Replication Prices in a Non Black-Scholes-Merton World

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Abstract

In this article, we will look at the effect of discrete hedging and other violations of Black-Scholes and Merton framework. The replication theory in finance stipulates that the fair value of a financial instrument is equal to another financial device with an identical payoff under all possible future scenarios. In reality, we cannot play according to theoretical rules for multiple reasons. We will review the theory of Black-Scholes-Merton to understand all the assumptions made to get their results. We will then review different hedging strategies and outcomes when hedging with different volatilities. Afterward, we will derive some lower bound errors of the replication prices mathematically when violating some assumptions of the BSM framework and investigate another source of hedge error when the stock dynamics follows a Jump Diffusion Process instead of a Geometric Brownian Motion. Finally, we will look at some simulation results to observe the effects on replication prices when hedging under a more realistic setup than BSM. When combining multiple violations, we follow that many options should not be priced under the replication scheme. For many OTM options, the ratio of the expected price to the standard deviation of that price is much too large to have any statistical significance. For these regions of the surface, the noise-to-signal ratio is too large. A practitioner that would like to historically price options under the $\mathbb{P}$ measure from market data should be aware of the differences of the surface where the replication price might not be reliable. It may be more prudent to keep a set of observations where the forecasts would be more reliable and extrapolate the values from known values for those observations that may prove to be too tricky to projections. In addition, a good volatility forecast, or a methodology to choose which volatility to compute the delta to be used in the hedging, is very useful to reduce the amount of noise and negative replication prices.

1 Introduction

The primary law of quantitative finance is the law of one price. In the absence of trade frictions and under conditions of free competition and price flexibility, identical goods sold in different locations must sell for the same price when prices are expressed in a common currency. This concept was used to build a theory around pricing an option by replicating via dynamic hedging of the underlying and investing the cash component in a bond at the risk-free rate. To derive the Black-Scholes Merton (BSM) [6] formula, we need to make several assumptions, namely:

- The movement of the underlying stock price is continuous, with constant volatility and no jumps (one-factor geometric Brownian motion).
- Traders can hedge continuously by taking on arbitrarily large long or short positions.
- No bid-ask spreads.
- No transaction costs.
- No forced unwinding of positions.

In this article, we will investigate the different causes and effects of violating the BSM framework's assumptions on an option's replication price. We will start by going over the derivation of the fair price of an option under BSM, will help build intuition in the model's assumptions and potential weaknesses. Afterward, we will go over an analytical approach to the hedging error when violating the rules of replication price. It will require us to make some assumptions that will give a lower bound of the replication price error. Finally, we'll run some numerical simulations to understand better the possible outcomes of trying to price an option via market data replication.

2 Black-Scholes-Merton Theory

Consider at time t a stock with an underlying price S a known constant volatility σ_S, and an expected return, μ_S, together with a risk's bond with price B that yields r, assumed constant through time. The stochastic evolution of the stock and bond prices is given by

$$dS = \mu_S S dt + \sigma_S S dW_t$$
$$dB = Brdt \tag{2.1}$$

where dW_t is the standard Weiner process. The dynamics of an option $V(t,S)$ can be given using Itô's Lemma. If we define the operator $\langle Q \rangle$ to be the Quadratic Variation of a process Q. From stochastic calculus, one can prove that the Quadratic Variation of the components of the stock dynamics

$$\langle dt, dW_t \rangle = \langle dt, dt \rangle = 0, \ \ \langle dW_t, dW_t \rangle = dt \tag{2.2}$$

This leads to being able to calculate the quadratic variation of the stock price

$$\langle S \rangle = (\sigma_S S)^2 dt \tag{2.3}$$

The evolution of V [4] can be described as

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\frac{\partial^2 S}{\partial S^2}(\sigma_S S)^2 dt \\ &= \left\{ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu_S S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(\sigma_S S)^2 \right\} dt + \frac{\partial V}{\partial S}\sigma_S S dW_t \\ &= \mu_V V dt + \sigma_V V dW_t \end{aligned} \tag{2.4}$$

We can create an instantaneously riskless portfolio by combining positions in S and V to cancel their risk. Let's define $\pi = \alpha S + V$, where α is the number of shares of stock required to hedge the risk of the option at time t. Then

$$\begin{aligned} d\pi &= \alpha(\mu_S S dt + \sigma_S S dW_t) + (\mu_V V dt + \sigma_V V dW_t) \\ &= (\alpha\mu_S S + \mu_V V)\, dt + (\alpha\sigma_S S + \sigma_V V)\, dW_t \end{aligned} \tag{2.5}$$

For the portfolio to be instantaneously riskless, the coefficient for stochastic term dW_t must be zero. We require

$$\begin{aligned} \alpha\sigma_S S + \sigma_V V &= 0 \\ \alpha &= -\frac{\sigma_V V}{\sigma_S S} \end{aligned} \tag{2.6}$$

in which case

$$d\pi = \alpha(\mu_S S dt + \sigma_S S dW_t) = \pi r dt \tag{2.7}$$

by the law of one price since it must earn riskless rate r. This leads to number of shares being equal to

$$\alpha = -\frac{V(\mu_V - r)}{S(\mu_S - r)} \tag{2.8}$$

Substituting Equation 2.8 into 2.6 and with 2.4 leads to the BSM equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 V}{dS^2} = rV \tag{2.9}$$

3 Hedging Strategies

3.1 Hedging under BSM framework

Consider an option V on an underlying stock S. If the option is hedged at discrete points in time, $t_0, t_1, t_2, \ldots, t_n$ such that $\delta t = t_i - t_{i-1}$. We'll simplify the notation $V_i = V(S_i, t_i)$ which is the market price of the option at time t_i when the stock price is S_i, and $\Delta_i = \Delta(S_i, t_i)$ denotes the number of shares of the stock S that is long/short at the start of each period i. Any cash received is invested at the riskless rate r, and any money borrowed is funded at the same arbitrary; it merely defines the hedged trading strategy [1].

We begin holding the option worth V_0 at time t_0 and taking the appropriate hedge in the underlying shares, which equals to $\Delta_0 S_0$. Cash is exchanged whenever the stock is shorted or repurchased, and the net cash balance grows at the riskless rate of r during each period. The initial cash position equals the value put into getting the shares. After one period, the value of the cash position is equal to $\Delta_0 S_0 e^{r\delta t}$. The stock price moved from S_0 to S_1, and the corresponding delta is now Δ_1. To rebalance the hedge, we must buy or sell $(\Delta_1 - \Delta_0)S_1$ shares. The value of the option went from $V_0 \rightarrow V_1$. The total cash flow so far is equal to

$$V_1 - \Delta_1 S_1 + (\Delta_1 - \Delta_0)S_1 + \Delta_0 S_0 e^{r\delta t} \tag{3.1}$$

After n steps, to a final value we get

$$V_n - \Delta_n S_n + \Delta_0 S_0 e^{nr\delta t} + \sum_{j=1}^{n} (\Delta_j - \Delta_{j-1}) S_j e^{(n-j)r\delta t} \tag{3.2}$$

In the limit, when $n \rightarrow \infty$ with $n\delta t = t_n - t_0 \equiv T$ remaining fixed, we can replace to integral

$$V_T - \Delta_T S_T + \Delta_0 S_0 e^{rT} + \int_0^T e^{r(T-t)} S_t [d\Delta_t]_b \tag{3.3}$$

The subscript b at the end of the formula denotes a backward Itô integral in which the increment $d\Delta_t$ is the infinitesimal change in Δ that occurred just before the stock price S was evaluated, in contrast to the usual forward Itô integral where $d\Delta$ occurs after S [2].

In the idealized BSM case, the option is perfectly hedged at every instance and therefore the final P&L is independent of the stock price path. Because the instantaneously hedged option is riskless, the hedging strategy replicates a riskless bond and therefore, by the law of one price, must have same final value. In that case V_0 is given by

$$V_0 e^{rT} = V_T - \Delta_T S_T + \Delta_0 S_0 e^{rT} + \int_0^T e^{r(T-t)} S_t [d\Delta_t]_b \tag{3.4}$$

You can integrate the last term in equation 3.4 by parts using the following relationship

$$e^{r(T-t)}S_t[d\Delta_t]_b = d[e^{r(T-t)}S_t\Delta_t] + e^{r(T-t)}\Delta_t r S_t dt - e^{r(T-t)}\Delta_t dS_t \tag{3.5}$$

to obtain

$$V_0 = V_T e^{-rT} - \int_0^T \Delta(S_t,t)[dS_t - S_t r dt]e^{rt} \tag{3.6}$$

Suppose you hedge perfectly and continuously with the BSM hedge ratio Δ_{BS} that exactly cancels out the exposure of the option to the stock. In that case, the hedged portfolio is riskless at every point in time and, therefore, independent of the path the stock takes to expiration. So far, we have not assumed the dynamics of stock movement. Now, assume that the underlying stock evolves according to Geometric Brownian motion and that stock drift is equal to the actual riskless rate r, so that $dS - Srdt = \sigma S dW_t$. Then

$$V_0 = V_T e^{-rT} - \int_0^T \Delta(S_t,t)\sigma S_t e^{rt} dW_t \tag{3.7}$$

The initial value of the call is path dependent unless the hedge ratio $\Delta = \Delta_{BS}$. However, suppose we take the call's expected value over all stochastic innovation dW_t of the stock price even when $\Delta \neq \Delta_{BS}$. Then

$$\mathbb{E}[V_0] = \mathbb{E}[V_T]e^{-rT} \tag{3.8}$$

Since the expected value of each increment, dW_t is zero for a Weiner process. We can conclude that provided that the stock undergoes geometric Brownian motion with drift r, irrespective of what hedge ratio Δ is used, no matter what hedging formula you use for delta, and even if you don't hedge at all, the expected value of the option is given by the BSM formula.

