Approximation Algorithms for the Fault-Tolerant Facility Placement Problem

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Outline

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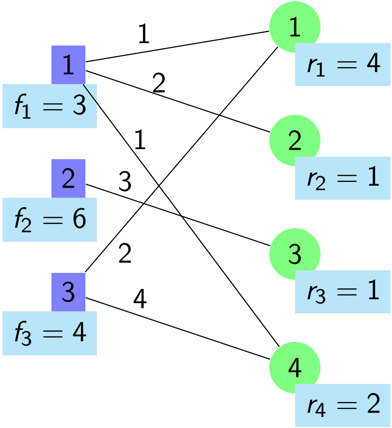
Fault-Tolerant Facility Placement Problem (FTFP)



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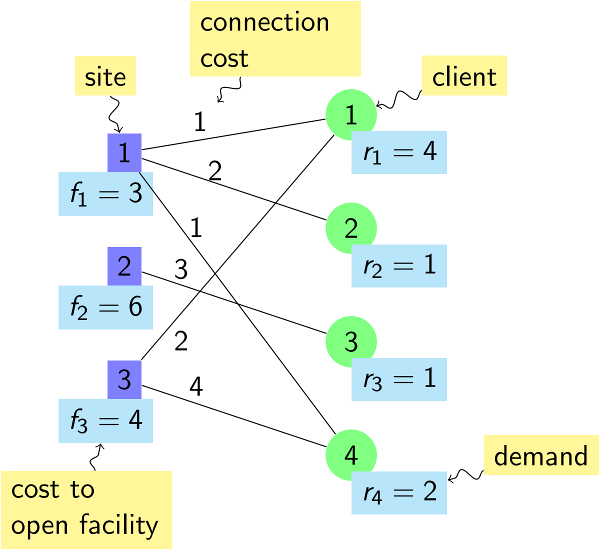

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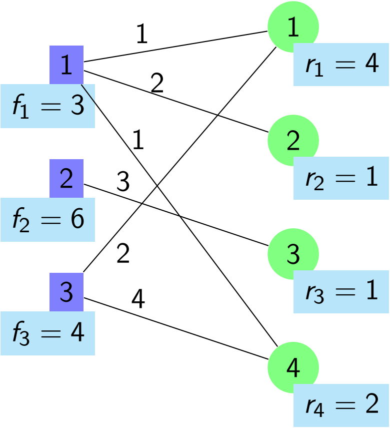
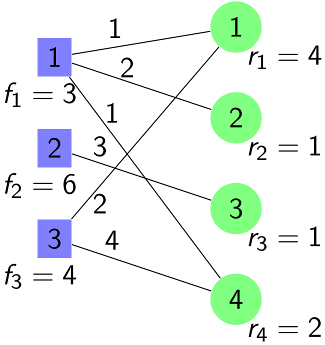
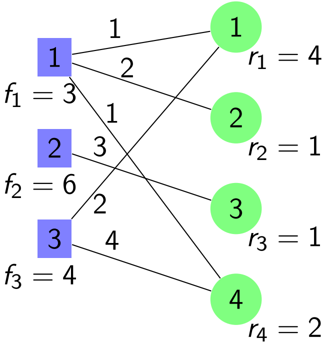


Feasible Integral Solution

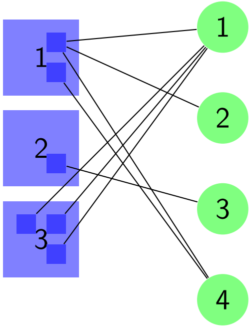


Instance

Feasible Integral Solution

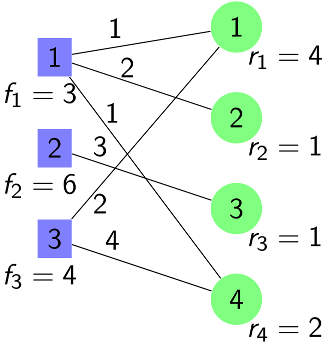


Instance



Solution

Feasible Integral Solution



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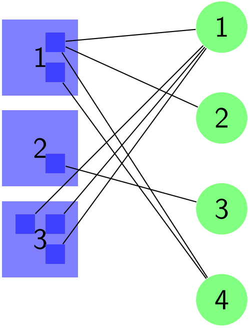
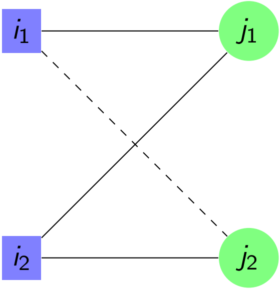


Solution

Cost

$$2f_1 + f_2 + 3f_3 + d_{11} + d_{12} + 2d_{14} + d_{23} + 3d_{31} = 38$$

Metric Distances: Triangle Inequality



$$d(i_1, j_2) \leq d(i_1, j_1) + d(i_2, j_1) + d(i_2, j_2)$$

Needed when estimating distances...

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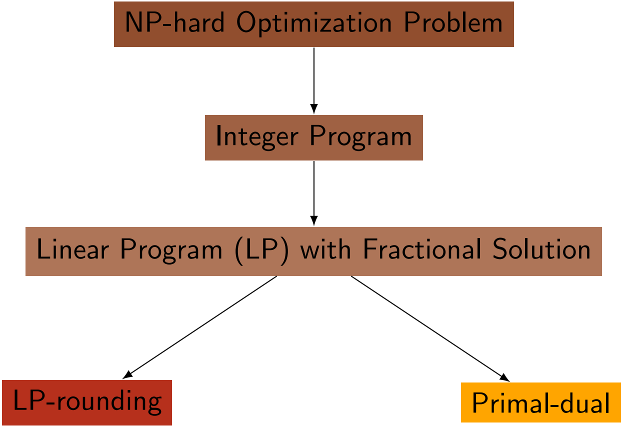
Hardness

How hard is FTFP?

