:**

$$X_{rc}(f) = \begin{cases} T & 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left\{1+\cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\} & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0 & |f| > \frac{1+\beta}{2T} \end{cases} \quad \Longleftrightarrow \quad \begin{aligned} x(t) &= \frac{\sin(\pi t/T)}{\pi t/T}\,\frac{\cos(\pi\beta t/T)}{1-4\beta^2 t^2/T^2} \\ &= \mathrm{sinc}(\pi t/T)\frac{\cos(\pi\beta t/T)}{1-4\beta^2 t^2/T^2} \end{aligned}$$

$\beta$: the *roll-off factor* in the range $0 \le \beta \le 1$






# Signal Design for Band-limited Channels

□ **The design of the raised cosine spectrum $X_{rc}(f)$:**

- When the channel is ideal, i.e., $C(f) = 1$, $|f| \le W$, we have

$$X_{rc}(f) = G_T(f)G_R(f)$$

$G_T(f)$ and $G_R(f)$: the frequency responses of the transmitter filter and receiver filter, respectively

If the receiver filter is matched to the transmitter filter, we have

$$X_{rc}(f) = G_T(f)G_R(f) = |G_T(f)|^2$$

$$G_R(f) = G_T^*(f) \quad \ggg \quad G_T(f) = \sqrt{|X_{rc}(f)|}e^{-j2\pi f t_0}$$

$t_0$: some nominal delay that is required to ensure physical realizability of the filter

# Signal Design for Channels with Distortion

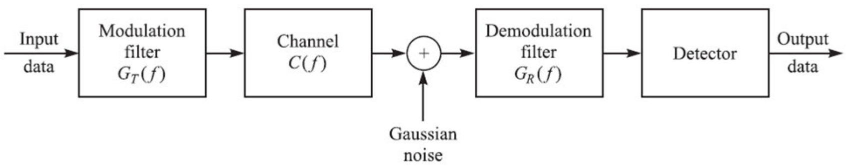
- When the channel is nonideal, It is assumed that the channel frequency-response $C(f)$ is known for $|f| \leq W$ and that $C(f) = 0$ for $|f| > W$.

  The output of the demodulator should satisfy

$$G_T(f)C(f)G_R(f) = X_d(f)e^{-j2\pi f t_0}, \qquad |f| \leq W$$

  $X_d(f)$: the desired frequency response of the cascade of the modulator, channel, and demodulator

  $t_0$: a time delay to ensure the physical realizability of the modulation and demodulation filters




System model for the design of the modulation and demodulation filters



# Signal Design for Channels with Distortion

## ☐ Consider the case of zero ISI by selecting $X_d(f) = X_{rc}(f)$

- **Solution 1:** We precompensate for the total channel distortion at the transmitter, so that the filter at the receiver is matched to the received signal.

  The magnitude characteristics are

$$|G_T(f)| = \frac{\sqrt{X_{rc}(f)}}{|C(f)|}, \qquad |f| \leq W$$

$$|G_R(f)| = \sqrt{X_{rc}(f)}, \qquad |f| \leq W$$

  The phase characteristic of the channel frequency response $C(f)$ may also be compensated at the transmitter filter.

## Signal Design for Channels with Distortion

- **Consider the case of zero ISI by selecting $X_d(f) = X_{rc}(f)$**
  - **Solution 2:** We split the channel compensation equally between the transmitter and receiver filters.

    The magnitude characteristics are

$$|G_T(f)| = \frac{\sqrt{X_{rc}(f)}}{|C(f)|^{1/2}}, \qquad |f| \le W$$

$$|G_R(f)| = \frac{\sqrt{X_{rc}(f)}}{|C(f)|^{1/2}} \qquad |f| \le W$$

The phase characteristic of $C(f)$ may also be split equally between the transmitter and receiver filters.



## Signal Design for Channels with Distortion

*Example:* Let us determine the transmitting and receiving filters for a binary communication system that transmits data at a rate of 4800 bits/s over a channel with frequency (magnitude) response

$$|C(f)| = \frac{1}{\sqrt{1+(f/W)^2}}, \qquad |f| \le W$$

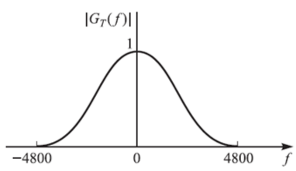
where $W = 4800$ Hz. The additive noise is zero-mean white Gaussian with spectral density $\frac{N_0}{2} = 10^{-15}$ W/Hz.

