

# Approximation Algorithms for the Fault-Tolerant Facility Placement Problem

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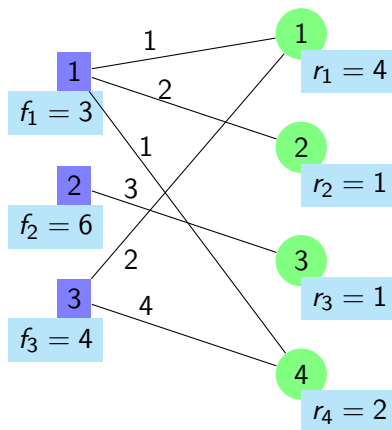
# Outline

- 1 The FTFP Problem
- 2 Results in Dissertation
- 3 Related Work
- 4 Techniques
- 5 Approximation Algorithms
- 6 Summary

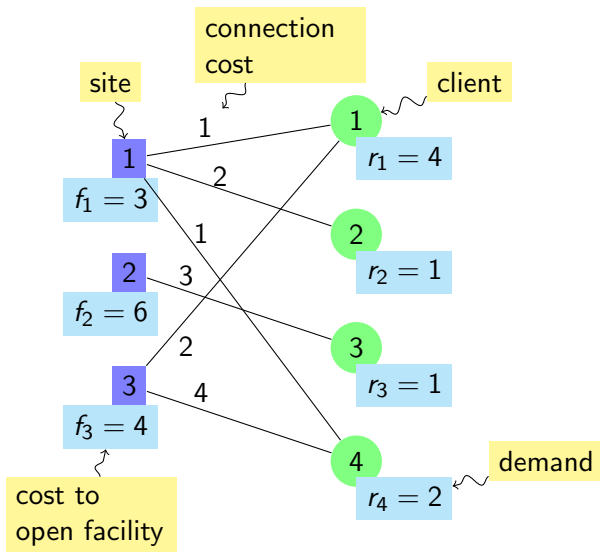
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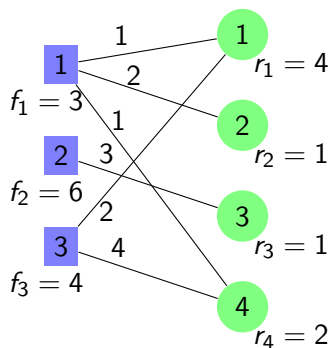
# Fault-Tolerant Facility Placement Problem (FTFP)



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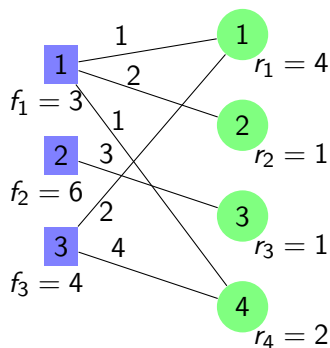


# Feasible Integral Solution

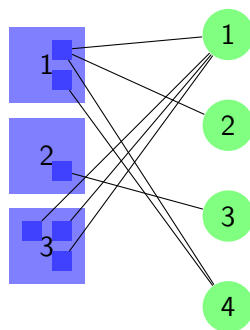


Instance

# Feasible Integral Solution

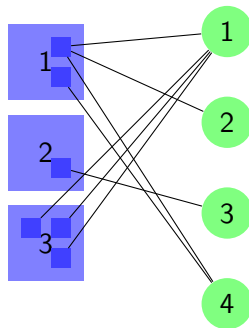
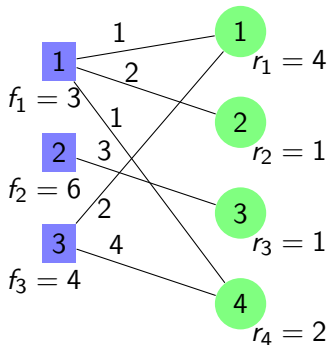


Instance



Solution

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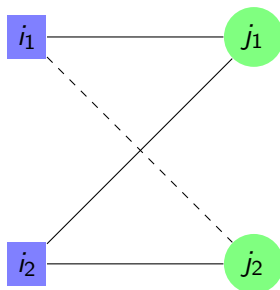


Cost

$$2f_1 + f_2 + 3f_3 + d_{11} + d_{12} + 2d_{14} + d_{23} + 3d_{31} = 38$$



# Metric Distances: Triangle Inequality



$$d(i_1, j_2) \leq d(i_1, j_1) + d(i_2, j_1) + d(i_2, j_2)$$

Needed when estimating distances...

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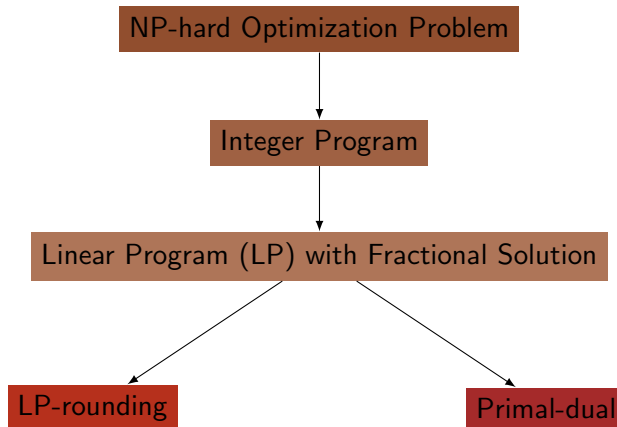
# Hardness

