Q1.

$p(a) = \sum_{b\in\{0,1\}}\sum_{c\in\{0,1\}} p(a,b,c)$ and $p(b) = \sum_{a\in\{0,1\}}\sum_{c\in\{0,1\}} p(a,b,c)$

$$p(b|c) = \frac{\sum_{a\in\{0,1\}} p(a,b,c)}{\sum_{a\in\{0,1\}}\sum_{b\in\{0,1\}} p(a,b,c)} \qquad p(c|a) = \frac{\sum_{b\in\{0,1\}} p(a,b,c)}{\sum_{b\in\{0,1\}}\sum_{c\in\{0,1\}} p(a,b,c)}$$

Table 2

a	b	c	p(a,b|c)	p(a|c)p(b|c)
0	0	0	0.400	0.400
0	1	0	0.100	0.100
1	0	0	0.400	0.400
1	1	0	0.100	0.100
0	0	1	0.277	0.277
0	1	1	0.415	0.415
1	0	1	0.123	0.123
1	1	1	0.185	0.185

Table 3

a	p(a)
0	600.000
1	400.000

c	a	p(c|a)
0	0	0.400
1	0	0.600
0	1	0.600
1	1	0.400

b	c	p(b|c)
0	0	0.800
1	0	0.200
0	1	0.400
1	1	0.600

Multiplying the three distributions together we recover the joint distribution $p(a,b,c)$ given in Table 1. thereby allowing us to verify the validity of the decomposition $p(a,b,c) = p(a)p(c|a)p(b|c)$ for this particular joint distribution. We can express this decomposition using the graph: $a \to c \to b$

Q2. From the figure 1, we see that $p(a,b,c,d) = p(a)p(b)p(c|a,b)p(d|c)$

Following the examples in before we see that $p(a,b) = \sum_c\sum_d p(a,b,c,d) = p(a)p(b)\sum_c p(c|a,b)\sum_d p(d|c)$
$= p(a)p(b)$

similarly, $p(a,b|d) = \dfrac{\sum_c p(a,b,c,d)}{\sum_a\sum_b\sum_c p(a,b,c,d)} = \dfrac{p(d|a,b)p(a)p(b)}{p(d)} \neq p(a|d)p(b|d)$

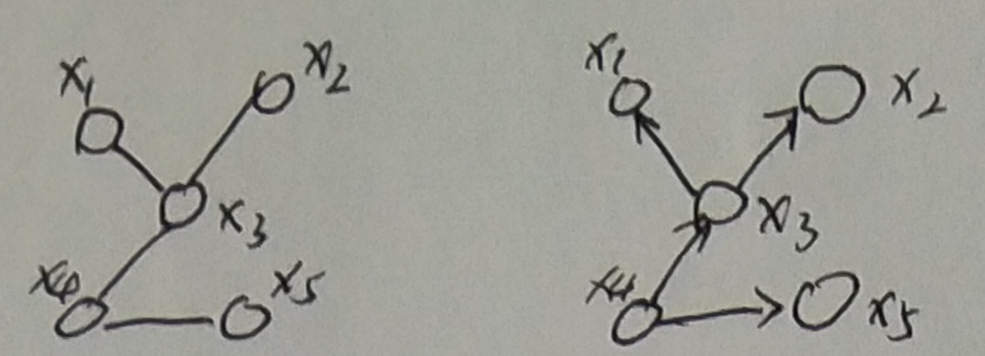
Q3. The described situation correspond to the graph shown in figure with $a=B$, $b=F$, $c=G$ and $d=D$. To evaluate the probability that the tank is empty given the driver's report that the gauge ~~reads~~ reads zero, we use Bayes' theorem. $p(F=0|D=0) = \dfrac{p(D=0|F=0)p(F=0)}{p(D=0)}$. To evaluate $p(D=0|F=0)$ we marginalize over B and G. $p(D=0|F=0) = \sum_{B,G} p(D=0|G)p(G|B,F=0)p(B) = 0.748$. and to evaluate ~~p(D=0|G)p~~ $p(D=0)$ we marginalize also over F, $p(D=0) = \sum_{B,G,F} p(D=0|G)p(G|B,F)p(B)p(F) = 0.352$. Combining these results with $p(F=0)$, we get $p(F=0|D=0) = 0.213$

If observe $B=0$, we longer marginalize over B, but ~~p(F|D~~ $p(F=0|D=0,B=0) = 0.110$. It's again lower.

