Approximation Algorithms for the Fault-Tolerant Facility Placement Problem

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Outline

The Fault-Tolerant Facility Placement Problem (FTFP)

Given

- $\mathbb{F}$, a set of sites can have facilities built,
- $\mathbb{C}$, a set of clients with demands,
- r_j, demand for client j,
- f_i, cost to build one facility at site i,
- d_{ij}, cost to connect one demand from client j to facility at site i. Distances form a metric.

Find

- y_i, number of facilities to build at each site,
- x_{ij}, number of connections between site i and client j.

Goal: Minimize the total cost of opening facilities, sum of $f_i y_i$ and connecting clients, sum of $d_{ij} x_{ij}$.

Main Result

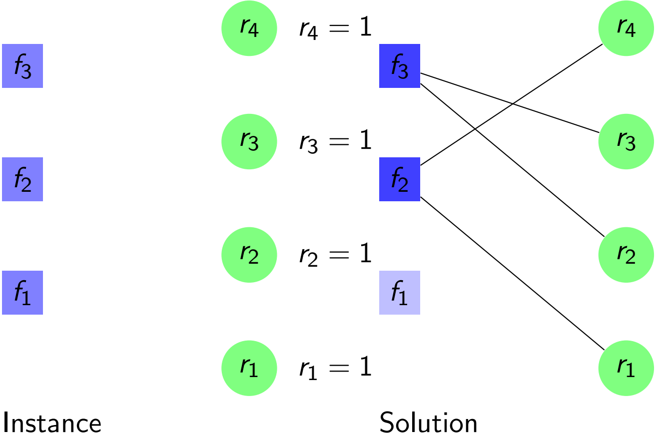
An LP-rounding algorithm with approximation ratio 1.575.

Related Problems

- The Uncapacitated Facility Location problem (UFL), all $r_j = 1$, best approximation ratio 1.488 (Li'12).
- The Fault-tolerant Facility Location problem (FTFL), all $y_i \leq 1$, best approximation ratio 1.7245 (Byrka *et al.*'10).
- The Fault-tolerant Facility Placement problem (FTFP), best approxiamtion ratio 1.575 (this paper), matching the best known LP-based approximation ratio for UFL.

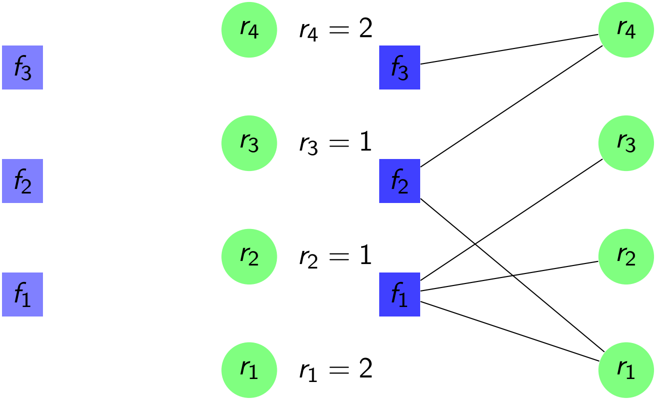
The Uncapacitated Facility Location Problem (UFL)

All demands are 1, each site can open only one facility.



The Fault-tolerant Facility Location Problem (FTFL)

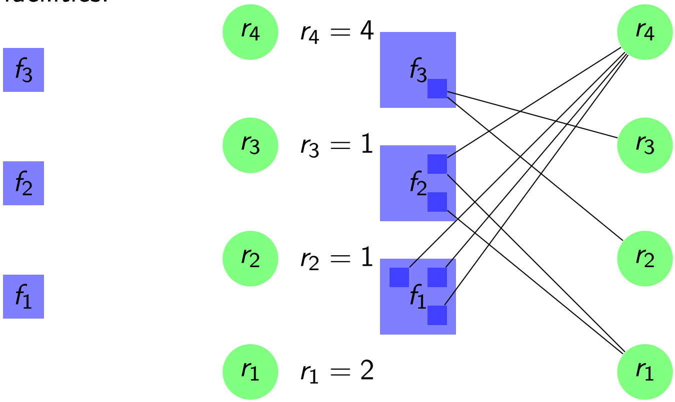
Demands may be more than 1, each site can open only one facility.