FTFP is NP-hard

FTFP is MaxSNP-hard

Best ratio ≥ 1.463 unless P $=$ NP


NP-hard Optimization Problem
Integer Program
Linear Program (LP) with Fractional Solution
LP-rounding
Primal-dual

Results Highlight

- LP-rounding: 1.575-approximation
- LP-rounding: asymptotic ratio of 1 when all demands large
- Primal-dual: H_n-approximation
- Primal-dual: Example of $\Omega(\log n / \log \log n)$ for dual-fitting

Relation between Problems

FTFP	$r_j \geq 1$	$< \infty$ facility per site
UFL	$r_j = 1$	≤ 1 facility per site
FTFL	$r_j \geq 1$	≤ 1 facility per site

Relation between Problems

FTFP	$r_j \geq 1$	$< \infty$ facility per site
UFL	$r_j = 1$	≤ 1 facility per site
FTFL	$r_j \geq 1$	≤ 1 facility per site

$$\text{UFL} \preceq \text{FTFP} \preceq \text{FTFL}$$

Relation between Problems

FTFP	$r_j \geq 1$	$< \infty$ facility per site
UFL	$r_j = 1$	≤ 1 facility per site
FTFL	$r_j \geq 1$	≤ 1 facility per site

UFL $\preceq$ FTFP $\preceq$ FTFL

LP-rounding	
UFL	1.575
FTFP	1.575
FTFL	1.7245

Relation between Problems

FTFP	$r_j \geq 1$	$< \infty$ facility per site
UFL	$r_j = 1$	≤ 1 facility per site
FTFL	$r_j \geq 1$	≤ 1 facility per site

UFL $\preceq$ FTFP $\preceq$ FTFL

LP-rounding	
UFL	1.575
FTFP	
FTFL	1.7245

Primal-dual	
UFL	1.52
FTFP	$O(\log n)$
FTFL	

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Related Work for UFL

Approximation Results for UFL

Shmoys, Tardos and Aardal	1997	3.16	LP-rounding
Chudak	1998	1.736	LP-rounding
Sviridenko	2002	1.58	LP-rounding
Jain and Vazirani	2001	3	primal-dual
Jain *et al.*	2002	1.61	greedy
Mahdian *et al.*	2002	1.52	greedy
Arya *et al.*	2004	3	local search
Byrka	2007	1.5	hybrid
Li	2011	1.488	hybrid

Lower Bound

Guha and Khuller	1998	1.463

Related Work for FTFL

Approximation Algorithms for FTFL

Jain and Vazirani	2000	$3 \ln \max_j r_j$	primal-dual
Guha *et al.*	2001	4	LP-rounding
Swamy, Shmoys	2008	2.076	LP-rounding
Byrka *et al.*	2010	1.7245	LP-rounding

No primal-dual algorithms for FTFL with constant ratio.

Work on FTFP (Dissertation Topic)

Approximation Algorithms for FTFP

Xu and Shen	2009		Introduced FTFP
Liao and Shen	2011	1.861	Dual-fitting (for special case)
Yan and Chrobak	2011	3.16	LP-rounding
Yan and Chrobak	2012	1.575	LP-rounding
Yan and Chrobak	preliminary results		Dual-fitting (for general case)

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Techniques

- Demand Reduction
  - Reduce all r_j to polynomial values (to ensure polynomial time of rounding)
  - ρ-approx for reduced instance $\Rightarrow$ ρ-approx for original instance
- Adaptive Partitioning
  - Split sites into facilities and clients into unit demands
  - Split associated fractional values
  - Properties ensure rounding similar to UFL can be applied

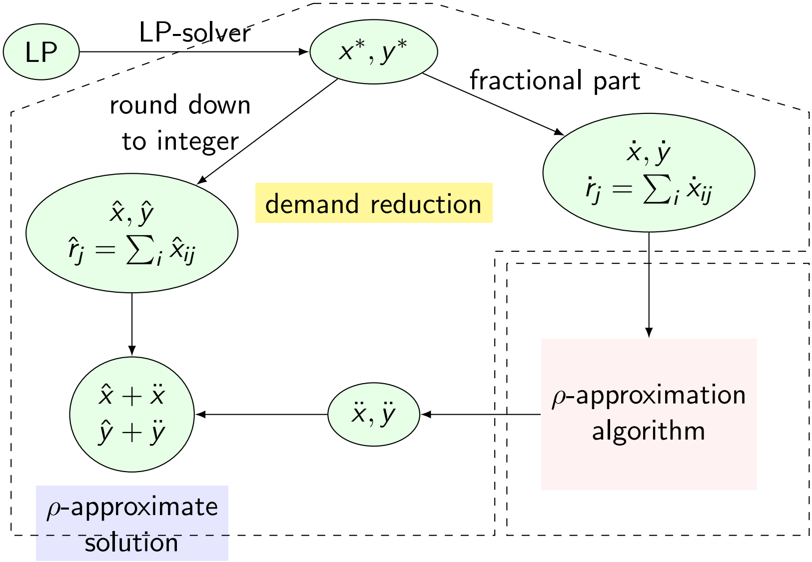
LP Formulation for FTFP

- y_i = number of facilities open at site $i \in \mathbb{F}$
- x_{ij} = number of connections from client $j \in \mathbb{C}$ to site $i \in \mathbb{F}$

$$
\begin{array}{llll}
\text{(Primal)} & \text{minimize} & \sum f_i y_i + \sum d_{ij} x_{ij} & \\
& \text{subject to} & y_i - x_{ij} \geq 0 & \forall i, j \\
& & \sum x_{ij} \geq r_j & \forall j \\
& & x_{ij} \geq 0, y_i \geq 0 & \forall i, j
\end{array}
$$

$$
\begin{array}{llll}
\text{(Dual)} & \text{maximize} & \sum r_j \alpha_j & \\
& \text{subject to} & \sum \beta_{ij} \leq f_i & \forall i \\
& & \alpha_j - \beta_{ij} \leq d_{ij} & \forall i, j \\
& & \alpha_j \geq 0, \beta_{ij} \geq 0 & \forall i, j
\end{array}
$$

