

Approximation Algorithms for the Fault-Tolerant Facility Placement Problem

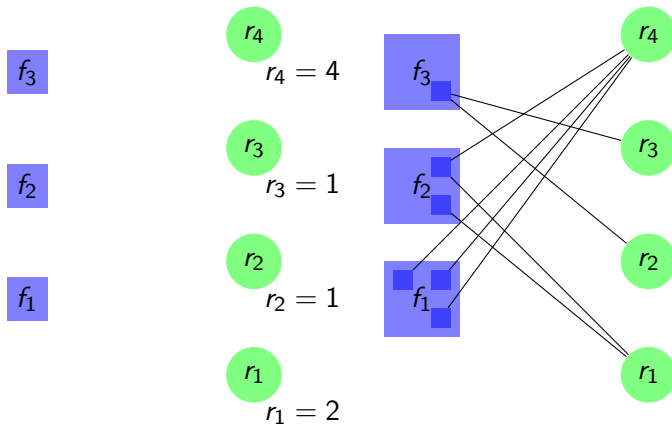
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 - Main result
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 - The Fault-tolerant Facility Location problem (FTFL)
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The Fault-tolerant Facility Placement Problem (FTFP)



Instance

Solution

Cost is $3f_1 + 2f_2 + f_3 + 2d_{21} + d_{32} + d_{33} + 3d_{14} + d_{24}$.

The Fault-Tolerant Facility Placement Problem (FTFP)

Given

- \mathbb{F} , a set of sites can have facilities built,
- \mathbb{C} , a set of clients with demands,
- r_j , demand for client j ,
- f_i , cost to build one facility at site i ,
- d_{ij} , cost to connect one demand from client j to facility at site i . Distances form a metric.

Find

- y_i , number of facilities to build at each site,
- x_{ij} , number of connections between site i and client j .

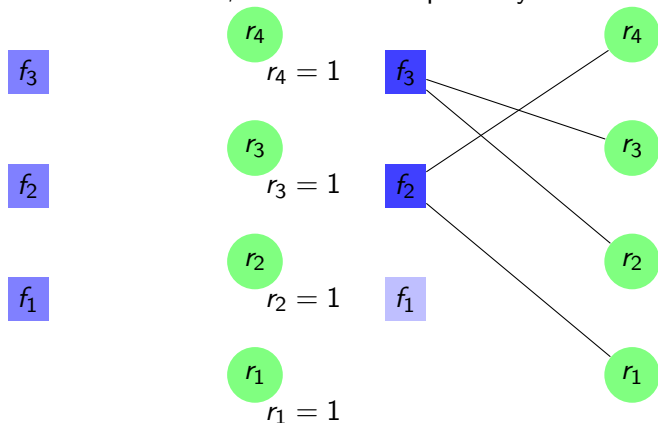
Goal: Minimize the total cost of opening facilities, sum of $f_i y_i$ and connecting clients, sum of $d_{ij} x_{ij}$.

An LP-rounding algorithm with approximation ratio 1.575.

- The Uncapacitated Facility Location problem (UFL), all $r_j = 1$, best approximation ratio 1.488 (Li'12).
- The Fault-tolerant Facility Location problem (FTFL), all $y_i \leq 1$, best approximation ratio 1.7245 (Byrka *et al.*'10).
- The Fault-tolerant Facility Placement problem (FTFP), best approximation ratio 1.575 (this paper), matching the best known LP-based approximation ratio for UFL.

The Uncapacitated Facility Location Problem (UFL)

All demands are 1, each site can open only one facility.

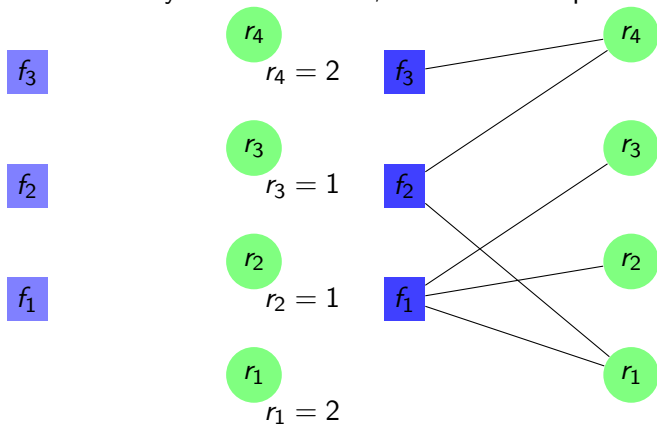


Instance

Solution

The Fault-tolerant Facility Location Problem (FTFL)

Demands may be more than 1, each site can open only one facility.

