FINM 32000: Homework 4

Due Wednesday February 17, 2021 at 11:59pm

Problem 1 of 2

Let S be a futures price. Assume that under risk-neutral probability measure, S has *CEV* dynamics

$$\mathrm{d}S_t = \sigma S_t^{1+\alpha}\mathrm{d}W_t, \qquad S_0 = 100 \tag{1}$$

with constants σ, α. The superscript on S_t is an exponent (power). The interest rate on the bank account is r. The S here is a futures price, so it has drift coefficient 0. Using the wrong drift coefficient on S will lose credit. (But prices of European options on S still have drift coefficient r).

(a) Let $C(S_t, t)$ be the time-t no-arbitrage price of an European put on S, with strike K and expiry T. Write down a PDE, with terminal condition, for $C(S, t)$. Leave your answer in terms of r, σ, α, K, T.

(b) Let $r = 0.05$, $\sigma = 3$, $\alpha = -0.5$. Use Crank-Nicolson to find the time-0 price of an *American* put on S with strike $K = 100$ and expiry $T = 0.25$. Partial code is provided in `hw4.ipynb`.

You may use the boundary conditions implemented in `pricer_put_CEV_CrankNicolson`. At the low-S boundary, it assumes the put value equals intrinsic value (exercise value). At the high-S boundary, it approximates the put value as zero.

You may use the FD grid given in `hw4.ipynb`.

(c) Compute numerically the time-0 delta and gamma of the put in (b).

(d) Using exactly the same `pricer_put_CEV_CrankNicolson` function as in (b) – meaning that you can change the input passed into the function, but cannot change the function's code – find the time-0 price of the American put in (b), but assuming *Black-Scholes* dynamics for S with volatility 0.30 and interest rate 0.05 and $S_0 = 100$.

Problem 2 of 2

Suppose that a non-dividend-paying stock S follows Black-Scholes dynamics, with interest rate r and volatility σ.

(a) Given Δ where $0 < \Delta < 1$, and given $T > 0$, solve for the strike K such that a call option on S with strike K and expiry T has time-0 delta equal to Δ. Express your answer in terms of the inverse cdf N^{-1} of the normal distribution, and any or all of $\Delta, S_0, r, \sigma, T$.

(b) Let $S_0 = 300$ and $T = \frac{1}{12}$ and $\sigma = 0.4$ and $r = 0.01$. Calculate the strikes and the premiums (meaning, the prices) of a 25-delta call and a 75-delta call. This means that $\Delta = 0.25$ and $\Delta = 0.75$ respectively.

A way to get the inverse of the normal cdf in Python is: `from scipy.stats import norm` and then use `norm.ppf()`

If you spend S_0 dollars on stock, you can buy one share, and thereby obtain delta 1.

If you spend the same amount of S_0 dollars on, instead, an option of a given strike and expiry, then you can buy S_0/C_0 contracts, where C_0 is the price of 1 contract. Each contract has delta Δ, so you obtain a total delta of

$$\Delta \frac{S_0}{C_0}.$$

This is sometimes denoted by the Greek letter lambda, and sometimes called the "leverage" or "gearing" of the option. Another way to derive/explain this lambda is that it's the *percentage* change in the option price, per *percentage* change in the underlying; this can be described as the *elasticity* of the option.

(c) Calculate the lambdas of the two options in part (b). Which one gives you more leverage?

Comment (no response required from you): This gives another answer to the question of why at a given strike, the OTM option (call or put) is usually expected to be more liquid (more activity, narrower spreads) than the ITM option (put or call) at that strike; see the second point below:

- Popularity of option strategies involving OTM options (for example, buying OTM protective puts, and selling OTM covered calls, as you implemented in the IB TWS exercises last quarter)
- Popularity of using options to obtain a bullish or bearish exposure to the underlying. More leverage is obtained by using OTM than ITM options. Thus OTM options require less capital commitment to obtain a desired exposure. (To say it another way, OTM options provide more exposure for a given allocation of capital).
- ITM options mainly have higher absolute deltas than OTM options, so ITM options are more exposed (in dollar terms, not percentage terms) to the movements of the underlying. This greater risk is a reason for market-makers to quote wider spreads for the ITM option, compared to narrower spreads for the OTM option.

Note that this OTM vs. ITM comparision does not compare OTM/ITM to *ATM* options, which are particularly active.