3.2 P&L Hedging with Realized Volatility

Let's assume that we have a crystal ball and know in advance what the realized volatility σ_R will be, and it is less than the implied volatility σ_I. We can make money as an option trade by selling the option with an implied volatility of σ_I and continuously hedging the underlying with Δ_R, calculated from σ_R. The hedged portfolio at any time t is given by

$$\pi(I,R) = \Delta_R S - V_I \tag{3.9}$$

Clearly, the present value of the total P&L generated over the life of the option from this trade should be [2]

$$\mathbb{PV}\left[\text{P\&L}(I,R)\right] = V_I - V_R \tag{3.10}$$

Assume that the stock price, S, evolves with drift μ and volatility σ_R so that

$$dS = \mu S dt + \sigma_R S dW_t \tag{3.11}$$

Let's examine the incremental profit $d\text{P\&L}(I,R)$ generated by this hedging strategy during a subsequent time interval dt when the stock price change by dS. We get

$$d\text{P\&L}(I,R) = \Delta_R dS - dV_I - (\Delta_R S - V_I)rdt \tag{3.12}$$

The first term is the increase in the value of the long position in the stock; the second term is the decrease in the value of the a short position in the option, the third term represents the interest on the cost of borrowing $(\Delta_R S - V_I)$ used to set up the initial hedge portfolio.

We can regroup implied and realized terms to obtain

$$d\text{P\&L}(I,R) = \Delta_R(dS - rSdt) + rV_I dt - dV_I \tag{3.13}$$

With the riskless hedging strategy, the increase in value of the hedge portfolio should be no different from the interest earned on the position at the riskless rate, so that $d\text{P\&L}(R,R) = 0$.

$$d\text{P\&L}(R,R) = dV_R - V_R r dt - \Delta_R[dS - rSdt] \tag{3.14}$$

We can re-write equation 3.13 to get

$$d\text{P\&L}(I,R) = dV_R - dV_I - (V_R - V_I)rdt \tag{3.15}$$

Using the product rule we get

$$d\text{P\&L}(I,R) = e^{rt} d\left[e^{-rt}(V_R - V_I)\right] \tag{3.16}$$

We can obtain the present value of the entire P&L of our hedging strategy to obtain

$$\begin{aligned}
\mathbb{PV}\left[\text{P\&L}(I,R)\right] &= e^{rT}\int_0^T d\left[e^{-rt}(V_R - V_I)\right] &(3.17)\\
&= e^{rT}\left[e^{-rt}(V_R - V_I)\right]_0^T &(3.18)\\
&= V_I - V_R &(3.19)
\end{aligned}$$

Provided that we know the future realized volatility and provided that we can hedge continuously, the final P&L at the expiration of the option is known and deterministic and is equal to the difference between the value of the option based on implied volatility and the value of the option based on implied volatility.

Although the terminal payoff is deterministic, we will show that the path to the payoff is stochastic. Recalling equation 3.12, we can use Itô's Lemma to expand dV_I and use the BSM equation to get

$$d\text{P\&L}(I,R) = -\left[\frac{\partial V_I}{\partial t}dt + \frac{\partial V_I}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V_I}{\partial S^2}S^2\sigma_R^2 dt\right] + \Delta_R dS - (\Delta_R S - V_I)rdt \tag{3.20}$$

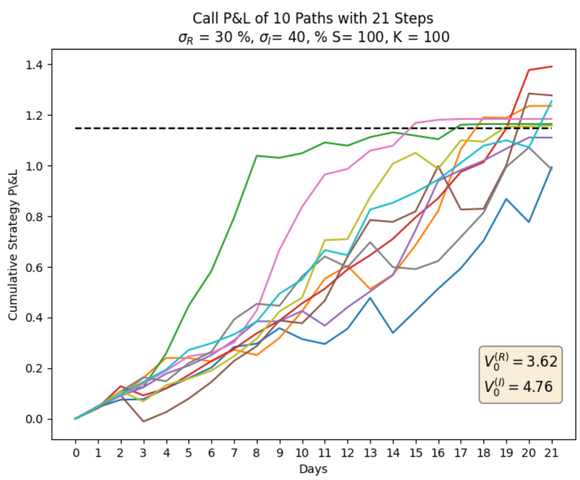
The value of V_I hedge at σ_I can be written as

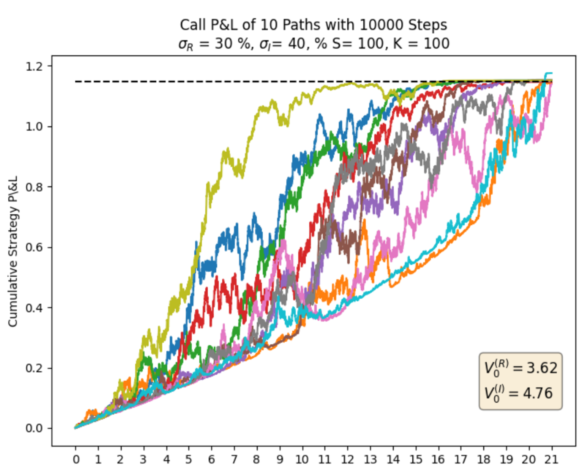
$$\frac{\partial V_I}{\partial t} = -\frac{1}{2}\frac{\partial^2 V_I}{\partial S^2} S^2 \sigma_I^2 + rV_I - rS\frac{\partial V_I}{\partial S} \tag{3.21}$$

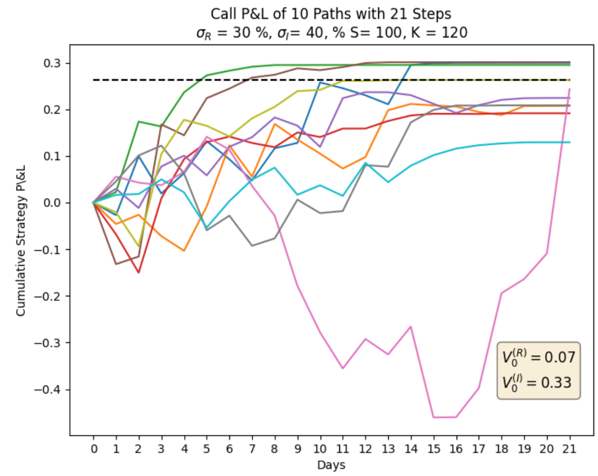
Putting 3.21 into 3.20 leads to

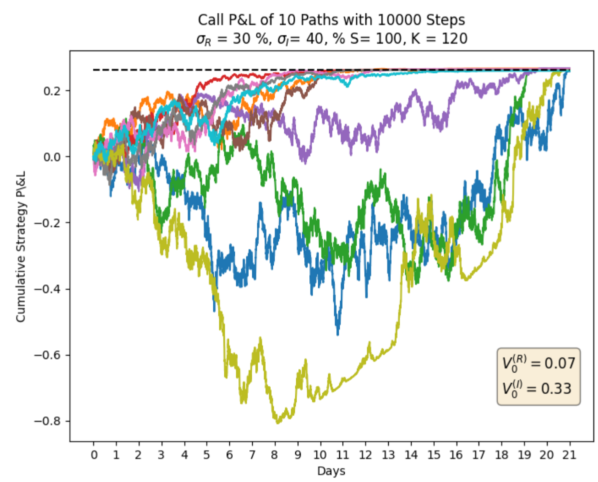
$$\begin{aligned} d\text{P\&L}(I,R) &= \frac{1}{2}\frac{\partial^2 V_I}{\partial S^2} S^2 (\sigma_I^2 - \sigma_R^2) dt \\ &\quad + \left(\frac{\partial V_R}{\partial S} - \frac{\partial V_I}{\partial S}\right) [(\mu - r) S dt + \sigma_R S dW_t] \end{aligned} \tag{3.22}$$

Thus, even though the final P&L is deterministic, the increments in the P&L when you value at I and hedge at R, have a random component dW_t proportional to the mismatch between Δ_R and Δ_I.







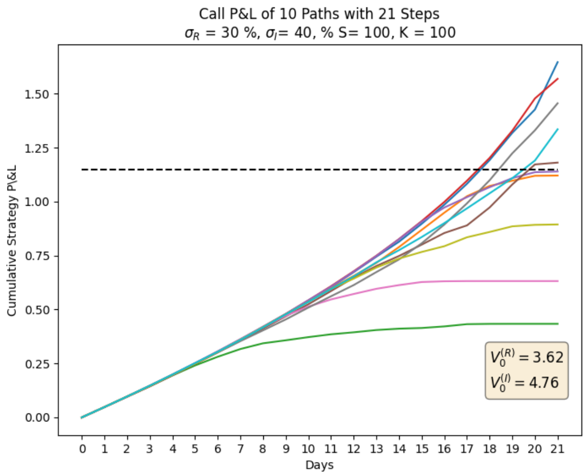


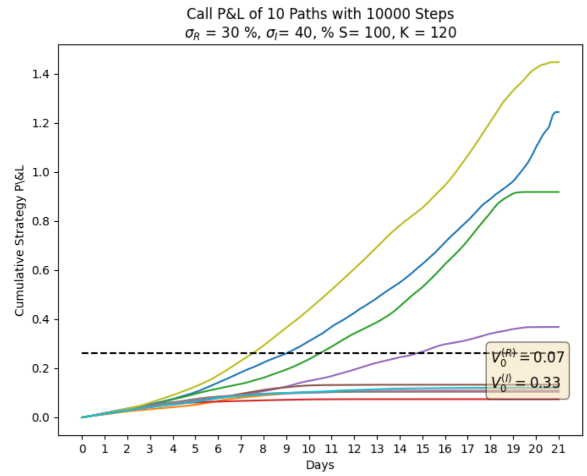
3.3 P&L Hedging with Implied Volatility

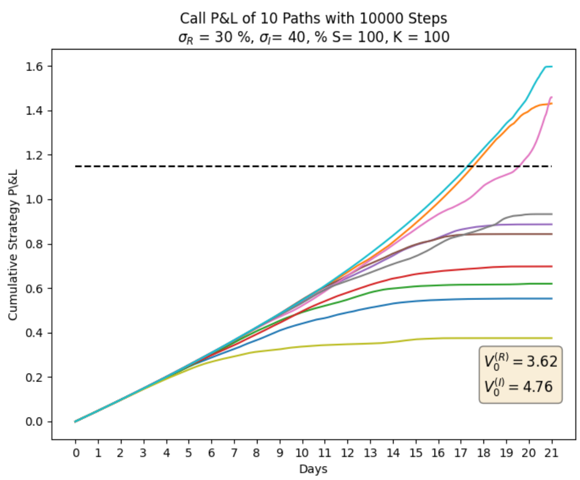
If, in this case, we hedged with volatility σ_I, from equation 3.22. We can replace delta from $\Delta_R \to \Delta_I$ so the stochastic term is now 0, and we have. [3]

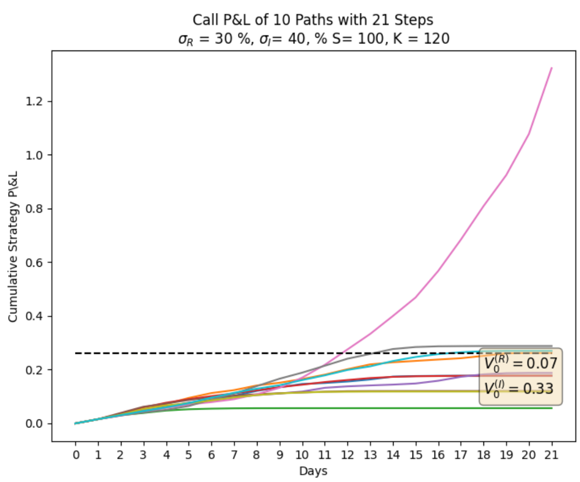
$$d\text{P\&L}(I,I) = \frac{1}{2}\frac{\partial^2 V_I}{\partial S^2} S^2 (\sigma_I^2 - \sigma_R^2) dt \tag{3.23}$$

The change in the P&L is determined by the difference in realized to implied variance, which we can treat as a constant over dt. But, even if we know σ_R and σ_I, $\frac{\partial^2 V_I}{\partial S^2} S^2$ will change over the life of the option as the time to expiration decreases and S changes. This will lead to the final value of the P&L that is path dependent.









