FINM37601:Mathematical Market Microstructure Homework 1

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**Question 1.** Derive the formulas to the mean $\mathbb{E}[p]$ and the variance $\mathbb{V}[p]$ of the transaction price $p$ with the variables mentioned.

In this model we have 3 possible states in our sample so we have $\Omega := \left\{a, \frac{a+b}{2}, b\right\}$ with an associated probability measure $\mathbb{P} := \{\rho_u, \rho_m, \rho_l\}$ respectively. From this model we can compute it's first two moments as below.

$$
\begin{aligned}
\mathbb{E}[p] &= \sum_{i\in\mathbb{P}} \mathbb{P}(p_i = \omega_i)\rho_i \\
&= b\rho_l + \frac{a+b}{2}\rho_m + a\rho_u \\
&= b\rho_l + \frac{a+b}{2}\left(1-\rho_u-\rho_l\right) + a\rho_u \\
&= \frac{a}{2}\left(1+\rho_u-\rho_l\right) + \frac{b}{2}\left(1+\rho_l-\rho_u\right).
\end{aligned}
$$

Let's compute $\left(\mathbb{E}[p]\right)^2$ next.

$$
\begin{aligned}
\left(\mathbb{E}[p]\right)^2 &= \frac{a^2}{4}\left(1+\rho_u^2+\rho_l^2+2\rho_u-2\rho_l-2\rho_u\rho_l\right) \\
&+ \frac{b^2}{4}\left(1+\rho_u^2+\rho_l^2+2\rho_l-2\rho_u-2\rho_u\rho_l\right) \\
&+ \frac{ab}{2}\left(1-\rho_l^2-\rho_u^2+2\rho_l\rho_u\right).
\end{aligned}
$$

Let's now compute the second moment $\mathbb{E}[p^2]$.

$$
\begin{aligned}
\mathbb{E}[p^2] &= \sum_{i\in\mathbb{P}} \mathbb{P}(p_i = \omega_i)\rho_i^2 \\
&= b^2\rho_l + \frac{(a+b)^2}{4}\left(1-\rho_u-\rho_l\right) + a^2\rho_u \\
&= \frac{a^2}{4}\left(3\rho_u+1-\rho_l\right) + \frac{b^2}{4}\left(3\rho_l+1-\rho_u\right) + \frac{ab}{4}\left(2-\rho_u-\rho_l\right).
\end{aligned}
$$

We are now equipped with all the elements to compute the variance using the well known formula $\mathbb{V}[p] = \mathbb{E}[p^2] - \left(\mathbb{E}[p]\right)^2$.

$$
\begin{aligned}
\mathbb{V}[p] &= \mathbb{E}[p^2] - (\mathbb{E}[p])^2 \\
&= \frac{a^2}{4}\left[3\rho_u + 1 - \rho_l - \left(1 + \rho_u^2 + \rho_l^2 + 2\rho_u - 2\rho_l - 2\rho_u\rho_l\right)\right] \\
&+ \frac{b^2}{4}\left[3\rho_l + 1 - \rho_u - \left(1 + \rho_u^2 + \rho_l^2 + 2\rho_l - 2\rho_u - 2\rho_u\rho_l\right)\right] \\
&- \frac{2ab}{4}\left[1 - \rho_u - \rho_l - \left(1 - \rho_l^2 - \rho_u^2 + 2\rho_l\rho_u\right)\right] \\
&= \frac{a^2}{4}\left(\rho_u + \rho_l - \rho_u^2 - \rho_l^2 + 2\rho_u\rho_l\right) \\
&+ \frac{b^2}{4}\left(\rho_u + \rho_l - \rho_u^2 - \rho_l^2 + 2\rho_u\rho_l\right) \\
&- \frac{2ab}{4}\left(\rho_u + \rho_l - \rho_u^2 - \rho_l^2 + 2\rho_u\rho_l\right) \\
&= \left(\frac{a-b}{2}\right)^2\left(\rho_u + \rho_l - (\rho_u - \rho_l)^2\right) \quad \square
\end{aligned}
$$

**Question 2.** Simplify your results in Question 1 by assuming that $|\rho_l - \rho_u| = \delta \ll 1$ so that higher than first order terms of $|\rho_l - \rho_u$ can be omitted. In practice, this is the situation when order flow has no significant directional upward or downward movements.

Since $\mathbb{E}[p]$ has no 2$^{\text{nd}}$ order terms, it will be the same for Question 2 and as well true for Question 3. As for the variance, we can omit the term $(\rho_u - \rho_l)^2 \to 0$, so the variance becomes

$$
\mathbb{V}[p] = \left(\frac{a-b}{2}\right)^2(\rho_u + \rho_l) \quad \square
$$

Note, in the case $\rho_l = \rho_u$ we have the symmetric binary case and the variance does match that scenario.