*Solution:* Since $W = 1/T = 4800$, we use a signal pulse with a raised cosine spectrum and $\beta = 1$.

Thus, we obtain

$$X_{rc}(f) = \tfrac{1}{2}T[1+\cos(\pi T|f|)]$$
$$= T\cos^2\left(\frac{\pi|f|}{9600}\right)$$

$$|G_T(f)| = |G_R(f)| = \left[1+\left(\frac{f}{4800}\right)^2\right]^{\frac{1}{4}} T^{\frac{1}{2}} \cos\left(\frac{\pi|f|}{9600}\right), \quad |f| \le 4800$$



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# Optimum Receiver for Channels with ISI and AWGN

- **The optimum detector for digital transmission through a nonideal band-limited channel with additive Gaussian noise:**
  - The received (equivalent lowpass) signal is expressed as

$$r_l(t) = \sum_n I_n h(t - nT) + z(t)$$

$h(t) = g(t) * c(t)$: the response of the channel to the input signal pulse $g(t)$

$z(t)$: the additive white Gaussian noise

## Optimum Receiver for Channels with ISI and AWGN

**□ Vector form representation of signal $r_l(t)$**

$$r_l(t) = \sum_n I_n h(t - nT) + z(t)$$

- Given a complete set of orthonormal functions $\{\phi_k(t)\}$, By further projecting $h(t-nT)$, $z(t)$ onto the orthonormal basis $\{\phi_k(t)\}$, we obtain

$$r_k = \sum_n I_n h_{kn} + z_k$$

$h_{kn}$ and $z_k$: the results of projecting $h(t-nT)$, $z(t)$ onto the set $\phi_k(t)$

$r_k$: the projection of $r_l(t)$ on the set of orthonormal function $\phi_k(t)$



## Optimum Receiver for Channels with ISI and AWGN

Denoting $\boldsymbol{r}_N \equiv [r_1 \ r_2 \cdots r_N]$ $\boldsymbol{I}_p \equiv [I_1 \ I_2 \cdots I_p]$ (p ≤ N) and $p(z_k) = \frac{1}{2\pi N_0}\exp\left(-\frac{|z_k|^2}{2N_0}\right)$

we obtain the likelihood function

$$p(\boldsymbol{r}_N|\boldsymbol{I}_p) = \left(\frac{1}{2\pi N_0}\right)^N \exp\left(-\frac{1}{2N_0}\sum_{k=1}^{N}\left|r_k - \sum_n I_n h_{kn}\right|^2\right)$$

- **The ML detector:**

$$\hat{\boldsymbol{I}}_p = arg\max_{\boldsymbol{I}_p} p(\boldsymbol{r}_N|\boldsymbol{I}_p)$$

equivalently

$$\hat{\boldsymbol{I}}_p = arg\max_{\boldsymbol{I}_p} \underbrace{-\sum_{k=1}^{N}\left|r_k - \sum_n I_n h_{kn}\right|^2}_{PM(\boldsymbol{I}_p)} = arg\max_{\boldsymbol{I}_p} PM(\boldsymbol{I}_p)$$

## Optimum Receiver for Channels with ISI and AWGN

- **The ML detector:** $\hat{\boldsymbol{I}}_p = arg\max_{\boldsymbol{I}_p} PM(\boldsymbol{I}_p)$

we obtain
$$PM(\boldsymbol{I}_p) = -\sum_{k=1}^{N}\left|r_k - \sum_n I_n h_{kn}\right|^2$$
$$= -\int_{-\infty}^{\infty}\left|r_l(t) - \sum_n I_n h(t-nT)\right|^2 dt$$
$$= \boxed{-\int_{-\infty}^{\infty}|r_l(t)|^2 dt} + 2\mathrm{Re}\sum_n\left[I_n^*\int_{-\infty}^{\infty} r_l(t)h^*(t-nT)\,dt\right]$$
$$-\sum_n\sum_m I_n^* I_m \int_{-\infty}^{\infty} h^*(t-nT)h(t-mT)\,dt$$

