FINM33601:Fixed Income Derivatives Homework 5

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Question 1: Hull-White Formulas. To derive the **Volatility functions** under Hull-White model, let's start with the HJM framework where

$$df_t^T = -\Sigma_t^T \sigma_t^T dt + \sigma_t^T dW_t,$$
$$\sigma_t^T = -\frac{\partial}{\partial T}\sigma_t^T, \qquad \Sigma_T^T = 0.$$

Under the Hull-White model, the volatility follows an exponentially decaying deterministic function:

$$\sigma_t^T = \sigma e^{-a(T-t)}.$$

Therefore, we have

$$\begin{aligned}
\Sigma_t^T &= -\int_t^T \sigma_t^\mu d\mu = -\int_t^T \sigma e^{-a(\mu-t)} d\mu, \\
&= -\sigma e^{-at} \int_t^T e^{-a\mu} d\mu = -\sigma e^{at} \left[\frac{e^{-a\mu}}{-a}\right]_t^T, \\
&= -\sigma e^{at} \left[\frac{e^{-at} - e^{-aT}}{a}\right] = -\sigma \frac{1}{a}\left(1 - e^{-a(T-t)}\right), \\
&= -\sigma b(t,T) \longrightarrow b(t,T) \triangleq \frac{1}{a}\left(1 - e^{-a(T-t)}\right).d\mu
\end{aligned} \tag{1}$$

As for the **Short rate state**, we have

$$\begin{aligned}
df_t^T &= -\Sigma_t^T \sigma_t^T dt + \sigma_t^T dW_t, \\
&= \sigma^2 b(t,T) e^{-a(T-t)} dt + \sigma e^{-a(T-t)} dW_t, \\
f_t^T &= f_0^T + \underbrace{\sigma^2 \int_0^t b(s,T) e^{-a(T-s)} ds}_{\text{A}} + \underbrace{\sigma \int_0^t e^{-a(T-s)} dW_s}_{\text{B}}
\end{aligned}$$

Let's solve for (B) first.

$$\begin{aligned}
\sigma \int_0^t e^{-a(T-s)} dW_s &= \sigma \int_0^t e^{-a(T-t+t-s)} dW_s, \\
&= \sigma \int_0^t e^{-a(T-t)} e^{-a(t-s)} dW_s, \\
&= \sigma e^{-a(T-t)} \int_0^t e^{-a(t-s)} dW_s, \\
&= e^{-a(T-t)} x_t \longrightarrow x_t = \sigma \int_0^t e^{-a(t-s)} dW_s.
\end{aligned} \tag{2}$$

Now, let's solve for (A).

$$
\begin{aligned}
\sigma^2 \int_0^t b(s,T) e^{-a(T-s)} ds &= \sigma^2 \int_0^t \frac{1}{a}\left(1 - e^{-a(T-s)}\right) e^{-a(T-s)} ds, \quad \mu = T - s,\\
&= \frac{-\sigma^2}{a} \int_T^{T-t} \left(1 - e^{-a\mu}\right) e^{-a\mu} d\mu, \quad \nu = 1 - e^{-a\mu},\\
&= -\frac{\sigma^2}{a} \int_{1-e^{-aT}}^{1-e^{-a(T-t)}} \frac{\nu}{a} d\nu,\\
&= -\frac{\sigma^2}{a^2} \left[\nu^2\right]_{1-e^{-aT}}^{1-e^{-a(T-t)}},\\
&= -\frac{\sigma^2}{a^2} \left[\frac{\left(1 - e^{-a(T-t)}\right)^2 - \left(1 - e^{-aT}\right)^2}{2}\right],\\
&= \sigma^2 \left[\frac{\left(1 - e^{-aT}\right)^2}{2a^2} - \frac{\left(1 - e^{-a(T-t)}\right)^2}{2a^2}\right],\\
&= \frac{1}{2}\sigma^2 \left(b(0,T)^2 - b(t,T)^2\right).
\end{aligned}
$$

Which if we plug both (A) and (B) into the forward rate equation before solving for (B), leads us to:

$$
f_t^T = f_0^T + \frac{1}{2}\sigma^2 \left(b(0,T)^2 - b(t,T)^2\right) + e^{-a(T-t)} x_t. \tag{3}
$$

Taking the limit to find the short rate:

$$
r_t = \lim_{T \to t^+} f_t^T = f_0^t + \frac{1}{2}\sigma^2 \left[b(0,t)^2 - b(t,t^+)^2\right] + e^{-a(t^+-t)} x_t.
$$

But, given that

$$
\lim_{T \to t^+} e^{-a(t^+-t)} = 1,
$$

we have

$$
r_t = f_0^t + \frac{1}{2}\sigma^2 b(0,t)^2 + x_t \tag{4}
$$

Now, let us find the results for the **Bond prices** under the Hull-White model.

$$
\begin{aligned}
P_t^T &= e^{-\int_t^T f_t^S dS},\\
&= \exp\left\{-\int_t^T \left[f_0^S + \frac{1}{2}\sigma^2 \left(b(0,S)^2 - b(t,S)^2\right) + e^{-a(S-t)} x_t\right] dS\right\},\\
&= \underbrace{\exp\left\{-\int_t^T f_0^S dS\right\}}_{\text{A}} \cdot \underbrace{\exp\left\{-\frac{1}{2}\sigma^2 \int_t^T \left(b(0,S)^2 - b(t,S)^2\right) dS\right\}}_{\text{B}} \cdot \underbrace{\exp\left\{x_t \int_0^t e^{-a(S-t)} dS\right\}}_{\text{C}}.
\end{aligned}
$$