4 Hedging Error Analytically

Let's assume that the implied and realized volatility are different. Again we will assume that $\sigma_R < \sigma_I$. Suppose that over a discrete time step dt the price of stock at time t evolves according

$$dS_t = \mu S_t dt + \sigma S\sqrt{dt}Z_t \tag{4.1}$$

where $Z \sim N(0,1)$ is normally distributed with mean zero and standard deviation 1. The value of the instantaneously delta-hedged option portfolio given by

$$\pi = \frac{\partial V_R}{\partial S} S - V_I \tag{4.2}$$

where the option V_I is valued at the implied volatility and hedged at the realized volatility σ_R. The hedging error accumulated over a discrete time dt owing to the mismatch between a continuous hedge ratio and the discrete time step is given by

$$\begin{aligned}
HE_{dt} &= \pi + d\pi - \pi e^{rdt} \\
&\approx d\pi - r\pi dt \\
&\approx \left[\frac{\partial V_R}{\partial S} dS - \frac{\partial V_I}{\partial t} dt - \frac{\partial V_I}{\partial S} dS - \frac{\partial^2 V_I}{\partial S^2}\sigma_I^2 S^2 Z^2 dt\right] \\
&- rdt\left[\frac{\partial V_R}{\partial S} S - V_I\right] \\
&\approx \left[-\frac{\partial V_I}{\partial t} dt - \frac{\partial^2 V_I}{\partial S^2}\sigma_I^2 S^2 Z^2 dt - rdt\left(\frac{\partial V_R}{\partial S} S - V_I\right)\right] \\
&+ \left(\frac{\partial V_R}{\partial S} - \frac{\partial V_I}{\partial S}\right) dS
\end{aligned} \tag{4.3}$$

From equation 2.9, we know that we can express V_I using BSM formula

$$rV_I = \frac{\partial V_I}{\partial t} + \frac{1}{2}\frac{\partial^2 V_I}{\partial S^2}\sigma_I^2 S^2 + \frac{\partial V_I}{\partial S} rS \tag{4.4}$$

Substitutting into equation 4.3, we obtain

$$\begin{aligned}
HE_{dt} &\approx \left[-\frac{\partial V_I}{\partial t}dt - \frac{\partial^2 V_I}{\partial S^2}\sigma_I^2 S^2 Z^2 dt\right] \\
&\quad - dt\left(\frac{\partial V_R}{\partial S}rS - \frac{\partial V_I}{\partial t} - \frac{1}{2}\frac{\partial^2 V_I}{\partial S^2}\sigma_I^2 S^2 - \frac{\partial V_I}{\partial S}rS\right) \\
&\quad + \left(\frac{\partial V_R}{\partial S} - \frac{\partial V_I}{\partial S}\right)dS \\
&\approx \frac{1}{2}\frac{\partial^2 V_I}{\partial S^2}\sigma_I^2 S^2(1-Z^2)dt + rS\left(\frac{\partial V_I}{\partial S} - \frac{\partial V_R}{\partial S}\right)dt \\
&\quad + \left(\frac{\partial V_R}{\partial S} - \frac{\partial V_I}{\partial S}\right)dS \\
&\approx \frac{1}{2}\frac{\partial^2 V_I}{\partial S^2}\sigma_I^2 S^2(1-Z^2)dt + rS\left(\frac{\partial V_I}{\partial S} - \frac{\partial V_R}{\partial S}\right)dt \\
&\quad + \left(\frac{\partial V_R}{\partial S} - \frac{\partial V_I}{\partial S}\right)\left(\mu S dt + \sigma_R S Z\sqrt{dt}\right) \\
&\approx \frac{1}{2}\frac{\partial^2 V_I}{\partial S^2}\sigma_I^2 S^2(1-Z^2)dt \\
&\quad + \left(\frac{\partial V_R}{\partial S} - \frac{\partial V_I}{\partial S}\right)\left((\mu-r)S dt + \sigma_R S Z\sqrt{dt}\right)
\end{aligned} \tag{4.5}$$

If we hedge at the implied volatility, the last 2 terms would be zero. Another observation is that even if the volatilities were different if the drift of the stock were equal to the riskless rate, we would also have zero expected error. We'll revisit this later when making an approximation. Still, for now, the hedging error is a non-generalized χ^2 distribution with a drift component due to the difference of deltas from our two volatilities. Over n steps to expiration, the total Hedging Error is

$$\begin{aligned}
HE \approx \sum_{j=1}^{n}\Bigg\{&\frac{1}{2}\frac{\partial^2 V_I}{\partial S^2}_{(j)}\sigma_I^2 S_{(j)}^2(1-Z_{(j)}^2)dt \\
&+ \left(\frac{\partial V_R}{\partial S}_{(j)} - \frac{\partial V_I}{\partial S}_{(j)}\right)\left((\mu-r)S_{(j)}dt + \sigma_R S_{(j)}Z_{(j)}\sqrt{dt}\right)\Bigg\}
\end{aligned} \tag{4.6}$$

Because $Z \sim N(0,1)$ we know that $\mathbb{E}[Z]=0$ and $\mathbb{E}[Z^2]=1$, which allows us to compute the following expected hedge error when discretely hedging a geometric brownian motion stock with a known realized volatility different than the implied volatility

$$\mathbb{E}[HE] \approx \sum_{j=1}^{n}\mathbb{E}\left[\left(\frac{\partial V_R}{\partial S}_{(j)} - \frac{\partial V_I}{\partial S}_{(j)}\right)(\mu-r)S_{(j)}dt\right] \tag{4.7}$$

If we were to hedge with realized volatility, the expected value of the hedge error is 0, as we should expect. But the formula above proves that there is a non-zero expected error from hedging with implied volatility when the stock dynamics have a different volatility under a discrete hedging scheme.

Let's make our life more straightforward now. Let's assume we hedge with realized volatility but in a discrete world. We know the expected hedge error is 0, but what is the variance? The hedge error is now.

$$\begin{aligned}
HE_{dt} &\approx \frac{1}{2}\frac{\partial^2 V_I}{\partial S^2}\sigma_I^2 S^2(1-Z^2)dt \\
\sigma_{HE}^2 &= \mathbb{E}[HE^2] - \underbrace{\mathbb{E}[HE]^2}_{=0} \\
&\approx \mathbb{E}\left[\sum_{j=1}^{n}\frac{1}{4}\left[\frac{\partial^2 V}{\partial S^2}\right]_{(j)}^2 \sigma_I^4 S_{(j)}^4\left(Z_{(j)}^4 - 2Z_{(j)}^2 + 1\right)dt^2\right]
\end{aligned} \tag{4.8}$$

Because kurtosis of $\mathbb{E}[Z^4]=3$, we get

$$\sigma_{HE}^2 \approx \mathbb{E}\left[\sum_{j=1}^{n}\frac{1}{2}\left(\frac{\partial^2 V}{\partial S^2}_{(j)}\sigma_I^2 S_{(j)}^2\right)^2 dt^2\right] \tag{4.9}$$

For an at-the-money option, integration over the normal distribution of stock returns leads to [3]

$$\mathbb{E}\left[\left(\frac{\partial^2 V}{\partial S^2}_{(j)} S_{(j)}^2\right)^2\right] = S_{(0)}^4 \frac{\partial^2 V}{\partial S^2}_{(0)}^2\sqrt{\frac{T^2}{T^2 - t_{(j)}^2}} \tag{4.10}$$

where S_0 is the initial stock price at the start of hedging strategy.

$$\begin{aligned}
\sigma_{HE}^2 &\approx \sum_{j=1}^{n}\frac{1}{2}S_{(0)}^4\frac{\partial^2 V}{\partial S^2}_{(0)}^2\sqrt{\frac{T^2}{T^2 - t_{(j)}^2}}(\sigma_I^2 dt)^2 \\
&\approx \frac{1}{2}S_{(0)}^4\frac{\partial^2 V}{\partial S^2}_{(0)}^2(\sigma_I^2 dt)^2\frac{1}{dt}\int_{\tau}^{T}\sqrt{\frac{T^2}{T^2-\tau^2}}d\tau \\
&\approx S_{(0)}^4\frac{\partial^2 V}{\partial S^2}_{(0)}^2(\sigma_I^2 dt)^2\frac{\pi(T-t)}{4dt} \\
&\approx \frac{\pi}{4}n\left(S_{(0)}^2\frac{\partial^2 V}{\partial S^2}_{(0)}\sigma_I^2 dt\right)^2
\end{aligned} \tag{4.11}$$

where $n := \frac{T-t}{dt}$. From BSM we know that the Gamma-Vega relationship is

$$S_{(0)}^2\frac{\partial^2 V}{\partial S^2}_{(0)}^2 = \frac{1}{\sigma(T-t)}\frac{\partial V}{\partial \sigma} \tag{4.12}$$

so that we can write

$$\begin{aligned}
\sigma_{HE}^2 &\approx \frac{\pi}{4}n\left(\frac{1}{\sigma(T-t)}\frac{\partial V}{\partial \sigma}\sigma_I^2 dt\right)^2 \\
&\approx \frac{\pi}{4n}\left(\sigma_I\frac{\partial V}{\partial \sigma}\right)^2
\end{aligned} \tag{4.13}$$

