Math 401, Exam 1, Feb 28.

Instructions: Show all work as appropriate for the methods taught in this course. Partial credit will be given for any work, words or ideas which are relevant to the problem. **Write everything on the answer booklet, including your name**.

1 [12pts]. Suppose a consumption matrix has

$$M = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0 & 0.15 & 0.5 \\ 0.05 & 0.25 & 0 \end{bmatrix} \text{ and } (I-M)^{-1} = \begin{bmatrix} 1.1422 & 0.4332 & 0.5593 \\ 0.0394 & \alpha & 0.7089 \\ 0.0670 & 0.3702 & 1.2052 \end{bmatrix}$$

(a) Write down the meaning of the value 0.5 in the matrix M.

(b) What can you say about α and why?

(c) How many product 2 additionally need to be produced, if the demand for product 1 increases by 1 unit.

(a) To produce 1 unit of product 3. it requires 0.5 unit of product 2.

(b) $\alpha \geq 1$. when the external demand for product 2 raises by 1 unit. we must produce at least 1 product 2

(c) 0.0394

2 [10pts]. Given an example of a matrix equation $A\overline{x} = \overline{b}$ which does not have a solution and for which the least-squares solution is unique. You do not need to solve, just give the equation.

$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

A have only 1 column. so the column of A is linearly independent. thus least square solution is unique. it is obvious that

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot t = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ have no solution

3 [14pts]. Write down the matrix product which will do each of the following in 2D. You can leave the matrices in function form and not write out or evaluate.

(a) Rotate by $\pi/3$ radians counterclockwise around the point (1, 0).

(b) The point (2,1) is moved to the origin, and the point (2, 2) is moved to the negative x axis. (Hint: use a tranlation followed by a rotation.)

(a) $T(1,0)\, R(\frac{\pi}{3}) \cdot T(-1,0)$

(b) $R(\frac{\pi}{2}) \cdot T(-2,-1)$.

part (b) is equivalent to asking moving the line segement connecting (2,1) and (2,2) to negative x axis, where (2,1) is moved to origin.

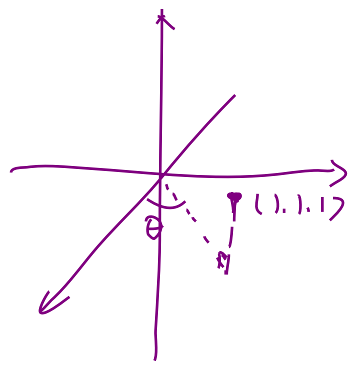
4 [14pts]. Write down the matrix or matrix product which will do each of the following in 3D. You can leave the matrices in function form and not write out or evaluate.

(a) Rotate by $\pi/2$ radians about the axis $y = 1, z = 2$ with positive x axis orientation.

(b) Use a single rotation around the z-axis, move the point (1, 1, 1) to the xz-plane.

(a) $T(0,1,2)\, R_x\left(\frac{\pi}{2}\right) \cdot T(0,-1,-2)$

(b)



The angle θ satisfy

$\tan\theta = \frac{1}{1} = 1$. $\Rightarrow \theta = \frac{\pi}{4}$

so $R_z\left(-\frac{\pi}{4}\right)$ works

(Actually $R_z\left(\frac{3}{4}\pi\right)$ also work)

5 [16pts]. The the points (-1, 0), (1, 2) and (2, 2) do not lie on a function of form $y = ax^2 + b$. However if you plug the pointss into this function you can construct a matrix equation which is solvable by least squares. Write down this matrix equation and find the least square solution.

$$(-1)^2 a + b = 0$$
$$(1)^2 \cdot a + b = 2 \quad \Leftrightarrow \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$
$$2^2 \cdot a + b = 2$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 6 \\ 6 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}^T \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 18 & 6 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix}$$

$$\begin{cases} x_1 = \frac{1}{3} \\ x_2 = \frac{2}{3} \end{cases}$$

$$y = \frac{1}{3} x^2 + \frac{2}{3}$$

6 [4pts]. True or false: For any $m \times n$ matrix A and $m \times 1$ vector $\overline{b}$, $(A^T A)\overline{x} = A^T\overline{b}$ always have at least one solution. (No need to explain)

True. There is always at least one solution for least square method.