FINM33601:Fixed Income Derivatives Homework 4

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Question 1.1. Given the following dynamics for the forward rate, we can easily find what the dynamics of the short rate is.

$$\begin{aligned}
f_t^T &= f_0^T + \int_0^t \sigma_s^T dW_s^{\mathbb{P}} + \int_0^t \alpha_s^T ds && \text{physical measure} \\
f_t^T &= f_0^T - \int_0^t \sigma_s^T \Sigma_s^T ds + \int_0^t \sigma_s^t dW_s^{\mathbb{Q}} && \text{risk-neutral measure} \\
r_t = \lim_{T\to t^+} f_t^T &= f_0^t - \int_0^t \sigma_s^t \Sigma_s^t ds + \int_0^t \sigma_s^t dW_s^{\mathbb{Q}} \\
r_t &= r_0 - \int_0^t \sigma_s^t \Sigma_s^t ds + \int_0^t \sigma_s^t dW_s^{\mathbb{Q}} \\
\mathbb{E}^{\mathbb{Q}}\left[r_t\right] &= r_0 - \int_0^t \sigma_s^t \Sigma_s^t ds && (1)
\end{aligned}$$

Question 1.2. Given that under the physical measure we have

$$\begin{aligned}
f_t^T &= f_0^T + \int_0^t \sigma_s^T dW_s^{\mathbb{P}} + \int_0^t \alpha_s^T ds, \\
df_t^T &= \alpha_t^T dt + \sigma_t^T dW_t^{\mathbb{P}}.
\end{aligned}$$

We can also express it

$$\begin{aligned}
d\left(-\int_t^T f_t^v dv\right) &= f_t^T dt - \int_t^T df_t^v dv, \\
&= r_t dt - \int_t^T df_t^v dv, \\
&= r_t dt - \int_t^T \left[\alpha_t^v dt + \sigma_t^v dW_t^{\mathbb{P}}\right] dv, \\
&= r_t dt - A_t^T dt - \Sigma_t^T dW_t^{\mathbb{P}}.
\end{aligned}$$

We also have that the dynamics of the bond depends on the forward rate dynamics

$$P_t^T = \exp\left\{-\int_t^T f_t^u du\right\}$$

Using Itô's Lemma with $g(x) = e^x$ leads to:

$$\begin{aligned} dP_t^T &= P_t^T \left[r_t dt - A_t^T dt - \Sigma_t^T dW_t^{\mathbb{P}} \right] + \frac{1}{2} P_t^T \left(\Sigma_t^T\right)^2 dt, \\ &= P_t^T \left[r_t - A_t^T + \frac{1}{2}\left(\Sigma_t^T\right)^2 \right] dt - \Sigma_t^T P_t^T dW_t^{\mathbb{P}}. \end{aligned}$$

Let B_t be discounted factor at time t. The No-Arbitrage condition implies that

$$\begin{aligned} B_t P_t^T &= \exp\left\{ -\int_0^t r_u du \right\} P_t^T, \\ dB_t &= -r_t B_t dt, \\ d\left(B_t P_t^T\right) &= B_t P_t^T \left[\left(-A_t^T + \frac{1}{2}\left(\Sigma_t^T\right)^2 \right) dt + \Sigma_t^T dW_t^{\mathbb{P}} \right]. \end{aligned}$$

We also know from the literature that $A_t^T = \frac{1}{2}\left(\Sigma_t^T\right)^2 - \Sigma_t^T \gamma_t$, with sufficient condition that $\Sigma_t^s \geq 0, \forall\, s \geq 0$. Thus

$$d\left(B_t P_t^T\right) = B_t P_t^T \left[\Sigma_t^T \gamma_t dt + \Sigma_t^T dW_t^{\mathbb{P}} \right].$$

We can use Girsanov's Theorem to tilt the Brownian Motion under a new probability measure $\mathbb{Q}$:

$$dW_t^{\mathbb{P}} = dW_t^{\mathbb{Q}} - \gamma_t dt.$$

Under this new measure, we can find the SDE of our discounted bond price:

$$d\left(B_t P_t^T\right) = \Sigma_t^T B_t P_t^T dW_t^{\mathbb{Q}},$$

which is a local-Martingale under the new measure $\mathbb{Q}$. Because $d(B_t^{-1}) = r_t B_t^{-1}$ the differential undiscounted bond price is:

$$\begin{aligned} dP_t^T &= d\left(B_t^{-1} B_t P_t^T\right), \\ &= r_t B_t^{-1} B_t P_t^T dt - \Sigma_t^T B_t^{-1} B_t P_t^T dW_t^{\mathbb{Q}}, \\ &= r_t P_t^T dt + \Sigma_t^T P_t^T dW_t^{\mathbb{Q}}. \end{aligned}$$

