FINM37601:Mathematical Market Microstructure Homework 2

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Question 1. Prove that the response function can be written as

$$R_l \sim \bar{V}^\alpha \left[\sum_{0<\tau\le l} \mathcal{G}(\tau)c(\tau-l) + \sum_{\tau>l} \mathcal{G}(\tau)c(\tau-l) - \sum_{0<\tau} \mathcal{G}(\tau)c(\tau) \right]$$

Starting with the definition of response function

$$\begin{aligned} R_l &= \langle (p_{t+l} - p_t)\epsilon_t \rangle \\ &= \langle p_{t+l}\epsilon_t - p_t\epsilon_t \rangle \\ &= \left\langle \sum_{\eta<t+l} \left(\mathcal{G}(t+l-\eta)V_\eta^\alpha \epsilon_\eta\epsilon_t\right) + \epsilon_{t+l}\epsilon_t - \sum_{\eta<t} \left(\mathcal{G}(t-\eta)V_\eta^\alpha\epsilon_\eta\epsilon_t\right) - \epsilon_t\epsilon_t \right\rangle \end{aligned}$$

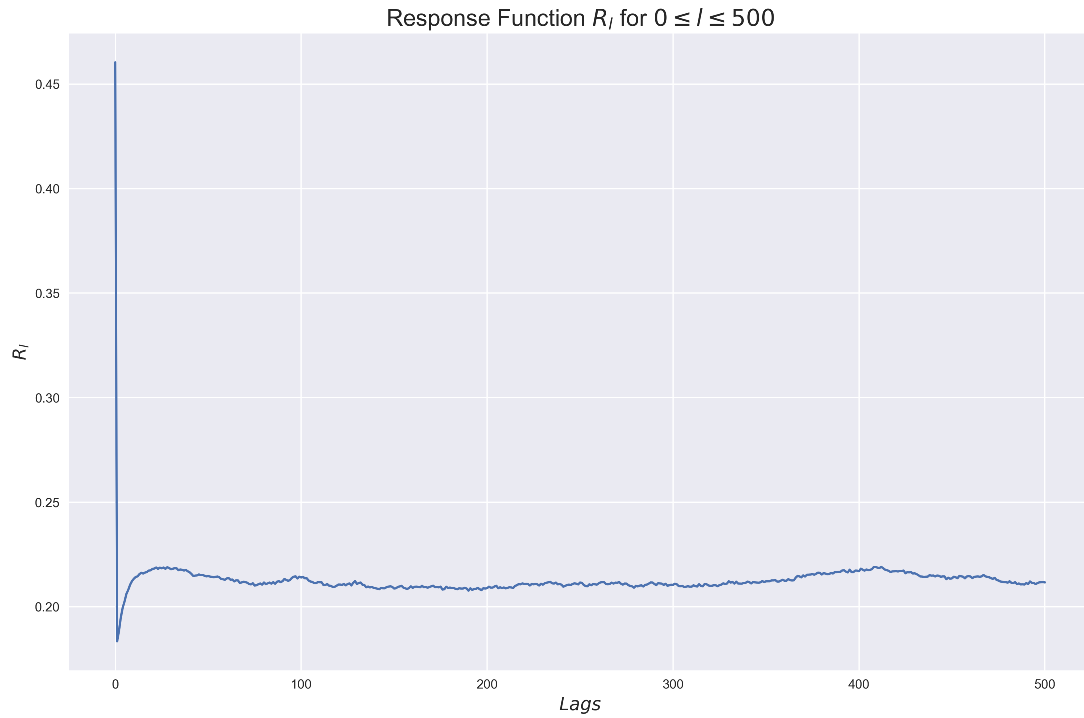
Define $\tau = t+l-\eta$ and $\tau^* = t-\eta$, we have

