FINM 32000: Homework 3

Due Wednesday February 10, 2021 at 11:59pm

Problem 1

The short rate (the instantaneous spot rate of interest) follows the process

$$\mathrm{d}r_t = \alpha(r_t, t)\mathrm{d}t + \beta(r_t, t)\mathrm{d}W_t$$

where W_t is Brownian motion under risk-neutral probabilities.

(a) Consider an interest rate derivative which pays $F(r_T)$ at time T, and has an arbitrage-free price $C_t = C(r_t, t)$. Apply Ito's rule to find the risk-neutral dynamics of C. Use this to derive (without proving) a PDE for $C(r, t)$.

Suppose, in particular, that the risk-neutral dynamics of r are given by

$$\mathrm{d}r_t = \kappa(\theta - r_t)\mathrm{d}t + \sigma\mathrm{d}W_t,$$

with parameters $\kappa = 3$, $\theta = 0.05$, $\sigma = 0.03$. Consider a $T = 5$-year discount bond (a zero-coupon bond which pays 1 at maturity).

(b) Write code to find the time-0 price of bond by applying a standard central-difference explicit finite difference scheme to the PDE in (a). (Therefore C_n^j will be determined by C_{n+1}^{j+1}, C_{n+1}^{j}, and C_{n+1}^{j-1}.)

Complete the code in the file `hw3.ipynb`.

(c) Also write code to price the bond using an explicit *upwind* approximation to $\frac{\partial C}{\partial r}$ instead of the usual central difference. Specifically, for those r_j such that $\kappa(\theta - r_j) \geqslant 0$, approximate $\frac{\partial C}{\partial r}(r_j, t_{n+1})$ using the points C_{n+1}^{j+1} and C_{n+1}^{j}. For those r_j such that $\kappa(\theta - r_j) < 0$, use the points C_{n+1}^{j} and C_{n+1}^{j-1}. (For $\frac{\partial^2 C}{\partial r^2}$, use the usual approximation).

In (b) and (c), to approximate the PDE's rC term, use the values of r and C at node (n, j). (As we said in class, node $(n+1, j)$ would also be a natural choice, but let's use n instead of $n+1$). At the grid's upper and lower boundaries r_{max} and r_{min}, impose for all $t < T$ the "linearity" boundary conditions

$$C(r_{max}, t) = 2C(r_{max} - \Delta r, t) - C(r_{max} - 2\Delta r, t)$$
$$C(r_{min}, t) = 2C(r_{min} + \Delta r, t) - C(r_{min} + 2\Delta r, t)$$

(This technique can help in some situations where it is not obvious what boundary conditions to use.) Thus, in each "column" of the grid, first solve for C in the interior nodes; then deal with the top and bottom nodes.

Now let us do some comparison of the central-difference and upwind schemes.

(d) Suppose $f : \mathbb{R} \to \mathbb{R}$ is smooth in some open neighborhood of x. Show that as $h \downarrow 0$,

$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = O(h) \quad \text{and} \quad \left| \frac{f(x+h) - f(x-h)}{2h} - f'(x) \right| = O(h^2).$$

(e) For all part (e) calculations: Use the grid spacings $\Delta r = 0.01$ and $\Delta t = 0.01$. Use $r_{max} = 0.35$ and $r_{min} = -0.25$ for the upper and lower boundaries of the grid, respectively.

Run a central-difference calculation and an upwind calculation of the bond price for $r_0 = 0.10$. Which is more accurate? The more accurate of the two solutions should agree, to three significant digits, with the true bond price in this model: 0.7661. The less accurate of the two solutions will be *very* inaccurate.

(f) Based on your answers to (d) and (e), insert either "greater" or "less" in each blank space in the following rule-of-thumb. No explanation necessary.

> Ignoring stability issues and considering only consistency (i.e. "truncation error," also known as "local discretization error"), the upwind explicit scheme, which uses one-sided spatial differences, discretizes the PDE with ____ accuracy than the standard explicit scheme, which uses central spatial differences.
>
> However, to actually guarantee convergence, the grid spacing must satisfy certain stability constraints. In a PDE exhibiting strong drift, we have just seen that these constraints may allow the upwind scheme ____ freedom in choosing grid spacing, compared to the standard scheme.

(g) Find the yield-to-maturity of a 5-year discount bond, in the case that $r_0 = 0.12$, and in the case that $r_0 = 0.02$. (The "good" results from part (e) may be used here. The "bad" results should not be used, unless you want to fix them by modifying the grid spacings).

Why, intuitively, is the yield for $r_0 = 0.12$ smaller than 0.12, whereas the yield for $r_0 = 0.02$ is greater than 0.02?

Comment: Under these short-rate dynamics, there do exist analytic pricing formulas for bonds. So we do not need finite difference methods to value the simple payoff that we have here. But the finite difference scheme can be modified to handle contracts for which exact pricing formulas do not exist.

Problem 2, next page

Same weight as Problem 1

Problem 2

Assume that some unconditionally stable finite difference scheme satisfies an error bound

$$E := a(\Delta t)^p + b(\Delta x)^q$$

where $a, b, p, q > 0$ are positive constants. (For example, as we will see, the implicit scheme has $(p, q) = (1, 2)$, and the Crank-Nicolson scheme has $(p, q) = (2, 2)$.)

Assume that the FD scheme's computation time is proportional to the number of points in the grid (which is true of all the usual FD methods). Therefore the FD scheme's computation time is Constant$/((\Delta t)(\Delta x))$.

Consider the question of how to choose $(\Delta t, \Delta x)$ to minimize the error bound, subject to a constant computation-time constraint (or "budget"). Therefore, the problem is to minimize E, subject to the constraint

$$(\Delta t)(\Delta x) = c$$

where $c > 0$ is some constant. You may assume that the minimum is attained at a point where the usual first-order constrained optimization condition holds.

(a) Show that the optimal $(\Delta t, \Delta x)$ satisfy

$$\Delta x = h(\Delta t)^{p/q}$$

where h is some constant. Express h in terms of any or all of a, b, p, q, c.

(b) How much does the optimized error bound E change, per unit of increase in the computational budget c? Your answer should be negative if E decreases, positive if E increases. Express your answer in terms of any or all of a, b, p, q, c, h. If you're not sure what to do, look up "shadow price" and "Lagrange" or "Lagrangian".