§ 5: Integration

* These notes streamline §5.2, 5.3, 5.5 *

Obj: We will use the machinery of derivatives to approximate total change given the rate of change.

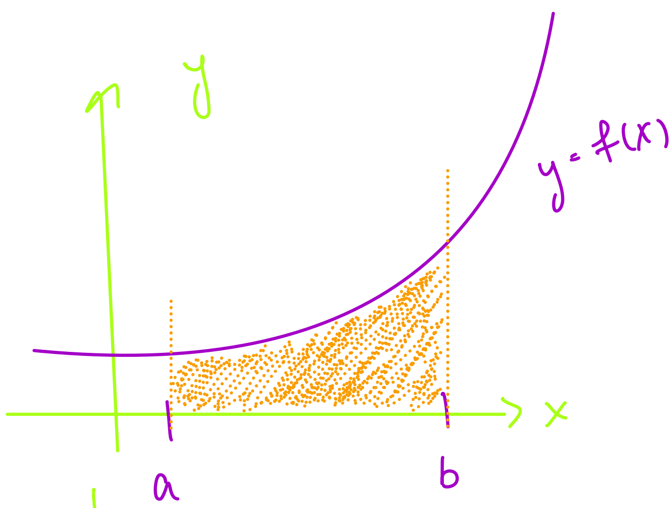
Warm up: Find any global min/max and infl. pt fn x^2+2x+1.

$f'(x) = 2x+2 \implies f''(x) = 2$

$f'(x) = 0 \implies x = -1 \implies$ by 2^{nd} der test, $f''(-1) = 2 \implies$ local min at -1

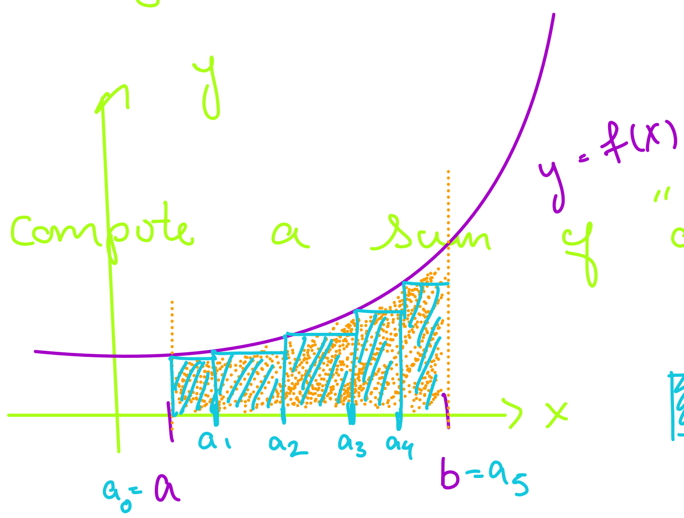
Test for sign on $f''(x)$: f'' const. $\implies$ no infl. pts

Motivation: Let $f(x)$ be a function. How do we compute the area of the shaded region



by using rectangles.

Idea: We compute a sum of "approximations"



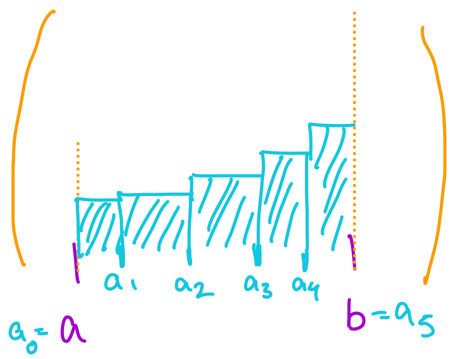
- Divide $[a,b]$ into smaller sub intervals, i.e.

$$[a,b] = [a_0,a_1] \cup [a_1,a_2] \cup \cdots \cup [a_{n-1},a_n]$$

where $a_0 = a \le a_1 \le \cdots \le a_{n-1} \le a_n = b$.