Algorithm for FTFP



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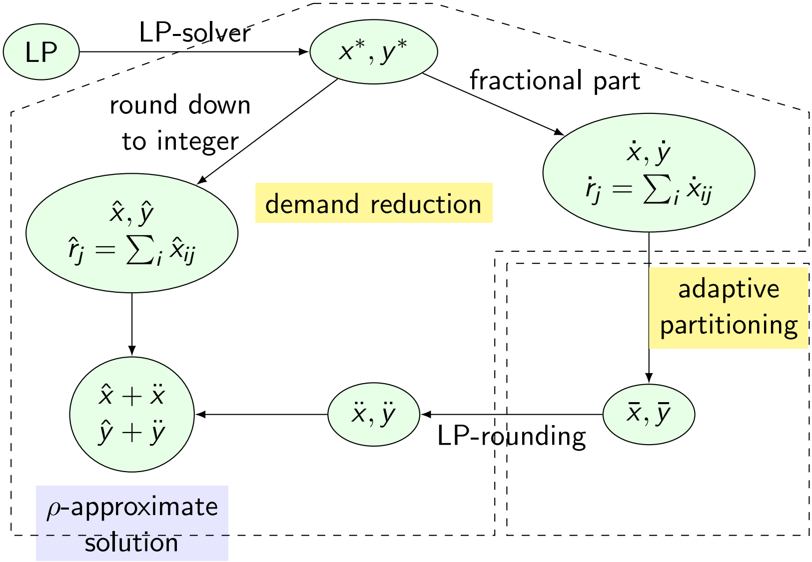

Techniques

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Demand Reduction

Implementation

- Solving LP for $(\mathbf{x}^*, \mathbf{y}^*)$.
- $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = (\mathbf{x}^*, \mathbf{y}^*)$ round down to integer
- $(\dot{\mathbf{x}}, \dot{\mathbf{y}}) = (\mathbf{x}^*, \mathbf{y}^*) - (\hat{\mathbf{x}}, \hat{\mathbf{y}})$, fractional part
- $\hat{r}_j = \sum_i \hat{x}_{ij}$ for $\hat{\mathcal{I}}$, $\dot{r}_j = r_j - \hat{r}_j$ for $\dot{\mathcal{I}}$
- $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ (integral) feasible and optimal for $\hat{\mathcal{I}}$
- $(\dot{\mathbf{x}}, \dot{\mathbf{y}})$ (fractional) feasible and optimal for $\dot{\mathcal{I}}$

Properties

- $\dot{r}_j = \text{poly}(|\mathbb{F}|)$
- ρ-approx for $\dot{\mathcal{I}}$ implies ρ-approx for $\mathcal{I}$

Demand Reduction: Consequences

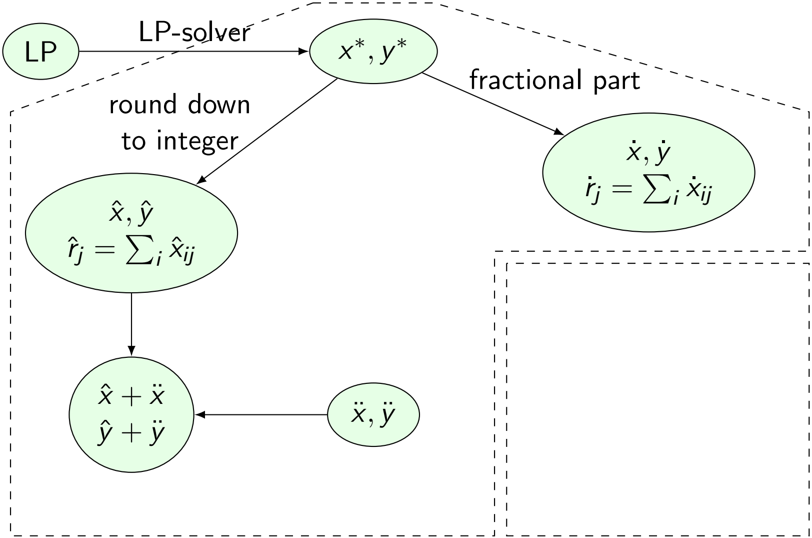
FTFP to FTFL, 1.7245-approximation

- Sites into facilities
- Clients with demand r_j
- FTFL size polynomial because demand reduction

