2D Modelling Pathlines of Planar Vortices

Landon Knipp

Abstract

Vortices are a natural occurring phenomena that are all around us: from the whirlpools that form at the drain in your tub to turbulent flow on airplanes. It is crucial to understand how vortices behave with themselves and other objects so we can innovate current technologies, such as design more efficient air crafts. A great deal of understanding how they behave is calculating their trajectories. Their paths can be extremely complex because they influence each other. A way to compute their behavior, we can use numerical integration.

1 Problem Description

The goal is to determine the path-lines of vortices in the presence of each other. The motion of a single object in the velocity field induced by a single vortex is simple: it orbits around in a circle. However, the reason this becomes more complicated as you place additional vortices near each other is consequential of how they each independently affect the overall velocity field of the fluid they reside in. Imagine having 30 vortices placed at random locations in a plane. It is nearly impossible to calculate how they will behave by hand. This is where the beauty of code comes in: we can break down this complex behavior into tiny simple steps, and string them all together to give us an accurate result.

2 Equation Formulation & Code Implementation

2.1 Equation Derivations

It is given that a point vortex induces a velocity field about its center:

$$\mathbf{u} = (0\hat{\mathbf{u}}_r, \frac{\Gamma}{2\pi r}\hat{\mathbf{u}}_\theta) \tag{1}$$

Γ is the circulation of the vortex: how strong it is and which direction it rotates; r is the distance away from the vortex. You can see that only the direction of the vortices will change. Since we are using MATLAB to solve for the path-lines, we want to make use of $r^2 = x^2 + y^2$ and that we know any distance of one vortex to any other: $r_i - r_j$

In order to be able to find a velocity field from all of the vortices present, we need to prove that there is a stream-function that describes the flow (which can satisfy the Laplace equation: $\nabla^2\psi = 0$). If one exists, then we can superimpose all of the velocity fields to form an overall velocity field:

$$u_\theta = -\frac{\partial\psi}{\partial r} \tag{2}$$

$$\partial\psi = -\frac{\Gamma}{2\pi r}\partial r \tag{3}$$

$$\psi = -\frac{\Gamma}{2\pi}ln(r) \tag{4}$$

Thus, there is a stream-function that exists, and we can superimpose the velocity fields induced by each vortex together. From 1 and breaking down r into components, the velocity a vortex will feel is:

$$\mathbf{u}_i = [\sum_{j=1,j\neq i}^{N} \frac{\Gamma_j}{2\pi[(x_i-x_j)^2+(y_i-y_j)^2]}[-(y_i-y_j)]\mathbf{i}, \sum_{j=1,j\neq i}^{N} \frac{\Gamma_j}{2\pi[(x_i-x_j)^2+(y_i-y_j)^2]}[(x_i-x_j)]\mathbf{j}] \quad (5)$$

N is the total number of vortices, the i subscripts represent the vortex being analyzed, and the j subscripts represent the rest of the vortices. Based on Helmhotz' Law, we know that the vortices will flow with the fluid: $\frac{d\mathbf{r}_i}{dt} = (\frac{d\mathbf{x}_i}{dt}, \frac{d\mathbf{y}_i}{dt}) = \mathbf{u}_i$. Thus, from kinematics, if we integrate Helmholtz' Law, we can approximate the future position after a given time period to be:

$$x_{i+1} = x_i + (\mathbf{u}_i \cdot \mathbf{i})\Delta t \quad (6)$$

$$y_{i+1} = y_i + (\mathbf{u}_i \cdot \mathbf{j})\Delta t \quad (7)$$

2.2 Coding Methods

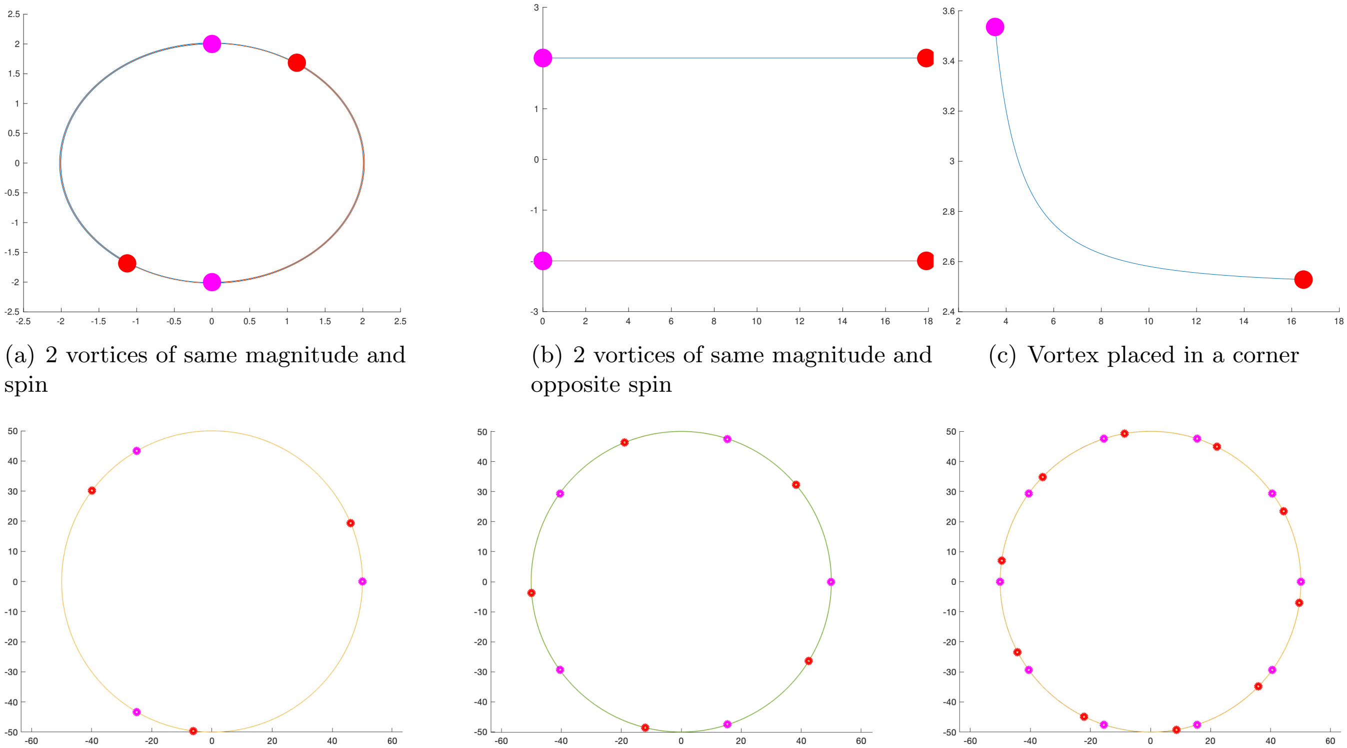
The way I decided to structure my code was by storing all of the data for in a single three dimensional matrix: the first dimension holding the time steps, the second dimension determining the x or y component, and the third dimension containing the positional coordinates. The number of time steps is determined by the total time interval, and how small we want the time step to be. The results will be more accurate as the time steps approach zero because the velocity field will be updated more often.

A major task to think about was assigning the values of the circulation for each test run. I decided to have the value of Γ be somewhat proportional to the distances between the vortices at the initial conditions. For example, I made $\Gamma = 50\frac{m^2}{s}$ for $r = 50m$; thus, $\frac{\Gamma}{r} \simeq 1\frac{m}{s}$ for $t = 0$ I did this so the path-lines of the vortices would be clearer to observe, especially for the cases that have patterns.

The length of the time intervals depended more on the problems individually, but also depended on what value of Γ I chose. Most of them were around 30 seconds, just long enough for the behavior of the vortices to be easily identified. For the time steps, I kept them small for all of the scenarios: 0.01s (except for when modeling the wake turbulence, which was 0.001s). This way the results are more accurate, which can be seen in 6 and 7. I plotted the initial points of each vortex with a magenta circle, and the final positions with red circles.