We can use Itô's Lemma on $f(x) = \ln(x)$, which leads to:

$$\begin{aligned} d\left(\log(P_t^T)\right) &= \frac{P_t^T}{d} P_t^T - \frac{1}{2}\frac{1}{(P_t^T)^2} d\langle P_t^T \rangle, \\ d\langle P_t^T \rangle &= \left(\Sigma_t^T\right)^2 (P_t^T)^2 dt, \\ d\log(P_t^T) &= r_t dt + \Sigma_t^T dW_t^{\mathbb{Q}} - \frac{1}{2}\left(\Sigma_t^T\right)^2 dt, \\ P_t^T &= P_0^T \exp\left\{ \int_0^t r_u du - \frac{1}{2}\int_0^t \left(\Sigma_u^t\right)^2 du + \int_0^t \Sigma_u^t dW_u^{\mathbb{Q}} \right\}, \\ &= B_t^{-1} P_o^T \exp\left\{ -\frac{1}{2}\int_0^t \left(\Sigma_u^t\right)^2 du + \int_0^t \Sigma_u^t dW_u^{\mathbb{Q}} \right\}. \end{aligned}$$

If we let

$$X_t = -\frac{1}{2}\int_0^t \left(\Sigma_u^t\right)^2 du + \int_0^t \Sigma_u^t dW_u^{\mathbb{Q}}$$

we then have:

$$P_t^T = B_t^{-1} P_0^T e^{X_t}.$$

Because we don't know the form of σ_t^T other than it is deterministic or non-stochastic, we cannot simplify more. We can take Itô's lemma again on P_t^T to have a different form. We already did it earlier where $g(x) = e^x = g'(x) = g''(x)$. We get:

$$\begin{aligned} P_t^T &= B_t^{-1} P_0^T \left\{ e^{X_0} + \int_0^t e^{X_u}\left(-\frac{1}{2}\left(\Sigma_u^t\right)^2 + \frac{1}{2}\left(\Sigma_u^t\right)^2\right) du + \int_0^t e^u \Sigma_u^t dW_u^{\mathbb{Q}} \right\}, \\ &= B_t^{-1} P_0^T \left\{ e^{X_0} + \int_0^t e^{X_u} \Sigma_u^t dW_u^{\mathbb{Q}} \right\}. \end{aligned}$$

We can now solve for the expectation and the variance.

$$\mathbb{E}^{\mathbb{Q}}\left[P_t^T\right] = P_0^T \mathbb{E}^{\mathbb{Q}}\left[B_t^{-1}\right]$$

We have that

$$B_t^{-1} = e^{-\int_0^t r_u du} = \exp\left\{-\int_0^t \left(r_0 - \int_0^u \sigma_s^u \Sigma_s^u ds + \int_0^u \sigma_s^u dW_s^{\mathbb{Q}}\right) du\right\}$$

The first 2 terms in the first integral are deterministic so we can separate them from the stochastic part of the integral.

$$B_t^{-1} = \exp\left\{-\Pi_t + \int_0^u \sigma_s^u dW_s^{\mathbb{Q}} du\right\} = e^{-\Pi_t} \cdot \exp\left\{\int_0^t \int_0^u \sigma_s^u dW_s^{\mathbb{Q}} du\right\}$$

Since the stochastic part is lognormal, we can using the property that $\mathbb{E}[e^{\Phi_t}] = e^{(\mathrm{Var}(\Phi_t)/2)}$. We have that

$$\mathrm{Var}\left(\int_0^t \int_0^u \sigma_s^u dW_s^{\mathbb{Q}} du\right) = \int_0^t \left(\int_0^u \sigma_s^u ds\right)^2 du$$

Finally leads us to:

$$\mathbb{E}^{\mathbb{Q}}\left[P_t^T\right] = P_0^T \exp\left\{-\Pi_t + \frac{1}{2}\int_0^t \left(\int_0^u \sigma_s^u ds\right)^2 du\right\} \tag{2}$$

Question 1.3. Using

$$\mathrm{Var}^{\mathbb{Q}}\left[P_t^T\right] = \mathbb{E}^{\mathbb{Q}}\left[(P_t^T)^2\right] - \left(\mathbb{E}^{\mathbb{Q}}\left[P_t^T\right]\right)^2$$

