Approximation Algorithms for the Facility Location prolems

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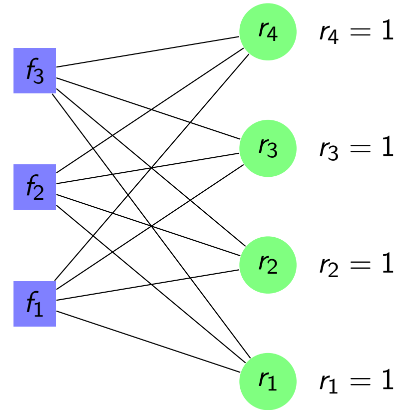
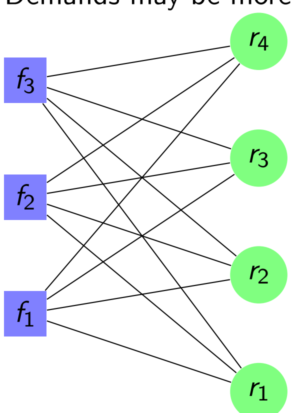
The Uncapacitated Facility Location Problem (UFL)

All demands are 1, each site can open only one facility.



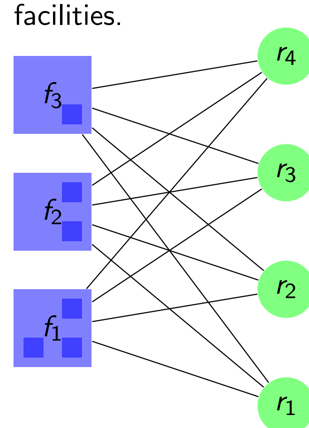
The Fault-tolerant Facility Location Problem (FTFL)

Demands may be more than 1, each site can open only one facility.



The Fault-tolerant Facility Placement Problem (FTFP)

Demands may be more than 1, each site can open multiple facilities.

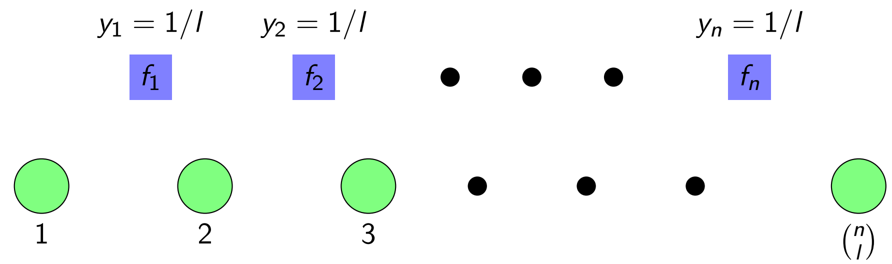


Best Known Approximation Results

- UFL: 1.488, a combination of LP-rounding and greedy, by Li (Princeton)
- FTFL: 1.7245, dependent rounding and laminar clustering, by Byrka, Srinivasan and Swamy (U Maryland)
- FTFP: 1.575, LP-rounding (UCR)

Lower Bound on Approximability

- No ratio better than 1.463 unless P = NP. Reduction from Set Cover, by Guha, Khuller, and Sviridenko.
- Integrality Gap is also 1.463, the example uses n facilities and $\binom{n}{l}$ clients. The fractional solution is each $y_i = 1/l$.



The LP Formulation

- y_i represent the number of facilities built at site i.
- x_{ij} represent the number of connections from client j to facilities at site i.

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_{i \in \mathbb{F}} f_i y_i + \sum_{i \in \mathbb{F}, j \in \mathbb{C}} d_{ij} x_{ij} & & (1) \\ \text{subject to} \quad & y_i - x_{ij} \geq 0 & \forall i \in \mathbb{F}, j \in \mathbb{C} & \\ & \textstyle\sum_{i \in \mathbb{F}} x_{ij} \geq r_j & \forall j \in \mathbb{C} & \\ & x_{ij} \geq 0, y_i \geq 0 & \forall i \in \mathbb{F}, j \in \mathbb{C} & \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{j \in \mathbb{C}} r_j \alpha_j & & (2) \\ \text{subject to} \quad & \textstyle\sum_{j \in \mathbb{C}} \beta_{ij} \leq f_i & \forall i \in \mathbb{F} & \\ & \alpha_j - \beta_{ij} \leq d_{ij} & \forall i \in \mathbb{F}, j \in \mathbb{C} & \\ & \alpha_j \geq 0, \beta_{ij} \geq 0 & \forall i \in \mathbb{F}, j \in \mathbb{C} & \end{aligned}$$

UFL Background:

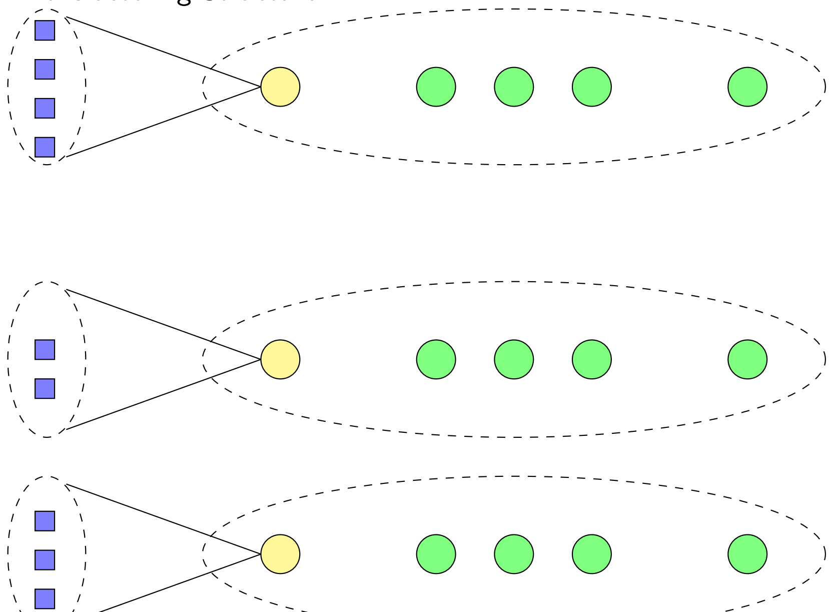
The Shmoys, Tardos and Ardal's Algorithm (STA97)

- If all $N(j)$ disjoint, then easy.
- To bound F^A, need neighborhood of chosen clients be disjoint.
- To bound C^A, need non-primary clients having a fail-over connection.

- The greedy clustering: iteratively find the best client and assign some other clients to it.
- Estimate $\max_{i \in N(j)} d_{ij}$, either cut the neighborhood $N(j)$ or use dual solution.

UFL background: Cont.

The Clustering Structure



UFL Background: End.

Chudak, Svi, Byrka and Li's improvement

- Chudak: randomized rounding, estimate on the expected connection cost
- Sviridenko: use a concave function to upper bound distance and to guide rounding
- Byrka: boost facility opening probability and use $N_{\mathrm{cls}}(j)$ for overlapping
- Li: find the right distribution for probability boost

Demand Reduction

Given an FTFP instance, we can reduce it to an instance such that $R = \max_j r_j$ is bounded by $|\mathbb{F}|$.

- $\hat{x}_{ij} = \lfloor x^*_{ij} \rfloor, \hat{y}_i = \lfloor y^*_i \rfloor$
- $\dot{x}_{ij} = x^*_{ij} - \hat{x}_{ij}, \dot{y}_i = y^*_i - \hat{y}_i$
- $\hat{r}_j = \sum_{i \in \mathbb{F}} \hat{x}_{ij}$
- $\dot{r}_j = r_j - \hat{r}_j$

Claim

$\hat{x}_{ij}, \hat{y}_i$ is feasible and optimal for $\hat{I}$, and $\dot{x}_{ij}, \dot{y}_i$ is feasible and optimal for $\dot{I}$.

Theorem

Given any $\rho \geq 1$ approximation algorithm $\mathcal{A}$ for solving restricted FTFP, we can obtain an algorithm with ρ-approximation for general FTFP.

Proof.

Solve LP and obtain $\hat{\mathcal{I}}$ and $\dot{\mathcal{I}}$. For $\hat{I}$ we have ratio 1, and use $\mathcal{A}$ to solve $\dot{\mathcal{I}}$ with ratio ρ. Final ratio is $\max\{1, \rho\}$. $\square$

Corollary

There is a 1.7245-approximation algorithm for the FTFP problem.

Proof.

We simply reduce the FTFP problem into the FTFL problem. The $\hat{\mathcal{I}}$ instance already has an integral solution $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$. Solving the instance $\dot{\mathcal{I}}$ using the 1.7245-approximation algorithm for FTFL by Byrka *et al.*. $\square$

Improve from 1.7245 to 1.575-approximation

- We have shown that FTFP can be reduced to FTFL while preserving the approximation ratio.
- Next step is to show FTFP can be approximated with a better ratio than FTFL.
- Simple case is when all r_j's are equal, then we can apply any UFL approximation results to FTFP as the uniform FTFP is simply a scaled version of UFL.
- For general FTFP, we need *Adaptive Partition*.