## How hard is FTFP?

FTFP is NP-hard

FTFP is MaxSNP-hard

Best ratio  $\geq 1.463$  unless  $P = NP$



# Results Highlight

- LP-rounding: 1.575-approximation
- LP-rounding: asymptotic ratio of 1 when all demands large
- Primal-dual:  $H_n$ -approximation
- Primal-dual: Example of  $\Omega(\log n / \log \log n)$  for dual-fitting

# Relation between Problems

|      |              |                              |
|------|--------------|------------------------------|
| FTFP | $r_j \geq 1$ | $< \infty$ facility per site |
| UFL  | $r_j = 1$    | $\leq 1$ facility per site   |
| FTFL | $r_j \geq 1$ | $\leq 1$ facility per site   |

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$$\text{UFL} \preceq \text{FTFP} \preceq \text{FTFL}$$

# Relation between Problems

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$$\text{UFL} \preceq \text{FTFP} \preceq \text{FTFL}$$

LP-rounding

UFL 1.575

FTFP

FTFL 1.7245



# Relation between Problems

|      |              |                              |
|------|--------------|------------------------------|
| FTFP | $r_j \geq 1$ | $< \infty$ facility per site |
| UFL  | $r_j = 1$    | $\leq 1$ facility per site   |
| FTFL | $r_j \geq 1$ | $\leq 1$ facility per site   |

$$\text{UFL} \preceq \text{FTFP} \preceq \text{FTFL}$$

| LP-rounding |        |
|-------------|--------|
| UFL         | 1.575  |
| FTFP        | 1.575  |
| FTFL        | 1.7245 |

| Primal-dual |             |
|-------------|-------------|
| UFL         | 1.52        |
| FTFP        | $O(\log n)$ |
| FTFL        | $O(\log n)$ |

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# Related Work for UFL

## Approximation Results for UFL

|                           |      |       |              |
|---------------------------|------|-------|--------------|
| Shmoys, Tardos and Aardal | 1997 | 3.16  | LP-rounding  |
| Chudak                    | 1998 | 1.736 | LP-rounding  |
| Sviridenko                | 2002 | 1.58  | LP-rounding  |
| Jain and Vazirani         | 2001 | 3     | primal-dual  |
| Jain <i>et al.</i>        | 2002 | 1.61  | greedy       |
| Mahdian <i>et al.</i>     | 2002 | 1.52  | greedy       |
| Arya <i>et al.</i>        | 2004 | 3     | local search |
| Byrka                     | 2007 | 1.5   | hybrid       |
| Li                        | 2011 | 1.488 | hybrid       |

## Lower Bound

|                  |      |       |
|------------------|------|-------|
| Guha and Khuller | 1998 | 1.463 |
|------------------|------|-------|

# Related Work for FTFL

## Approximation Algorithms for FTFL

|                     |      |                    |             |
|---------------------|------|--------------------|-------------|
| Jain and Vazirani   | 2000 | $3 \ln \max_j r_j$ | primal-dual |
| Guha <i>et al.</i>  | 2001 | 4                  | LP-rounding |
| Swamy, Shmoys       | 2008 | 2.076              | LP-rounding |
| Byrka <i>et al.</i> | 2010 | 1.7245             | LP-rounding |

No primal-dual algorithms for FTFL with constant ratio.

# Work on FTFP (Dissertation Topic)

## Approximation Algorithms for FTFP

|                 |                     |       |                                 |
|-----------------|---------------------|-------|---------------------------------|
| Xu and Shen     | 2009                |       | Introduced FTFP                 |
| Liao and Shen   | 2011                | 1.861 | Dual-fitting (for special case) |
| Yan and Chrobak | 2011                | 3.16  | LP-rounding                     |
| Yan and Chrobak | 2012                | 1.575 | LP-rounding                     |
| Yan and Chrobak | preliminary results |       | Dual-fitting (for general case) |

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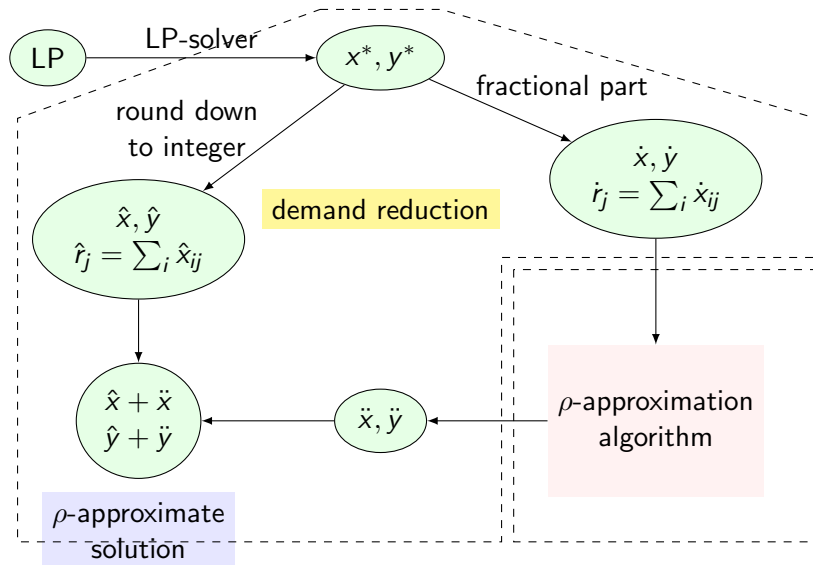
# Algorithm for FTFP — LP