Q4. With $N=5$ and x_3 and x_5 observed, the graph is shown like figure 3. This graph is undirected, but from Figure we see that the equivalent directed graph can be obtained by simply directing all the edges from left to right. In this directed graph, the edges on the path from x_2 to x_5 meet head-to-tail at x_3 and since x_3 is observed, by d-separation $x_2 \perp\!\!\!\perp x_5 | x_3$;

$p(x_2) = \frac{1}{Z}\mu_\alpha(x_2)\mu_\beta(x_2)$. $\mu_\alpha(x_2)$ is given by the figure, while for $\mu_\beta(x_2)$

$\mu_\beta(x_2) = \sum_{x_3}\psi_{2,3}(x_2,x_3)\mu_\beta(x_3) = \psi_{2,3}(x_2,\hat{x}_3)\mu_\beta(\hat{x}_3)$



since x_3 is observed and we denote that observed value $\hat{x}_3$. Thus, any influence that x_5 might ~~have~~ have on $\mu_\beta(\hat{x}_3)$ will be in terms of a scaling factor that is independent of x_2 and which will be absorbed into the normalization constant Z and so $p(x_2|\hat{x}_3,x_5)=p(x_2|\hat{x}_3)$

Q5. We do the induction over the size of the tree and we grow the tree one node at a time while, at the same time, we update the message passing schedule. Note that we can build up any tree this way. For a single root node, the required condition holds trivially true, since there are no message to be passed. We then assume that it holds for a tree with N nodes. In the induction step we add a new leaf node to such a tree. This new leaf node need not to wait for any messages from other nodes in order to send its outgoing message and so it can be scheduled to send it first, before any other message are sent. Its parent node will receive this message, whereafter the message propagation will follow the schedule for the original tree with N nodes, for which the condition is assumed to hold.

For the propagation of the outward messages from the root back to the leaves, we first follow the propagation schedule for the original tree with N nodes, for which the condition is assumed to hold. When this has completed, the parent of the new leaf node will be ready to send its outgoing message to the new leaf node, thereby completing the propagation for the tree with N+1 nodes.

Q6. ~~In the 1st~~

$$\tilde{p}(x_1)=\mu_{f_a\to x_1}(x_1)=\sum_{x_2} f_a(x_1,x_2)\mu_{x_2\to f_a}(x_2)=\sum_{x_2} f_a(x_1,x_2)\mu_{f_b\to x_2}(x_2)\mu_{f_c\to x_2}(x_2)$$

$$=\sum_{x_2} f_a(x_1,x_2)\sum_{x_3} f_b(x_2,x_3)\sum_{x_4} f_c(x_2,x_4)=\sum_{x_2}\sum_{x_3}\sum_{x_4} f_a(x_1,x_2)f_b(x_2,x_3)f_c(x_2,x_4)$$

$$=\sum_{x_2}\sum_{x_3}\sum_{x_4}\tilde{p}(x)$$

$$\tilde{p}(x_3)=\mu_{f_b\to x_3}(x_3)=\sum_{x_2} f_b(x_2,x_3)\mu_{x_2\to f_b}(x_2)=\sum_{x_2} f_b(x_2,x_3)\mu_{f_a\to x_2}(x_2)\mu_{f_c\to x_2}(x_2)$$

$$=\sum_{x_2} f_b(x_2,x_3)\sum_{x_1} f_a(x_1,x_2)\sum_{x_4} f_c(x_2,x_4)=\sum_{x_1}\sum_{x_2}\sum_{x_4} f_a(x_1,x_2)f_b(x_2,x_3)f_c(x_2,x_4)=\sum_{x_1}\sum_{x_2}\sum_{x_4}\tilde{p}(x)$$

$$\tilde{p}(x_1,x_2)=f_a(x_1,x_2)\mu_{x_1\to f_a}(x_1)\mu_{x_2\to f_a}(x_2)=f_a(x_1,x_2)\mu_{f_b\to x_2}(x_2)\mu_{f_c\to x_2}(x_2)$$

$$=f_a(x_1,x_2)\sum_{x_3} f_b(x_2,x_3)\sum_{x_4} f_b(x_2,x_4)=\sum_{x_3}\sum_{x_4} f_a(x_1,x_2)f_b(x_2,x_3)f_b(x_2,x_4)=\sum_{x_3}\sum_{x_4}\tilde{p}(x)$$