We also have

$$\mathbb{E}^{\mathbb{Q}}\left[(P_t^T)^2\right]$$

and

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[(P_t^T)^2\right] &= (P_0^T)^2 \exp\left\{-2\Pi_t + \frac{1}{2}\int_0^t \left(\int_0^u 2\sigma_s^u ds\right)^2 du\right\} \\
&= (P_0^T)^2 \exp\left\{-2\Pi_t + 2\int_0^t \left(\int_0^u \sigma_s^u ds\right)^2 du\right\} \\
(\mathbb{E}^{\mathbb{Q}}\left[P_t^T\right])^2 &= (P_0^T)^2 \exp\left\{-2\Pi_t + \int_0^t \left(\int_0^u \sigma_s^u ds\right)^2 du\right\}
\end{aligned}$$

$$\mathrm{Var}^{\mathbb{Q}}\left[P_t^T\right] = (P_0^T)^2 \exp\left\{-2\Pi_t + \int_0^t \left(\int_0^u \sigma_s^u ds\right)^2 du\right\} \cdot \left(\exp\left\{\int_0^t \left(\int_0^u \sigma_s^u ds\right)^2 du\right\} - 1\right) \tag{3}$$

Question 2.1. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and

$$\begin{aligned}
\xi_t &= \exp\left\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t (\theta_s)^2 ds\right\}, \\
W_t^{\mathbb{Q}} &= W_t^{\mathbb{P}} + \int_0^t \theta_s ds
\end{aligned}$$

First, we must show $\mathbb{Q}$ is a measure in $(\Omega, \mathcal{F})$. For a disjoint and countably infinite sequence $\{\mathcal{G}_i\}_{i=1}^{\infty} \subset \mathcal{F}$, we have

$$\mathbb{Q}\left(\cup_{i=1}^{\infty}\mathcal{G}_i\right) = \int_{\cup_{i=1}^{\infty}\mathcal{G}_i} \xi_T d\mathbb{P} = \sum_{i=1}^{\infty}\int_{\mathcal{G}_i} \xi_t d\mathbb{P} = \sum_{i=1}^{\infty}\mathbb{Q}(\mathcal{G}_i)$$

Since $\{\mathcal{G}_i\}$ are countable and disjoint, so $\mathbb{Q}$ is a measure. The Weiner process $W_t^{\mathbb{Q}}$ is a brownian motion under $\mathbb{Q}$ such that $\forall A \subset \mathcal{F} \in \Omega$ we have that

$$\mathbb{Q}(A) = \int_A \xi_t d\mathbb{P} = \int_A d\mathbb{Q}.$$

Another important property is that $\{\xi_t\}$ is a local martingale, indeed

$$d\xi_t = \xi_t\left[\left(-\theta_t dW_t - \frac{1}{2}(\theta_t)^2 dt\right) + \frac{1}{2}(\theta_t)^2 dt\right] = -\theta_t \xi_t dW_t.$$

Considering $\xi_0 = 1$ and a local martingale, this means that $\mathbb{E}^{\mathbb{P}}[\xi_t] = 1 \quad \forall t \geq 0$.

$$\mathbb{Q}(\Omega) = \int_{\Omega} \xi_T d\mathbb{P} = \mathbb{E}^{\mathbb{P}}[\xi_T] = 1 \tag{4}$$

So, $\mathbb{Q}$ is a probability measure.

Question 2.2. To show $\mathbb{Q}$ is equivalent to $\mathbb{P}$, we must show that $\mathbb{Q} << \mathbb{P}$ and $\mathbb{P} << \mathbb{Q}$. Or, in other words, that they are both absolutely continuous to each other.

Let our two probability space be $(\Omega, \mathcal{F}, \mathbb{Q})$ and $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\theta \in \mathcal{F}$ such that $\mathbb{P}(\theta) = 0$. Then

$$\int_{\omega \in \theta} d\mathbb{P}(\omega) = 0$$

By definition

$$\mathbb{Q}(\omega) = \int_{\omega \in \theta} \xi_T(\omega) d\mathbb{P}(\omega).$$

But since $\xi_T(\omega) \neq 0 \ \forall \omega \in \Omega$, we have that

$$\mathbb{Q}(\omega) = 0 \iff \mathbb{P}(\omega) = 0.$$

The reason it is bidirectional is because by Radon-Nikodym theorem, there exists a derivative ν in a measurable space $(\Omega, \mathcal{F}, \mathbb{Q})$ such that $\forall A \in \mathcal{F}$ we have

$$\mathbb{P}(A) = \int_A \nu d\mathbb{Q}.$$

But since we already showed that there exists a derivative ξ_T in $\mathbb{P}$ to go from $\mathbb{P} - \rightarrow \mathbb{Q}$ and now a derivative ν to go from $\mathbb{Q} \rightarrow \mathbb{P}$, and that ξ_T is of exponential form, then ν must also been in exponential form thus $\nu(\omega) \neq 0 \ \forall \omega \in \Omega$.

$$(\mathbb{P} << \mathbb{Q}) \,\&\, (\mathbb{Q} << \mathbb{P}). \tag{5}$$