We derive one of the BSM greek known as vega for a call which is the same for a put

$$\frac{\partial C}{\partial \sigma} = S\phi(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}\phi(d_2)\frac{\partial}{\partial \sigma}\left(d_1 - \sigma_I\sqrt{T-t}\right)$$
$$\frac{\partial d_1}{\partial \sigma} = -\frac{\log(S/K) - r(T-t) + \sigma^2/2(T-t)}{\sigma^2\sqrt{T-t}} = \frac{-d_2}{\sigma}$$
$$\frac{\partial C}{\partial \sigma} = -\frac{S}{\sigma}\phi(d_1)d_2 - \frac{Ke^{r(T-t)}}{\sigma}\phi(d_2)\underbrace{\left(-d_2 - \sigma\sqrt{(T-t)}\right)}_{=-d_1}$$
$$= \phi(d_1)\left(-\frac{Sd_2}{\sigma} + \frac{Sd_1}{\sigma}\right)$$
$$= S\phi(d_1)\sqrt{T-t} = S\frac{\sqrt{T-t}}{\sqrt{2\pi}}e^{-1/2d_1^2} \tag{4.14}$$

When the underlying price of an option is equal to the strike, the riskless rate is low, and the time to expiry is small enough we can approximate the price of an option and the vega of a vanilla European option via

$$\frac{\partial V}{\partial \sigma} \approx \frac{S\sqrt{(T-t)}}{\sqrt{2\pi}}$$
$$V \approx \frac{S\sigma_I\sqrt{(T-t)}}{\sqrt{2\pi}} \tag{4.15}$$

The volatility of the hedging error is

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}}\frac{\sigma_I}{\sqrt{n}}\frac{\partial V}{\partial \sigma}$$
$$\approx \sqrt{\frac{\pi}{4}}\frac{\sigma_I}{\sqrt{n}}\frac{V}{\sigma_I}$$
$$\frac{\sigma_{HE}}{V} \approx \sqrt{\frac{\pi}{4n}}$$
$$\approx \frac{0.89}{\sqrt{n}} \tag{4.16}$$

As we see in this example, the hedging error decrease with the square root of n. Say we would look at replicating an option by hedging for five steps. The estimated error for the hedge error in relation to the underlying price is about 40% of the option price for one standard deviation.

This is the result of a simplified world where the realized volatility is equal to the implied. The dynamics of the stock allow for no jumps and are governed by geometric Brownian motion. We've also assumed no transaction cost and no bid-ask leakage. We've already seen that the expected value of the hedge error is non-zero if we do not hedge with realized volatility, and the drift of the stock under geometric Brownian motion is not equal to the riskless rate.

We've also seen that this applies to an at-the-money option where the math simplifies into a nice analytical solution. We don't know if the variance of the hedge error actually increases the further out in the wings we try to replicate. Another challenge trying to price options via replications using data is the error around the data itself needing to be better, which can introduce even more uncertainty in the forecast.

5 Jump Risk

The risk embedded in the jump diffusion model can only be removed completely by using an infinite number of hedging instruments . With a finite number of instruments in the hedge, the diffusion risk can still be eliminated by imposing delta neutrality, but the compound Poisson process governing the arrival and magnitude of jumps precludes the removal of the associated jump risk.

We will lay down the mathematics assuming we are only hedging with the underlying so we cannot construct a perfect hedge even in the event of continuous dynamic hedging. Let's start by defining the dynamics of the underlying asset price [5]

$$dS = (r - \kappa\lambda)Sdt + \sigma SdW_t + (J-1)Sd\xi_t \tag{5.1}$$

where we've introduce a new variable ξ_t which is a Poisson counting process, λ is the jump intensity, and J is a random variable representing the jump amplitude with $\kappa = \mathbb{E}(J-1)$. $\log J$ is assumed to be normally distributed with constant mean ζ and standard deviation γ.

Let's setup a hedge portfolio containing an amount B in cash, a long position of α units of the underlying asset S, and a short position in the target option -V

$$\pi = -V + \alpha S + B \tag{5.2}$$

We must therefore consider the infinitesimal change in the value of π off the overall hedged position. Since we are concerned with the real work evolution of this portfolio, the underlying jump diffusion process of interest is governed by the objective measure $\mathbb{P}$. We have:

$$dS = (\mu - \kappa\lambda)^{\mathbb{P}}Sdt + \sigma SdW_t^{\mathbb{P}} + (J-1)Sd\xi_t^{\mathbb{P}}$$
$$dV = \left[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (\mu - \kappa\lambda)^{\mathbb{P}}S\frac{\partial V}{\partial S}\right]dt$$
$$+ \sigma S\frac{\partial V}{\partial S}dW_t^{\mathbb{P}} + (V(JS) - V(S))d\xi^{\mathbb{P}}$$
$$dB = rBdt \tag{5.3}$$

This implied that the instantaneous change in the value of the overall hedged position is

$$\begin{aligned} d\pi &= -dV + \alpha dS + dB \\ &= -\left[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}\right] dt + rBdt \\ &+ \sigma S\left[-\frac{\partial V}{\partial S} + \alpha\right] dW_t^{\mathbb{P}} + (\mu - \kappa\lambda)^{\mathbb{P}} S\left[-\frac{\partial V}{\partial S} + \alpha\right] dt \\ &+ \left[\alpha S(J-1) - (V(JS) - V(S))\right] d\xi^{\mathbb{P}} \end{aligned} \tag{5.4}$$

If the portfolio is delta neutral, then $\frac{\partial \pi}{\partial S} = 0$, i.e.

$$\frac{\partial \pi}{\partial S} = -\frac{\partial V}{\partial S} + \alpha = 0 \rightarrow \quad \underbrace{\alpha = \frac{\partial V}{\partial S}}_{\text{like Black-Scholes}} \tag{5.5}$$

this eliminates the middle two terms of equation 5.2 so we end up with

$$\begin{aligned} d\pi &= -dV + \alpha dS + dB \\ &= -\left[\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}\right] dt + rBdt \\ &+ \left[\alpha S(J-1) - (V(JS) - V(S))\right] d\xi^{\mathbb{P}} \end{aligned} \tag{5.6}$$

Following the work of Merton the value of a European option under jump diffusion process is given by

$$\begin{aligned} V_T =& \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial V} + (r - \kappa\lambda) S\frac{\partial V}{\partial S} - rV \\ &+ \lambda\left(\int_0^\infty V(S\eta, T) g^{\mathbb{Q}}(\eta) d\eta - V(S,T)\right), \end{aligned} \tag{5.7}$$

where $g^{\mathbb{Q}}(\cdot)$ is the risk-adjusted jump size density. We can rearrange the equation to get

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2} &= rV + \left\{\lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(S(J-1)) - rS\right\}\frac{\partial V}{\partial S} \\ &- \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}(V(JS) - V(S)) \end{aligned} \tag{5.8}$$

Substituting 5.8 into 5.7 yields

$$\begin{aligned} d\pi =& r\left[-V + S\frac{\partial V}{\partial S} + B\right] dt \\ &+ \lambda^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}(V(JS) - V(S)) - \frac{\partial V}{\partial S}\mathbb{E}^{\mathbb{Q}}(S(J-1))\right] dt \\ &+ \left[\frac{\partial V}{\partial S} S(J-1) - (V(JS) - V(S))\right] d\xi^{\mathbb{P}}. \end{aligned} \tag{5.9}$$

Using the delta neutral constraint gives

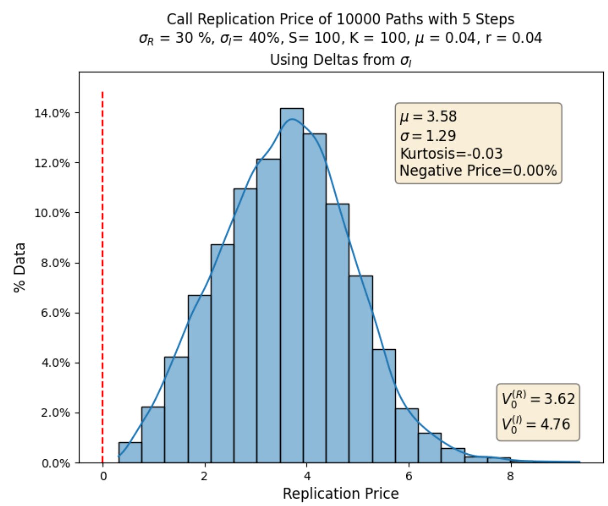
$$\begin{aligned} \pi =& r\pi dt + \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}\left[V(JS) - V(S) - \frac{\partial V}{\partial S} S(J-1)\right] dt \\ &+ \left[\frac{\partial V}{\partial S} S(J-1) - (V(JS) - V(S))\right] d\xi^{\mathbb{P}}. \end{aligned} \tag{5.10}$$

This equation indicates that the value of the overall hedged position grows at the risk free rate, but has additional terms due to the jump components both deterministic and stochastic. If the jump processes under $\mathbb{P}$ and $\mathbb{Q}$ are the same, the real world expected value of the instataneous jump risk is zero.

