Math 401, Practice Exam 1

1. Suppose a consumption matrix has

$$M = \begin{bmatrix} 0.0500 & 0.0200 & 0.0200 \\ 0.1000 & 0.0800 & 0 \\ 0.1200 & 0.0400 & 0.0100 \end{bmatrix} \text{ and } (I-M)^{-1} = \begin{bmatrix} 1.0578 & 0.0239 & 0.0214 \\ 0.1150 & 1.0896 & \alpha \\ 0.1329 & 0.0469 & \beta \end{bmatrix}$$

(a) Write down the meaning of the value 0.1200 in the matrix M.

(b) Write down the meaning of the value 0.0214 in the matrix $(I-M)^{-1}$.

(c) Which of α or β must be bigger than or equal to 1, why? Could the other one great than or equal to 1, yes or no?

(a) 0.12 means to produce 1 unit of product 1, it takes 0.12 unit of product 3

(b) 0.0214 means if the external demand for product 3 increase by 1 unit, we need 0.0214 more product 1 to be produced

(c) β must be bigger than or equal to 1.

a can be either bigger or smaller than 1, as long as non-negative

2. Given an example of a matrix equation $A\vec{x} = \vec{b}$ which does not have a solution and for which the least-squares solution would not be unique. You do not need to solve, just give the equation.

Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$. $\vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

then if $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. $A\vec{x} = \begin{pmatrix} 0 \\ x_1 + x_2 \end{pmatrix}$

Thus, $A\vec{x} = \vec{b}$ have no solution

Since the column vectors of A

are not linearly independent.

least square solutions must be not unique.

3. Write down the matrix product which which will do each of the following in 2D. You can leave the matrices in function form and not write out or evaluate.

(a) Rotate by $\pi/5$ radians clockwise around the point (2, 7).

(b) Reflect in the line $y = -x + 1$.

(a) First move (2.7) to (0.0)

Then rotate,

Then move (0.0) to (2.7)

so it is

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(-\frac{\pi}{5}) & -\sin(-\frac{\pi}{5}) \\ \sin(-\frac{\pi}{5}) & \cos(-\frac{\pi}{5}) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) First move $y=-x+1$ to $y=-x$

Then rotate $y=-x$ to the line $y=0$

Then reflect WRST $y=0$

Then rotate $y=0$ to $y=-x$

Then move $y=-x$ back to $y=-x+1$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) & 0 \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Write down the matrix product which which will do each of the following in 3D. You can leave the matrices in function form and not write out or evaluate.

(a) Rotate by $\pi/4$ radians about the axis $x = 1, z = 2$ with positive y axis orientation.

(b) Only using rotations around the x, y and z-axes, move the point $(1, \sqrt{3}, 2)$ to the positive z-axis

(a) First move the line $x=1, z=2$ to $x=0, z=0$

Then rotate around y axis

Then move the y axis back to $x=1, z=2$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{\pi}{4} & 0 & \sin\frac{\pi}{4} & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) Note that in spherical coordinates

$(1, \sqrt{3}, 2) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$

where $\rho = 2\sqrt{2}$, $\theta = \frac{\pi}{3}$, $\phi = \frac{\pi}{4}$

Thus, we can first rotate $(1, \sqrt{3}, 2)$ to the xz plane. then to z axis

$$RY\left(-\frac{\pi}{4}\right) RZ\left(-\frac{\pi}{3}\right) = \begin{pmatrix} \cos(-\frac{\pi}{4}) & 0 & -\sin(-\frac{\pi}{4}) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(-\frac{\pi}{4}) & 0 & \cos(-\frac{\pi}{4}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(-\frac{\pi}{3}) & \sin(-\frac{\pi}{3}) & 0 & 0 \\ \sin(-\frac{\pi}{3}) & \cos(-\frac{\pi}{3}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5. Find the least-squares line $y = mx + b$ for the points (-1,0), (1,2) and (2,2).

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad A^T A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \qquad = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

so

$$\begin{cases} 6m + 2b = 6 \\ 2m + 3b = 4 \end{cases} \Rightarrow \begin{cases} m = \frac{5}{7} \\ b = \frac{6}{7} \end{cases}$$

$$y = \frac{5}{7} x + \frac{6}{7}$$