- $y_i$  = number of facilities open at site  $i \in F$
- $x_{ij}$  = number of connections from client  $j \in \mathbb{C}$  to site  $i \in F$

$$\begin{aligned}
 (\text{Primal}) \quad & \text{minimize} \quad \sum f_i y_i + \sum d_{ij} x_{ij} \\
 & \text{subject to} \quad y_i - x_{ij} \geq 0 \quad \forall i, j \\
 & \quad \quad \quad \sum x_{ij} \geq r_j \quad \forall j \\
 & \quad \quad \quad x_{ij} \geq 0, y_i \geq 0 \quad \forall i, j
 \end{aligned}$$

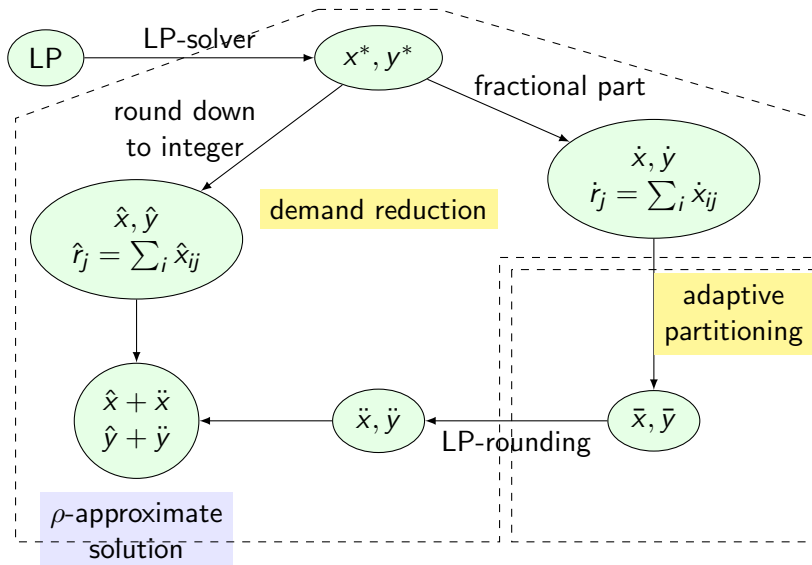
$$\begin{aligned}
 (\text{Dual}) \quad & \text{maximize} \quad \sum r_j \alpha_j \\
 & \text{subject to} \quad \sum \beta_{ij} \leq f_i \quad \forall i \\
 & \quad \quad \quad \alpha_j - \beta_{ij} \leq d_{ij} \quad \forall i, j \\
 & \quad \quad \quad \alpha_j \geq 0, \beta_{ij} \geq 0 \quad \forall i, j
 \end{aligned}$$

# Algorithm for FTFP — Demand Reduction





# Algorithm for FTFP — Adaptive Partitioning



# Techniques

- Demand Reduction

- Reduce all  $r_j$  to polynomial values (to ensure polynomial time of rounding)
- $\rho$ -approx for reduced instance  $\Rightarrow$   $\rho$ -approx for original instance

- Adaptive Partitioning

- Split sites into facilities and clients into unit demands
- Split associated fractional values
- Properties ensure rounding similar to UFL can be applied

# Demand Reduction

## Implementation

- Solving LP for  $(\mathbf{x}^*, \mathbf{y}^*)$
- $(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = (\mathbf{x}^*, \mathbf{y}^*)$  round down to integer
- $(\dot{\mathbf{x}}, \dot{\mathbf{y}}) = (\mathbf{x}^*, \mathbf{y}^*) - (\hat{\mathbf{x}}, \hat{\mathbf{y}})$ , fractional part
- $\hat{r}_j = \sum_i \hat{x}_{ij}$  for  $\hat{\mathcal{I}}$ ,  $\dot{r}_j = r_j - \hat{r}_j$  for  $\dot{\mathcal{I}}$
- $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$  (integral) feasible and optimal for  $\hat{\mathcal{I}}$
- $(\dot{\mathbf{x}}, \dot{\mathbf{y}})$  (fractional) feasible and optimal for  $\dot{\mathcal{I}}$

## Properties

- $\dot{r}_j = \text{poly}(|F|)$
- $\rho$ -approx for  $\dot{\mathcal{I}}$  implies  $\rho$ -approx for  $\mathcal{I}$