Instance

Solution

Demands may be more than 1, each site can open multiple facilities.



Instance

Solution

The LP Formulation for FTFP

- y_i represent the number of facilities built at site i.
- x_{ij} represent the number of connections from client j to facilities at site i.

$$
\begin{aligned}
\text{minimize} \quad & \textstyle\sum_{i \in \mathbb{F}} f_i y_i + \sum_{i \in \mathbb{F}, j \in \mathbb{C}} d_{ij} x_{ij} & & (1)\\
\text{subject to} \quad & y_i - x_{ij} \geq 0 & \forall i \in \mathbb{F}, j \in \mathbb{C} & \\
& \textstyle\sum_{i \in \mathbb{F}} x_{ij} \geq r_j & \forall j \in \mathbb{C} & \\
& x_{ij} \geq 0, y_i \geq 0 & \forall i \in \mathbb{F}, j \in \mathbb{C} &
\end{aligned}
$$

$$
\begin{aligned}
\text{maximize} \quad & \textstyle\sum_{j \in \mathbb{C}} r_j \, \alpha_j & & (2)\\
\text{subject to} \quad & \textstyle\sum_{j \in \mathbb{C}} \beta_{ij} \leq f_i & \forall i \in \mathbb{F} & \\
& \alpha_j - \beta_{ij} \leq d_{ij} & \forall i \in \mathbb{F}, j \in \mathbb{C} & \\
& \alpha_j \geq 0, \beta_{ij} \geq 0 & \forall i \in \mathbb{F}, j \in \mathbb{C} &
\end{aligned}
$$

Techniques

- Demand Reduction.
- Adaptive Partition.

Demand Reduction

- Reduce a general FTFP instance to a restricted FTFP instance with $r_j \leq |\mathbb{F}|$ for all clients j.
- Solving LP to obtain $(\mathbf{x}^*, \mathbf{y}^*)$.
- Round down $(\mathbf{x}^*, \mathbf{y}^*)$ to obtain integral part $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$. Define $\hat{r}_j = \sum_i \hat{x}_{ij}$.
- The rest form fractional part $(\dot{\mathbf{x}}, \dot{\mathbf{y}})$. Define $\dot{r}_j = r_j - \hat{r}_j$.
- Both parts are feasible and optimal for their respective FTFP instances $\hat{\mathcal{I}}$ and $\dot{\mathcal{I}}$.

Claim

$\dot{r}_j \leq |\mathbb{F}|$ *for all clients j in $\dot{\mathcal{I}}$.*

Theorem

Theorem

Given any ρ-approximation algorithm $\mathcal{A}$ for the restricted FTFP problem with $r_j \leq |\mathbb{F}|$, if ρ is an upper bound on comparing algorithm's cost and the optimal fractional solution's cost, then we have a ρ-approximation algorithm for the general FTFP problem.

Corollary

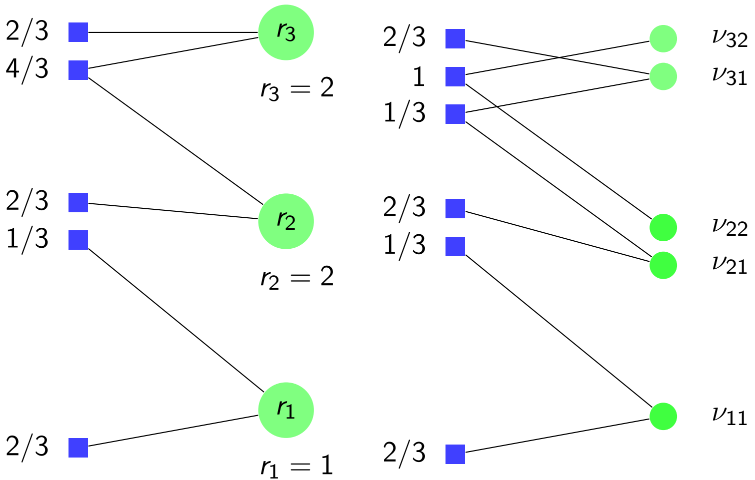
Using 1.7245*-approximation algorithm for FTFL, can have a* 1.7245*-approximation algorithm for FTFP.*