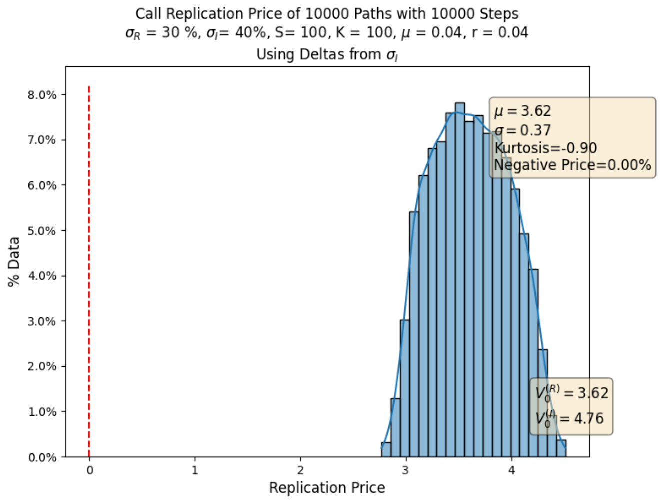
6 Simulations

6.1 Effect of Drift on the Replication Price

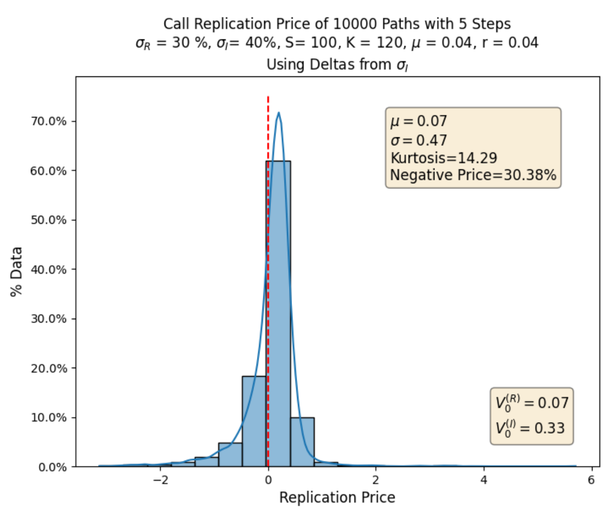
We will look at the effect of the Jump Diffusion Process in the section 6.3, but until then, Geometric Brownian Motion governs the stock dynamics. We will hedge using the implied volatility delta and investigate what happens when we hedge using the delta from the true realized volatility. We'll also look at how different frequency of hedging affects the distribution and the impact of our forecast if we are trying to replicate an option that is at-the-money(ATM) or out-the-money(OTM). For each graphic, $V_0^{(R)}$ and $V_0^{(I)}$ represent the option price under Black-Scholes-Merton using the realized and implied volatility respectively.

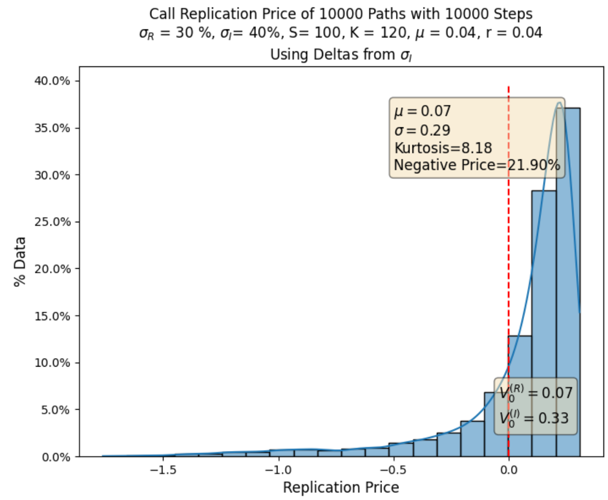


For this first graph, we can see that even if the realized volatility is different and we are replicating the option by hedging only five times during 21 days, we can, on average, capture the actual option price with a reasonable standard deviation and low kurtosis. Let's see what happens if we increase the frequency of hedging.

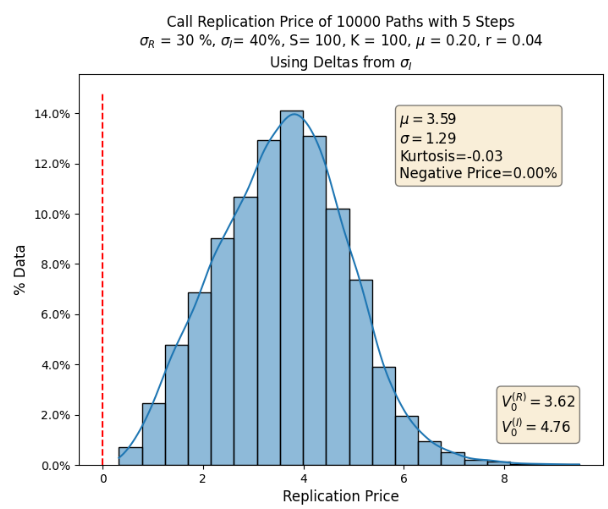


By hedging 2000 more times, we could only reduce the standard deviation by a factor of 4x. This seems to agree with the results of section 3.3 where if we hedge with a volatility different than the realized one, we are bound to have a terminal payoff distribution that doesn't converge. Before changing the drift of the underlying, let's look at the replication prices for OTM options.

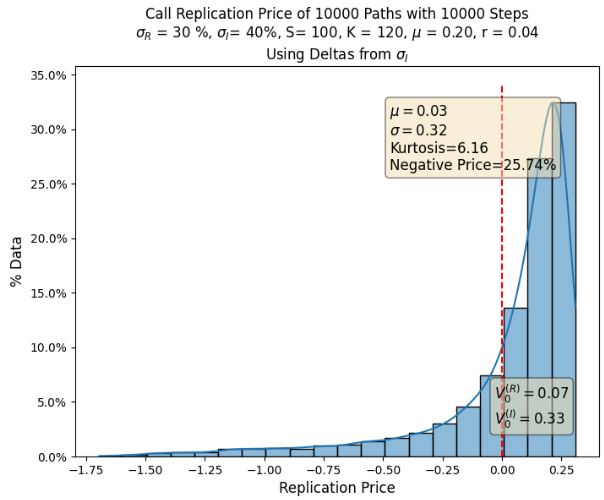




As we can see, the standard deviation this time has barely changed, and we are seeing many negative observations regardless of if we hedge five times within 21 days or 10000 times. The standard deviation is more than four times the option's price, making it very noisy. By including the hostile prices, on average, we pick up the correct mean, but any forecast would be unreliable due to the signal/noise ratio for deep OTM options when hedging under the risk-neutral deltas. Let's look at the effect of the drift terms now.



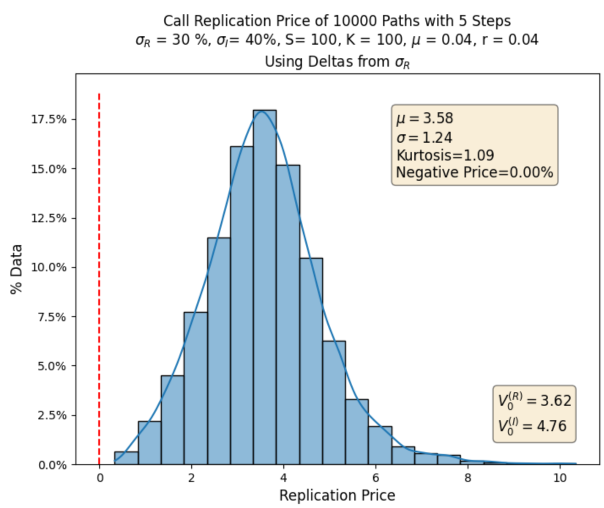
Here we are hedging only five times within 21 days, and the only thing we changed is the drift went from 4% to 20%, but the mean and standard deviation of the distribution were not affected for an ATM option. Let's look for OTM now.



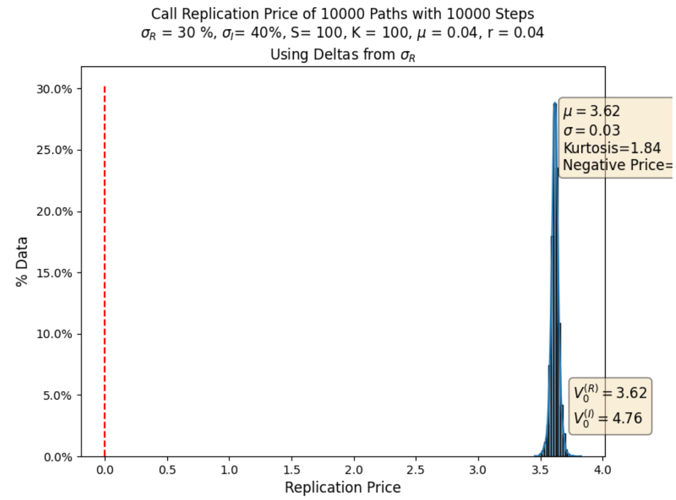
If we compare this plot to the previous OTM with 10000 paths, we see that the mean is now biased downward, and the standard deviation is higher with more negative replication prices. This is consistent across the different frequencies of hedging, suggesting that the drift impacts OTM replication prices.

6.2 Effect Volatility Delta on the Replication Price

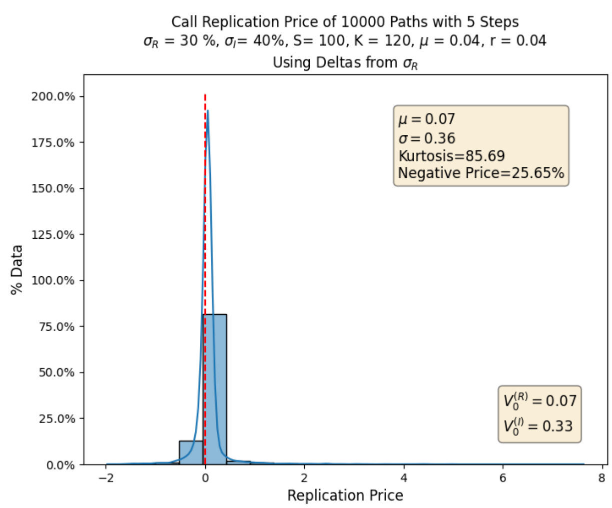
Let's start by looking at the ATM option hedging only 5 times, same underlying dynamics as the previous section, but this time, we are hedging on realized volatility.