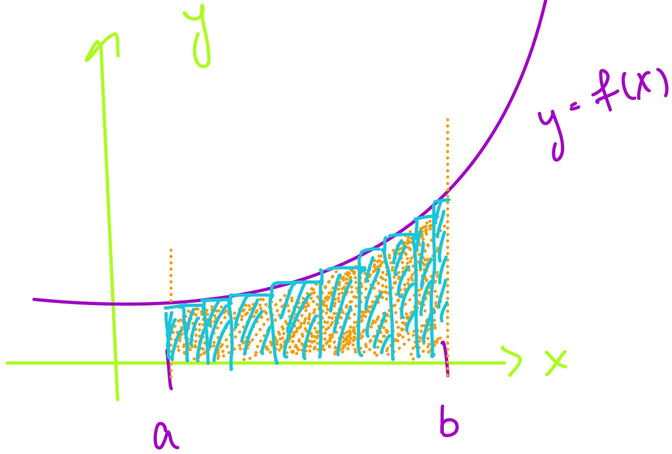
- Compute the area of the shaded region given by the rectangles, i.e.

$$\text{Area}\Bigg(\quad \Bigg) = \sum_{i=1}^{n} f(a_i)(a_i - a_{i-1})$$

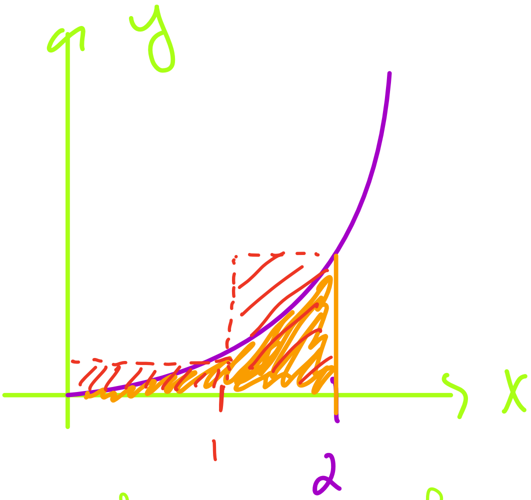


$$= f(a_1)(a_1 - a_0) + f(a_2)(a_2 - a_1) + \cdots + f(a_n)(a_n - a_{n-1}).$$

- Repeat the process above w/ "more rectangles", i.e.



Q2: $f(x) = x^2$, $[0,2]$



$[0,\tfrac{1}{2}] \cup [\tfrac{1}{2},1] \cup [1,2]$

$[0,2] = [0,1] \cup [1,2]$

Area $(/\!/) = f(1)(1-0) + f(2)(2-1)$

$= 1^2 \cdot 1 + 2^2 \cdot 1$

$= 1 + 4 = 5$

Def: Suppose f is a differentiable function on $[a,b]$ (i.e. $f'(p)$ exists for all p in $[a,b]$). The "definite integral" of f from a to b is given by

* $\int$ is also denoted by $\int$ *

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i)\,\Delta t_i \quad \text{(left sum)} \quad \Delta_i = t_{i+1} - t_i$$

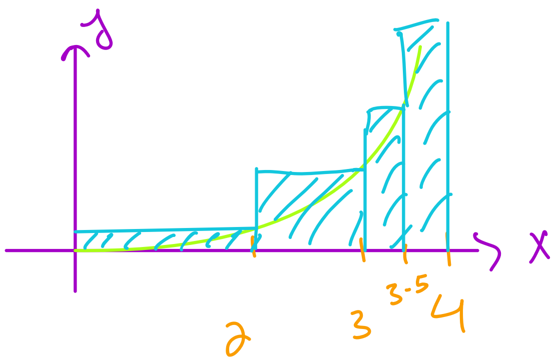
*Area of shaded region *

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\,\Delta t_i \quad \text{(right sum)} \quad \Delta_i = t_i - t_{i-1}$$

where $[a,b]$ is divided into sub intervals given by end pts $t_0 = a \le t_1 \le \dots \le t_n = b$ and Δt_i denotes the length of a subinterval. We call each sum a "Riemann sum", f an "integrand" and a, b the "bounds" (or "limits") of integration.

