FINM 32000: Homework 5

Due Thursday February 25, 2021 at 11:59pm

Problem 1

Let r be the constant interest rate. Let $0 < T_1 < T_2$.

(a) Let F_t be the time-t forward price for T_2-delivery of some arbitrary underlying S, not necessarily tradeable. By definition of *forward price*, a forward contract paying $S_{T_2} - F_t$ at time T_2 has time-t value 0.

Let f_t be the time-t value of a T_2-forward contract on the same underlying, but with some delivery price K (not necessarily equal to F_t).

Express f_t in terms of K and F_t and a discount factor.

Hint: consider a portfolio long one (K, T_2)-forward contract, and short one (F_t, T_2)-forward contract. The portfolio's time-t value can be expressed in two ways, so the two expressions must be equal.

(b) If S is a *stock* paying no dividends, the forward price must be $F_t = S_t e^{r(T_2-t)}$; otherwise, arbitrage would exist.

> If, say, $F_t > S_t e^{r(T_2-t)}$, then arbitrage would exist: at time t, borrow S_t dollars, buy the stock, and short the forward (with delivery price F_t and time-t value 0). At time T_2, deliver the stock, and receive F_t, which is more than enough to cover your accumulated debt of $S_t e^{r(T_2-t)}$ dollars.

However, if S is the spot price of a barrel of crude oil (so, for all t, the time-t price for time-t delivery is S_t per barrel), then this argument fails. Explain briefly (one or two sentences, no math) why *this specific arbitrage* does not apply to crude oil. Vague generalities will not suffice. You must pinpoint precisely, in the specific quote above, why we cannot simply replace "stock" with "crude oil".

Hint: Consider practical complications.

So we need more assumptions to relate F_t and S_t (here and in (c,d,e,f,g), the S denotes spot crude oil, and F_t denotes the time-t forward price for T_2-delivery crude oil). One approach is to model the risk-neutral dynamics of S. Under risk-neutral measure, assume that S satisfies

$$\begin{aligned} S_t &= \exp(X_t) \\ \mathrm{d}X_t &= \kappa(\alpha - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t. \end{aligned}$$

Then, since r is constant and $\mathbb{E}_t(e^{-r(T_2-t)}(S_{T_2} - F_t))$ must be 0, one can calculate

$$F_t = \mathbb{E}_t(S_{T_2}) = \exp\left[e^{-\kappa(T_2-t)}\log S_t + \left(1 - e^{-\kappa(T_2-t)}\right)\alpha + \frac{\sigma^2}{4\kappa}\left(1 - e^{-2\kappa(T_2-t)}\right)\right],$$

where $\mathbb{E}_t$ is time-t conditional expectation. Suppose $\kappa = 0.472$, $\alpha = 4.4$, $\sigma = 0.368$, $r = 0.05$, and the time-0 spot price is $S_0 = 106.9$.

Let C be the time-0 price of a K-strike T_1-expiry European call on F. So this call pays $(F_{T_1} - K)^+$. Let the call option have strike $K = 103.2$ and expiration $T_1 = 0.5$. Let the forward mature at $T_2 = 0.75$. See `hw5p1.ipynb`

(c) Estimate $C(S_0)$ using Monte Carlo simulation of S with 100 timesteps on $[0, T_1]$. Choose the number of paths large enough that the standard error [the sample standard deviation, divided by the square root of the number of paths] is less than 0.05. Report the standard error. Don't use any variance reduction technique.

(d) Estimate $\partial C/\partial S$ by using Monte Carlo simulation to calculate $(C(S_0 + 0.01) - C(S_0))/0.01$. For the $C(S_0+0.01)$ calculation, *reuse* the same normal random variables which you generated for the $C(S_0)$ calculation. (Do not re-*generate* random variables to compute $C(S_0 + 0.01)$)

(e) Calculate analytically $\partial f_0/\partial S$, where f_0 is the time-0 value of a position long one forward contract on a barrel of crude oil, with maturity T_2 and some fixed delivery price K.

(f) Suppose you want to hedge a position short one call (so your hedge portfolio should replicate a position long one call), by continuously rebalancing a position in T_2-maturity forward contracts. Your hedge portfolio at time 0 should be long how many forward contracts? Your final answer should be a number.