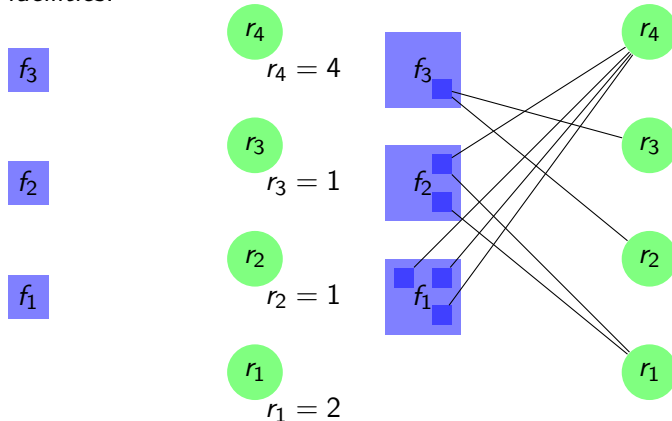


Instance

Solution

The Fault-tolerant Facility Placement Problem (FTFP)

Demands may be more than 1, each site can open multiple facilities.



Instance

Solution

The LP Formulation for FTFP

- y_i represent the number of facilities built at site i .
- x_{ij} represent the number of connections from client j to facilities at site i .

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in \mathbb{F}} f_i y_i + \sum_{i \in \mathbb{F}, j \in \mathbb{C}} d_{ij} x_{ij} & (1) \\ \text{subject to} \quad & y_i - x_{ij} \geq 0 & \forall i \in \mathbb{F}, j \in \mathbb{C} \\ & \sum_{i \in \mathbb{F}} x_{ij} \geq r_j & \forall j \in \mathbb{C} \\ & x_{ij} \geq 0, y_i \geq 0 & \forall i \in \mathbb{F}, j \in \mathbb{C} \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & \sum_{j \in \mathbb{C}} r_j \alpha_j & (2) \\ \text{subject to} \quad & \sum_{j \in \mathbb{C}} \beta_{ij} \leq f_i & \forall i \in \mathbb{F} \\ & \alpha_j - \beta_{ij} \leq d_{ij} & \forall i \in \mathbb{F}, j \in \mathbb{C} \\ & \alpha_j \geq 0, \beta_{ij} \geq 0 & \forall i \in \mathbb{F}, j \in \mathbb{C} \end{aligned}$$

- Demand Reduction.
- Adaptive Partition.

- Reduce a general FTFP instance to a restricted FTFP instance with $r_j \leq |\mathbb{F}|$ for all clients j .
- Solving LP to obtain $(\mathbf{x}^*, \mathbf{y}^*)$.
- Round down $(\mathbf{x}^*, \mathbf{y}^*)$ to obtain integral part $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$. Define $\hat{r}_j = \sum_i \hat{x}_{ij}$.
- The rest form fractional part $(\dot{\mathbf{x}}, \dot{\mathbf{y}})$. Define $\dot{r}_j = r_j - \hat{r}_j$.
- Both parts are feasible and optimal for their respective FTFP instances $\hat{\mathcal{I}}$ and $\dot{\mathcal{I}}$.

Claim

$\dot{r}_j \leq |\mathbb{F}|$ for all clients j in $\dot{\mathcal{I}}$.

Theorem

Given any ρ -approximation algorithm \mathcal{A} for the restricted FTFP problem with $r_j \leq |\mathbb{F}|$, if ρ is an upper bound on comparing algorithm's cost and the optimal fractional solution's cost, then we have a ρ -approximation algorithm for the general FTFP problem.