Independent of $\boldsymbol{I}_p$



## Optimum Receiver for Channels with ISI and AWGN

$$PM(\boldsymbol{I}_p) = -\int_{-\infty}^{\infty}|r_l(t)|^2 dt + 2\mathrm{Re}\sum_n\left[I_n^*\int_{-\infty}^{\infty} r_l(t)h^*(t-nT)\,dt\right]$$
$$-\sum_n\sum_m I_n^* I_m \int_{-\infty}^{\infty} h^*(t-nT)h(t-mT)\,dt$$

- Define $y_n \equiv y(nT) = \int_{-\infty}^{\infty} r_l(t)h^*(t-nT)\,dt$

  $y_n$ could be generated by passing $r_l(t)$ through a matched filter $h^*(-t)$ and sampling the output at the symbol rate $1/T$

- Define $x_n \equiv x(nT) = \int_{-\infty}^{\infty} h^*(t)h(t+nT)\,dt$

  $x(t)$ represents the output of a filter having an impulse response $h^*(-t)$ and an excitation $h(t)$

- **The ML detector:**

$$\hat{\boldsymbol{I}}_p = arg\max_{\boldsymbol{I}_p} CM(\boldsymbol{I}_p) \quad \text{with} \quad CM(\boldsymbol{I}_p) = 2\mathrm{Re}\left(\sum_n I_n^* y_n\right) - \sum_n\sum_m I_n^* I_m x_{n-m}$$

## Optimum Receiver for Channels with ISI and AWGN

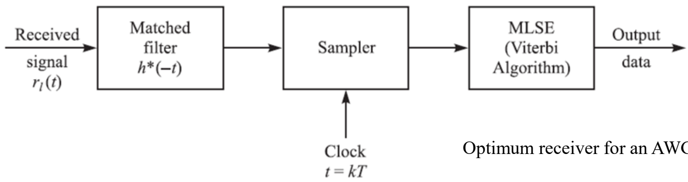
In practical system, it is reasonable to assume that the ISI affects a finite number of symbols, i.e., that $x_n = 0$ for $|n| > L$:

$$CM(\boldsymbol{I}_p) = 2\mathrm{Re}\left(\sum_n I_n^* y_n\right) - \sum_n \sum_m I_n^* I_m x_{n-m}$$

Recursive computation according to the relation

$$CM_n(\boldsymbol{I}_n) = CM_{n-1}(\boldsymbol{I}_{n-1}) + \mathrm{Re}\left[I_n^*\left(2y_n - x_0 I_n - 2\sum_{m=1}^{L} x_m I_{n-m}\right)\right]$$

The Viterbi algorithm could solve the maximization of $CM_n(\boldsymbol{I}_n)$ efficiently.




Optimum receiver for an AWGN channel with ISI



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# Equalization

- The ML sequence estimation for a channel with ISI has a computational complexity that grows **exponentially** with the length of the channel time dispersion. If the size of the symbol alphabet is $M$ and the number of interfering symbols contributing to ISI is $L$, the Viterbi algorithm computes $M^{L+1}$ metrics for each new received symbol.

- In the following, equalization approaches are adopted to compensate for the ISI.



# Equalization

**The purpose is to reduce intersymbol interference to allow the recovery of the transmit symbols.**

- Typical equalizers
  - **linear equalizer:** a linear filter
  - **decision-feedback equalizer**: a nonlinear filter with a feedforward filter and a feedback filter
  - **blind equalizer**: adjusts the equalizer coefficients without a training sequence
  - **turbo equalizer**: applies turbo decoding while treating the channel as a convolutional code

# Equalization

For $x_n = 0$ for $|n| > L$, the discrete-time sequence $\{y_k\}$, i.e., the output of the matched filter:

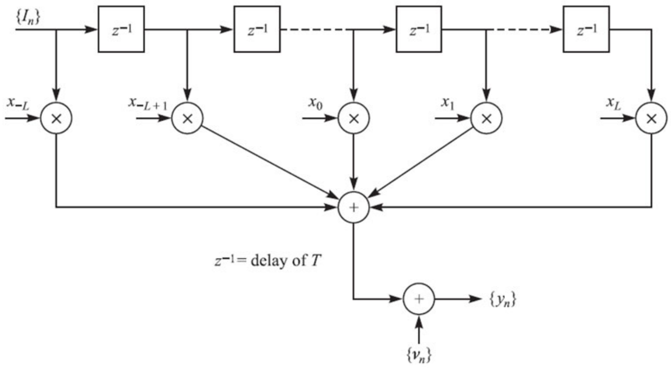
$$y_k = \sum_n I_n x_{k-n} + v_k = \sum_{n=-L}^{L} x_n I_{k-n} + v_k$$

$$v_k = \int_{-\infty}^{\infty} z(t) h^*(t - kT)\, dt$$

- The set of noise variables $\{v_k\}$ is a Gaussian-distributed sequence with zero-mean and autocorrelation function

$$E\left(v_k^* v_j\right) = \begin{cases} 2N_0 x_{j-k} & (|k-j| \le L) \\ 0 & (\text{otherwise}) \end{cases}$$

**The noise sequence is correlated** unless $x_k = 0$, $k \neq 0$.




Equivalent discrete-time model of channel with intersymbol interference



# Equalization

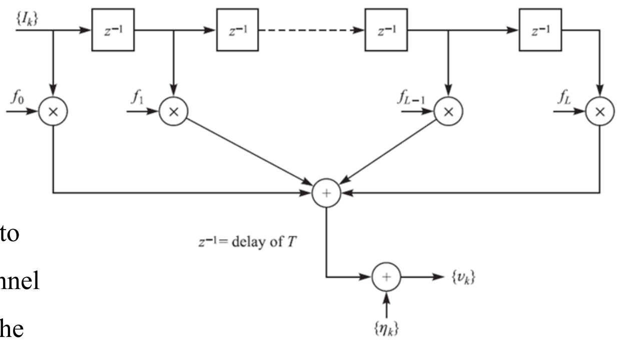
- The **correlations** of the noise sequence $\{v_k\}$ at the output of the matched filter causes the difficulty in the evaluation of performance of the various equalization methods.

- Passing $\{y_k\}$ through a **noise-whitening filter**, the output of the noise-whitening filter:

$$v_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k$$

$\{\eta_k\}$: **a white Gaussian noise**

$\{f_k\}$: a set of tap coefficients of a filter equivalent to the cascade of the transmitting filter $g(t)$, the channel $c(t)$, the matched filter $h^*(-t)$, the sampler, and the discrete-time noise-whitening filter $1/F^*(\frac{1}{z^*})$ and $F(z)F^*\left(\frac{1}{z^*}\right) = X(z)$




Equivalent discrete-time model of intersymbol interference channel with AWGN

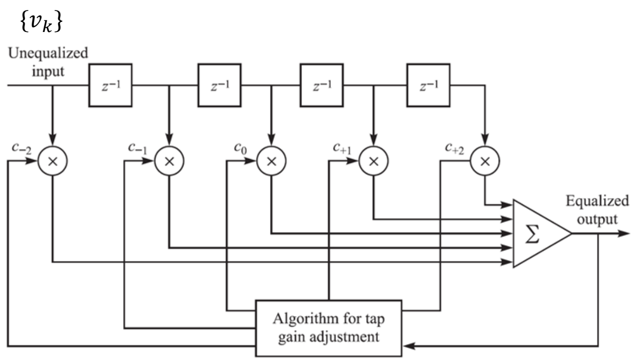
# Linear Equalization

- **Linear Transversal Filter:**

  the estimate of the *k*-th symbol:

$$\hat{I}_k = \sum_{j=-K}^{K} c_j v_{k-j}$$

  $\{c_j\}$: the $2K+1$ complex-valued tap weight coefficients of the filter




Linear transversal filter with $K = 2$



# Linear Equalization

☐ **Question**: what is the criterion to obtain the optimal coefficients $\{c_j\}$?

The most meaningful measure of performance for a digital communication system is the **average probability of error**. However, the probability of error is a **highly non-linear** function of $\{c_j\}$, and thus optimizing the equalizer coefficients under this criterion is **computationally costly**.