Let's start with (A) which is straight forward by definition

$$\exp\left\{-\int_t^T f_0^S dS\right\} = P_0^{t,T}.$$

Now let's solve the integral inside of (B):

$$\begin{aligned}
\int_t^T \left(b(0,S)^2 - b(t,S)^2\right) dS &= \frac{1}{a^2}\left[\int_t^T \left(1-e^{-aS}\right)^2 dS - \int_t^T \left(1-e^{-a(S-t)}\right)^2 dS\right], \\
&= \frac{1}{a^2}\left[\int_t^T \left(1-2e^{-aS}+e^{-2aS}\right) dS - \int_t^T \left(1-2e^{-a(S-t)}+e^{-2a(S-t)}\right) dS\right], \\
\eta &= S-t, \\
&= \frac{1}{a^2}\left[\int_t^T \left(-2e^{-aS}+e^{-2aS}\right) dS + \int_0^{T-t} \left(2e^{-a\eta}-e^{-2a\eta}\right) dS\right], \\
&= \frac{1}{a^2}\left(\left[\frac{e^{-2aS}}{-2a}\right]_t^T - \left[\frac{2e^{-aS}}{-a}\right]_t^T + \left[\frac{2e^{-a\eta}}{-a}\right]_0^{T-t} - \left[\frac{e^{-2a\eta}}{-2a}\right]_0^{T-t}\right), \\
&= \frac{1}{a^3}\left[-\frac{1}{2}\left(e^{-2aT}-e^{-2at}\right) + \left(2e^{-aT}-2e^{-at}\right)\right. \\
&\left. - \left(2e^{-a(T-t)}-2\right) + \frac{1}{2}\left(e^{-2a(T-t)}-1\right)\right], \\
&= \frac{1}{a^3}\left[-\frac{1}{2}e^{-2at}\left(e^{-2a(T-t)-1}\right) + 2e^{-at}\left(e^{-aT}-1\right)\right. \\
&\left. -2\left(e^{-a(t-t)}-1\right) + \frac{1}{2}\left(e^{-2a(T-t)}-1\right)\right], \\
&= \frac{1}{a^2}\left(\frac{1-e^{-a(T-t)}}{a}\right)\left[\frac{\left(1-e^{-2at}\right)\left(1-e^{-a(T-t)}\right)}{2} + \left(1-e^{-at}\right)\left(1-e^{-at}\right)\right], \\
&= \left(\frac{1-e^{-a(T-t)}}{a}\right)\left[\left(\frac{1-e^{-a(T-t)}}{a}\right)\left(\frac{1-e^{-2at}}{2a}\right) + \left(\frac{1-e^{-at}}{a}\right)^2\right], \\
&= b(t,T)\left(b(t,T)\frac{1-e^{-2at}}{2a} + b(0,t)^2\right).
\end{aligned}$$

Now, let's solve for (C):

$$\begin{aligned}
\exp\left\{-x_t\int_t^T e^{-a(S-t)} dS\right\} &= \exp\left\{-x_t\left(\frac{1}{-a}\left[e^{-a(S-t)}\right]_t^T\right)\right\}, \\
&= \exp\left\{-x_t\frac{1}{a}\left(1-e^{-a(T-t)}\right)\right\}, \\
&= e^{-b(t,T)x_t}.
\end{aligned}$$

Putting everything together leads to:

$$
\begin{aligned}
P_t^T &= P_0^{t,T} \exp\left\{-\frac{1}{2}\sigma^2 b(t,T)\left(b(t,T)\frac{1-e^{-2at}}{2a} + b(0,t)^2\right)\right\} e^{-b(t,T)x_t}, \\
P_t^T &= P_0^{t,T} A(t,T) b^{-b(t,T)x_t}, && (5)\\
A(t,T) &\triangleq \exp\left\{-\frac{1}{2}\sigma^2 b(t,T)\left(b(t,T)\frac{1-e^{-2at}}{2a} + b(0,t)^2\right)\right\}. && (6)
\end{aligned}
$$

Question 2.1. Let's show the variance of the short rate state x_t:

$$
\begin{aligned}
\mathrm{Var}\left[x_t\right] &= \sigma^2 \int_0^t e^{-2a(t-S)} dS, \\
&= \sigma^2 \frac{1}{2a}\left[e^{-2a(t-S)}\right]_0^t, \\
&= \sigma^2 \frac{1-e^{-2at}}{2a}. && (7)
\end{aligned}
$$

Question 2.2. Finally, let's show log-normal variance of bond prices under Hull-White model.

$$
\begin{aligned}
\log P_t^T &= \log P_0^{t,T} + \log A(t,T) - b(t,T)x_t, \\
\mathrm{Var}\left[\log P_t^T\right] &= b(t,T)^2 \mathrm{Var}\left[x_t\right], \\
&= \sigma^2 b(t,T)^2 \frac{1-e^{-2at}}{2a}. && (8)
\end{aligned}
$$