1. Leontief Input-Output Model

1. Know the definition of consumption matrix M, the meaning of each element in the matrix. Note that if one of the diagonal elements of M is bigger than 1, it would be problematic.

2. Given $(I-M)^{-1}$, where M is the consumption matrix, be able to find the increase in production vector needed in order to feed an increase in external demand. Or, explain the meaning of a particular entry of $(I-M)^{-1}$.

3 The diagonals elements of $(I-M)^{-1}$ must be bigger than 1, for other entries of $(I-M)^{-1}$, we only know they should be bigger than 0.

4. If consumption matrix M is simple 2×2 matrix, be able to solve $\vec{p} = M\vec{p} + \vec{d}$ by hand.

2. Computer graphics

1. Know the translation matrix in 2D (3x3 matrix) and 3D(4x4 matrix) and know their symbols, i.e., $T(a, b)$ and $T(a, b, c)$.

2. Know the rotation matrix in 2D around the origin, for example, counterclockwise by angle θ.

3. Know how to get the rotation matrix in 2D around the a general point, by translation, rotation and translation back.

4. You do not need to know the rotation matrix in 3D explicitly, you can just use symbols like $RX(\theta)$, $RY(\theta)$ and $RZ(\theta)$.

5. Know how to get the rotation matrix in 3D, around the a given line parallel to one of the coordinate axes, by using combinations of $RX(\theta)$, $RY(\theta)$ $RZ(\theta)$ and translation matrices. Also rotations around a given line, for example, in the xy plane, where you need to first rotate the line to be one of the coordinate axes.

6. Be able to use a single rotation to move a given point in 2D or 3D, to a given axis.

3. Least Squares and Curve Fitting

1. Know the definition of the column space of a matrix A.

2. Be able to find the least square solution to $A\vec{x} = \vec{b}$ by hand for simple matrices A.

3. The solution of least square equation $A^T A\vec{x} = A^T\vec{b}$ is unique, if the all column vectors of A are linearly independent.

4. The equation $A^T A\vec{x} = A^T\vec{b}$ always have infinitely many solutions if the column vectors of A are not linearly independent. For example, if A is $m \times n$ matrix with $n > m$, there are always infinitely many solutions to least square.

5.Given a sequences of points, be able to use least square to find the "best" curve among a given class of functions. For example, assume $y = ax^2 + b$, find a and b using least square.

4. Team ranking

1. Given a series of games played and outcomes, be able to write down the Massey matrix equation(where the last row is all 1s and the last entry of right hand side is 0).

2. Given the Massey matrix equation, with certain elements missing, be able to recover the missing values using the fact that the sum of each row is 0, and the matrix is symmetric before we replace the last row.

3. Given the Massey matrix equation, we are able to tell the total number of games played between each two teams, but we can not infer the scores of each game.

4. Any additional game played may affect the ranking, even when the result of the additional game is a tie between two teams.

5. Linear Discrete Dynamical Systems

1. Be able to diagonalize a simple 2×2 matrix, that is, write $A = PDP^{-1}$

2. Given a recurrence relation, be able to turn it into a matrix equation. For example, the Fibonacci sequence of numbers.

6. Markov Chains

1. Know the definition of probability vector, transition matrix, steady-state vector and regular transition matrix.

2. Given a transition matrix, be able to draw a diagram for a given Markov chain.

3. Given a diagram, be able to write down the transition matrix.

4. If all entries of the transition matrix is strictly positive, then transition Matrix is regular.

5. A Markov chain can have more than one steady vector, but if it is regular, the steady vector is unique.

6. If the transition matrix is not regular, the Markov chain can keep switching between states, and never approach steady state vector, even though it have steady vectors. For example, draw a Markov chain with 5 states, and for every 5 iterations, it return to itself.

7. Know the meaning of a given element in T^k for fixed k.

8. Be able to compute a given element of T^k, by finding all the possible paths and add the product of probabilities along all paths.

9. If there is a one-sided arrow pointing from one group of states to another group, then eventually all population will end up in the group that the arrow is pointed toward. In this case, of course the markov chain is not regular, but it is possible that given any initial population distribution, eventually it will approach the same steady vector.

7. Google page rank

1. Given a set of links between pages, be able to write down the transition matrix T.

2. Note that the weight is always 0.15 and 0.85. And note that if there is no outbound link on a given page, then the surfer is equally likely to visit any page, including the page he/she is current at.

8. Absorbing Markov chain

1. Given some word descriptions, be able to write down the diagram and transition matrix for the Markov chain.

2. Assume we the matrix $(I - Q)^{-1}$ is given, be able to find the time before absorption, given that initially we are at a given state. And be able to find the probability of ending up in a given absorbing state, given that initially we are at another state.

3. Be able to find $(I - Q)^{-1}$ for simple 2×2 matrix Q.

9. Differential Equations

1. Given a heat diffusion model, be able to write down the system of differential equation for the model, and organize it as a matrix equation.

2. If one of the nodes is connect to outside, then eventually the temperature of all nodes will equal to the outside.

3. If none of the nodes is connect to outside, then eventually the temperature of all nodes will be averaged.

4. The matrix A for heat diffusion model is always symmetric. Recall that symmetric matrix is always diagonalizable.

5. Given a simple matrix A, be able to find e^A and e^{tA}, including the cases when A is diagonalizable, and when A is not diagonalizable, but only have very few nonzero entries.

10. Exponentials and Rotations

1. Know the general form a skew symmetric matrix, 2×2, 3×3, 4×4.

2. If A is 2×2 skew symmetric, know what e^A looks like.

3. Given a arbitrary direction $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, be able to find a find a 3×3 skew symmetric matrix A such that e^A corresponds to the rotation about the line through the origin in the direction of v by θ radians. Remember to check whether v is a unit vector.

4. For any matrix A, e^A is invertible, the inverse is e^{-A}.

5. A must be a square matrix in order for e^A to be defined.

6. If $AB = BA$, then $e^{A+B} = e^A \cdot e^B = e^B \cdot e^A$

7. e^A can be identity matrix, even A is not zero matrix, we can create such examples by using rotations by 2π around an arbitrary direction.

11. Singular Value Decomposition

1. Be able to find the SVD by hand, given a simple matrix A.

2. $A = U\Sigma V^T$, note that the column u_i of U, and column v_i of V corresponding to the positive singular value σ_i must satisfy $v_i = \frac{1}{\sigma_i}A^T u_i$ and $u_i = \frac{1}{\sigma_i}Av_i$, so usually we just find one of them, for example, find u_i by solving $AA^T u_i = \sigma_i^2 u_i$, and solve the other using the relation. For zero eigenvalues, the eigenvectors u_i and v_i does not satisfy the relation, so we just solve $AA^T u_i = 0$ and $A^T Av_i = 0$.

3. Make sure your u_i are all unit vectors and mutually perpendicular, and same for v_i.

4. Make sure your Σ only have non zero entries in the diagonal, and they are nonnegative numbers arranged in a nonincreasing order.

5. If A is symmetric matrix, then A is orthogonally diagonalizable, that is $A = QDQ^T$, here Q is a orthogonal matrix, D is diagonal matrix. Recall that orthogonal means $QQ^T = Q^T Q = I$. Note that here D may have negative values in the diagonal.