

# FINM 32000: Homework 6

Due Friday March 5, 2021 at 11:59pm

## Problem 1

Let  $\mathbf{S}$  be the column vector with components  $S^{[1]}, S^{[2]}$ , where the stock prices  $S^{[j]}$  have risk-neutral dynamics

$$dS_t^{[j]} = rS_t^{[j]}dt + \sigma_{[j]}S_t^{[j]}dW_t^{[j]} \quad j = 1, 2$$

with risk-free interest rate  $r = 0.05$ , and constant volatilities  $\sigma_{[1]} = 0.3$ ,  $\sigma_{[2]} = 0.2$ .

The time-0 prices are  $S_0^{[1]} = 100$ ,  $S_0^{[2]} = 110$ . The  $\mathbb{P}$ -Brownian motions  $W^{[i]}$  and  $W^{[j]}$  have correlation  $\rho = 0.8$ .

- (a) Let  $\mathbf{X}$  be the column vector with components  $X^{[1]}, X^{[2]}$  where  $X^{[j]} := \log S^{[j]}$ . Find the covariance matrix of  $\mathbf{X}_T$ .

Hint: One possible solution is to write  $\mathbf{X}_T$  as a nonrandom vector plus  $\mathbf{\Sigma}\mathbf{W}_T$  where  $\mathbf{\Sigma}$  is the nonrandom diagonal matrix with diagonal elements  $\sigma_{[1]}, \sigma_{[2]}$ , and  $\mathbf{W}$  is the random column vector with components  $W^{[1]}, W^{[2]}$ . Then  $\text{Cov}(\mathbf{X}_T) = \mathbf{\Sigma}\text{Cov}(\mathbf{W}_T)\mathbf{\Sigma}^\top$ .

Consider a basket  $H := \frac{1}{2}S^{[1]} + \frac{1}{2}S^{[2]}$  of one-half of a share of each stock.

- (b) Using 10000 standard Monte Carlo simulations, estimate the time-0 price  $C$  of an option that pays  $(H_T - 110)^+$  at time  $T = 1.0$ . Also give the standard error [the sample standard deviation, divided by the square root of the number of simulations] of your Monte Carlo estimate.

Each of the 10000 simulations should use just *one*  $\mathbb{R}^2$ -valued random vector  $\mathbf{Z}$  of simulated independent standard normal random variables.

- (c) Use 10000 antithetic pairs  $(\mathbf{Z}, -\mathbf{Z})$  to estimate  $C$ , together with a standard error (as given in L5.28).

Consider the “geometric basket”  $G := (S^{[1]}S^{[2]})^{1/2}$ .

- (d) The random variable  $\log G_T$  is normally distributed (because it’s a linear transformation of a multivariate normal vector). Show that  $\log G_T$  has expectation

$$\frac{1}{2} \log(S_0^{[1]}S_0^{[2]}) + \left( r - \frac{\sigma_{[1]}^2 + \sigma_{[2]}^2}{4} \right) T$$

and variance

$$\frac{\sigma_{[1]}^2 + 2\rho\sigma_{[1]}\sigma_{[2]} + \sigma_{[2]}^2}{4} T.$$

- (e) Let  $C^G$  be the time-0 price of a geometric basket option paying  $(G_T - K)^+$  at time  $T$ .

Express  $C^G$  in terms of the function  $C^{BS}$  defined in FINM 33000 L6.3-6.4. Specifically, fill in the blanks:

$$C^G = C^{BS}(\_, 0, K, T, \_, r, \_).$$

Your answer should be a general formula, in which you have not substituted 0.8 for  $\rho$ , etc. (You may *also* do the substitutions, but don't neglect the general formula).

- (f) Using a geometric basket option as a control variate, run  $M = 10000$  Monte Carlo simulations to estimate  $C$ , together with a standard error. Use the control variate estimate  $\hat{C}_M^{cv, \hat{\beta}}$  from L6.6 or L6.7. Use the (asymptotically valid) standard error  $\hat{\sigma}_M^{cv, \hat{\beta}} / \sqrt{M}$ .

Use `hw6.ipynb`.

## Problem 2

Let the bank account and non-dividend paying stock have risk-neutral dynamics

$$\begin{aligned} dB_t &= rB_t dt & B_0 &= 1 \\ dS_t &= rS_t dt + \sigma S_t dW_t & S_0 &> 0 \end{aligned}$$

where  $\sigma > 0$  and  $W$  is a  $\mathbb{P}$ -Brownian motion.

Consider a  $K$ -strike  $T$ -expiry vanilla call option, and let  $C$  denote its time-0 price.

- (a) Let  $S_0 = 100$ ,  $\sigma = 0.2$ ,  $r = 0.02$ ,  $K = 150$ ,  $T = 1$ .

Run 100000 ordinary Monte Carlo simulations to estimate  $C$ , together with a standard error.

- (b) Suppose that we sample from a new probability measure  $\mathbb{P}^*$ , under which  $W$  now has constant drift  $\lambda$  instead of drift 0. Thus  $W_t = W_t^* + \lambda t$  where  $W^*$  is a standard  $\mathbb{P}^*$ -BM.

Find the  $\mathbb{P}^*$ -expectation  $\mathbb{E}^* S_T$  in terms of  $S_0$ ,  $r$ ,  $\sigma$ ,  $\lambda$ , and  $T$ .

Calculate  $\lambda$  such that  $\mathbb{E}^* S_T = 165$ .

(Why did we choose 165? The picture in L6.16 shows that the optimal distribution from which to sample will have a mean that is greater than the strike  $K$ . So let's choose 10% higher than  $K$ . This will not be optimal, but we expect that it will be an improvement over ordinary Monte Carlo. There are more systematic ways to determine a reasonable drift adjustment, not utilized here.)

- (c) Run 100000 importance sampling simulations, using the specific drift adjustment calculated in (b), to estimate  $C$ , together with a standard error. Be aware that your zero-mean normal random draws, here, simulate increments of  $W^*$  not  $W$ .

Each simulation should require only one number to be generated by `randn`.

Use `hw6.ipynb`.

Comment: of course, we do not need Monte Carlo to price a call under GBM. However, suppose you wanted to price a deep OTM option under an intractable *stochastic volatility* model, using importance sampling. You could still use a similar approach.