# Demand Reduction: Consequences

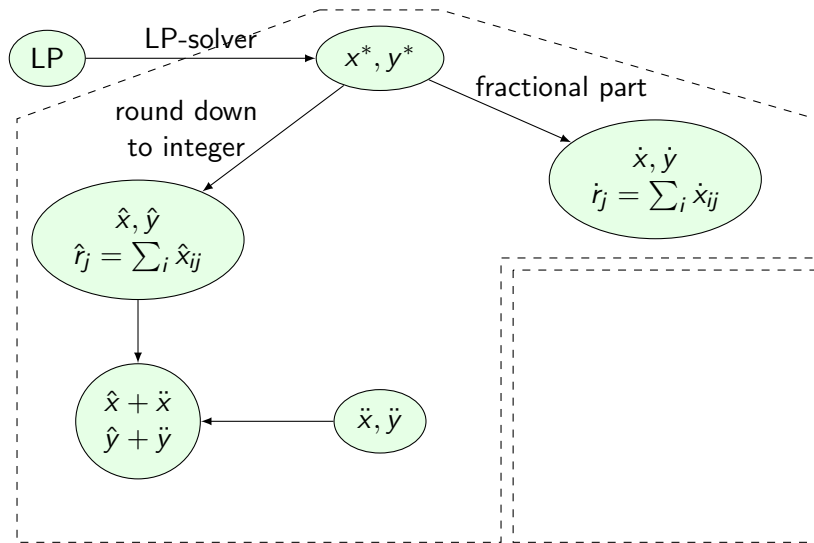
FTFP to FTFL, 1.7245-approximation

- Sites into facilities
- Clients with demand  $r_j$
- FTFL size polynomial because of demand reduction

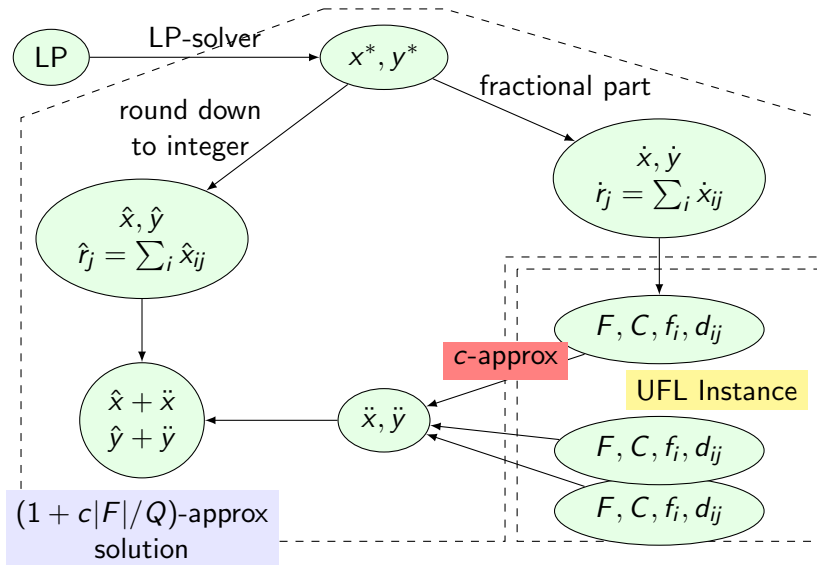
Ratio  $1 + O(|F|/Q)$  for  $Q = \min_j r_j$ , approaches 1 when  $Q$  is large

- Next slide

# Ratio $1 + O(|F|/Q)$ for FTFP



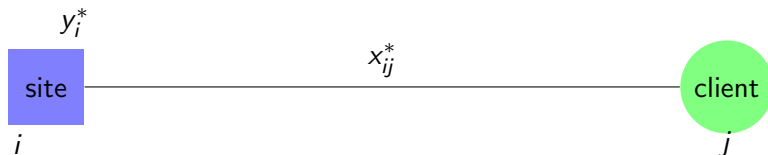
# Ratio $1 + O(|F|/Q)$ for FTFP



# Techniques

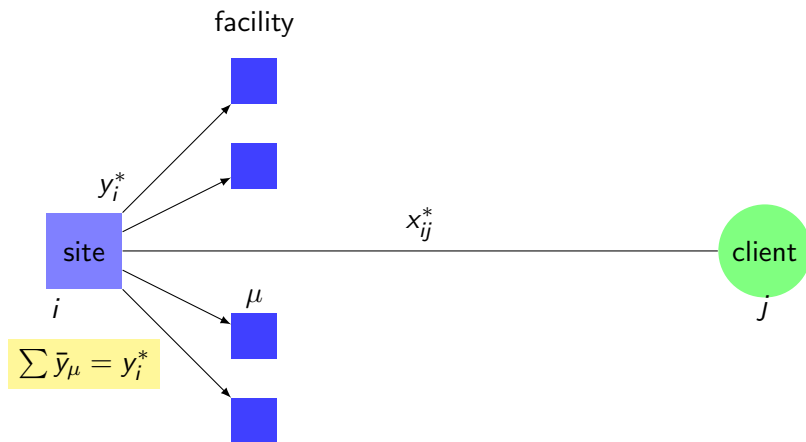
- Demand Reduction
  - Reduce all  $r_j$  to polynomial values (to ensure polynomial time of rounding)
  - $\rho$ -approx for reduced instance  $\Rightarrow$   $\rho$ -approx for original instance
- Adaptive Partitioning
  - Split sites into facilities and clients into unit demands
  - Split associated fractional values
  - Properties ensure rounding similar to UFL can be applied

