Math 401, Practice Exam 2

1. Suppose we have the following Massey matrix equation

$$\begin{bmatrix} 4 & -1 & -2 & \alpha \\ -1 & 2 & -1 & \beta \\ -2 & -1 & 4 & \gamma \\ 1 & 1 & 1 & 1 \end{bmatrix} \hat{r} = \begin{bmatrix} 5 \\ -2 \\ -4 \\ 0 \end{bmatrix}$$

(a) Write down the values of α, β, γ.

(b) How many games in total haven been played? Which two teams have not played against each other?

(a) $\alpha = -1$ $\beta = 0.$ $\gamma = -1$

(b) Team 4 have played 2 games

Totally : $\dfrac{4+2+4+2}{2} = 6$ games

Team 2 have not played with team 4.

2. Assume F_n is defined recursively as $F_n = 4F_{n-1} - 3F_{n-2}$.

(a) Let $\bar{x}_n$ be the column vector

$$\bar{x}_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

Find the 2×2 matrix M such that $\bar{x}_{n+1} = M\bar{x}_n$.

(b) Find a matrix P and a diagonal matrix D such that $M = PDP^{-1}$.

(a) $M = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}$

(b) $\det(\lambda - M) = 0 \iff (\lambda - 4) \cdot \lambda + 3 = 0$

$(\lambda - 1)(\lambda - 3) = 0$.

If $\lambda = 1$. we solve $M \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

one of the solution is $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

If $\lambda = 3$ we solve $M \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 3 \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

one of the solution is $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

so $P = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$M = PDP^{-1}$

3. Give an example of a 5×5 transition matrix T such that $T \neq I$, $T^2 \neq I$, $\ldots$ $T^5 \neq I$, but $T^6 = I$. (Here I denote the 5×5 identity matrix.)

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

diagram:

The first 3 states return to itself for every 3 iterations.

The state 4 and 5 return to itself for every 2 iterations.

Totally for every 6 iterations. all states return to itself, but if iterations is less than 6. at least one state will not return.

4. Given the transition matrix

$$T = \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.1 & 0.75 & 0 & 0 \\ 0.1 & 0.05 & 0.6 & 0.2 \\ 0 & 0 & 0.4 & 0.8 \end{bmatrix}$$

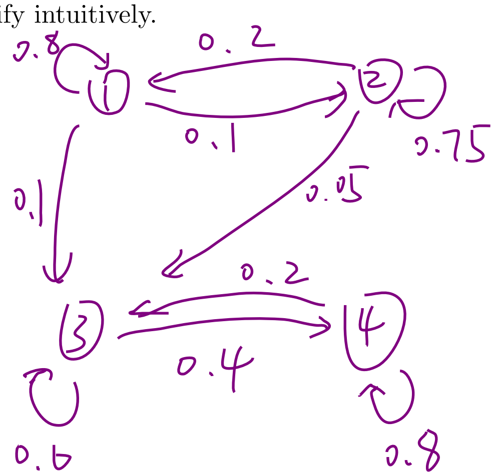
(a) Draw the corresponding population movement diagram.

(b) Is the transition matrix regular? Why.

(c) Find the $(3, 2)$ entry of T^2.

(d) Do you think the Markov chain has a steady state to which all other states converge? Justify intuitively.

(a)



(b) No, for example. state 3 can never go to state 2

(c) There are 3 possible paths

$2 \to 1 \to 3$ $\quad 0.2 \cdot 0.1$

$2 \to 2 \to 3$ $\quad 0.75 \cdot 0.05$

$2 \to 3 \to 3$ $\quad 0.05 \cdot 0.6$

so $(T^2)_{32} = 0.2 \cdot 0.1 + 0.75 \cdot 0.05 + 0.05 \cdot 0.6$

(d) Yes. the steady state is $\vec{X}_* = \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{pmatrix}$

where $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ is the steady state of the Markov chain

with $T = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}$, since all the population will eventually move to state 3 and 4.

5. Suppose we have an internet of 4 pages, with the following links

$P_1 \to P_2$, $P_1 \to P_3$, $P_1 \to P_4$, $P_3 \to P_4$, $P_4 \to P_2$.

Write down the corresponding transition matrix T. You can leave this in the expanded sum form.

$$T = \frac{0.15}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + 0.85\begin{bmatrix} 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{3} & \frac{1}{4} & 0 & 1 \\ \frac{1}{3} & \frac{1}{4} & 0 & 0 \\ \frac{1}{3} & \frac{1}{4} & 1 & 0 \end{bmatrix}$$