Math 401, Practice Exam 3

1. Three fair coins are tossed. The coins randomly fall heads or tails. At each subsequent step, all the coins that fall tails are picked up and tossed again until all coins show heads. Assume initially all three coins are tails.

(a)Define a Markov chain with 4 states (for example, you can define the states by how many heads currently), and draw the diagram for this Markov chain.

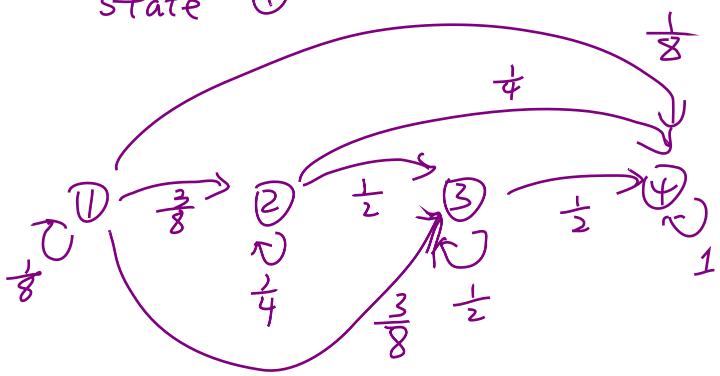
(b)Write down the transition matrix T.

(a) State 1 3 tails

State 2 2 tails

State 3 1 tail

State 4 0 tail



(b) $T = \begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 \\ \frac{3}{8} & \frac{1}{4} & 0 & 0 \\ \frac{3}{8} & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 1 \end{pmatrix}$

2. Give an example of a 3×3 skew-symmetric matrix A such that A is not zero matrix and $e^A = I_3$.

We know that if $A = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix}$

$$e^A = \begin{pmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So we can simply take

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2\pi \\ 0 & 2\pi & 0 \end{pmatrix}$$

$$e^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\pi & -\sin 2\pi \\ 0 & \sin 2\pi & \cos 2\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. Find a 3×3 matrix A such that e^A corresponds to the rotation about the line through the origin in the direction of $\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ by π radians.

$$\|\vec{u}\| = \sqrt{1 + (-2)^2 + 2^2} = 3$$

so $$A = \frac{1}{3} \cdot \begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

4. Find the SVD of $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

$A \cdot A^T = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\det(\lambda I - \cancel{A}AA^T) = 0 \iff (\lambda-2)^2 - 1 = 0$, $\lambda = 3$ or $\lambda = 1$

so $\sigma_1 = \sqrt{3}$, $\sigma_2 = 1$

If $\lambda = 3$. $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot 3$

The unit vector solution is $\vec{u}_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, or $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

let us choose $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, Then $\vec{v}_1 = \frac{1}{\sqrt{3}} \cdot A^T \cdot \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{-\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{pmatrix}$

If $\lambda = 1$. $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot 1$

The unit vector solution is $\vec{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

$\vec{v}_2 = A^T \cdot \vec{u}_2 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

since V is 3×3 matrix, we still need $\vec{v}_3$. which is the unit eigenvector of A^TA with eigenvalue 0.

$A^TA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ $\quad A^TA \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \vec{v}_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$

Over all $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix}^T$

5. Consider the third order differential equation

$$x'''(t) + x(t) = 0.$$

Define the vector $\vec{u}(t) = \begin{bmatrix} x(t) \\ x'(t) \\ x''(t) \end{bmatrix}$.

(a) Find a matrix A such that $\vec{u}'(t) = A\vec{u}(t)$

(b) Is A skew symmetric?

(a) We know $\vec{u}'(t) = \begin{pmatrix} x'(t) \\ x''(t) \\ x'''(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ x''(t) \\ -x(t) \end{pmatrix}$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x'(t) \\ x''(t) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

(b) A is not skew symmetric

6. True of False: If A is symmetric matrix, and A have a SVD equals $U\Sigma V^T$, then U and V are symmetric.

False.

If A is symmetric,

we may $U=V$ (not necessarily)

but we do not know if U is symmetric