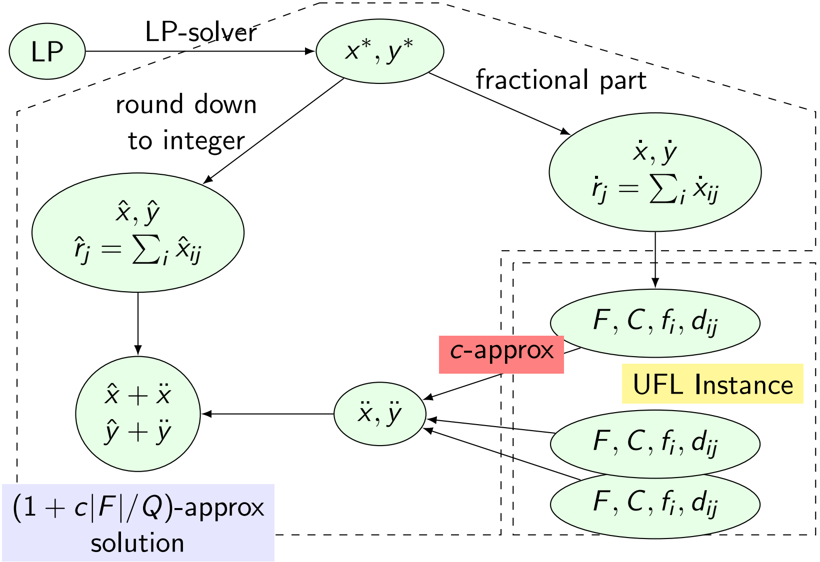
Ratio $1 + O(|F|/Q)$ for $Q = \min_j r_j$, approaches 1 when Q is large

- Next slide

Ratio $1 + O(|F|/Q)$ for FTFP



Ratio $1 + O(|F|/Q)$ for FTFP



Techniques

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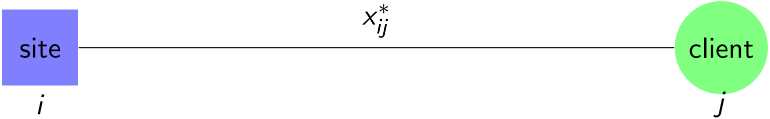
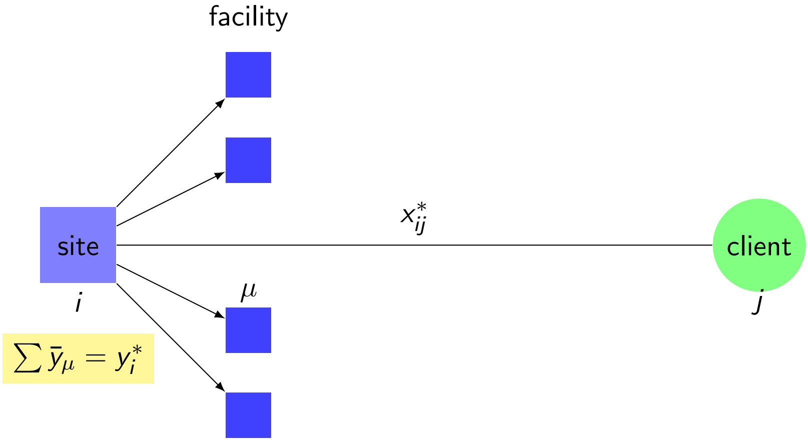
Adaptive Partitioning



Adaptive Partitioning



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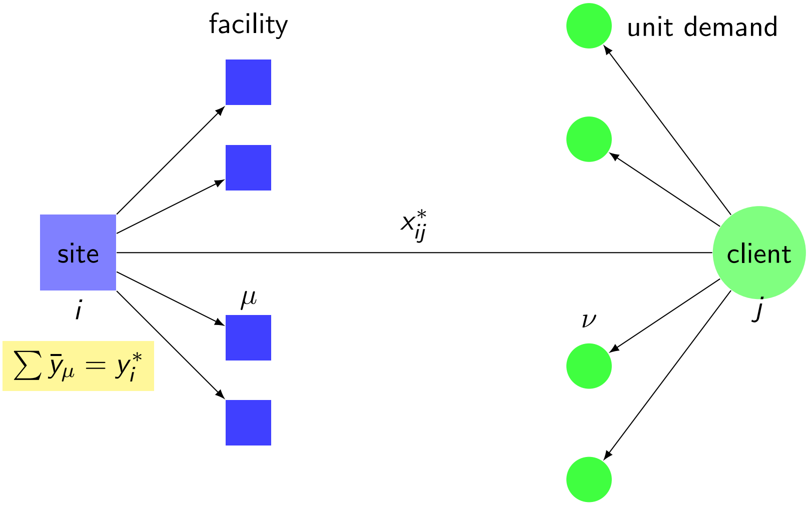

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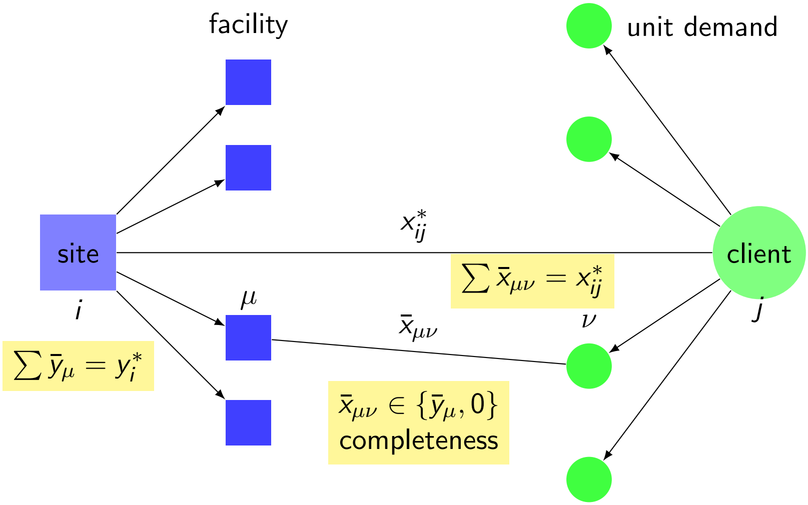
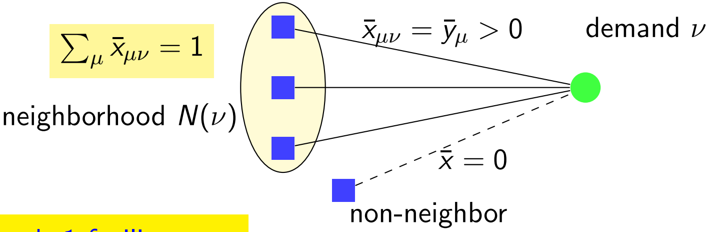