- Two widely used criteria:
  - **peak distortion criterion**
  - **mean-square-error criterion**

# Peak Distortion Criterion

- The cascade of the discrete-time linear filter model having an impulse response $\{f_n\}$ and an equalizer having an impulse response $\{c_n\}$ can be represented by a single equivalent filter having the impulse response

$$q_n = \sum_{j=-\infty}^{\infty} c_j f_{n-j} \qquad \text{the convolution of } \{c_n\} \text{ and } \{f_n\}$$

- Consider the equalizer with an infinite number of taps, the output at the $k$-th sampling instant can be expressed in the form

$$\hat{I}_k = q_0 I_k + \underbrace{\sum_{n \neq k} I_n q_{k-n}}_{\text{intersymbol interference}} + \underbrace{\sum_{j=-\infty}^{\infty} c_j \eta_{k-j}}_{\text{the convolution of } \{c_n\} \text{ and noise sequence } \{\eta_n\}}$$



# Peak Distortion Criterion

- **The peak distortion criterion is to minimize the peak distortion, i.e., the worst-case intersymbol interference at the output of the equalizer.**

The peak value of the intersymbol interference, i.e., *peak distortion*, is written as

$$\begin{aligned} \mathcal{D}(\boldsymbol{c}) &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |q_n| \\ &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left| \sum_{j=-\infty}^{\infty} c_j f_{n-j} \right| \end{aligned}$$

It is to minimize $D(\boldsymbol{c})$, which is a function of the equalizer tap weights.

# Peak Distortion Criterion

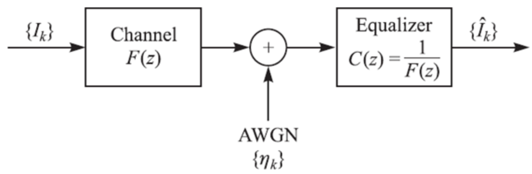
- **Infinite-length equalizer:**
  - If an equalizer having an infinite number of taps, it is possible to select the tap weights so that $D(\boldsymbol{c}) = 0$, i.e., $q_n = 0$ for all $n$ except $n = 0$, that is

$$q_n = \sum_{j=-\infty}^{\infty} c_j f_{n-j} = \begin{cases} 1 & (n = 0) \\ 0 & (n \neq 0) \end{cases}$$

***zero-forcing filter***

  - Taking the $z$ transform, the above condition is equivalent to

$$Q(z) = C(z)F(z) = 1 \quad \ggg \quad C(z) = \frac{1}{F(z)}$$




Block diagram of channel with zero-forcing equalizer

**Complete elimination of the intersymbol interference requires the use of an inverse filter to $F(z)$.**



# Peak Distortion Criterion

- **Finite-length equalizer:**
  - For a finite-length equalizer having $2K + 1$ taps, $c_j = 0$ for $|j| > K$, the convolution of $\{f_n\}$ with $\{c_n\}$ is zero outside the range $-K \le n \le K + L - 1$. That is, $q_n = 0$ for $n < -K$ and $n > K + L - 1$.
  - With $q_0$ normalized to unity, the peak distortion is

$$\mathcal{D}(\boldsymbol{c}) = \sum_{\substack{n=-K \\ n\neq 0}}^{K+L-1} |q_n| = \sum_{\substack{n=-K \\ n\neq 0}}^{K+L-1} \left| \sum_j c_j f_{n-j} \right|$$

  - The equalizer has $2K + 1$ adjustable parameters $\{c_j\}$, there are $2K + L$ nonzero values in the response $\{q_n\}$. Thus it is generally **impossible to completely eliminate the intersymbol interference** at the output of the equalizer.
  - The peak distortion is a **convex function** of the coefficients $\{c_j\}$. Its minimization can be carried out **numerically** using, for example, the method of steepest descent.

## Mean-Square-Error (MSE) Criterion

- **MSE criterion**: the equalizer coefficients $\{c_j\}$ are adjusted to minimize the mean square value

$$J = E|\varepsilon_k|^2 = E|I_k - \hat{I}_k|^2$$

$I_k$: the information symbol transmitted in the *k*th signaling interval

$\hat{I}_k$:the estimate of information symbol at the output of the equalizer

- ***Infinite-length equalizer***

$$\hat{I}_k = \sum_{j=-\infty}^{\infty} c_j v_{k-j} \quad \ggg \quad J = E\left|I_k - \sum_{j=-\infty}^{\infty} c_j v_{k-j}\right|^2$$

This could be solved by the **orthogonality principle** in mean square estimation.