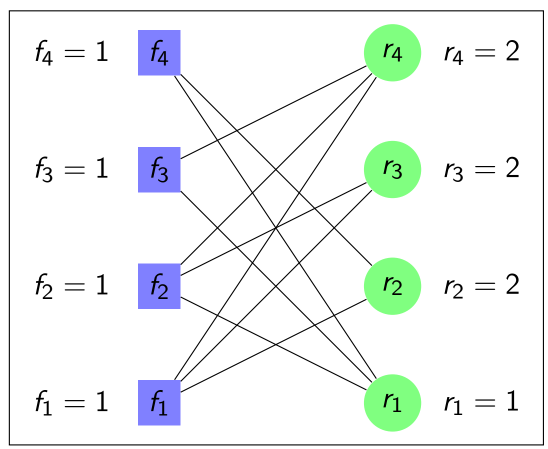
Adaptive Partition

Given an instance of FTFP, with its fractional optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$, w.l.o.g. we assume *completeness*, i.e. $x^*_{ij} > 0$ implies $x^*_{ij} = y^*_i$. Then we can partition the instance into unit demands and facilities, with fractional solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ such that

- x^*_{ij} is spread among its demands.
- y^*_i is spread among its facilities.
- Each demand ν has a neighborhood $\overline{N}(\nu)$ with total connectoin value of 1.
- Primary demands have a smaller cost than non-primary demands assigned to them.

An Example of Adaptive Partition

The instance has 4 sites and 4 clients.
Only $d_{ij} = 1$ edges are shown.



The Fractional Optimal Solution

y_i^*	1	2	3	4
	4/3	1/3	1/3	1/3

(a)

x_{ij}^*	$j=1$	2	3	4
$i=1$	0	4/3	4/3	4/3
2	1/3	0	1/3	1/3
3	1/3	1/3	0	1/3
4	1/3	1/3	1/3	0

(b)

Table: An optimal fractional solution to the FTFP instance.

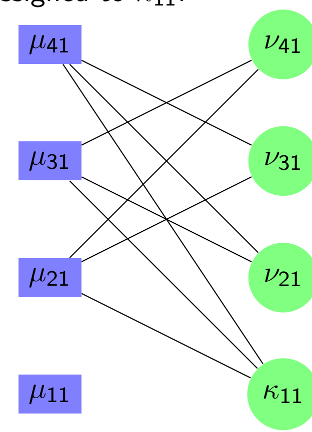
The dual solution has all $\alpha_j^* = 4/3$.

Phase 1: Iteration 1

Choose client 1 and create a primary demand κ_{11}. Each of client 2,3,4 creates a demand and assigned to κ_{11}.

$\bar{y}$	1	2	3	4
	1/3	1/3	1/3	1/3

$\bar{x}$	1	2	3	4
1	0	0	0	0
2	1/3	0	1/3	1/3
3	1/3	1/3	0	1/3
4	1/3	1/3	1/3	0

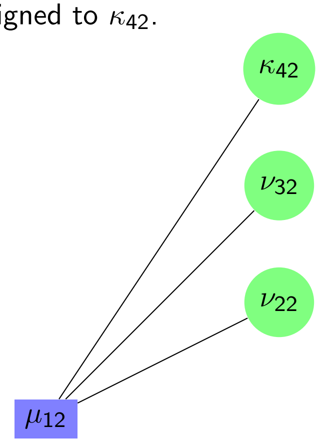


Phase 1: Iteration 2

Choose client 4 and create a primary demand κ_{42}. Each of client 2,3 creates a demand and assigned to κ_{42}.

$\bar{y}$	1	2	3	4
	1			

$\bar{x}$	1	2	3	4
2	1	0	0	0
3	1	0	0	0
4	1	0	0	0

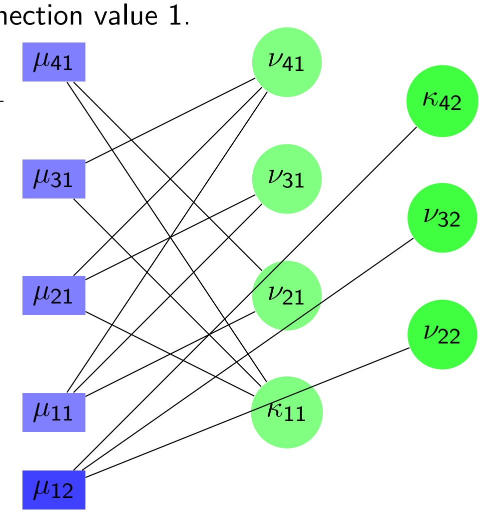


Phase 2: Augment to Unit

Notice all demands have connection value 1.