The delivery price K of the forward contracts is irrelevant to the answer here; it would affect only how many units of the bank account to carry in the portfolio (which I am not asking you to compute).

(g) Consider the following "purchase agreement" contract. The holder of this contract receives time-T_2 delivery of θ barrels of crude oil, and pays, at time T_2, a delivery price of K dollars per barrel. The θ is chosen at time T_1 by the holder of the purchase agreement, subject to the restriction that $4000 \le \theta \le 5000$; in particular, $\theta = 0$ is not a valid choice, because the contract is a commitment to purchase at least 4000 barrels. Using your answer to (c), without running any new simulations, find the time-0 value of this contract.

Here K, T_1, T_2 have the same values as on the previous page.

Hint: Assume the holder acts optimally; thus θ is either 4000 or 5000, depending on F_{T_1}.

Problem 2

Complete the coding of `hw5p2.ipynb`. Here the objective is not to price an option, but rather to simulate the distribution (under physical measure) of the profit or loss (abbreviated P&L or PNL) obtained by "gamma scalping". Assume zero interest rates: $r = 0$ in all parts of this problem.

Assume that, under physical probability measure, a non-dividend paying stock S has dynamics

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t, \qquad S_0 = 100$$

The σ_t is stochastic but let us assume it is independent of the Brownian motion W. We will be taking examples of simple realizations of σ that do not depend on t.

Assume that at all times $t \leq T$, the time-t market price of a European call option with strike K and expiry T is given by the Black-Scholes formula with implied volatility $\sigma_{\text{imp}} = 30\%$.

Consider the following trading strategy. At time 0 we go long 1 call option, and we delta hedge it by trading stock discretely in time. Specifically, for some $\Delta t > 0$ and some integer $N > 0$, such that $N(\Delta t) < T$, we rebalance the portfolio discretely in time by buying or selling stock at times

$$0, \Delta t, 2\Delta t, \ldots, (N-1)\Delta t,$$

transacting whatever quantity is needed, to make the portfolio delta-neutral (meaning, the combined delta of all portfolio holdings is to be zero) immediately after the trade. In other words, the number of shares held in your portfolio in the time period starting at time $(n-1)\Delta t$ and ending at time $n\Delta t$ is the negative of the delta of the option at time $(n-1)\Delta t$, for each $n = 1, \ldots, N$. To be precise, delta here means the Black-Scholes delta, using the market's implied volatility σ_{imp}, and the (mark-to-market) PNL of the stock for that particular period is, as usual, quantity times change in price.

Finally, at time $N\Delta t$, we exit the trade by closing all positions (thus, selling the option, and buying stock to cover the short stock position). Therefore the option PNL from time 0 to time T equals the change in the option price from time 0 to time T; the stock PNL from time 0 to time T is the sum, from $n = 1$ to $n = N$, of the nth period stock PNL; and the PNL of the full portfolio is the sum of the options PNL and stock PNL .

The cash (bank account) holdings are whatever they need to be, in order for the portfolio to self-finance. Because $r = 0$, we don't need to explicitly keep track of this cash amount.

Let $\Delta t = 1/252$ to rebalance once per trading day, let $N = 20$ to run the strategy for 20 trading days. Let $K = 100$ and $T = 25/252$. Assume that options are trading at a market implied volatility of $\sigma_{\text{imp}} = 30\%$ in all cases, and in part (a) assume $\mu = 0.05$ (unlike an *option pricing* simulation, which would use drift r, the purpose of our simulations here is to display the physical distribution of hedging error, therefore the simulations here use physical drift μ).

(a) Estimate, using 10000 paths, the expecion of the total PNL from time 0 to time T from this gamma scalping strategy, if the realized volatility σ turns out to be 40%. Estimate the expectation of total PNL if the realized volatility σ turns out to be only 20%.

(b) Suppose that instead of physical drift $\mu = 0.05$, we simulate the paths using physical drift $\mu = -0.05$. How much does this change your answers from part [(a)]? Explain briefly why changing μ does / does not have a big impact.

Again, we are assuming that the market prices of options have implied volatility $\sigma_{\text{imp}} = 30\%$ in *all* cases in (a) and (b). These cases in (a) and (b) are examining various scenarios for the *realized* volatility and drift of the S paths, holding fixed the implied volatility of the options.