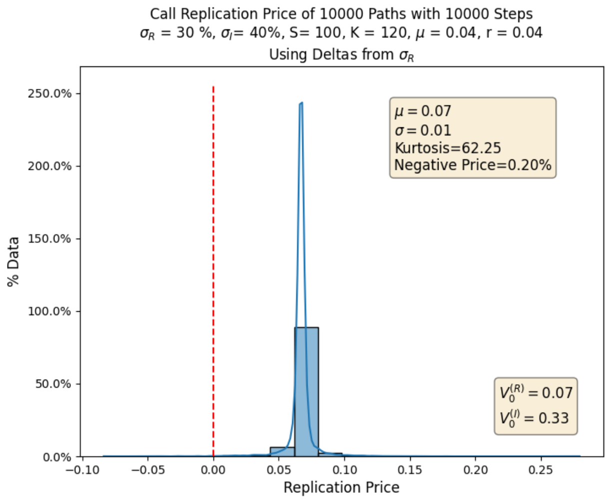


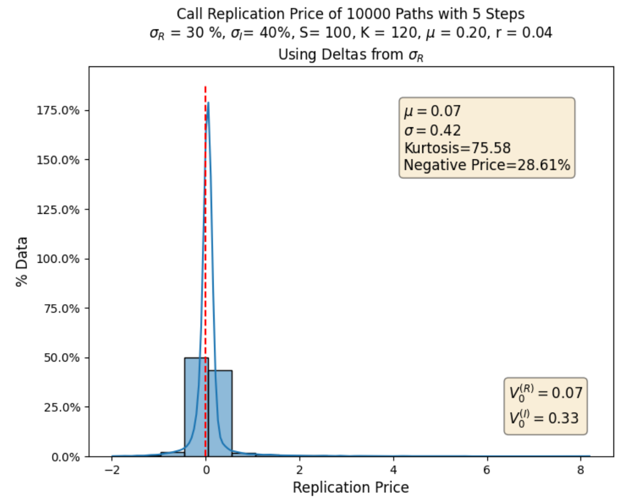
No noticable differences, let's see what happens if we increase the frequency of hedging.



We almost get perfect convergence! Previously, the standard deviation was reduced by only a factor of 4x this time it is about 40x, which agrees with the result we have from section 3 since 40x is about $\sqrt{2000}$. This also implies for OTM options, see below.



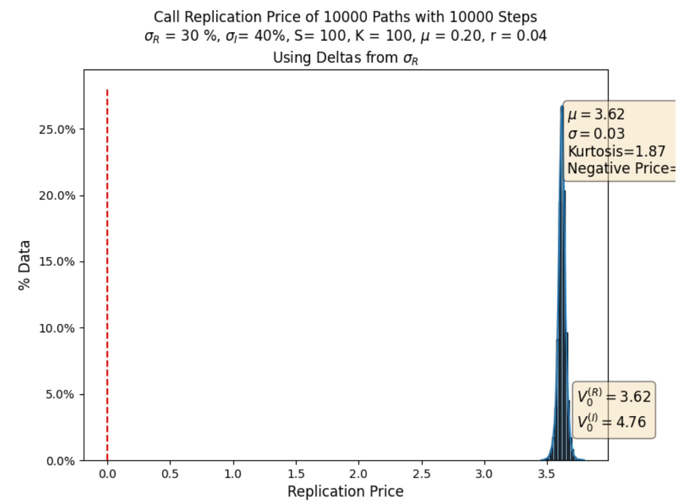


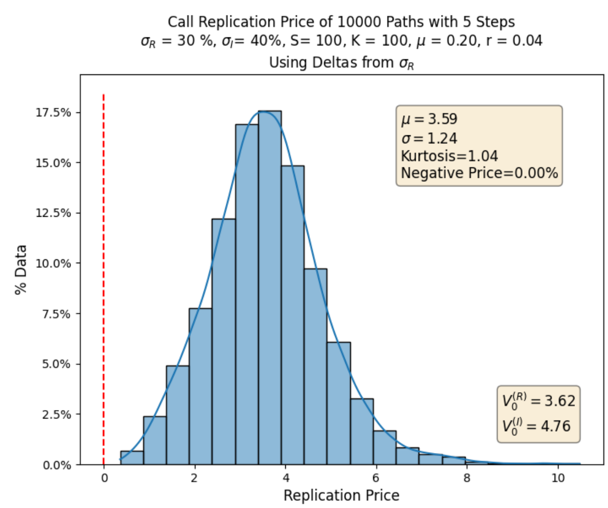


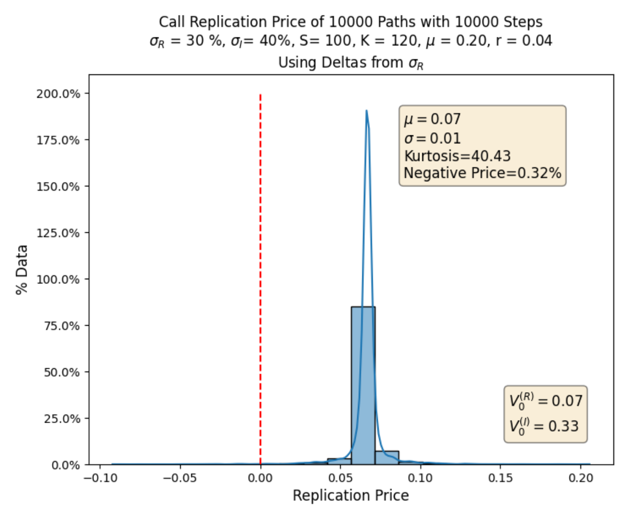
Again, increasing the frequency of hedging helps reduce the number of negative prices and the standard deviation of our replication prices. Even if we hedge with correct volatility if we only hedge a few times, we are left with a significant amount of our negative replication prices. Less than when hedging with the implied volatility but not that significantly more. With a good volatility forecast frequently hedging with the right volatility leads to better convergence than when hedging on implied volatility, whether the option is ATM or OTM.

As for the drift, when we hedge with the realized volatility, the distribution of replication price doesn't affect much. Again, this agrees with the results we have from section 3. When we have only five times, we get similar distribution than when the drift equals risk-free rate.

Same if we increase the hedging frequency as one would except since we have the right underlying volatility.



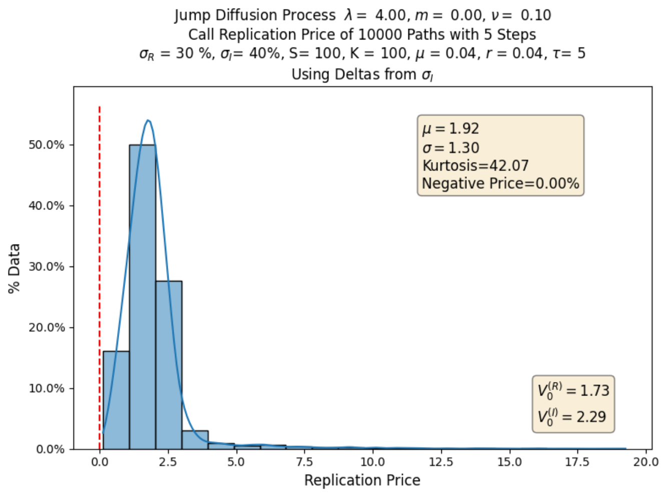


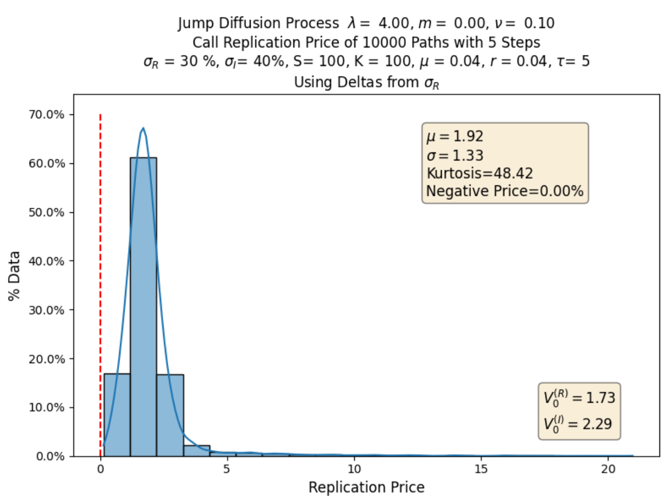


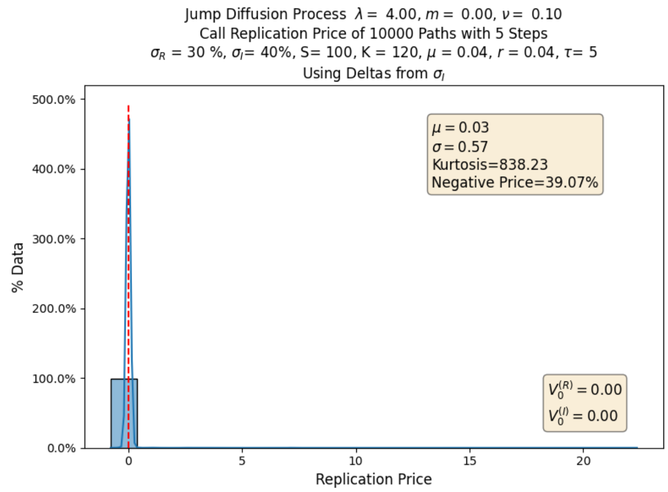
6.3 Effect on Jump Diffusion Process on the Replication Price

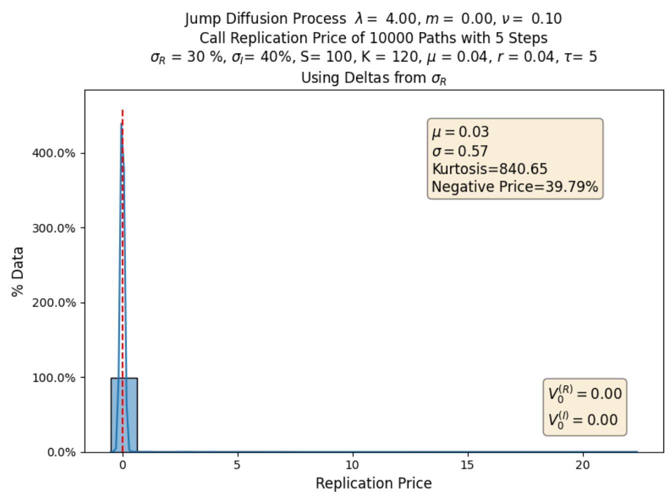
Let's now look at the effect of Jumps on the replication price. We will assume four jumps a year (simulate earnings) and each jump size will be driven by a Normal Distribution with mean 0 and standard deviation of 12%, so an annualized jump volatility of 192%. This may look high, but it happens in practice in some of the more volatile names.

Let's start by looking at the effect of hedging on implied volatility and realized volatility for five days to maturity options, hedging once a day for ATM and OTM when the drift is equal to the risk-free rate.