$\bar{x}$	1	2	3	4
1	0	1/3	1/3	1/3
2	1/3	0	1/3	1/3
3	1/3	1/3	0	1/3
4	1/3	1/3	1/3	0

$\bar{x}$	2	3	4
1	1	1	1
2	0	0	0
3	0	0	0
4	0	0	0



3-approximation Algorithm

- Each primary demand open $\mu \in \overline{N}(\kappa)$ with probability $\bar{y}_\mu$.
- Each primary demand connects to the only open facility $\phi(\kappa)$ in $\overline{N}(\kappa)$.
- Each non-primary demand connects to $\phi(\kappa)$.
- Expected facility cost at most F^*.
- Expected connection cost at most $C^* + 2\,\mathrm{LP}^*$.

1.736-approximation Algorithm

- Improve connection cost estimate: For non-primary demands use μ in $\overline{N}(\nu)$ if one is open.
- Expected facility cost at most F^*.
- Expected connection cost at most $C^* + 2/e\,\mathrm{LP}^*$.

1.575-approximation

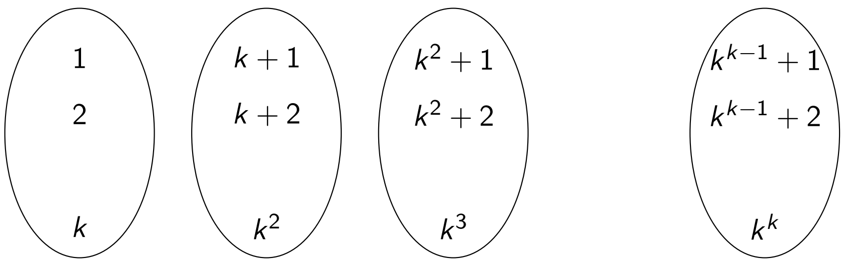
- Need a more refined partition to deal with close and far neighborhood.
- $\overline{N}_{\mathrm{cls}}(\nu)$ has total connection value $1/\gamma$.
- $\overline{N}_{\mathrm{far}}(\nu)$ has total connection value $1 - 1/\gamma$.
- Assignment implies overlap of $\overline{N}_{\mathrm{cls}}$ of ν and κ.
- Expected facility cost at most γF^*.
- Expected connection cost at most $\max\{\frac{1+1/e^\gamma}{1-1/e^\gamma}, 1 + 2/e^\gamma\}\ C^*$.
- Picking $\gamma = 1.575$ gives ratio 1.575.

Primal-dual Algorithms

- A Simple $O((\log R/\log\log R)^2)$ Algorithm.
- Greedy Algorithm with Dual-fitting Analysis.

A Simple $O((\log R/\log\log R)^2)$ Algorithm

- Let $r_1, \ldots, r_n$ be demands of the n clients.
- Group clients by $[k^{l-1}+1,\ k^l]$ for k such that $k^k = R = \max_j r_j$.
- Solve each group by treating each client with $r_j = k^l$.
- Combine all solution to each group to obtain final integral solution.



Performance Analysis

Theorem

There is a primal-dual $O((\log R/\log\log R)^2)$-approximation algorithm for FTFP.

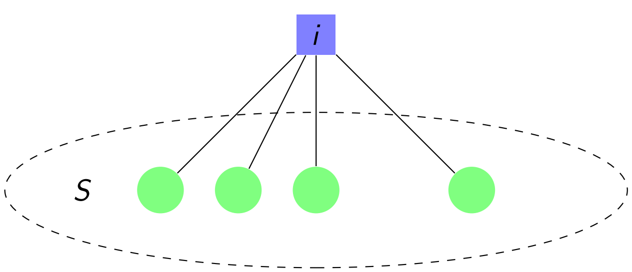
Proof.

- Solving each group individually, by treating it as uniform demand instance with all $r_j = k^l$ for the l^{th} group. We pay a factor of k for each group, since each r_j is within a factor of k of k^l.
- When combining solutions, we pay a factor of k since each facility can be over counted at most k times. Notice we have k groups because $k^k = R$.

$\square$

The Greedy Algorithm

- Repeatedly picking the star with minimum average cost.
- A star is a facility i and a set of clients S.
- Average cost is $(f_i + \sum_{j\in S} d_{ij})/|S|$.



Known Facts about Greedy

- Runs in polynomial time as the best star remains best until exhausted so can combine iterations into phases.
- $O(H_n)$-approximation by dual-fitting analysis.
- Open question: Is it $O(1)$-approximation?

Summary

We studied the fault-tolerante facility placement problem (FTFP) on approximation algorithms.

- Known results
  - LP-rounding algorithms achieve a best ratio of 1.575, matching the best LP-based ratio for its special case, UFL.
  - Primal-dual algorithms achieve $O(\log R / \log\log R)$, better than the $O(\log R)$ ratio for primal-dual algorithm for FTFL.
  - The greedy algorithm has ratio no more than $O(H_n)$.
- Work in progress: Resolve whether Greedy is $O(1)$-approximation or not.