$$\begin{aligned} R_l &= \left\langle \sum_{\tau<t+l} \mathcal{G}(\tau) V^\alpha_{t+l-\tau}\epsilon_{t+l-\tau}\epsilon_t \right\rangle + \langle \epsilon_{t+l}\epsilon_t\rangle + \langle \epsilon_t\epsilon_t \rangle - \left\langle \sum_{\tau^*<t} \mathcal{G}(\tau^*) V^\alpha_{t-\tau^*}\epsilon_{t-\tau^*}\epsilon_t \right\rangle \\ R_l &\sim \left\langle \sum_{\tau<t+l} \mathcal{G}(\tau) V^\alpha_{t+l-\tau}\epsilon_{t+l-\tau}\epsilon_t \right\rangle - \left\langle \sum_{\tau^*<t} \mathcal{G}(\tau^*) V^\alpha_{t-\tau^*}\epsilon_{t-\tau^*}\epsilon_t \right\rangle \\ &\sim \left\langle \sum_{0\le\tau\le l} \mathcal{G}(\tau) V^\alpha_{t+l-\tau}\epsilon_{t+l-\tau}\epsilon_t \right\rangle + \left\langle \sum_{l<\tau<l+t} \mathcal{G}(\tau) V^\alpha_{t+l-\tau}\epsilon_{t+l-\tau}\epsilon_t \right\rangle - \left\langle \sum_{\tau^*<t} \mathcal{G}(\tau^*) V^\alpha_{t-\tau^*}\epsilon_{t-\tau^*}\epsilon_t \right\rangle \\ &\sim \left\langle \sum_{0\le\tau\le l} \mathcal{G}(\tau) V^\alpha_{t+l-\tau}\epsilon_{t+l-\tau}\epsilon_t \right\rangle + \left\langle \sum_{\tau>l} \mathcal{G}(\tau) V^\alpha_{t+l-\tau}\epsilon_{t+l-\tau}\epsilon_t \right\rangle \\ &- \left\langle \sum_{\tau>l+t} \mathcal{G}(\tau) V^\alpha_{t+l-\tau}\epsilon_{t+l-\tau}\epsilon_t \right\rangle + \left\langle \sum_{\tau^*\ge t} \mathcal{G}(\tau^*) V^\alpha_{t-\tau^*}\epsilon_{t-\tau^*}\epsilon_t \right\rangle - \left\langle \sum_{0<\tau^*} \mathcal{G}(\tau^*) V^\alpha_{t-\tau^*}\epsilon_{t-\tau^*}\epsilon_t \right\rangle \\ &\sim \left\langle \sum_{0\le\tau\le l} \mathcal{G}(\tau) V^\alpha_{t+l-\tau}\epsilon_{t+l-\tau}\epsilon_t \right\rangle + \left\langle \sum_{\tau>l} \mathcal{G}(\tau) V^\alpha_{t+l-\tau}\epsilon_{t+l-\tau}\epsilon_t \right\rangle - \left\langle \sum_{0<\tau^*} \mathcal{G}(\tau^*) V^\alpha_{t-\tau^*}\epsilon_{t-\tau^*}\epsilon_t \right\rangle \\ &\sim \bar{V}^\alpha \left[\sum_{0\le\tau\le l} \mathcal{G}(\tau)c\left(l-\tau\right) \right] + \bar{V}^\alpha \left[\sum_{\tau>l} \mathcal{G}(\tau)c\left(l-\tau\right) \right] + \bar{V}^\alpha \left[\sum_{0<\tau} \mathcal{G}(\tau^*)c\left(\tau^*\right) \right] \end{aligned}$$

Redefining τ^* as τ and knowing that $c(l) \sim c(-l)$ leads use to

$$R_l \sim \bar{V}^\alpha \left[\sum_{0<\tau\le l} \mathcal{G}(\tau)c(\tau-l) + \sum_{\tau>l} \mathcal{G}(\tau)c(\tau-l) - \sum_{0<\tau} \mathcal{G}(\tau)c(\tau) \right]$$

Question 2. See plot below.



Question 3. The response function has very fluctuating shape based on the slice of data we are representing. For the first 6 groups, we see a sharp decline after $l = 0$ then a slight increase towards the end. As for group 4 to 6, there is some movement in the response function prior to higher level of lags, no definite direction but more movement than group 1 to 3.