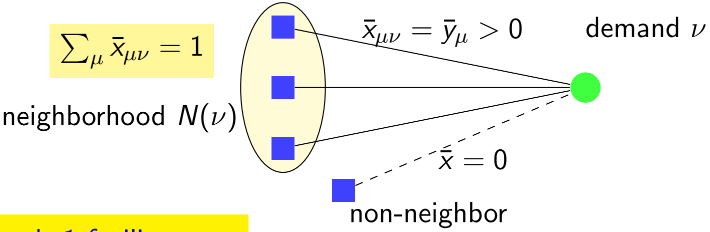
Now round each $\bar{y}_\mu$ and $\bar{x}_{\mu\nu}$ to 0 or 1...

Neighborhood of Demand



ν needs 1 facility...

Neighborhood of Demand



ν needs 1 facility...

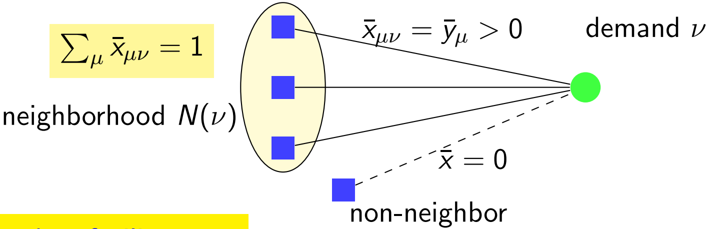
Strategy 1: for each ν, open one $\mu \in N(\nu)$ with prob. $\bar{y}_\mu$

- optimal connection cost
- large facility cost

Strategy 2: do this for demands with disjoint neighborhoods

- optimal facility cost
- large connection cost

Neighborhood of Demand



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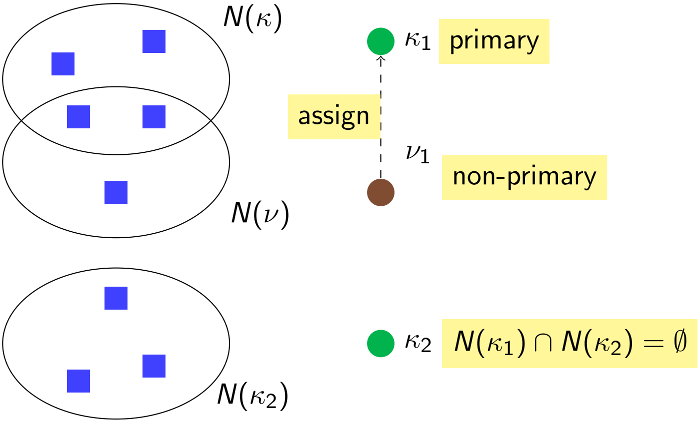
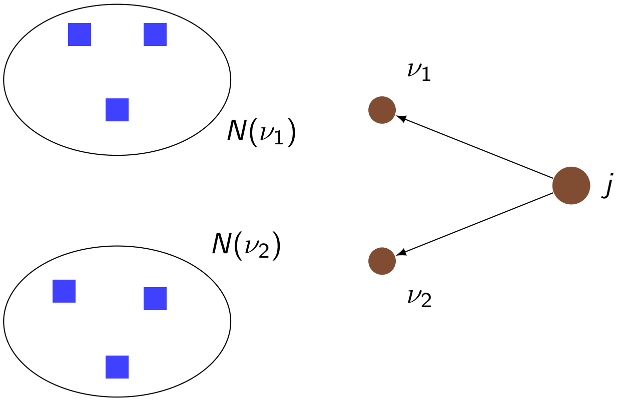
How to balance these two strategies?

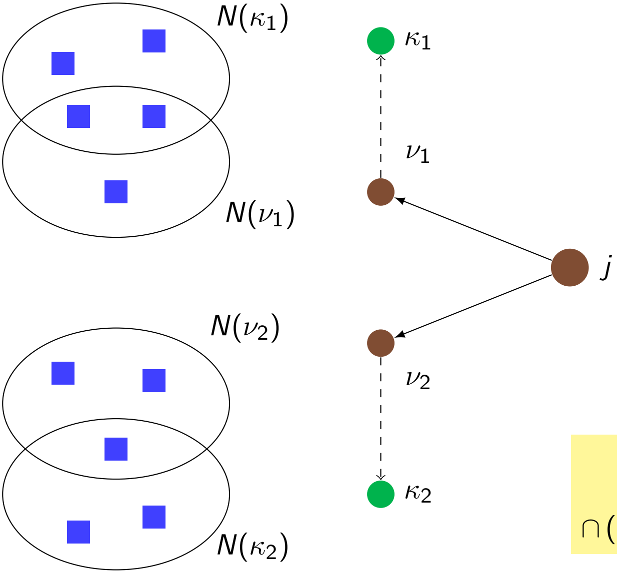
Two Types of Demands: Primary and Non-primary



