Assignment 7

1. $\begin{bmatrix} 4-1 & 0 & 1 \\ -2 & 1-1 & 0 \\ -2 & 0 & 1-1 \end{bmatrix}$ $\begin{bmatrix} 4-2 & 0 & 1 \\ -2 & 1-2 & 0 \\ -2 & 0 & 1-2 \end{bmatrix}$ $\begin{bmatrix} 4-3 & 0 & 1 \\ -2 & 1-3 & 0 \\ -2 & 0 & 1-3 \end{bmatrix}$

det = 0 det = 0 det = 0

$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$

reduce $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\downarrow$ $\downarrow$ $\downarrow$

$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 1, \quad 2, \quad 3$

eigenspace = $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

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2. done as program

3. done as program

$$\left[\begin{bmatrix}2\\2\\1\end{bmatrix}\begin{bmatrix}1/2\\-1\\1\end{bmatrix}\begin{bmatrix}-2\\2\\1\end{bmatrix}\right]$$

4. $\det(A-\lambda I) = (0-\lambda)(1-\lambda)(-1-\lambda) + 0 + 0 - 4(1-\lambda) - 0 - 4(-1-\lambda)$

$\boxed{A = A^{-1}}$ √

$(-\lambda+\lambda^2)(-1-\lambda) - 4 + 4\lambda + 4 + 4\lambda$

$-\lambda^3 + 9\lambda$

$(\lambda)(-\lambda^2+9)$ $\lambda = \pm 3, 0$

$\lambda = 3$:
$$\begin{bmatrix}-3&2&2\\2&-2&0\\2&0&-4\end{bmatrix} = \begin{bmatrix}1&0&-2\\0&1&-2\\0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}2\\2\\1\end{bmatrix}$$

$\lambda = 0$:
$$\begin{bmatrix}0&2&2\\2&1&0\\2&0&-1\end{bmatrix} \quad \begin{bmatrix}1&0&-\frac{1}{2}\\0&1&1\\0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}\frac{1}{2}\\-1\\1\end{bmatrix}$$

$\lambda = -3$:
$$\begin{bmatrix}3&2&2\\2&4&0\\2&0&2\end{bmatrix} \quad \begin{bmatrix}1&0&2\\0&1&-2\\0&0&0\end{bmatrix} \rightarrow \begin{bmatrix}-2\\2\\1\end{bmatrix}$$

5. $\det(A-\lambda I) \rightarrow \lambda^2(9-\lambda)$

$\lambda = 0, 0, 9$

$\lambda = 0$
$$\begin{bmatrix}1&-2&2\\-2&4&-4\\2&-4&4\end{bmatrix} \downarrow \begin{bmatrix}1&-2&2\\0&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}-8&-2&2\\-2&-5&-4\\2&-4&-5\end{bmatrix} \downarrow \begin{bmatrix}1&0&-1/2\\0&1&1\\0&0&0\end{bmatrix}$$

$P_1 = \begin{bmatrix}2\\1\\0\end{bmatrix}$ $u_2 = \begin{bmatrix}-2\\0\\1\end{bmatrix}$ $P_2 = \begin{bmatrix}1/2\\-1\\1\end{bmatrix}$

$$P = \begin{bmatrix}2\\1\\0\end{bmatrix}, \frac{1}{5}\begin{bmatrix}-2\\4\\5\end{bmatrix}, \begin{bmatrix}1/2\\-1\\1\end{bmatrix}$$

normalize P $= \boxed{\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\\0\end{bmatrix}, \frac{5}{3\sqrt{5}}\begin{bmatrix}-2\\4\\5\end{bmatrix}, \frac{2}{3}\begin{bmatrix}1/2\\-1\\1\end{bmatrix}}$

$$D = \begin{bmatrix}0&0&0\\0&0&0\\0&0&9\end{bmatrix}$$

$P_2 = u_2 - \text{proj}_{P_1} u_2$

$$= \begin{bmatrix}-2\\0\\1\end{bmatrix} - \left(\frac{-4}{5} P_1\right) = \begin{bmatrix}-2+\frac{4}{5}\cdot 2\\0+\frac{4}{5}\\1\end{bmatrix} = \begin{bmatrix}-2/5\\4/5\\1\end{bmatrix}$$

6.

$$A^TA = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$$

$$\det(A-\lambda I) = (4-\lambda)(13-\lambda)-36 = 0$$
$$= \lambda^2 - 17\lambda + 16$$
$$= (\lambda-16)(\lambda-1) = 0$$
$$\lambda = 16, 1$$

$$\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{V_1 = \frac{2}{\sqrt{5}}\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \qquad V_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

$$\boxed{\begin{aligned} \sigma_1 &= \sqrt{16} = 4 \\ \sigma_2 &= \sqrt{1} = 1 \end{aligned}} \qquad A = u\Sigma V^T$$

$$u_i = \frac{1}{\sigma} Av_i \qquad \boxed{u_1 = \frac{1}{2\sqrt{5}}\begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

$$U = [u_1, u_2, 0 \ldots 0]$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ & & 0 \\ 0 & & \ddots & 0 \end{bmatrix} \qquad A = u\Sigma V^T$$

$$V = [V_1, V_2, 0 \ldots 0]$$

7. $X^T X v_i = \lambda v_i$

add X to both sides

$X X^T X v_i = \lambda X v_i$

$X X^T (X v_i) = \lambda (X v_i)$

$X v_i$ is a eigenvector for $X X^T$