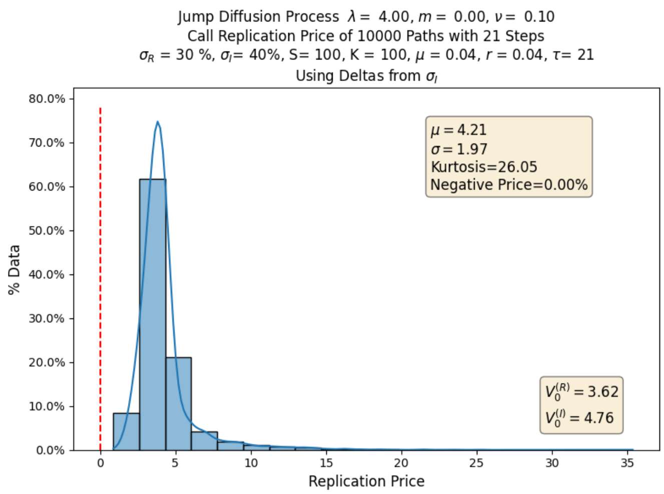


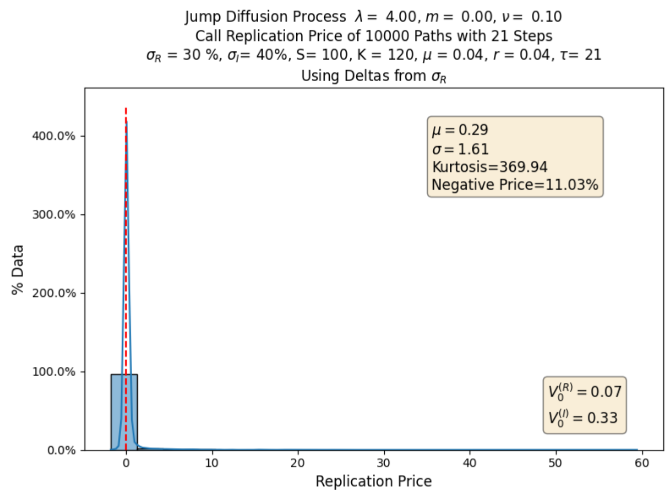


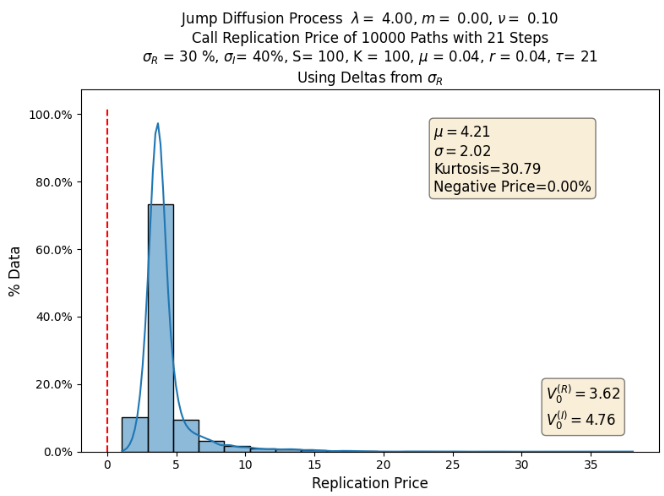


The difference between realized and implied for both of these is negligible. Although, if we compare to the previous section, the mean-to-standard deviation ratio for hedging five times is higher with jumps. For the OTM options, the number of negative replication prices is very high and the replication prices have very high kurtosis.

Let's now look at twenty one business days again with the drift equals to the the riskfree rate.

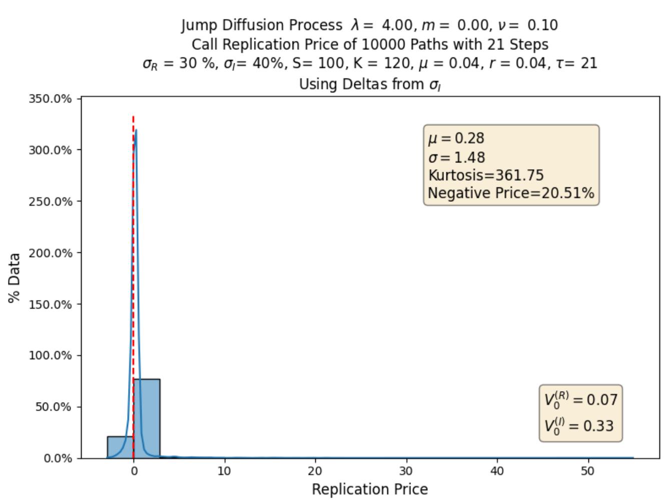


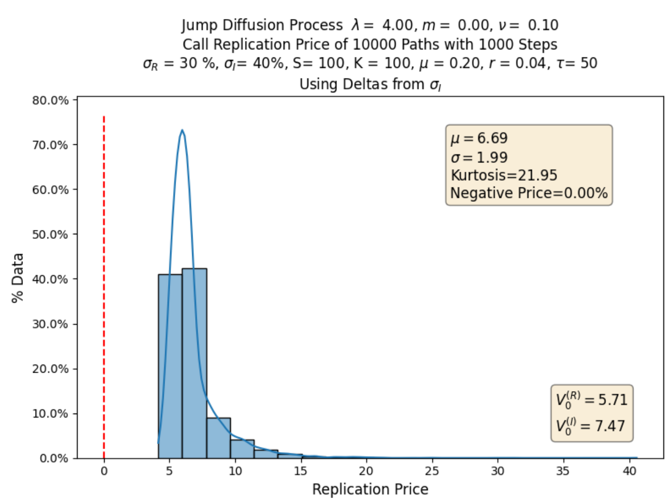


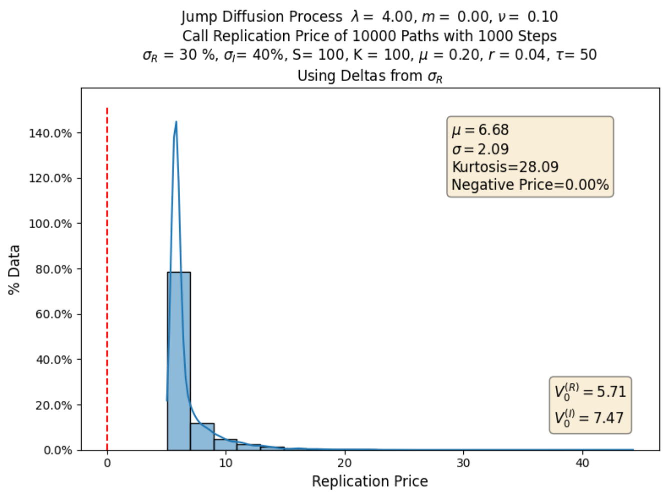


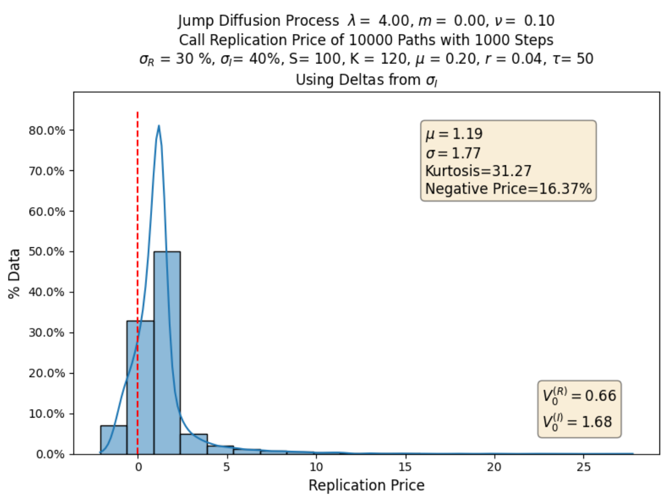
Again, in terms of the expected value and standard deviation. We see something much closer in this example than with geometric Brownian motion. It seems like hedging with the correct volatility has a negligible effect in reducing the variance of the replication price when we allow jumps in the simulation. In all cases, the forecasted mean is between the European Black-Scholes prices' under-realized and implied volatility. One of the main reasons there is a volatility premium to account for those jumps and we see the effect more dominant here in the wing, suggesting that implied volatility curves should have convexity.

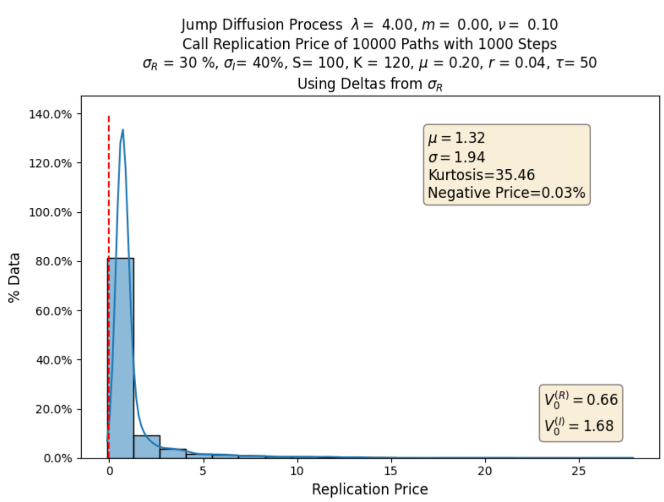
Let's now look at what happens when we hedge under-realized and implied volatility deltas more frequently. We'll add some drift as well, significantly higher than the risk-free rate. The option will now expire in fifty business days.









