Proof of Chudak98 Connection Cost Bound

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1 Chudak98 Connection Cost

This section provides a proof, or more of an explanation, for the expected connection cost bound for the LP-rounding algorithm of Chudak IPCO'98 paper. The following argument is from Chudak and Shmoys SICOMP'03.

In the algorithm, each primary client connects to the only facility opened in its neighborhood. For non-primary clients, it connects to the nearest facility open in its neighborhood, if none opens, it uses the target facility.

The claim is that the connection cost can be bounded by a random process that opens each f_i independently with probability y_i and connects to the nearest facility, if none, connects to the target facility. In this document we use y_i to refer to y_i^*.

Consider a non-primary client j, we prove this claim by upper bounding the expected connection cost using a rounding process with provably worse connection cost. Let $g_1, g_2, \dots, g_k$ be the group of facilities in the neighborhood of primary clients $p_1, p_2, \dots, p_k$ such that $\bar{d}_1 \leq \bar{d}_2 \leq \dots, \leq \bar{d}_k$, where $\bar{d}_s = \sum_{i \in K_s} d_{ij} y_i / \sum_{i \in K_s} y_i$ and $K_s = N(j) \cap N(p_s)$. We also use the notation that $z_s = \sum_{i \in K_s} y_i$. The rounding process we use is to connect to a facility in g_1 if any facility in g_1 opens (this happens with probability z_1), otherwise we connect to a facility in g_2 (this happens with probability $z_2(1 - z_1)$), and so on. Notice that each group g_s can have at most one facility open, and the opening of a facility in each group is indpendent of other groups. We assume there are l facilities in $N(j)$, and both $\bar{d}_{k+1}$ and d_{l+1} refer to the expected cost of the connection to the target facility, that is $\bar{d}_{k+1} = d_{l+1}$. Clearly this kind of rounding with group preference is no better than simply connecting to the nearest open facility, which is what our algorithm does.

Now alg's cost is no more than:

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\begin{aligned}
E[C_j] &= \bar{d}_1 z_1 + \bar{d}_2 z_2 (1 - z_1) + \dots + \bar{z}_k (1 - z_i) \dots (1 - z_{k-1}) + \bar{d}_{k+1} (1 - z_1) \dots (1 - z_k) \\
&\leq (1 - \prod_{s=1}^{k} (1 - z_s)) \sum_{s=1}^{k} \bar{d}_s z_s + \prod_{s=1}^{k} (1 - z_s) \bar{d}_{k+1},
\end{aligned}
$$

with the inequality follows from another lemma proved with Chebyshev's sum inequality.

We know that $\sum_{s=1}^{k} \bar{d}_s z_s = \sum_{i=1}^{l} d_i y_i$ by definition of $\bar{d}_s$ and z_s. Moreover, $\prod_{s=1}^{k} (1 - z_s) \leq \prod_{i=1}^{l} (1 - y_i)$, by noticing $1 - (y_1 + y_2) \leq (1 - y_1)(1 - y_2)$ for any $y_1 + y_1 \leq 1, 0 \leq y_1, y_2 \leq 1$. In other words, splitting a facility makes the probability that none of them open higher. It follows that

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E[C_j] \leq (1 - \prod_{s=1}^{k} (1 - z_s)) \sum_{s=1}^{k} \bar{d}_s z_s + \prod_{s=1}^{k} (1 - z_s) \bar{d}_{k+1} \leq (1 - \prod_{i=1}^{l} (1 - y_i)) \sum_{i=1}^{l} d_i y_i + \prod_{i=1}^{l} (1 - y_i) d_{l+1}
$$

since we are decreasing the probability of the first term, which has a smaller distance, while increasing the probability of the second term, the sum of the probabilities of the two terms remains equal to 1, so the overal distance can only increase. The right side is what we need since $\sum_{i=1}^{l} d_i y_i = C_j^*$.

At a high level, we can think about the provably worse rounding process as treating the facilities in the same group as an aggregated facility and apply indepdent rounding on those aggregated facilities. The above didn't mention non-clustered facilities, that is, facilities that do not belong to the neighborhood of any primary client, for those, it is clear that each of them behave exactly like one aggregated facility and the same argument applies.