3 Discussion

3.1 Test Cases



(a) 2 vortices of same magnitude and spin

(b) 2 vortices of same magnitude and opposite spin

(c) Vortex placed in a corner

(d) Circle with 3 vortices

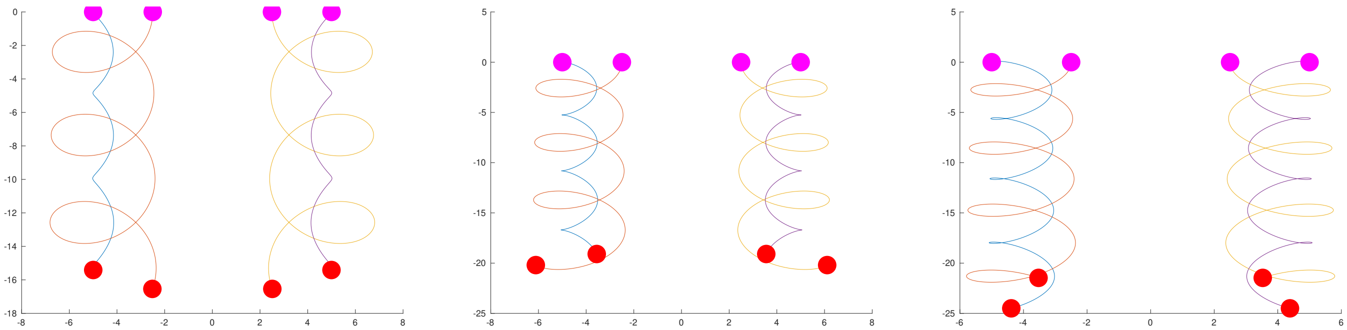
(e) Circle with 5 vortices

(f) Circle with 10 vortices

Figure 1: Test Cases

For test case (a), it is clear that the two vortices orbit around a center point. In this case, they form an elliptical shape about their centroid. Case (b) shows that the two vortices push each other along in a straight line, where they remain parallel to each other. Case (c) shows the a vortex influencing its own position when placed in a corner. It pushes itself down and away from the vertical wall. Figures (d), (e), and (f) are all scenarios of vortices being placed equidistant on the circumference of a circle. No matter how many point vortices are present, they all stay along the circumference.

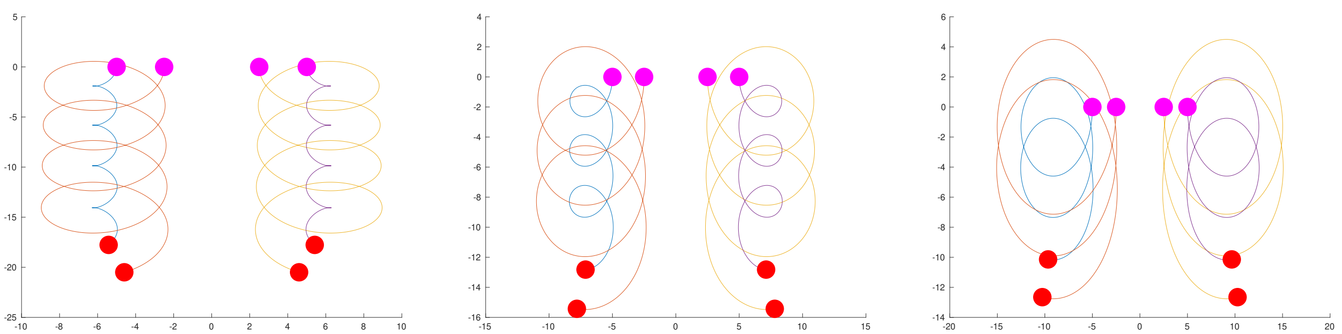
3.2 Required Calculations



(a) $\gamma = 0.2$ (b) $\gamma = 0.4$ (c) $\gamma = 0.6$

Figure 2: First Set of Conditions:
$[(-b/2, 0, -\Gamma), (-b/4, 0, -\gamma\Gamma), (b/4, 0, +\gamma\Gamma), (b/2, 0, +\Gamma)]$

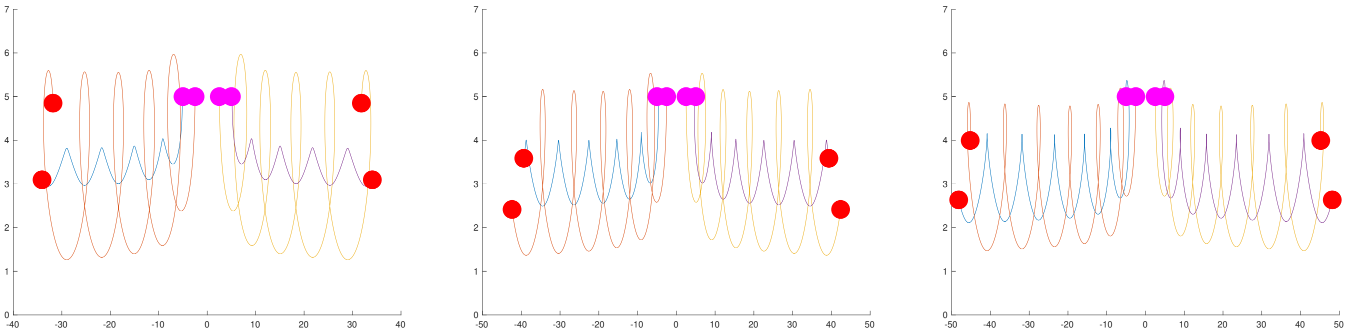
The vortices with the extra factor do loops around the vortices with no factor because they are of greater strength and influenced less by the other vortex. All of them have similar patterns, but as γ got larger and larger, the looping patter got tighter and tighter, and this is because the vortex with γ gets closer to the circulation magnitude of its partner. They then go up in a linear fashion because there opposite counter parts are mirrored. This makes sense because it is similar to adding the orbital and linear patterns as seen in the test cases.



