Math 401, Quiz 2, 03/14/2023

Instructions: Show all work as appropriate for the methods taught in this course. Partial credit will be given for any work, words or ideas which are relevant to the problem. **Write everything on the answer booklet, including your name**.

1. Suppose four teams play a number of games with the following results:

- T1 plays T2 and wins by 3.
- T2 plays T3 and wins by -2.
- T3 plays T4 and wins by -1.
- T1 plays T4 and wins by 1.
- T1 plays T4 and wins by 2.

Write down the Massey matrix equation $M\bar{r} = \bar{p}$ (Do not solve $\bar{r}$, just write down what is M and $\bar{p}$, and remember you need to replace the whole last row of M by 1 and the last entry of $\bar{p}$ by 0).

$$A = \begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ 1 & & & -1 \\ 1 & & & -1 \end{pmatrix} \qquad \vec{P_0} = \begin{pmatrix} 3 \\ -2 \\ -1 \\ 1 \\ 2 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 3 & -1 & 0 & -2 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -2 & 0 & -1 & 3 \end{pmatrix} \qquad A^T\vec{P_0} = \begin{pmatrix} 6 \\ -5 \\ 1 \\ -2 \end{pmatrix}$$

replace last row of A^TA by 1s. and last entry of $A^T\vec{P_0}$. by 0. we have

$$M = \begin{pmatrix} 3 & -1 & 0 & -2 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \vec{P} = \begin{pmatrix} 6 \\ -5 \\ 1 \\ 0 \end{pmatrix}$$

2. True or false(no need to explain): Suppose four teams have played a number of games, and suppose recently team 1 and team 2 played one game which is a tie, and team 3 and team 4 played one game which is also a tie, then if we use the Massey Method, the ranking of the teams won't be affected by the two recent games.

False. the Massey equations will be different with the two extra games

3. Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.6 & 0 \\ 0 & 0.5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

(a)Let A^n be the n-th power of matrix A, find the matrix D_n such that

$$A^n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot D_n \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

(b)Find $\lim_{n\to\infty} A^n$.

(a) $A^n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.6 & \\ & 0.5 \end{bmatrix}^n \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$

so $D_n = \begin{bmatrix} 0.6^n & \\ & 0.5^n \end{bmatrix}$

(b) when $n \to +\infty$, $D_n \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

so $A^n \to \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

4. Let

$$M = \begin{bmatrix} 0 & 2 \\ -1 & 3 \end{bmatrix}$$

Find a matrix P and a diagonal matrix D such that $M = PDP^{-1}$.

$$(\lambda - M) = \begin{pmatrix} \lambda & -2 \\ 1 & \lambda-3 \end{pmatrix}$$

$$\det(\lambda - M) = \lambda(\lambda-3)+2 = (\lambda-1)(\lambda-2)$$

① $\lambda = 1$. Solve $\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

one of the solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

② $\lambda = 2$. Solve $\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

one of the solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so if we let $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Then $M = PDP^{-1}$

5. Assume $F_0 = 0, F_1 = 1$ and F_n is defined recursively as $F_n = F_{n-1} + 2F_{n-2}$ for $n \geq 2$.

(a) Find F_5.

(b) Let $\bar{x}_n$ be the column vector

$$\bar{x}_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

Find the 2×2 matrix M such that $\bar{x}_{n+1} = M\bar{x}_n$ for $n \geq 0$.

(a) $F_2 = 1 + 2 \cdot 0 = 1$

$F_3 = 1 + 2 \cdot 1 = 3$

$F_4 = 3 + 2 \cdot 1 = 5$

$F_5 = 5 + 2 \cdot 3 = 11$

(b) $$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_{n+1} + 2F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$$