

Homework Assignment: The Hull White Model

Course: Fixed Income Derivatives

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1 Hull-White Formulas

Derive the following results for the Hull-White model.

Volatility functions

$$\begin{aligned}b(t, T) &\triangleq \frac{1}{a} \left(1 - e^{-a(T-t)}\right) \\ \Sigma_t^T &= -\sigma \cdot b(t, T)\end{aligned}$$

Short rate state

$$\begin{aligned}x_t &= \sigma \int_0^t e^{-a(t-s)} dW_s \\ r_t &= f_0^t + \frac{1}{2} \sigma^2 b(0, t)^2 + x_t \\ f_t^T &= f_0^T + \frac{1}{2} \sigma^2 \left(b(0, T)^2 - b(t, T)^2 \right) + \sigma \int_0^t e^{-a(T-s)} dW_s \\ &= f_0^T + \frac{1}{2} \sigma^2 \left(b(0, T)^2 - b(t, T)^2 \right) + e^{-a(T-t)} x_t\end{aligned}$$

Bond prices

$$\begin{aligned}A(t, T) &\triangleq \exp \left\{ -\frac{1}{2} \sigma^2 b(t, T) \left(b(t, T) \frac{1 - e^{-2at}}{2a} + b(0, t)^2 \right) \right\} \\ P_t^T &= P_0^{t, T} A(t, T) e^{-b(t, T) x_t}\end{aligned}$$

Follow the same steps used in the notes for the Ho-Lee model.

2 Hull-White Variance Calculations

1. Show that the variance of the Hull-White short rate state x_t is given by

$$\text{Var}[x_t] = \sigma^2 \frac{1 - e^{-2at}}{2a}.$$

2. Show that under Hull-White bond prices have a log-normal variance given by

$$\text{Var}[\log P_t^T] = \sigma^2 b(t, T)^2 \frac{1 - e^{-2at}}{2a}.$$