(a) $\gamma = 0.2$ (b) $\gamma = 0.4$ (c) $\gamma = 0.6$

Figure 3: Second Set of Conditions
$[(-b/2, 0, -\Gamma), (-b/4, 0, +\gamma\Gamma), (b/4, 0, -\gamma\Gamma), (b/2, 0, +\Gamma)]$

Each vortex has a counterpart mirrored across $x = 0$ that have opposite spin, so as in Test Case (a), they move in a line. They still make orbital patterns because there are still some with the same spin. The pattern becomes more elliptical as γ is increased.

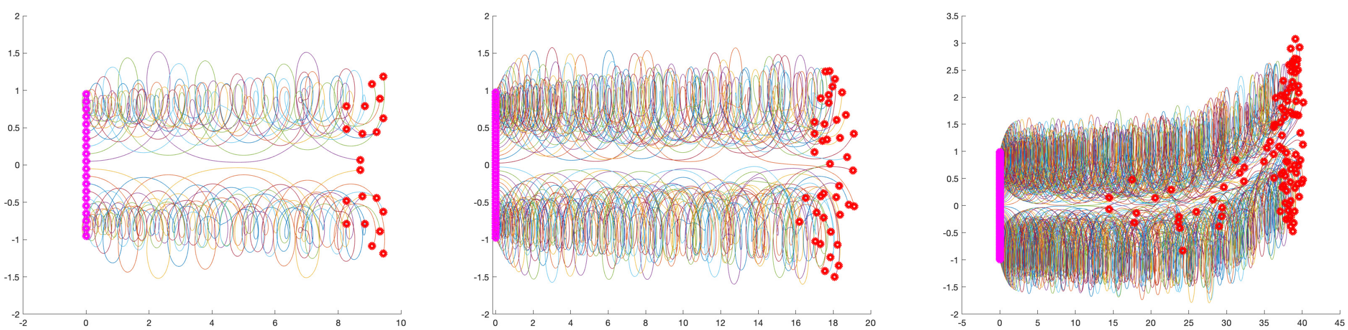


(a) $\gamma = 0.2$ (b) $\gamma = 0.4$ (c) $\gamma = 0.6$

Figure 4: Third Set of Conditions
$$[(-b/2, b/2, -\Gamma), (-b/4, b/2, -\gamma\Gamma), (b/4, b/2, +\gamma\Gamma), (b/2, /b/2, +\Gamma)]$$

These point vortices are located near the ground, simulating the turbulent flow of an aircraft during landing or takeoff. They all begin in a horizontal line, but then diverge away from the center in pairs. They move downwards a bit, as seen how the tops of the loops after the first one are lower. However, as in the corner case, they begin to affect their own trajectories since they get closer to the ground. They then spiral away to the left and the right. The point vortices with the smaller magnitude for Γ does larger loops around the vortex with a larger Γ.

3.3 Vortex Sheet Roll Up



(a) $N = 10$ (b) $N = 20$ (c) $N = 50$

Figure 5: Wake Turbulence Modelling
$$(y_i, \Gamma_1) = \left(\frac{i+\frac{1}{2}}{N}, \frac{\frac{2i+1}{N}}{\sqrt{1-(i+\frac{1}{2})^2/N^2}}\right);\ y \in (-1,1);\ i = -N : N-1$$

All of the vortices were placed close to each other in a vertical line. It is interesting to see how they split down the middle line into two main streams that each independently orbit a center point that continues to travel horizontally. As I added more point vortices, the pattern did not change much, except the behavior got more chaotic at an earlier point in time. The system then began to spiral off the center line, and some vortices got stuck in the path and did not continue with the rest.