Neighborhood Structure for Siblings

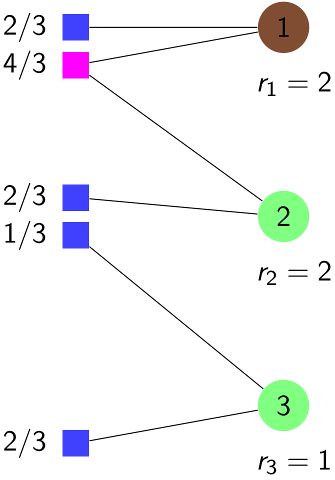


Neighborhood Structure for Siblings



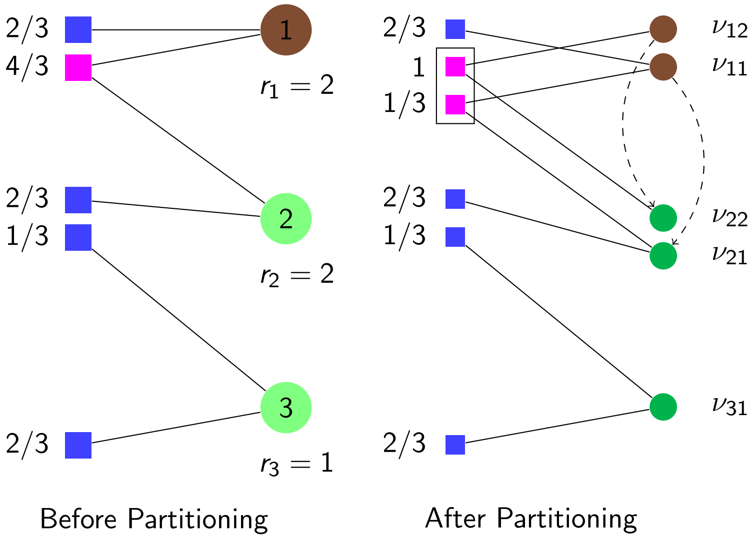
For siblings
$(N(\kappa_1) \cup N(\nu_1))$
$\cap (N(\kappa_2) \cup N(\nu_2)) = \emptyset$

Example of Partitioning



Before Partitioning

Example of Partitioning



Summary of Partitioning

Partitioning:

- Clients $\rightarrow$ demands
- Sites $\rightarrow$ facilities
- $(x^*, y^*) \rightarrow (\bar{x}, \bar{y})$
- $\sum_\mu \bar{x}_{\mu\nu} = 1$
- $\bar{x}_{\mu\nu} = \bar{y}_\mu$ or 0

Structure:

- If κ_1, κ_2 primary then $N(\kappa_1) \cap N(\kappa_2) = \emptyset$
- Each non-primary ν assigned to κ with
  - $N(\kappa) \cap N(\nu) \neq \emptyset$
  - priority$(\kappa) \leq$ priority (ν)
- $N(\kappa_1) \cup N(\nu_1) \ \cap \ N(\kappa_2) \cup N(\nu_2) = \emptyset$

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small facility cost

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small connection cost of ν

- $N(\kappa_1) \cup N(\nu_1) \;\cap\; N(\kappa_2) \cup N(\nu_2) = \emptyset$

Summary of Partitioning

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- Clients $\rightarrow$ demands
- Sites $\rightarrow$ facilities
- $(x^*, y^*) \rightarrow (\bar{x}, \bar{y})$
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- $N(\kappa_1) \cup N(\nu_1) \; \cap \; N(\kappa_2) \cup N(\nu_2) = \emptyset$

fault-tolerance

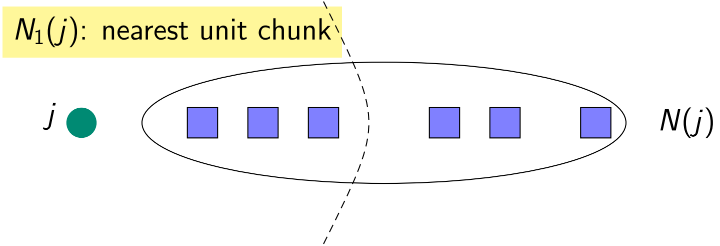
Partition Implementation

Partition implementation: two phases

- Phase 1, the partitioning phase
  - Define demands
  - Allocate facilities
- Phase 2, the augmenting phase
  - Add facilities to make neighborhood unit

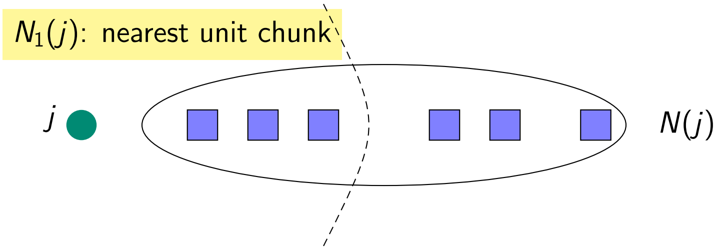
Phase 1

In each iteration, create one demand for best client



Phase 1

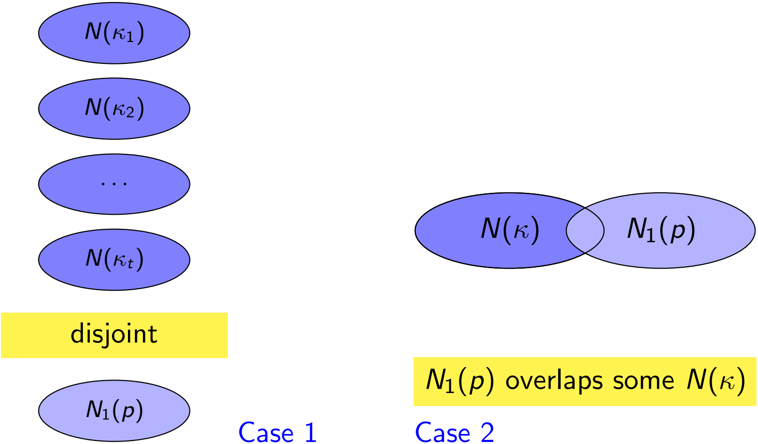
In each iteration, create one demand for best client