## Mean-Square-Error (MSE) Criterion

According to the orthogonality principle, we select the coefficients $\{c_j\}$ to make $\varepsilon_k = I_k - \sum_{j=-\infty}^{\infty} c_j v_{k-j}$ orthogonal to the signal sequence $\{v_{k-l}^*\}$ for $-\infty < l < \infty$, i.e.,

$$E\left(\varepsilon_k v_{k-l}^*\right) = 0, \qquad -\infty < l < \infty \quad \ggg \quad E\left[\left(I_k - \sum_{j=-\infty}^{\infty} c_j v_{k-j}\right) v_{k-l}^*\right] = 0$$

equivalently

$$\ggg \quad \sum_{j=-\infty}^{\infty} c_j E\left(v_{k-j} v_{k-l}^*\right) = E\left(I_k v_{k-l}^*\right)$$

with

$$E\left(v_{k-j} v_{k-l}^*\right) = \sum_{n=0}^{L} f_n^* f_{n+l-j} + N_0 \delta_{lj}$$

$$E\left(I_k v_{k-l}^*\right) = \begin{cases} f_{-l}^* & (-L \le l \le 0) \\ 0 & (\text{otherwise}) \end{cases}$$

equivalently

$$\ggg \quad \sum_{j=-\infty}^{\infty} c_j \sum_{n=0}^{L} f_n^* f_{n+l-j} + N_0 \delta_{lj} = f_{-l}^*$$

# Mean-Square-Error (MSE) Criterion

$$\sum_{j=-\infty}^{\infty} c_j \sum_{n=0}^{L} f_n^* f_{n+l-j} + N_0 \delta_{lj} = f_{-l}^* \qquad -\infty < l < \infty$$

- Taking the $z$ transform of both sides of the resulting equation, we obtain

$$C(z)[F(z)F^*(1/z^*) + N_0] = F^*(1/z^*)$$

- The transfer function of the equalizer based on the MSE criterion is

$$C(z) = \frac{F^*(1/z^*)}{F(z)F^*(1/z^*) + N_0}$$



# Mean-Square-Error (MSE) Criterion

- ***Finite-length equalizer*** $\hat{I}_k = \sum_{j=-K}^{K} c_j v_{k-j}$
  - The MSE for the equalizer having $2K + 1$ taps

$$J(K) = E|I_k - \hat{I}_k|^2 = E\left| I_k - \sum_{j=-K}^{K} c_j v_{k-j} \right|^2$$

  - Minimization of $J(K)$ with respect to the tap weights $\{c_j\}$ is equivalent to forcing the error $\varepsilon_k = I_k - \hat{I}_k$ to be orthogonal to the signal samples $\{v_{j-l}^*\}$, which leads to linear equations

$$\sum_{j=-K}^{K} c_j \Gamma_{lj} = \xi_l, \qquad l = -K, \ldots, -1, 0, 1, \ldots, K \quad ⋙ \quad \boldsymbol{\Gamma C} = \boldsymbol{\xi} \quad ⋙ \quad \boldsymbol{C}_{\text{opt}} = \boldsymbol{\Gamma}^{-1}\boldsymbol{\xi}$$

with $\Gamma_{lj} = \begin{cases} \sum_{n=0}^{L} f_n^* f_{n+l-j} + N_0 \delta_{lj} & (|l-j| \le L) \\ 0 & (\text{otherwise}) \end{cases} \qquad \xi_l = \begin{cases} f_{-l}^* & (-L \le l \le 0) \\ 0 & (\text{otherwise}) \end{cases}$

## Conclusions

I. Characterization of Band-Limited Channels
II. Signal Design for Band-limited Channels
III. Optimum Receiver for Channels with ISI and AWGN
IV. Linear Equalization



## Next Lecture

**Lecture 4: Carrier and Symbol Synchronization**

I. Signal Parameter Estimation
II. Carrier Phase Estimation
III. Symbol Timing Estimation
IV. Performance Characteristics of ML estimators