Corollary

Using 1.7245-approximation algorithm for FTFL, can have a 1.7245-approximation algorithm for FTFP.

Adaptive Partition

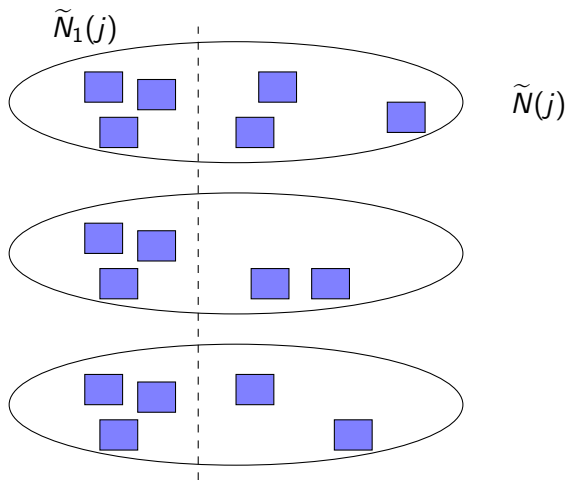
- Begin with a fractional complete solution (\mathbf{x}, \mathbf{y}) .
- In the partitioned solution,
 - Each site i has facilities μ .
 - Each client j has r_j demand points ν .
 - Each facility μ has fractional opening \bar{y}_μ .
 - Each demand point connects to each facility with value $\bar{x}_{\mu\nu}$.
- The partitioned solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies a number of properties.
 - y_i^* distributed among facilities at site i ,
 - x_{ij}^* distributed among sibling demands of client j ,
 - $\bar{x}_{\mu\nu} = \bar{y}_\mu$ or 0 (completeness),
 - Each demand ν is assigned to a primary demand κ with a low cost.

Two phases

- Phase 1, the partitioning phase, define demands and allocate facilities.
- Phase 2, the augmenting phase, allocate additional facilities to make total connection value unit.

Phase 1, Step 1

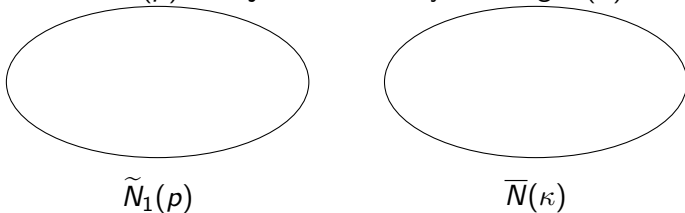
For each client j with residual demand $\bar{r}_j > 0$, arrange neighboring facilities from near to far. The nearest few with total connection value 1 defines $\bar{N}_1(j)$.



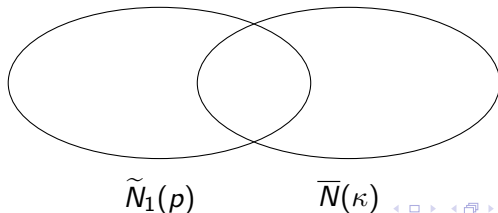
Phase 1, Step 2

Select client p such that the sum of the average distance to $\tilde{N}_1(p)$ and α_p^* is minimized. Now we have two cases to proceed.

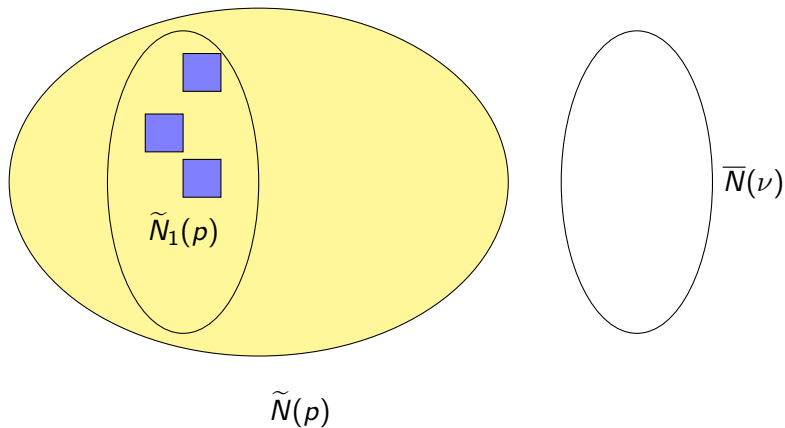
- Case 1: $\tilde{N}_1(p)$ is disjoint from every existing $\bar{N}(\kappa)$.