# Adaptive Partitioning

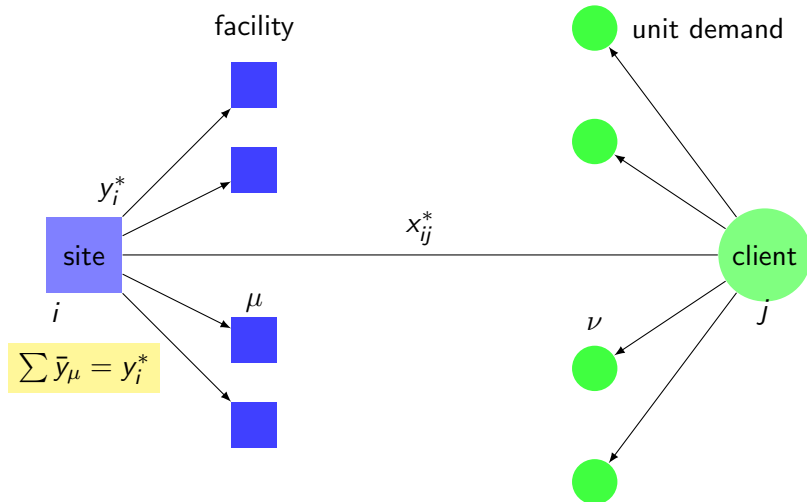




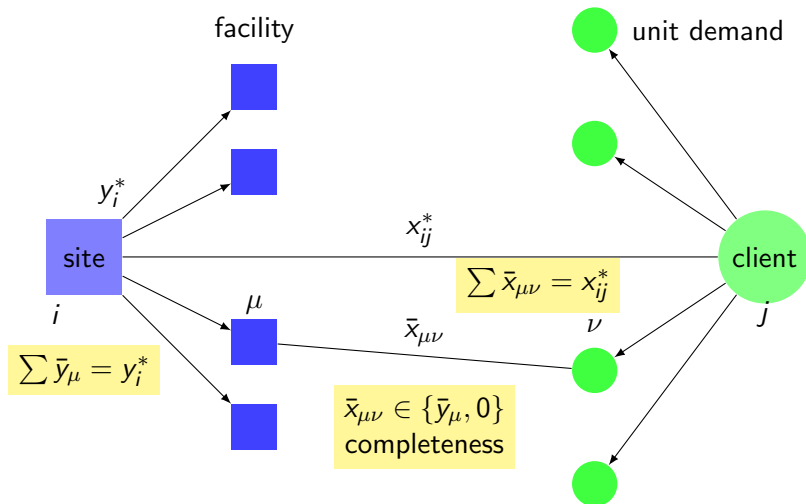
# Adaptive Partitioning



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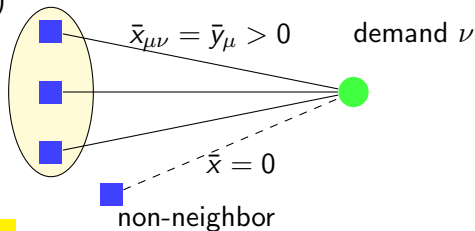
# Adaptive Partitioning



# Neighborhood of Demand

neighborhood  $N(\nu)$

$$\sum_{\mu} \bar{x}_{\mu\nu} = 1$$

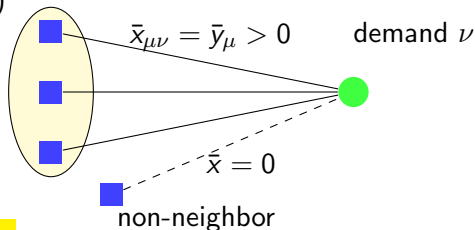


$\nu$  needs 1 facility...

# Neighborhood of Demand

neighborhood  $N(\nu)$

$$\sum_{\mu} \bar{x}_{\mu\nu} = 1$$



$\nu$  needs 1 facility...

**Strategy 1:** for each  $\nu$ , open one  $\mu \in N(\nu)$  with prob.  $\bar{y}_{\mu}$

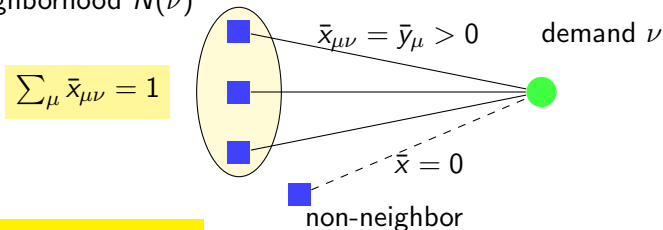
- optimal connection cost
- large facility cost

**Strategy 2:** open facility only for demands with disjoint neighborhoods

- optimal facility cost
- large connection cost

# Neighborhood of Demand

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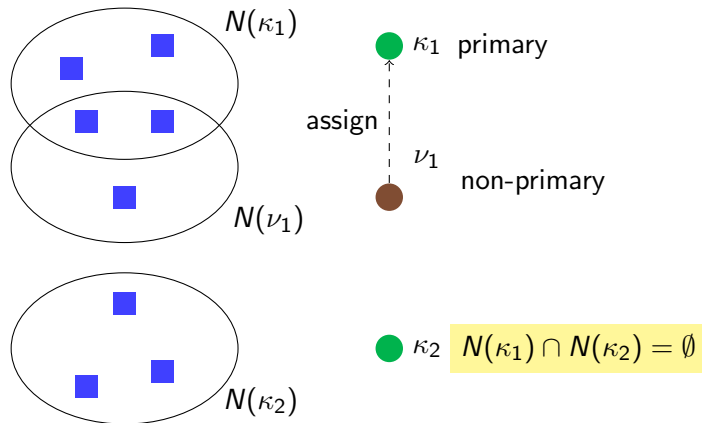
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**Strategy 2:** open facility only for demands with disjoint neighborhoods

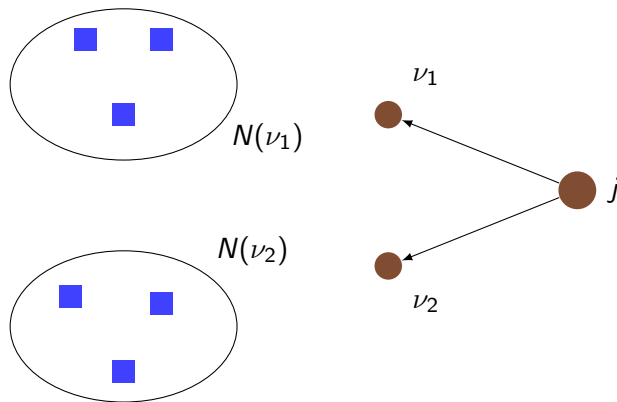
- optimal facility cost
- large connection cost

How to balance these two costs?

