

CSE 3313 - Homework #5 – z-transforms 1

Find the z-transform polynomial ratio and ROC for the following unit sample sequences:

$$\sum_{n=-\infty}^{\infty} h[n]z^{-n}, \quad z = re^{j\omega}$$

1. $h[n] = \left(\frac{4}{5}\right)^n u[n]$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{4}{5z}\right)^n = \frac{1}{1 - \frac{4}{5z}} = \frac{z}{z - \frac{4}{5}}$$

$$ROC: |z| > \frac{4}{5}$$

2. $h[n] = 4^n u[-n - 1]$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} 4^n u[-n - 1]z^{-n} = \sum_{n=-1}^{-\infty} 4^n z^{-n} = \sum_{n=1}^{\infty} \left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^{n+1} \\ &= \left(\frac{z}{4}\right) \sum_{n=0}^{\infty} \left(\frac{z}{4}\right)^n = \left(\frac{z}{4}\right) \frac{1}{1 - \frac{z}{4}} = \frac{z}{4 - z} \end{aligned}$$

$$ROC: |z| < 4$$

3. $h[n] = \left(\frac{1}{5}\right)^n u[-n]$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n z^{-n} = \sum_{n=0}^{\infty} (5z)^n = \frac{1}{1 - 5z}$$

$$ROC: |z| < \frac{1}{5}$$

4. $h[n] = 2^n u[n - 1]$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=1}^{\infty} \left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^{n+1} = \left(\frac{2}{z}\right) \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n = \left(\frac{2}{z}\right) \frac{1}{1 - \frac{2}{z}} = \frac{2}{z - 2}$$

$$ROC: |z| > 2$$

5. Which of the systems above has a Fourier Transform that exists?

A Fourier Transform exists when the z-transform converges at $|z|=1$. This means that the Fourier Transform exists for 1, and 2 but not 3 or 4.

Find the inverse z-transform of the following z-transforms:

$$\frac{z}{z-a} = \frac{1}{1-az^{-1}} = \sum_{n=0}^{\infty} a^n z^{-n}$$

6. $H(z) = \frac{z}{z-\frac{1}{3}}, |z| > \frac{1}{3}$

pole: $z = \frac{1}{3}$, ROC is right sided. Done by inspection.

$$\frac{z}{z-\frac{1}{3}} = \frac{1}{1-\frac{1}{3}z^{-1}} \rightarrow h[n] = \left(\frac{1}{3}\right)^n u[n]$$

7. $H(z) = \frac{z}{z-2}, |z| < 2$

pole: $z = 2$, ROC is left sided. Done by inspection.

$$\frac{z}{z-2} = \frac{1}{1-2z^{-1}} \rightarrow h[n] = 2^n u[-n-1]$$

8. $H(z) = \frac{1}{1-2z^{-1}}, |z| < 2$

Same as above question

$$h[n] = 2^n u[-n-1]$$

9. $H(z) = \frac{1}{1-\frac{2}{3}z^{-1}}, |z| > \frac{2}{3}$

pole: $z = \frac{2}{3}$, ROC is right sided. Done by inspection.

$$\frac{1}{1-\frac{2}{3}z^{-1}} \rightarrow h[n] = \left(\frac{2}{3}\right)^n u[n]$$

10. Which of the systems above has a Fourier Transform that exists?

A Fourier Transform exists when the series converges at $|z| = 1$. This means that all the questions have a Fourier transform that exists.

Find the poles and zeros of the following z-transform polynomial ratios:

$$11. H(z) = \frac{z(2z-6)}{(z-2)(z-4)}$$

zeros: $z=0, z=3$

poles: $z=2, z=4$

$$12. H(z) = \frac{2 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

$$\frac{2 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})} = \frac{z(2z - \frac{5}{6})}{(z - \frac{1}{2})(z - \frac{1}{3})}$$

zeros: $z = \frac{5}{12}, z = 0$

poles: $z = \frac{1}{2}, z = \frac{1}{3}$

Find the z-transform and ROC (region of convergence) of the following unit sample sequences:

$$\sum_{n=-\infty}^{\infty} h[n]z^{-n}, \quad z = re^{j\omega}$$

$$13. h[n] = \left(\frac{3}{4}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n]$$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{3}{4}\right)^n u[n]z^{-n} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{4}\right)^n u[n]z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{3}{4z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{4z}\right)^n = \frac{1}{1 - \frac{3}{4}z^{-1}} + \frac{1}{1 - \frac{1}{4}z^{-1}} = \frac{z}{z - \frac{3}{4}} + \frac{z}{z - \frac{1}{4}} \end{aligned}$$

$$ROC: |z| > \frac{3}{4} \text{ \& } |z| > \frac{1}{4} \rightarrow |z| > \frac{3}{4}$$

$$14. h[n] = 2^n u[n] + 4^n u[n]$$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=-\infty}^{\infty} 2^n u[n]z^{-n} + \sum_{n=-\infty}^{\infty} 4^n u[n]z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{4}{z}\right)^n = \frac{1}{1 - 2z^{-1}} + \frac{1}{1 - 4z^{-1}} = \frac{z}{z - 2} + \frac{z}{z - 4} \end{aligned}$$

$$ROC: |z| > 2 \text{ \& } |z| > 4 \rightarrow |z| > 4$$

$$15. h[n] = \left(\frac{3}{4}\right)^n u[n] - 2^n u[-n - 1]$$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{3}{4}\right)^n u[n]z^{-n} - \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{3}{4z}\right)^n - \left(\frac{z}{2}\right) \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \frac{z}{z - \frac{3}{4}} - \frac{z}{2 - z} \end{aligned}$$

$$ROC: |z| > \frac{3}{4} \text{ \& } |z| < 2 \rightarrow \frac{3}{4} < |z| < 2$$

$$16. h[n] = -2^n u[-n-1] - 5^n u[-n-1]$$

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h[n] z^{-n} = - \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n - \sum_{n=1}^{\infty} \left(\frac{z}{5}\right)^n \\ &= - \left(\frac{z}{2}\right) \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n - \left(\frac{z}{5}\right) \sum_{n=0}^{\infty} \left(\frac{z}{5}\right)^n = - \frac{z}{2-z} - \frac{z}{5-z} \end{aligned}$$

$$ROC: |z| < 2 \text{ \& } |z| < 5 \rightarrow |z| < 2$$

17. Which of the systems above has a Fourier Transform that exists?

A Fourier Transform exists when the z-transform converges at $|z|=1$. This means that the Fourier Transform exists for 13, 15, and 16.