- Case 2: $\tilde{N}_1(p)$ overlaps with some $\bar{N}(\kappa)$.

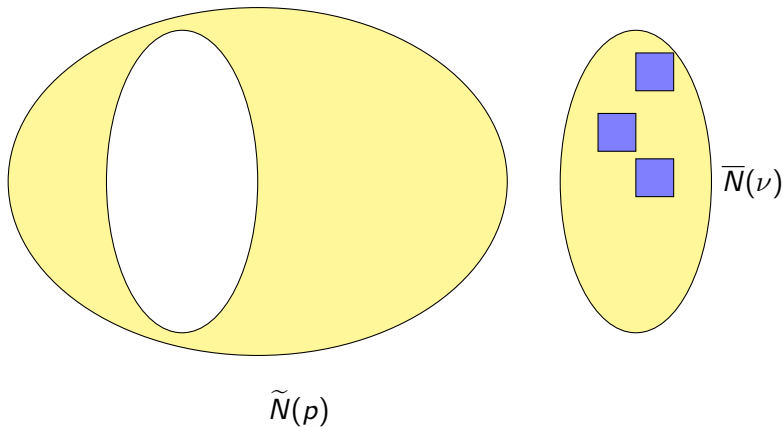


Phase 1, Step 2 (Cont. Case 1)



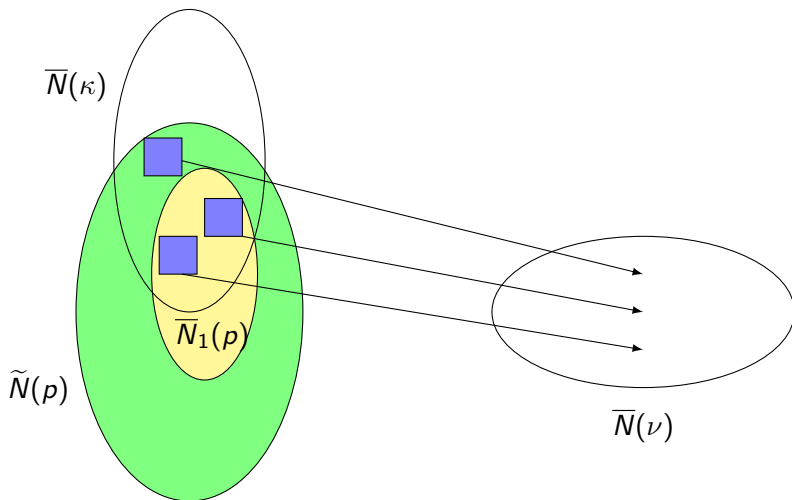
Phase 1, Step 2 (Cont. Case 1)

All facilities in $\tilde{N}_1(p)$ moved to $\overline{N}(\nu)$.



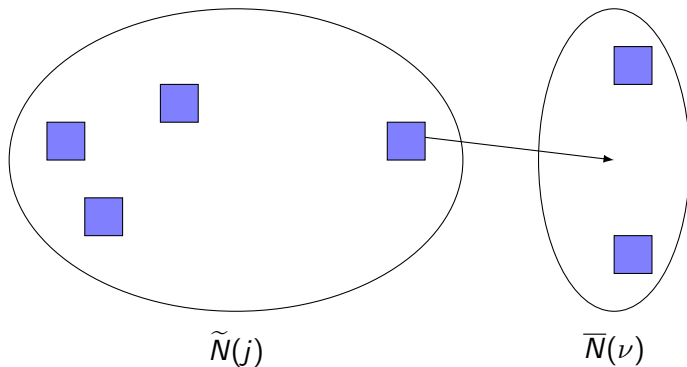
Phase 1, Step 2 (Cont. Case 2)

Move all overlapping facilities in $\tilde{N}(p) \cap \overline{N}(\kappa)$ into $\overline{N}(\nu)$.