# Two Types of Demands: Primary and Non-primary

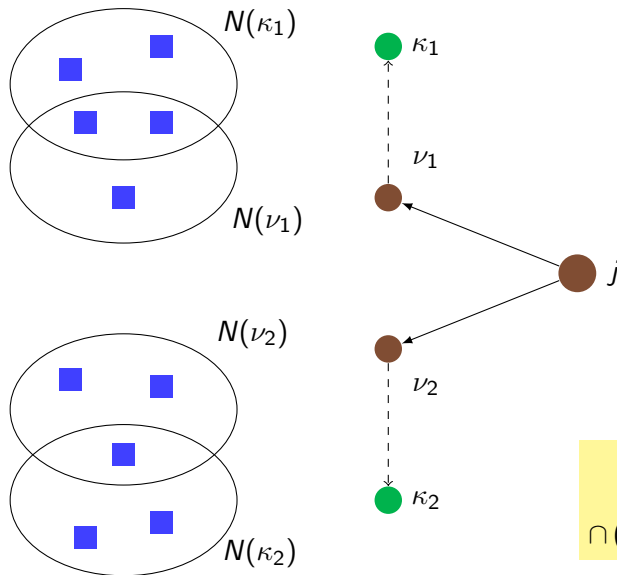


# Neighborhood Structure for Siblings



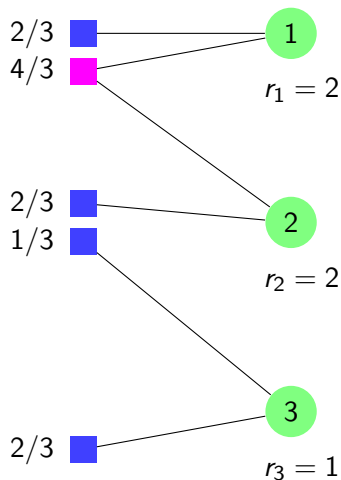


# Neighborhood Structure for Siblings



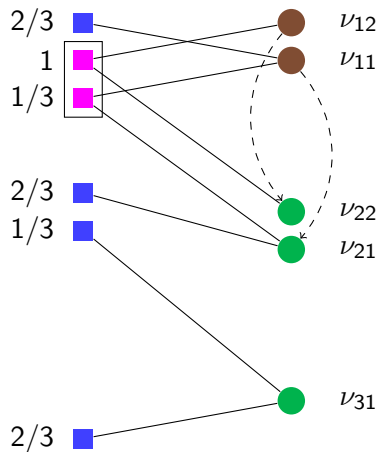
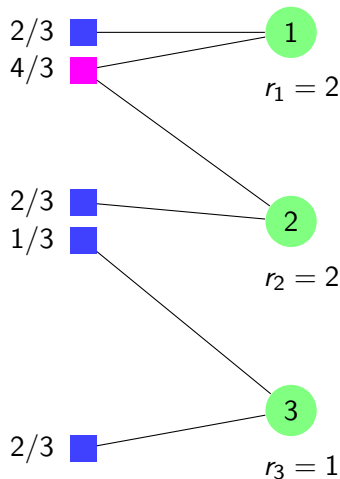
For siblings  
 $(N(\kappa_1) \cup N(\nu_1))$   
 $\cap (N(\kappa_2) \cup N(\nu_2)) = \emptyset$

# Example of Partitioning



Before Partitioning

# Example of Partitioning



# Summary of Partitioning

## Partitioning:

- Clients  $\rightarrow$  demands
- Sites  $\rightarrow$  facilities
- $(x^*, y^*) \rightarrow (\bar{x}, \bar{y})$
- $\sum_{\mu} \bar{x}_{\mu\nu} = 1$
- $\bar{x}_{\mu\nu} = \bar{y}_{\mu}$  or 0

## Structure:

- If  $\kappa_1, \kappa_2$  primary then  $N(\kappa_1) \cap N(\kappa_2) = \emptyset$
- Each non-primary  $\nu$  assigned to  $\kappa$  with
  - $N(\kappa) \cap N(\nu) \neq \emptyset$
  - $\text{priority}(\kappa) \leq \text{priority}(\nu)$
- $(N(\kappa_1) \cup N(\nu_1)) \cap (N(\kappa_2) \cup N(\nu_2)) = \emptyset$

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### small facility cost

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### small connection cost of $\nu$

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- $(N(\kappa_1) \cup N(\nu_1)) \cap (N(\kappa_2) \cup N(\nu_2)) = \emptyset$