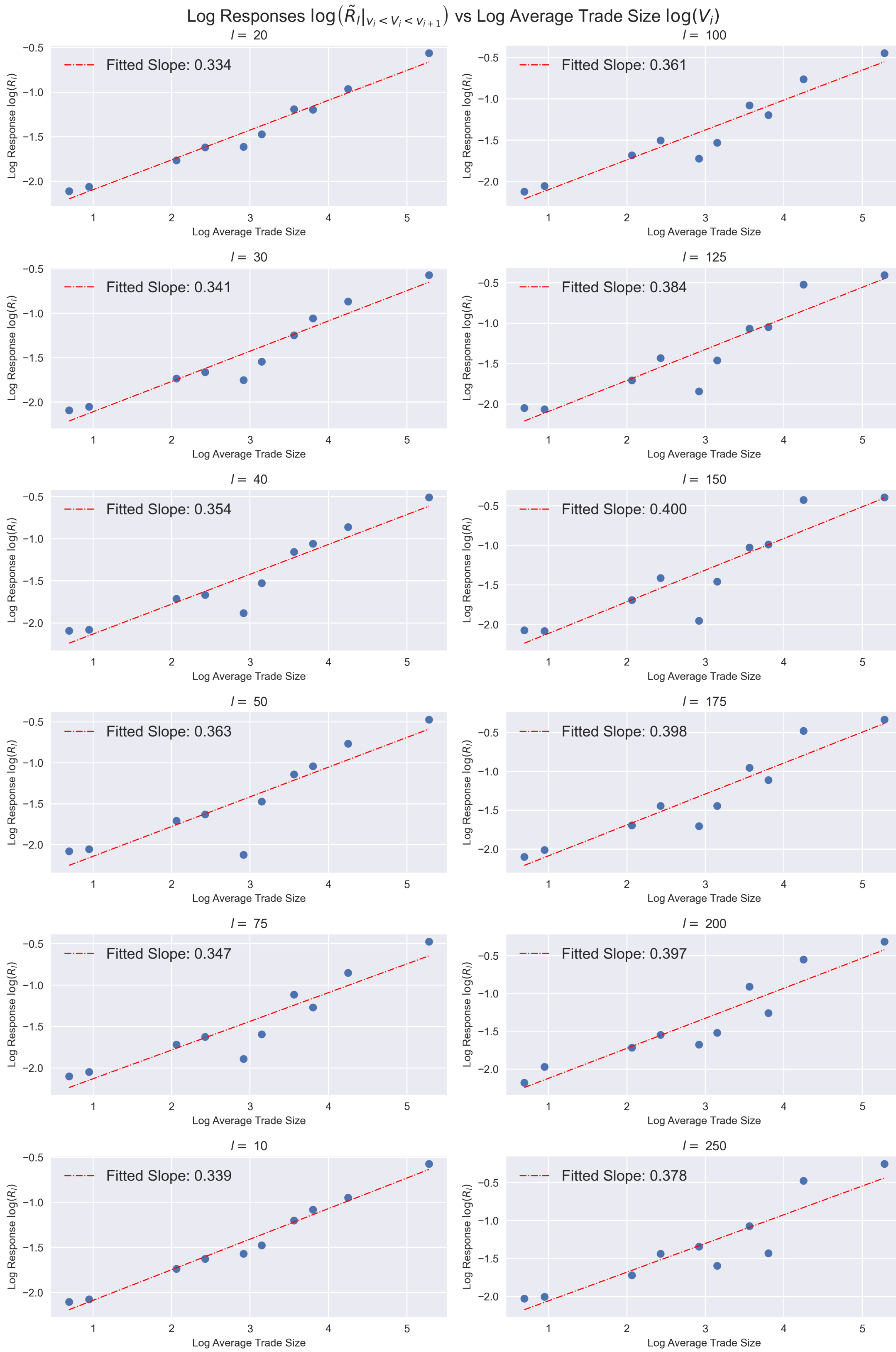
Group number 7 and 8 have very much different shapes than previous ones. We observe an initial increase in the response function for up to lag 200, then sharp decrease and a bounce back in the later lags.

Group 9 is very unique with a sharp increase in the response function followed by a decline at around the same lags as group 7 and 8. Then the response function spikes back up to a response level higher than any groups prior.

Group 10 is also very unique with an almost constantly increasing response function as the lag increases with no real period where the response function is decreasing.


R_l for different groups of trade sizes
Response Function R_l for Group 1 with $v_1 = 0$ and $v_2 = 2$
Response Function R_l for Group 2 with $v_1 = 2$ and $v_2 = 5$
Response Function R_l for Group 3 with $v_1 = 5$ and $v_2 = 10$
Response Function R_l for Group 4 with $v_1 = 10$ and $v_2 = 15$
Response Function R_l for Group 5 with $v_1 = 15$ and $v_2 = 20$
Response Function R_l for Group 6 with $v_1 = 20$ and $v_2 = 30$
Response Function R_l for Group 7 with $v_1 = 30$ and $v_2 = 40$
Response Function R_l for Group 8 with $v_1 = 40$ and $v_2 = 55$
Response Function R_l for Group 9 with $v_1 = 55$ and $v_2 = 90$
Response Function R_l for Group 10 with $v_1 = 90$ and $v_2 = 1000000$
R_l
Lags

Question 4.


Log Responses $\log\left(\tilde{R}_l|_{v_i<V_i<v_{i+1}}\right)$ vs Log Average Trade Size $\log(V_i)$
$l =$ 20
Fitted Slope: 0.334
$l =$ 100
Fitted Slope: 0.361
$l =$ 30
Fitted Slope: 0.341
$l =$ 125
Fitted Slope: 0.384
$l =$ 40
Fitted Slope: 0.354
$l =$ 150
Fitted Slope: 0.400
$l =$ 50
Fitted Slope: 0.363
$l =$ 175
Fitted Slope: 0.398
$l =$ 75
Fitted Slope: 0.347
$l =$ 200
Fitted Slope: 0.397
$l =$ 10
Fitted Slope: 0.339
$l =$ 250
Fitted Slope: 0.378
Log Response $\log(R_l)$
Log Average Trade Size