**Question 3.** Simplify your results in Question 1 by assuming that $\rho_m \ll 1$ so that higher than first order terms of $\rho_m$ can be omitted. In practice, this is the situation when "off exchanges" trades (such as those in "dark pools") are rare.

We have that $\rho_m = 1 - \rho_u - \rho_l$, therefore the second order term of $\rho_m$ we have

$$
\rho_m^2 = 1 + \rho_u^2 + \rho_l^2 - 2\rho_u - 2\rho_l + 2\rho_u\rho_l
$$

Let's now expand the terms inside the variance from Question 1 and express it in terms of $\rho_m^2$.

$$
\begin{aligned}
\mathbb{V}[p] &= \left(\frac{a-b}{2}\right)^2 \left(\rho_u + \rho_l - (\rho_u - \rho_l)^2\right) \\
&= \left(\frac{a-b}{2}\right)^2 \left(-\rho_u^2 - \rho_l^2 + 2\rho_u\rho_l + \rho_u + \rho_l\right) \\
&= \left(\frac{a-b}{2}\right)^2 \left(-1 + 1 - \rho_u^2 - \rho_l^2 + 2\rho_u - \rho_u + 2\rho_l - \rho_l - 2\rho_u\rho_l + 4\rho_u\rho_l\right) \\
&= \left(\frac{a-b}{2}\right)^2 \left(\cancel{\underbrace{-1 - \rho_u^2 - \rho_l^2 + 2\rho_u + 2\rho_l - 2\rho_u\rho_l}_{-\rho_m^2}} + 1 - \rho_u - \rho_l + 4\rho_u\rho_l\right) \\
&= \left(\frac{a-b}{2}\right)^2 \left(1 - \rho_u - \rho_l + 4\rho_u\rho_l\right) \quad \square
\end{aligned}
$$

**Question 4.** Let $T_*$ be the "market microstructure characteristic time scale" for this stock. It can be derived by relating the variance calculated from Question 1-3 with the variance of a continuous arithmetic Brownian price process with a constant volatility of $\sigma$; that is

$$\sqrt{\mathbb{V}[p]} \sim p_0 \sigma T_*^{\gamma} \tag{1}$$

Here $p_0$ is a base price so that the dimensions on both sides of Equation 1 becomes the same. In this assignment, let's assume $\gamma = \frac{1}{3}$ to reflect potential fat-tail distribution of price returns. Based on the simplified version of Question 2 and 3 respectively, give the formulas of $T_*$ as a function of the following variables : $b, a, \rho_l, \rho_u, \rho_m, \sigma, p_0$. Comment on how $T_*$ changes as the market becomes more or less "fat-tailed" in return distribution (i.e., $\gamma$ decreases or increases).

Let's start by defining the standard deviations of Question 2 and 3 respectively as follows:

$$
\begin{aligned}
\sigma_2 &= \left|\frac{a-b}{2}\right| \sqrt{\rho_u + \rho_l} \\
\sigma_3 &= \left|\frac{a-b}{2}\right| \sqrt{1 - \rho_u - \rho_l + 4\rho_u\rho_l}.
\end{aligned}
$$

Using the formula

$$T \sim \left[\frac{\sigma}{\sigma p_0}\right]^{\frac{1}{\gamma}},$$

leads us to

$$
\begin{aligned}
T_2 &= \left[\frac{\left|\frac{a-b}{2}\right|\sqrt{\rho_u+\rho_l}}{\sigma p_0}\right]^{\frac{1}{\gamma}} \\
T_3 &= \left[\frac{\left|\frac{a-b}{2}\right|\sqrt{1-\rho_u-\rho_l+4\rho_u\rho_l}}{\sigma p_0}\right]^{\frac{1}{\gamma}}.
\end{aligned}
$$

As the respective $T_*$ for Question 2 and 3.