Adaptive Partition

- Begin with a fractional complete solution $(\mathbf{x}, \mathbf{y})$.
- In the partitioned solution,
  - Each site i has facilities μ.
  - Each client j has r_j demand points ν.
  - Each facility μ has fractional opening $\bar{y}_\mu$.
  - Each demand point connects to each facility with value $\bar{x}_{\mu\nu}$.
- The partitioned solution $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ satisfies a number of properties.
  - y_i^* distributed among facilities at site i,
  - x_{ij}^* distributed among sibling demands of client j,
  - $\bar{x}_{\mu\nu} = \bar{y}_\mu$ or 0 (completeness),
  - Each demand ν is assigned to a primary demand κ with a low cost.

Animation of Partition

An Example of Partition



Before Partition

After Partition

Properties of Partition

Properties

- Each demand ν assigned to a primary demand κ with overlapping $\overline{N}(\nu)$ and $\overline{N}(\kappa)$.
- For sibling demands ν_1 and ν_2, $\overline{N}(\nu_1) \cup \overline{N}(\kappa_1)$ is disjoint from $\overline{N}(\nu_2) \cup \overline{N}(\kappa_2)$.
- For a certain cost specified by the approximation algorithm, κ always have a lower cost compared to the assigned ν.

Implication

- The fractional solution can be rounded to a fault-tolerant integral solution.
- The cost of the integral solution can be approximated.

A 3-approximation Algorithm

Given $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, rounded by

- For each primary κ, choose facility μ in neighborhood with probability $\bar{y}_\mu$.
- For each non-primary ν, connects to $\phi(\kappa)$, the facility chosen in the primary's neighborhood.

The rounded solution satisfies fault-tolerant requirement.
The rounded solution has cost at most $3\mathrm{LP}^*$.

- Facility cost is at most F^*.
- For each demand ν, connection cost is at most $\sum_{\mu \in \overline{N}(\nu)} d_{\mu\nu}\bar{x}_{\mu\nu} + 2\alpha^*_\nu$.

A 1.736-approximation Algorithm

- Change in rounding:
  - For facilities μ not in any $\overline{N}(\kappa)$, round indedendently.
  - each non-primary ν uses nearest neighboring facility if one is open, else use $\phi(\kappa)$.
- The expected connection cost for ν now reduced to $\sum_{\mu \in \overline{N}(\nu)} d_{\mu\nu} \bar{x}_{\mu\nu} + 2/e \cdot \alpha^*_\nu$.

Refined Partition for 1.575-approximation

Properties

- $\overline{N}(\nu)$ consists of $\overline{N}_{\text{cls}}(\nu)$ and $\overline{N}_{\text{far}}(\nu)$ and they are disjoint.
- $\overline{N}_{\text{cls}}(\nu)$ overlaps with $\overline{N}_{\text{cls}}(\kappa)$.
- For siblings ν_1, ν_2, $\overline{N}_{\text{cls}}(\kappa_1) \cup \overline{N}(\nu_1)$ disjoint from $\overline{N}_{\text{cls}}(\kappa_2) \cup \overline{N}(\nu_2)$.
- cost of κ is smaller than cost of ν.

Construction of Partition

- Allocation.
- Augmentation.

- Use Byrka's rounding.
- ν uses only facilities in $\overline{N}(\nu) \cup \overline{N}_{\rm cls}(\kappa)$. Thus no two sibling conflict.
- Cost analysis is similar to Byrka's for UFL.

The End.