**Math 401, Quiz 3, 04/18/2023**

**Instructions**: **Write everything on the answer booklet, including your name, and there are problems on the back of the page**.

1. An urn contains 4 balls, some red and some blue. One ball is selected at random one at a time from the urn. Each selected ball is replaced by a ball of the opposite color. The process continues until all balls in the urn are of the same color.

(a)Define a Markov chain with 5 states (for example, you can define the states by how many blue balls in the urn, and try to label the absorbing states as states 4 and 5), and draw the diagram for this Markov chain.

(b)Write down the transition matrix $T$.

(c)Yes or No, no need to explain: Assume initially we have 3 blue balls and 1 red ball in the urn, the probability that the process ends with 4 blue balls is higher than 4 red balls.

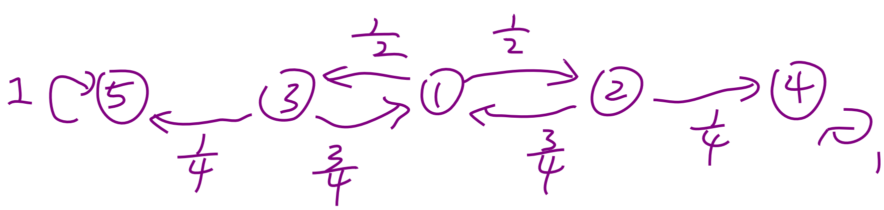
(a) state 1 : 2 blue balls

2 : 3 blue balls

3 : 1 blue ball

4 4 blue balls

5 : no blue balls




(b) $$T = \begin{bmatrix} & \frac{3}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 1 \end{bmatrix}$$

(c) Yes

2. A game is played as follows. At each step, a fair die is rolled. If 4, 5 or 6 is rolled, a ball is added to the urn. If 2 or 3 is rolled, no ball is added to the urn. If 1 is rolled, then we remove a ball from the urn. The player wins the game when the urn has 3 balls, and the player losses once the urn has no ball.

(a)Define a Markov chain which have 4 states(for example, you can define the states by how many balls in the urn, and try to label the absorbing states as states 3 and 4), and draw the diagram for this Markov chain.

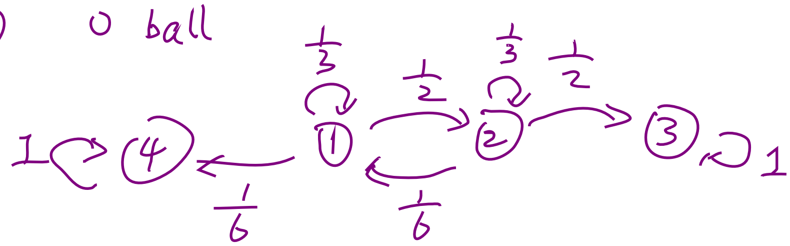
(b)Write down the transition matrix $T$.

(a) State ① 1 ball

② 2 balls

③ 3 balls

④ 0 ball




(b) $$T = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{6} & 0 & 0 & 1 \end{pmatrix}$$

3. Consider the Markov chain with the following transition probability matrix.

$$A = \begin{bmatrix} 0.5 & 0.1 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0.1 & 1 & 0 \\ 0 & 0.3 & 0 & 1 \end{bmatrix}$$

(a) Given that the process begins in state 1, find the expected time to reach an absorbing state. Hint: recall that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then, } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(b)Given that the process begins in state 1, find the probability that the process reaches state 4.

(a) $Q = \begin{pmatrix} 0.5 & 0.1 \\ 0.5 & 0.5 \end{pmatrix}$

$(I-Q)^{-1} = \begin{pmatrix} 0.5 & 0.1 \\ 0.5 & 0.5 \end{pmatrix} \cdot \frac{1}{0.2} = \begin{pmatrix} 2.5 & 0.5 \\ 2.5 & 2.5 \end{pmatrix}$

$2.5 + 2.5 = 5.$

so expected 5 steps before absorption

(b) $R = \begin{pmatrix} 0 & 0.1 \\ 0 & 0.3 \end{pmatrix}$

$R \cdot (I-Q)^{-1}_{(2,1)} = 0 \cdot 2.5 + 0.3 \cdot 2.5 = 0.75$

4.The matrix $N = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is not diagonalizable, but it still has a matrix exponential.

(a)Compute $e^N$ directly from the definition: (Hint: what is $N^2$?)

$$e^N = I_2 + N + \frac{N^2}{2!} + \frac{N^3}{3!} + \ldots + \frac{N^k}{k!} + \ldots$$

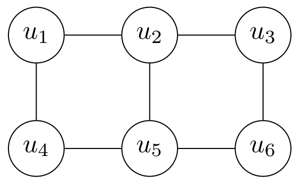
(b) Compute $e^{tN}$ directly from the definition.

(a) $N = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad N^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\quad N^k = \vec{0}$. $k \geq 2$.

$e^N = I + N = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $e^{tN} = I + tN = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5. Consider the heat diffusion graph below. Assume no heat can escape.




(a)Find the heat diffusion equation $\vec{u}'(t) = A\vec{u}(t)$.

(b) Without solving any equations, if the initial condition is $\vec{u}(0) = [0\ 6\ 0\ 0\ 0\ 0]^T$, guess what is the $\vec{u}(t)$ when $t \to \infty$ ? (Hint: will all states have same temperature in the end?)

(a) $u_1'(t) = -(u_1 - u_2) - (u_1 - u_4)$

$u_2'(t) = -(u_2 - u_1) - (u_2 - u_3) - (u_2 - u_5)$

$u_3'(t) = -(u_3 - u_2) - (u_3 - u_6)$

$u_4'(t) = -(u_4 - u_1) - (u_4 - u_5)$

$u_5'(t) = -(u_5 - u_2) - (u_5 - u_4) - (u_5 - u_6)$

$u_6'(t) = -(u_6 - u_3) - (u_6 - u_5)$

$\vec{u}'(t) = A\vec{u}(t)$. $\quad A = \begin{pmatrix} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & 0 & 1 & -2 \end{pmatrix}$

(b) $\vec{u}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ when $t \to +\infty$.

the temperature will be averaged