Let

$$f(x) = a^{1/x} = e^{\log\left(a^{1/x}\right)} = e^{1/x\log a}$$

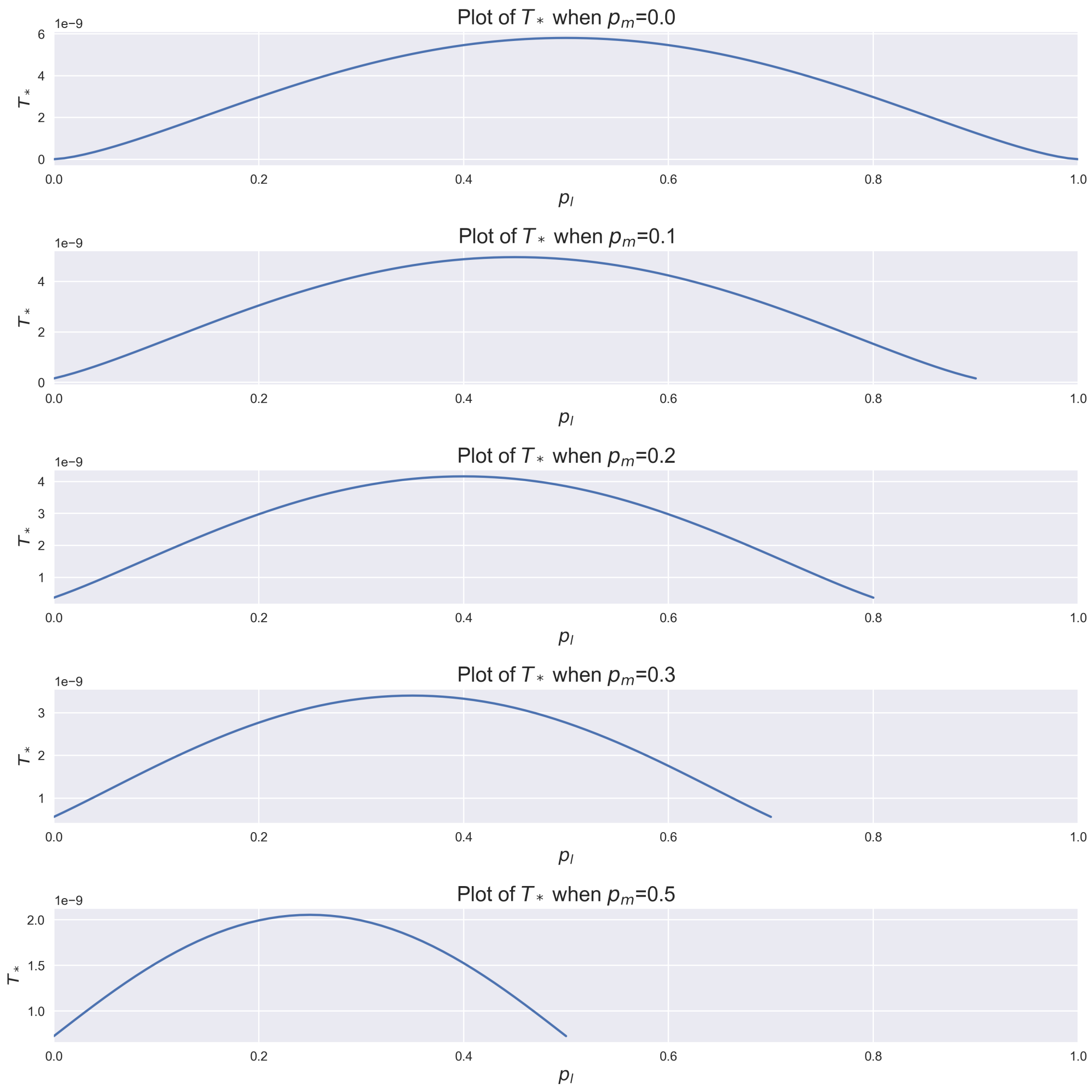
then it's derivative is

$$f^{'}(x) = \frac{\mathrm{d}}{\mathrm{dx}}\left(\log a x^{-1}\right)f(x) = -\frac{\log a \cdot a^{1/x}}{x^2}.$$

Under assumption that $\gamma > 0$. This means that when $a \in (0,1)$ then the derivative is increasing when $\gamma > 1$ or $(x > 1)$ and when $a \geq 1$ the derivative is decreasing for $\gamma < 1$. So depending on the inputs in the model, as the market becomes more or less "fat-tailed" the time scale will either decrease or increase.

Although, it is fair to assume that the denominator $\sigma p_0$ will be larger than the numerator. Under that scenario, this implied $a \in (0,1)$ so as the market becomes more and more fat-tailed (i.e. $\gamma \to 0$), we expect the microstructure time scale to decrease.

**Question 5.** See plot below for Question 5.




**Question 6.** From the above plot, we see that when the time scale spent of the process to move from a random bid-ask bounce into it's random walk process decreases. This seems to be in agreement with users of "Dark pool" where assets spends less time in random bid-ask prices and more time on their asset price dynamics which leads to lower variance.