**Math 401, Exam 2, Mar 30.**

**Instructions**: Show all work as appropriate for the methods taught in this course. Partial credit will be given for any work, words or ideas which are relevant to the problem. **Write everything on the answer booklet, including your name. And note that there are also problems on the back of the page**.

1. Suppose we have the following Massey matrix equation

$$\begin{bmatrix} \alpha & -1 & -2 & -2 \\ -1 & 2 & -1 & 0 \\ -2 & \gamma & \beta & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \hat{r} = \begin{bmatrix} 5 \\ -2 \\ -4 \\ 0 \end{bmatrix}$$

(a)[6 pts] Write down the values of $\alpha, \beta, \gamma$.

(b)[4 pts] Are you able to determine how many games Team 1 won? Just answer Yes or No, no need to explain.

(c)[6 pts] How many games in total Team 4 have played? Which team has not played against team 4?

(a) $\alpha = 5$ $\beta = 4$ $\gamma = -1$

(b) No,

(c) Totally 3 games. Have not played against team 2.

2. Let

$$M = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$$

(a)[10 pts] Find a matrix $P$ and a diagonal matrix $D$ such that $M = PDP^{-1}$.

(b)[4 pts] What can you say about the two column vectors of $P$? (Hint: compute the inner product.)

(a) $\det(\lambda I - M) = \lambda(\lambda-3) - 4 = (\lambda-4)(\lambda+1)$

If $\lambda = -1$ $\begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot (-1)$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

If $\lambda = 4$ $\begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot 4$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Thus $P = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ $D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$

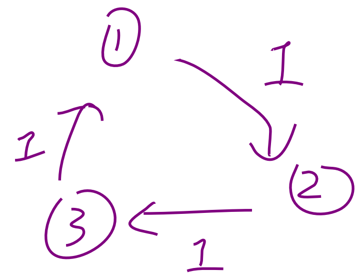
(b) $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is $\perp$ to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (perpendicular)

since $\begin{pmatrix} 2 \\ -1 \end{pmatrix}^T \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2 - 2 = 0.$

Actually, this holds whenever $M$ is symmetric.

3.[12 pts] Give an example of a $3 \times 3$ transition matrix $T$ such that $T \neq I$, and $T$ is not regular, but $T$ has a steady state vector (can have more than one steady vector). Write down an example of steady state vector for the transition matrix you gave.

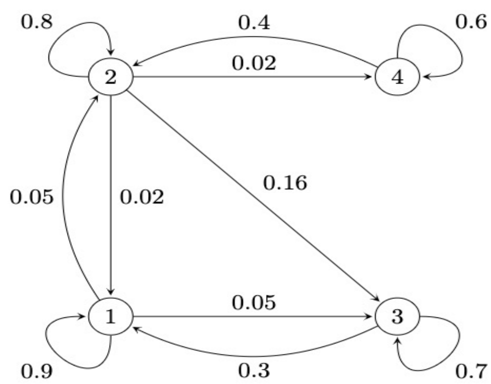
$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$




$$X_* = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

T is not regular, the populations keep switch among states.

4. Given the following transition diagram:




(See the back of the page.)

Suppose $T$ is the associated transition matrix.

(a)[8 pts] Is the transition matrix regular? If so, what is the smallest $k$ such that all entries of $T^k$ are nonzero.

(b)[6 pts] Find the $(4,1)$ entry of $T^3$. You can leave your answer as a sum of numbers, no need to simplify.

(a) Yes. $k=3$.

(b) There are 3 possible paths.

$1 \to 1 \to 2 \to 4$

$1 \to 2 \to 2 \to 4$

$1 \to 2 \to 4 \to 4$

$$(T^3)_{(4,1)} = 0.05 \cdot 0.02 \cdot (0.9 + 0.8 + 0.6)$$

5. Suppose we have an internet of 4 pages, with the following links

$P_1 \to P_2$, $P_1 \to P_3$, $P_1 \to P_4$, $P_3 \to P_4$, $P_2 \to P_4$.

(a)[10 pts] Write down the corresponding transition matrix $T$. You can leave this in the expanded sum form.

(b)[4 pts] Without solving the equations, guess which page will have the highest rank.

(a) $$T = \frac{0.15}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + 0.85\begin{bmatrix} 0 & 0 & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 & \frac{1}{4} \\ \frac{1}{3} & 0 & 0 & \frac{1}{4} \\ \frac{1}{3} & 1 & 1 & \frac{1}{4} \end{bmatrix}$$

(b) Page 4.