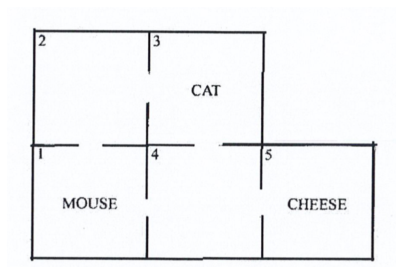
Math 401, Exam 3, May 2.

Instructions: Show all work as appropriate for the methods taught in this course. Partial credit will be given for any work, words or ideas which are relevant to the problem. **Write everything on the answer booklet, including your name. And note that there are also problems on the back of the page**.

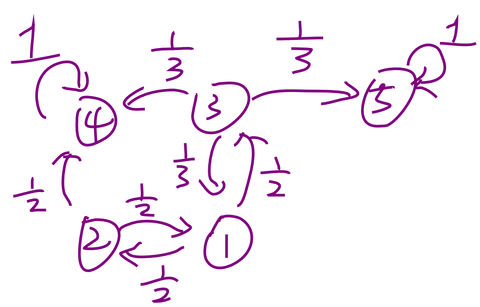
1.A mouse moves in a maze. At each step, if it is in a room with k horizontal or vertical adjacent rooms, it will move to one of the k adjacent rooms, choosing one at random, each with probability $\frac{1}{k}$. A fat lazy cat remains all the time in room 3, and a piece of cheese waits for the mouse in room 5. The mouse starts in room 1, see the following figure:



If the mouse enters the room inhabited by the cat, the cat will eat it. Also, if the mouse eats the cheese, it rests forever.

(a)[8 pts] Define a Markov chain with 5 states (and try to label the absorbing states as 4 and 5), and draw the diagram for this Markov chain.

(b)[6 pts] Write down the transition matrix T.



(a) State ①: room 1

State ②: room 2

State ③: room 4

State ④: room 3

State ⑤: room 5

(b) $$T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 1 \end{pmatrix}$$

2. Consider the second order differential equation

$$x''(t) + x'(t) - 2x(t) = 0.$$

Define the vector $\vec{u}(t) = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$.

(a)[6 pts] Find a matrix A such that $\vec{u}'(t) = A\vec{u}(t)$

(b)[8 pts] Find a matrix P and a diagonal matrix D such that $e^A = PDP^{-1}$.

(a) $\vec{u}'(t) = \begin{pmatrix} x'(t) \\ x''(t) \end{pmatrix} = \begin{pmatrix} x'(t) \\ 2x(t) - x'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$

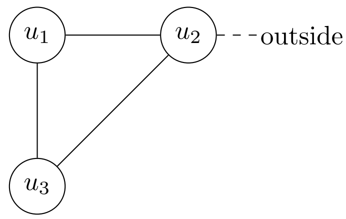
(b) $\det(\lambda I - A) = \lambda(\lambda+1) - 2 = (\lambda+2)(\lambda-1)$

If $\lambda = 1$ $\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we can chose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

If $\lambda = -2$. $\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2 \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we can choose $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

so $P = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$. $D = \begin{pmatrix} e & 0 \\ 0 & e^{-2} \end{pmatrix}$

3. Consider heat diffusion across the graph below



Heat can flow across any edge, and heat can escape from the second node to the outside world, whose temperature is fixed and equal 0.

(a)[8 pts] Write down the a differential equation for each of $u_1'(t), u_2'(t), u_3'(t)$. Then assemble your differential equations into a system $\vec{u}'(t) = A\vec{u}(t)$.

(b)[4 pts] Without solving any equations, if the initial condition is $\vec{u}(0) = [0\ 3\ 0]^T$, guess what is the $\vec{u}(t)$ when $t \to \infty$?

(a)

$$u_1'(t) = -(u_1 - u_2) - (u_1 - u_3)$$

$$u_2'(t) = -(u_2 - 0) - (u_2 - u_1) - u_2(-u_3)$$

$$u_3'(t) = -(u_3 - u_1) - (u_3 - u_2)$$

so $\vec{u}'(t) = A\vec{u}(t)$. where $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

(b) $\vec{u}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ since all the heat will escape and temperature with be same as outside

4[10 pts]. Give an example of a nonzero 3×3 matrix A such that $e^{5A} = I_3$. (Hint: use skew symmetric matrix, since they are related to rotations).

Take $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$e^{\theta A}$ is rotation around z axis by degree θ.

If we take $\theta = \frac{2\pi}{5}$. Let $\bar{A} = \frac{2\pi}{5} A$

we have $e^{5 \cdot \bar{A}} = e^{2\pi A} = I_3$

5[16 pts]. Find the singular value decomposition of $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}$.

$$A \cdot A^T = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\det(\lambda I - AA^T) = ((\lambda-1)^2 - 1) \cdot \lambda, \qquad \lambda = 0 \quad \text{or} \quad \lambda = 2$$

If $\lambda = 2$, $\sigma_1 = \sqrt{2}$. $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

A unit vector solution $\vec{u_1} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$

Thus $\vec{v_1} = \frac{1}{\sqrt{2}} \cdot A^T \vec{u_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

If $\lambda = 0$, $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

two mutually perpendicular solution is $\vec{u_2} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$ $\vec{u_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

We still need to find $\vec{v_2}$, which is unit vector solution to

$A^T A \vec{v_2} = \vec{0}$, $\qquad A^T A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

$\vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so overall

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

6[4 pts]. True of False, no need to explain: If A and B are two square matrices such that $A + B = \bar{\mathbf{0}}$, then $e^A \cdot e^B = e^B \cdot e^A$, in other words, e^A and e^B commute.

True. because $A = -B$

$A \cdot B = A \cdot (-A) = B \cdot A$

so $e^A \cdot e^B = e^B \cdot e^A = e^{A+B} = I$