- $\text{bid}(j) = \text{avgdist}(N_1(j)) + \alpha_j^*$
- Best bid client p creates a demand
- Two cases, depending on $N_1(p)$

Phase 1, Step 2

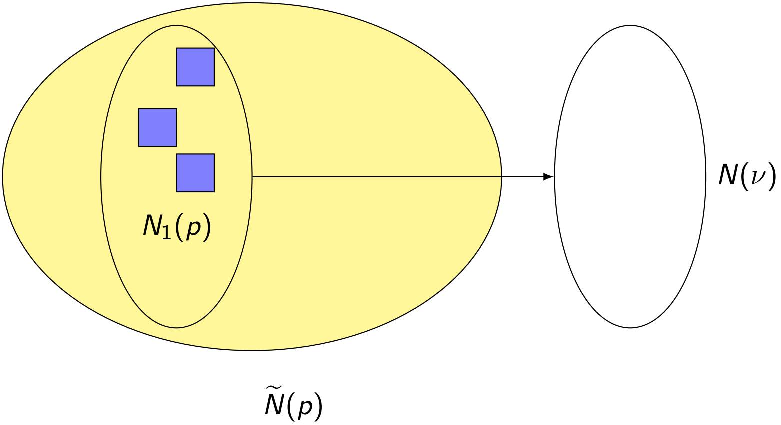
Selected best client p, two cases:



Case 1

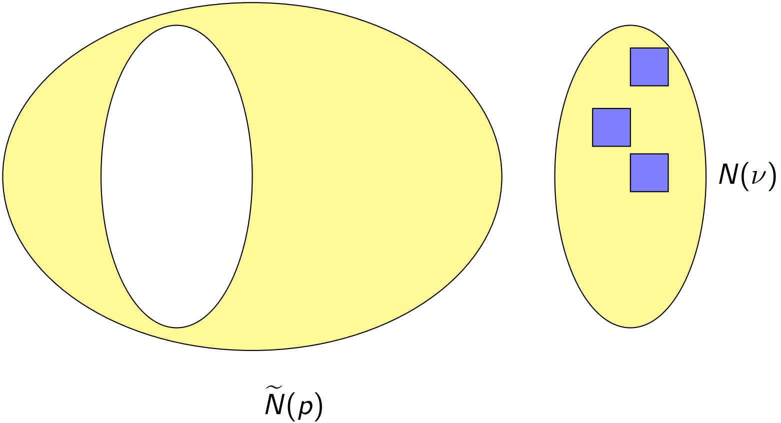
Case 2

Phase 1, Step 2 (Cont. Case 1)



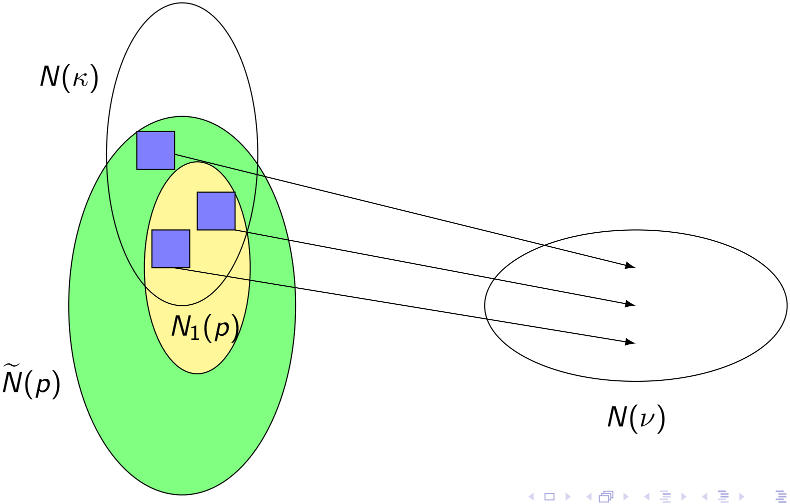
Phase 1, Step 2 (Case 1)

All facilities in $N_1(p)$ moved to $N(\nu)$



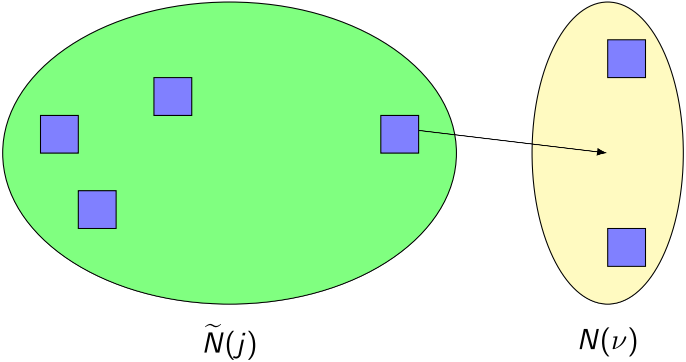
Phase 1, Step 2 (Cont. Case 2)

Move all overlapping facilities in $\widetilde{N}(p) \cap N(\kappa)$ into $N(\nu)$.