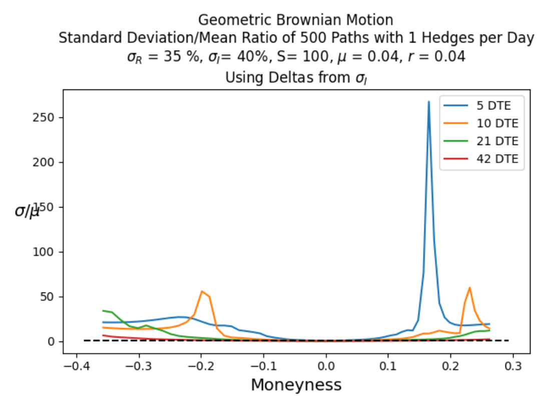


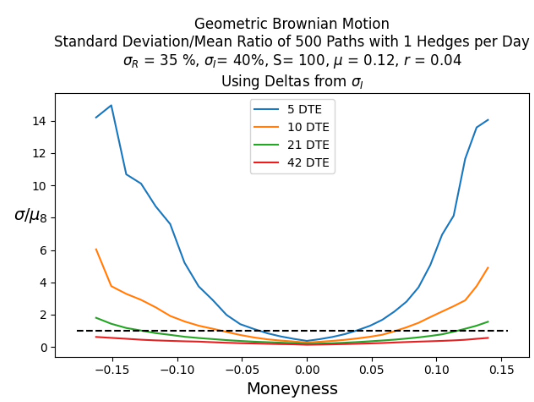
We notice that the number of replication prices for the ATM continues to be 0%. As for the OTM, it makes a significant difference to hedge using realized volatility now that we hedge more frequently. In the previous section, we saw that the convergence rate greatly improved if we frequently hedged under the realized volatility. In this case, we get much convergence whenever jumps don't occur in that fifty business day period.

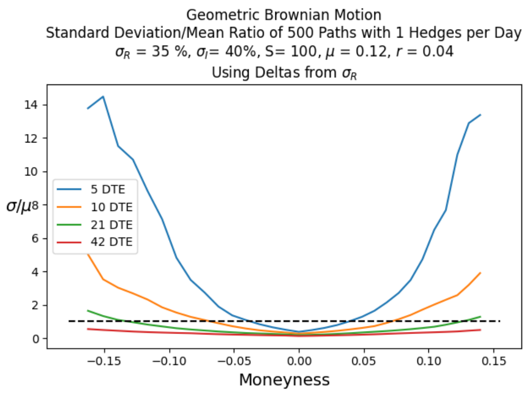
6.4 Error Distribution across Moneyness Geometric Brownian Motion

In the following two sections, we will look at the replication price standard deviation ratio to the mean price curve across moneyness. The black horizontal line represents when the standard deviation equals the price, so when the ratio equals 1. This is meant to be used to identify regions of moneyness where one might think that there is too much noise to get a sensible signal. We will also look on each graph at the distribution for different groups of expirations.

Let's start by looking at hedging once a day. The difference between the two plots is that one has a broader moneyness scale and a different drift. For obvious reasons, I will look at graphs with smaller moneyness moving forward because some outliers make the figures unreadable.

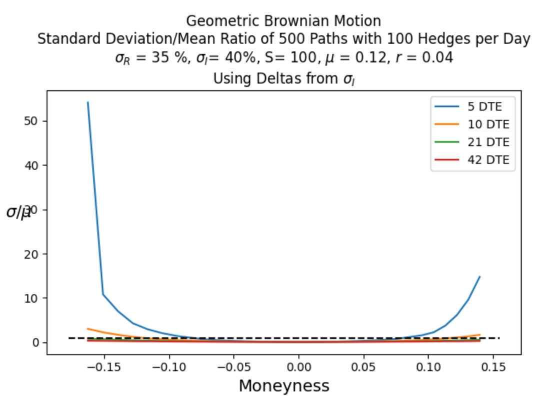


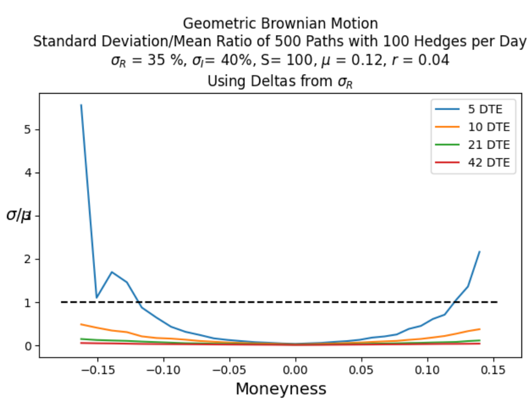




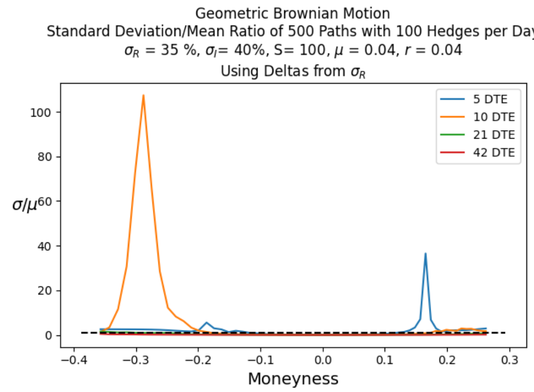
In the first figure, some of these points get so large for some short expiries that it makes the whole figure unreadable. The bottom two last figures display what the curves would look like between implied and realized volatility. They look quite similar, and in both cases, we can observe a flattening of the convexity for longer expiries.

Let's see what happens if we increase the hedging frequency to one hundred times a day.





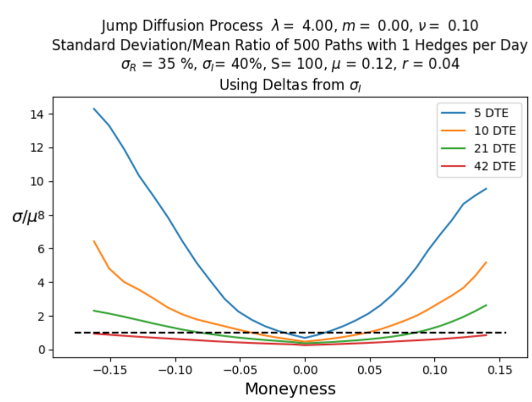
Here the curves look a lot flatter, and in the case of realized volatility, except for far OTM options, the ratio of mean to standard deviation is well below one. Let's see what happens if we increase the range of moneyness for the realized volatility.

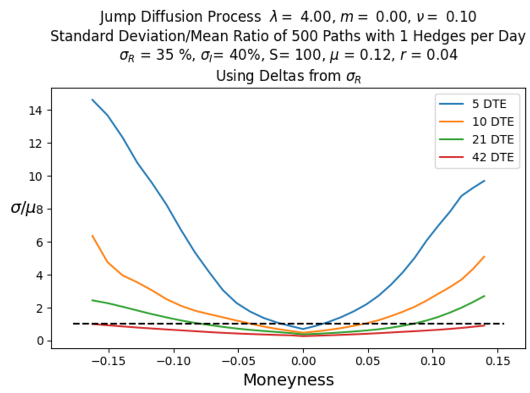


We continue to see some weirdness for OTM options even when hedging under-realized volatility and under Geometric Brownian Motion.

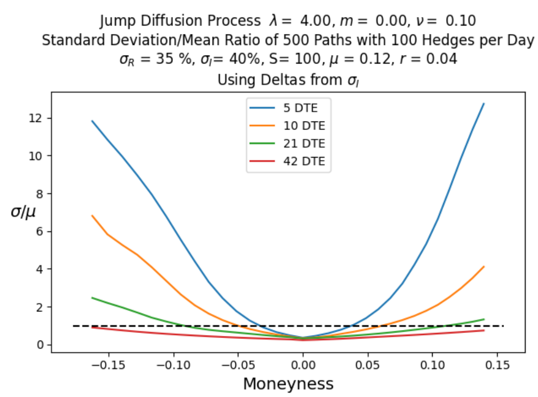
6.5 Error Distribution across Moneyness Jump Diffusion Process

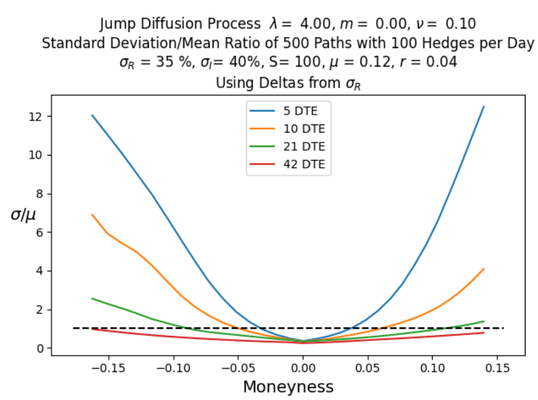
For jumps, the general behavior is similar to Geometric Brownian Motion but with some additional variance in the replication prices. Let's first look at hedging once a day under-realized volatility.





We get similar behavior and few noticeable differences between realized and implied volatility deltas. Let's see what we get if we increase the number of hedging to one hundred times daily. the number of hedging to one hundred-times a day.





The difference here is less significant than in the Geometric Brownian Motion case, but more observations fall under the ratio equal to 1 when increasing the frequency of hedging. But the effect is a lot less impactful, and we continue seeing a lot of OTM options with low signal-to-noise ratio for shorter-term expiries. The impact of hedging with realized versus implied is also less noticeable here.

7 Conclusion

- If you estimate future realized volatility correctly and hedge continuously at that volatility, your P&L will capture the spread of volatility between the implied and realized
- If you hedge discretely, the P&L will have a random component. If you predict the realized volatility well and hedge with it, your path is stochastic, but the terminal payoff is deterministic. If you hedge with implied volatility, the path won't be as stochastic, but the terminal payoff will be stochastic.
- In practice, traders hedge at implied volatility. The more implied volatility differs from realized volatility, the more you lose the benefit of the increasing number of hedging.
- The drift of the underlying doesn't seem to affect the replication prices of options when they are ATM, but it affects OTM, decreasing the strength of the signal when the option is hedged using implied volatility instead of realized volatility.
- In the absence of jumps, if one could forecast the realized volatility, the number of negative replication prices and the standard deviation of the replication prices are greatly reduced for ATM and OTM.
- Jumps will increase the number of negative replication prices, increase the variance in the replication prices, and also add a positive expected value to those replications from the realized implied volatility, especially in the wings. This is a good explanation and source of convexity in implied volatility surfaces.
- The effect of hedging with realized volatility and implied volatility is less drastic when considering jumps unless you hedge frequently. Regardless, you will still have a low signal-to-noise ratio in the wings.
- The ratio of noise to signal in replication prices grows exponentially in the wings and for short-term expires.

References

[1] *Pricing Models*, chapter 4, pages 41–61. John Wiley & Sons, Ltd, 2012.

[2] *The Solution of the Black-Scholes-Merton (BSM) Equation*, chapter 5, pages 63–88. John Wiley & Sons, Ltd, 2012.

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