<u>Ex</u>: $f(x) = x^2$, $[a,b] = [0,4]$, approx. $\int_0^4 x^2\,dx$

$[0,4] = [0,2] \cup [2,3] \cup [3,3.5] \cup [3.5,4]$

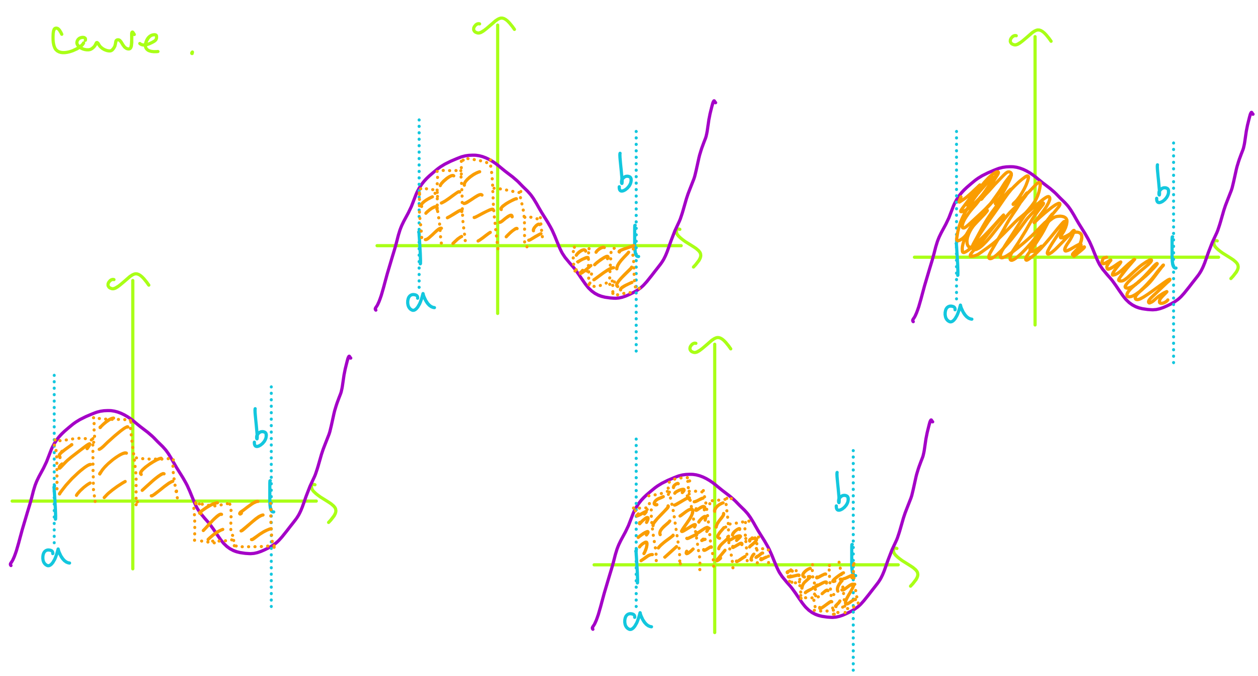


Area($\equiv$) = $f(2)(2-0) + f(3)(3-2) + f(3.5)(3.5-3) + f(4)(4-3.5)$

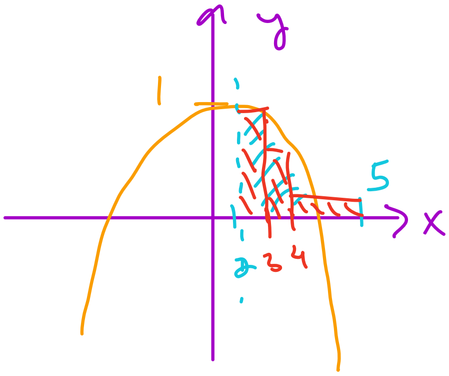
$= 2^2 \cdot 2 + 3^2 \cdot 1 + (3.5)^2(0.5) + 4^2(0.5)$

$= \ldots\ldots \simeq \int_0^4 x^2\,dx$

<u>Rmk</u>: The just in more refined divisions of $[a,b]$ give better approximations for area under the curve.



Ex: $f(x) = -x^2+1$, $[a,b] = [2,5]$, approx. $\int_2^5 (-x^2+1)\,dx$



$$[2,5] = [2,3] \cup [3,4] \cup [4,5]$$

$$\text{Area}(\;) = f(3)(\overset{1}{\cancel{3-2}}) + f(4)(\overset{1}{\cancel{4-3}}) + f(5)(\overset{1}{\cancel{5-4}})$$

$$= -3^2+1 + (-4^2)+1 + (-5^2)+1$$

$$= -9 + 3 - 16 - 25 = -47$$

$$\Rightarrow -47 \approx \int_2^5 (-x^2+1)\,dx$$

Th: (Fundamental Theorem of Calculus) Suppose $f(x)$ in a differentiable function on $[a,b]$. Then

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

Ex: $\int_0^1 2x\,dx =$ $\qquad\qquad$ $\int_0^2 3x^2\,dx =$

$$\int_1^2 \frac{1}{x}\,dx =$$

Thm: Suppose f, g are differentiable functions and $a, b, c \in (-\infty, \infty)$. Then

1. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (a \le c \le b)$

2. $\int_a^b \left(f(x) \pm g(x)\right) dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

3. $\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx.$