Phase 2

Add facilities from $\widetilde{N}(j)$ to $N(\nu)$ until total connection value is 1.



We are done with adaptive partitioning and got a structured fractional solution

Next: the rounding algorithms...

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3-Approximation for FTFP

Client priority values

- $\mathrm{tcc}(j) + \alpha_j^*$
  (average connection cost + dual value)

Rounding

- Facilities: Each primary κ opens random $\mu \in N(\kappa)$
- Connections: All demands assigned to κ connect to μ

Analysis

- Fault-Tolerance: ν uses only facilities in $N(\nu) \cup N(\kappa)$
- Cost: $\leq 3 \cdot \mathrm{LP}^*$, because
  - Facility cost $\leq F^*$
  - Connection cost $\leq C^* + 2 \cdot \mathrm{LP}^*$

1.736-Approximation for FTFP

Client priority values

- $\mathrm{tcc}(j) + \alpha_j^*$
  (average connection cost + dual value)

Rounding

- Facilities:
  - Each primary κ opens random $\mu \in N(\kappa)$
  - Other facilities open randomly independently
- Connections:
  - if a neighbor open, connect to nearest neighbor
  - else, connect via assigned primary demand

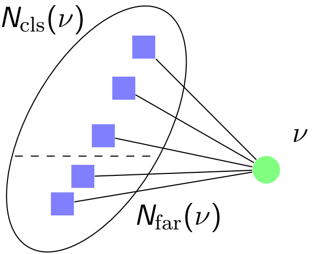
Analysis

- Fault-Tolerance: ν uses only facilities in $N(\nu) \cup N(\kappa)$
- Cost: $\leq (1 + 2/e)\,\mathrm{LP}^*$, because
  - Facility cost $\leq F^*$
  - Connection cost $\leq C^* + \frac{2}{e} \cdot \mathrm{LP}^*$

1.575-Approximation for FTFP – Idea

More intricate neighborhood structure

- Two neighborhoods: close and far, $N(\nu) = N_{\mathrm{cls}}(\nu) \cup N_{\mathrm{far}}(\nu)$
- $N_{\mathrm{cls}}(\nu) =$ nearest $(1/\gamma)$-fraction of $N(\nu)$
- $N_{\mathrm{cls}}(\nu) \cap N_{\mathrm{cls}}(\kappa) \neq \emptyset$, if ν assigned to κ
- For siblings ν_1, ν_2, $N_{\mathrm{cls}}(\kappa_1) \cup N(\nu_1)$ and $N_{\mathrm{cls}}(\kappa_2) \cup N(\nu_2)$ disjoint
- ...



1.575-Approximation for FTFP

Client priority values

- $\mathrm{tcc}_{\mathrm{cls}}(j) + \mathrm{dmax}_{\mathrm{cls}}(j)$
  (average + worst connection cost to close neighborhood)

Rounding (extension of Byrka's)

- Facilities:
  - Each primary κ opens random $\mu \in N_{\mathrm{cls}}(\kappa)$
  - Other facilities open randomly independently
- Connections:
  - if a neighbor open, connect to nearest neighbor
  - else, connect via assigned primary demand

Analysis

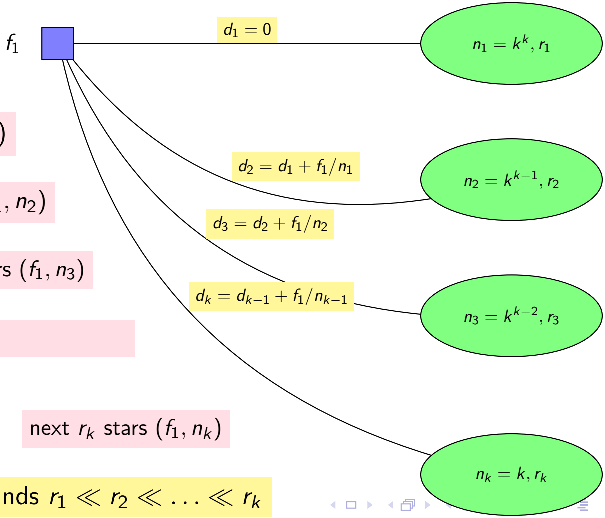
- Fault-Tolerance: ν uses only facilities in $N(\nu) \cup N_{\mathrm{cls}}(\kappa)$
- Cost: $\leq \gamma \cdot \mathrm{LP}$ for $\gamma = 1.575$, because
  - Facility cost $\leq \gamma \cdot F^*$
  - Connection cost $\leq \gamma \cdot C^*$

Greedy and Dual-fitting

- Greedy in polynomial time
  - Best star can be found quickly
  - Best star remains best
- Ratio H_n (Wolsey's result): Greedy is H_n-approx for
  - Minimizing a linear function
  - Subject to submodular constraint
- Lower bound $\Omega(\log n / \log \log n)$ for dual-fitting
  - Example has k groups, $n = k^k$
  - Shrinking factor is $k/2$

Dual-fitting Example

Dual feasibility forces a ratio of $k/2$, number of clients $n = k^k$



first r_1 stars (f_1, n_1)

next r_2 stars (f_1, n_2)

next r_3 stars (f_1, n_3)

next r_k stars (f_1, n_k)

demands $r_1 \ll r_2 \ll \ldots \ll r_k$

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- 1.575-approximation algorithm for FTFP
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Open Problems

- Can FTFL be approximated with the same ratio?
- LP-free algorithms for FTFP or FTFL with constant ratio?
- Close the 1.463 – 1.488 gap for UFL!