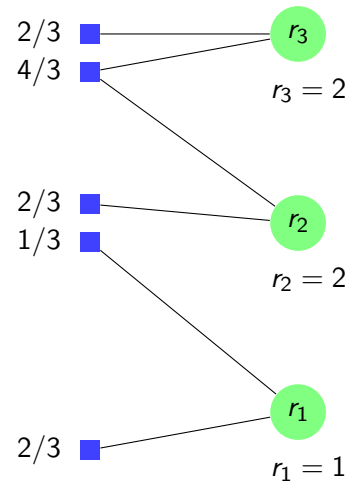


Phase 2

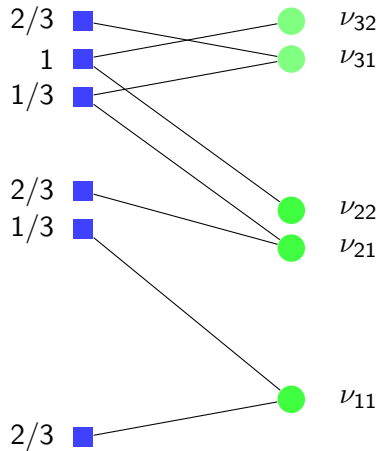
Add facilities from $\tilde{N}(j)$ to $\overline{N}(\nu)$ until total connection value is 1.



An Example of Partition



Before Partition



After Partition

Properties of Partition

Properties

- Each demand ν assigned to a primary demand κ with overlapping $\overline{N}(\nu)$ and $\overline{N}(\kappa)$.
- For sibling demands ν_1 and ν_2 , $\overline{N}(\nu_1) \cup \overline{N}(\kappa_1)$ is disjoint from $\overline{N}(\nu_2) \cup \overline{N}(\kappa_2)$.
- For a certain cost specified by the approximation algorithm, κ always have a lower cost compared to the assigned ν .

Implication

- The fractional solution can be rounded to a fault-tolerant integral solution.
- The cost of the integral solution can be approximated.

A 3-approximation Algorithm

Given $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, rounded by

- For each primary κ , choose facility μ in neighborhood with probability \bar{y}_μ .
- For each non-primary ν , connects to $\phi(\kappa)$, the facility chosen in the primary's neighborhood.

The rounded solution satisfies fault-tolerant requirement.

The rounded solution has cost at most $3LP^*$.

- Facility cost is at most F^* .
- For each demand ν , connection cost is at most $\sum_{\mu \in \bar{N}(\nu)} d_{\mu\nu} \bar{x}_{\mu\nu} + 2\alpha_\nu^*$.

A 1.736-approximation Algorithm

- Change in rounding:
 - For facilities μ not in any $\bar{N}(\kappa)$, round independently.
 - each non-primary ν uses nearest neighboring facility if one is open, else use $\phi(\kappa)$.
- The expected connection cost for ν now reduced to $\sum_{\mu \in \bar{N}(\nu)} d_{\mu\nu} \bar{x}_{\mu\nu} + 2/e \cdot \alpha_{\nu}^*$.

Refined Partition for 1.575-approximation

Properties

- $\overline{N}(\nu)$ consists of $\overline{N}_{\text{cls}}(\nu)$ and $\overline{N}_{\text{far}}(\nu)$ and they are disjoint.
- $\overline{N}_{\text{cls}}(\nu)$ overlaps with $\overline{N}_{\text{cls}}(\kappa)$.
- For siblings ν_1, ν_2 , $\overline{N}_{\text{cls}}(\kappa_1) \cup \overline{N}(\nu_1)$ disjoint from $\overline{N}_{\text{cls}}(\kappa_2) \cup \overline{N}(\nu_2)$.
- cost of κ is smaller than cost of ν .

Construction of Partition

- Allocation.
- Augmentation.

1.575-approximation

- Use Byrka's rounding.
- ν uses only facilities in $\overline{N}(\nu) \cup \overline{N}_{\text{cls}}(\kappa)$. Thus no two sibling conflict.
- Cost analysis is similar to Byrka's for UFL.

The End.