On primal-dual and dual-fitting of the FTFP problem

UCR theory lab

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1 Dual-fitting Analysis on FTFP

This section gives a simple example that the dual-fitting analysis of a greedy algorithm which repeatedly picking the most cost-effective star (the star with minimum average cost) is unlikely to give the same ratio as that for the UFL problem.

The example consists of 1 facility with cost $f_1 = n$, and n clients with demands $r_1 = r_2 = \ldots = r_{n-1} = 1$ and $r_n = n$. All $d_{ij} = 0$. Now running the star-greedy algorithm, we will first pick a star with all n clients and we open 1 copy of facility f_1. We then have only client n with residual demand $r'_n = n - 1$, and we have no other option but to open facility f_1 for another $n - 1$ copies.

Now the dual-fitting based analysis will associate each demand with a dual variable $\alpha_j^1, \ldots, \alpha_j^{r_j}$ and the proposed dual solution is $\bar{\alpha}_j = \sum_{l=1}^{r_j} \alpha_j^l / r_j$ and try to find a minimum γ such that $\{\bar{\alpha}_j/\gamma\}$ is a feasible dual, that is

$$\sum_{j=1}^{n} (\bar{\alpha}_j/\gamma - d_{1j})_+ \leq f_1 = n \tag{1}$$

which is

$$\sum_{j=1}^{n} \bar{\alpha}_j/\gamma \leq n \tag{2}$$

since all $d_{ij} = 0$ and $f_1 = n$.

From the greedy algorithm, we have $\alpha_j^1 = 1$ for $j = 1, \ldots, n$, and $\alpha_n^l = n$ for $l = 2, \ldots, n$. Therefore $\bar{\alpha}_j = 1$ for $j = 1, \ldots, n-1$ and $\bar{\alpha}_n = (1 + (n-1)n)/n = n - 1$. The shrinking factor γ we seek thus satisfies

$$\sum_{j=1}^{n-1} (\bar{\alpha}_j/\gamma - 0)_+ + (\bar{\alpha}_n/\gamma - 0)_+ \leq f_1 = n, \tag{3}$$

which is

$$(n-1)(1/\gamma) + (n/\gamma) \leq n \tag{4}$$

Simple algebra will show that γ can be made arbitrarily close to 2 when n is large. On the other hand we know that the same greedy algorithm with dual-fitting analysis gives a ratio of 1.81 for the UFL problem where all $r_j = 1$.

This example does not actually rule out the possibility to prove a constant ratio of the star-greedy algorithm on FTFP. In fact greedy gets exactly the same solution as the optimal integral solution for this example. All it says is that the dual-fitting analysis on greedy algorithm, when applied to the FTFP or FTFL problem, cannot possibly give a ratio much better than 2. And this partly explains why generalizing primal-dual or dual-fitting algorithms from UFL to fault-tolerant problems like FTFL or FTFP is not successful when r_j's are not equal, that is demands are not uniform. Intuitively there seems to be some issue fundamental to the dual-fitting approach as the proposed dual solution $\bar{\alpha}_j$'s can be very different between each other so shrinking all of them by a common factor γ might not give a strong upper bound on the approximation ratio. It is also quite possible that the example may be strengthened to show that dual-fitting cannot achieve a worse yet ratio.