Kronecker product

Definition. Let $A = (a_{ij})$ be a $n \times n$ matrix and let B be a $m \times m$ matrix. Then the *Kronecker product* of A and B is the $mn \times mn$ block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

The Kronecker product is also known as the *direct product* or the *tensor product* [1].

Fundamental properties [1, 2]

1. The product is bilinear. If k is a scalar, and A, B and C are square matrices, such that B and C are of the same order, then
$$A \otimes (B + C) = A \otimes B + A \otimes C,$$
$$(B + C) \otimes A = B \otimes A + C \otimes A,$$
$$k\,(A \otimes B) = (kA) \otimes B = A \otimes (kB)\,.$$
2. If A, B, C, D are square matrices such that the products AC and BD exist, then $(A \otimes B)\ (C \otimes D)$ exists and
$$(A \otimes B)\ (C \otimes D) = AC \otimes BD.$$
If A and B are invertible matrices, then
$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$
3. If A and B are square matrices, then for the transpose (A^T) we have
$$(A \otimes B)^T = A^T \otimes B^T.$$
4. Let A and B be square matrices of orders n and m, respectively. If $\{\lambda_i | i = 1, \ldots, n\}$ are the eigenvalues of A and $\{\mu_j | j = 1, \ldots, m\}$ are the eigenvalues of B, then $\{\lambda_i \mu_j | i = 1, \ldots, n,\ j = 1, \ldots, m\}$ are the eigenvalues of $A \otimes B$. Also,
$$\det(A \otimes B) = (\det A)^m (\det B)^n,$$
$$\operatorname{rank}(A \otimes B) = \operatorname{rank} A \ \operatorname{rank} B,$$
$$\operatorname{trace}(A \otimes B) = \operatorname{trace} A \ \operatorname{trace} B,$$

References

1 H. Eves, *Elementary Matrix Theory*, Dover publications, 1980.
2 T. Kailath, A.H. Sayed, B. Hassibi, *Linear estimation*, Prentice Hall, 2000

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Entry type	Definition
Classification	msc 15-00
Synonym	tensor product (for matrices)
Synonym	direct product