### fault-tolerance

# Partitioning Implementation

## Partitioning implementation: two phases

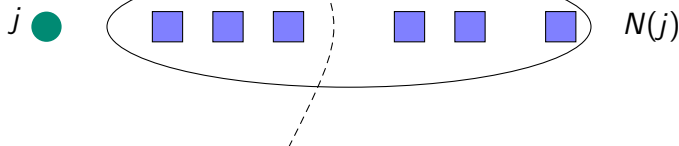
- Phase 1, the partitioning phase
  - Define demands
  - Allocate facilities
- Phase 2, the augmenting phase
  - Add facilities to make neighborhood unit



# Phase 1, Step 1

In each iteration, create one demand for best client

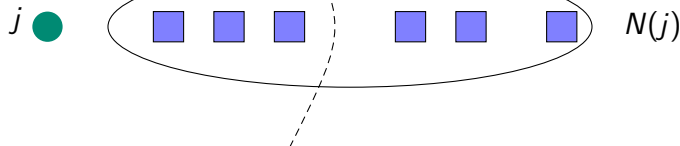
$N_1(j)$ : nearest unit chunk



# Phase 1, Step 1

In each iteration, create one demand for best client

$N_1(j)$ : nearest unit chunk



- $\text{bid}(j) = \text{avgdist}(N_1(j)) + \alpha_j^*(\text{dual value})$
- Best bid client  $p$  selected to create a demand

# Phase 1, Step 2

Best client  $p$  creates demand  $\nu$ , to decide  $N(\nu)$ , two cases:

$N(\kappa_1)$

$N(\kappa_2)$

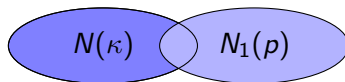
...

$N(\kappa_t)$

disjoint

$N_1(p)$

Case 1



$N_1(p)$  overlaps some  $N(\kappa)$

Case 2

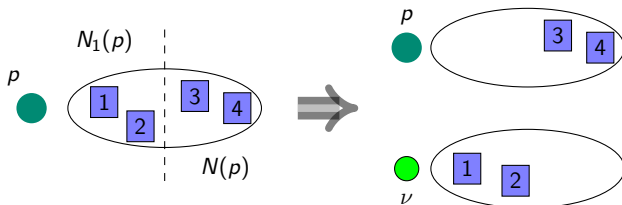
# Phase 1, Step 2 Contd.

Best client  $p$  creates demand  $\nu$ , to decide  $N(\nu)$ , two cases:

# Phase 1, Step 2 Contd.

Best client  $p$  creates demand  $\nu$ , to decide  $N(\nu)$ , two cases:

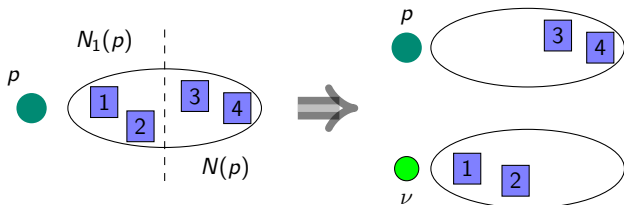
- Case 1: disjoint,  $N(\nu)$  gets  $N_1(p)$



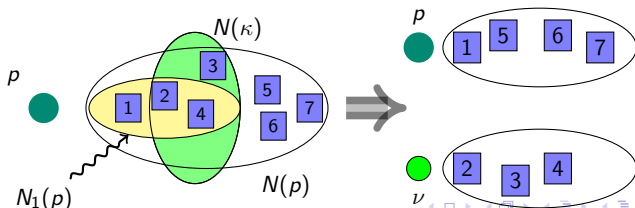
# Phase 1, Step 2 Contd.

Best client  $p$  creates demand  $\nu$ , to decide  $N(\nu)$ , two cases:

- Case 1: disjoint,  $N(\nu)$  gets  $N_1(p)$

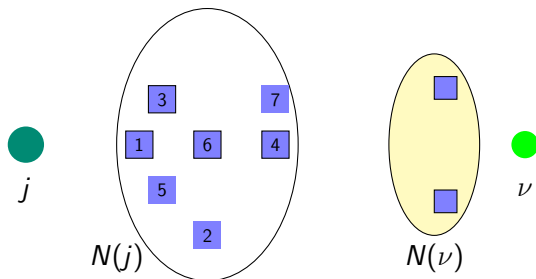


- Case 2: overlap,  $N(\nu)$  gets  $N(p) \cap N(\kappa)$



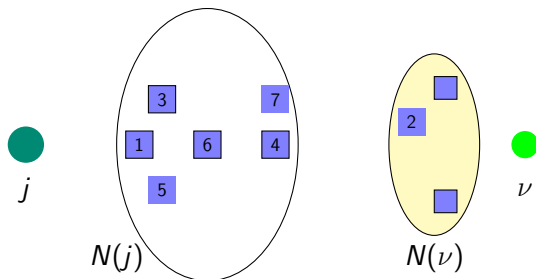
# Phase 2

Add facilities from  $N(j)$  to  $N(\nu)$  until total value 1



# Phase 2

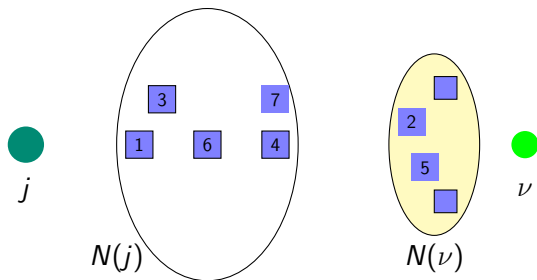
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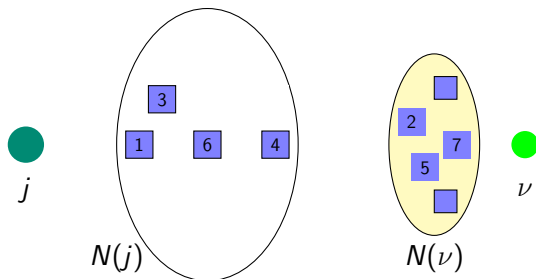
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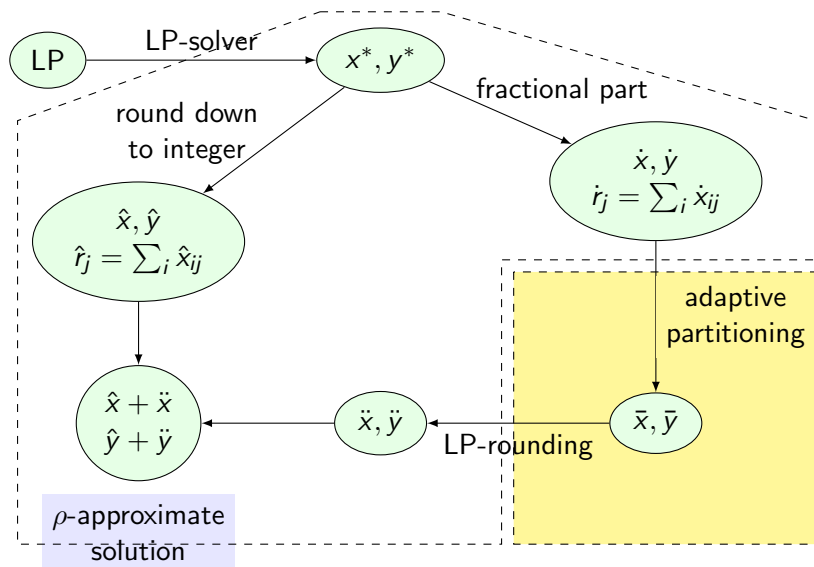
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# Phase 2

Add facilities from  $N(j)$  to  $N(\nu)$  until total value 1





Done with partitioning, next to rounding

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# 3-Approximation for FTFP

Rounding: round each  $\bar{y}_\mu$  and  $\bar{x}_{\mu\nu}$  to 0 or 1

- **Facilities:** each primary  $\kappa$  opens one  $\mu \in N(\kappa)$
- **Connections:** non-primary demands  $\nu$  assigned to  $\kappa$  connect to  $\mu$

## Analysis

- **Fault-Tolerance:**  $\nu$  uses only facilities in  $N(\nu) \cup N(\kappa)$
- **Cost:**  $\leq 3 \cdot \text{LP}^*$ , because
  - Facility cost  $\leq F^*$
  - Connection cost  $\leq C^* + 2 \cdot \text{LP}^*$

# 1.736-Approximation for FTFP

Rounding: round each  $\bar{y}_\mu$  and  $\bar{x}_{\mu\nu}$  to 0 or 1

- **Facilities:**
  - Each primary  $\kappa$  opens random  $\mu \in N(\kappa)$
  - Other facilities open randomly independently
- **Connections:**
  - If a neighbor open, connect to nearest neighbor
  - Else connect via assigned primary demand

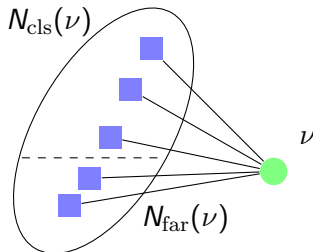
## Analysis

- **Fault-Tolerance:**  $\nu$  uses only facilities in  $N(\nu) \cup N(\kappa)$
- **Cost:**  $\leq (1 + 2/e) \text{LP}^*$ , because
  - Facility cost  $\leq F^*$
  - Connection cost  $\leq C^* + (2/e) \cdot \text{LP}^*$

# 1.575-Approximation for FTFP — Partitioning

## More intricate neighborhood structure

- Two neighborhoods: close and far,  $N(\nu) = N_{\text{cls}}(\nu) \cup N_{\text{far}}(\nu)$
- $N_{\text{cls}}(\nu) =$  nearest  $(1/\gamma)$ -fraction of  $N(\nu)$
- $N_{\text{cls}}(\nu) \cap N_{\text{cls}}(\kappa) \neq \emptyset$ , if  $\nu$  assigned to  $\kappa$
- For siblings  $\nu_1, \nu_2$ ,  $N_{\text{cls}}(\kappa_1) \cup N(\nu_1)$  and  $N_{\text{cls}}(\kappa_2) \cup N(\nu_2)$  disjoint
- ...



# 1.575-Approximation for FTFP — Rounding

Rounding: boost  $(\mathbf{x}^*, \mathbf{y}^*)$  by  $\gamma$  and apply demand reduction and adaptive partitioning, then round by

- **Facilities:**
  - Each primary  $\kappa$  opens random  $\mu \in N_{\text{cls}}(\kappa)$
  - Other facilities open randomly independently
- **Connections:**
  - If a neighbor open, connect to nearest neighbor
  - Else connect via assigned primary demand

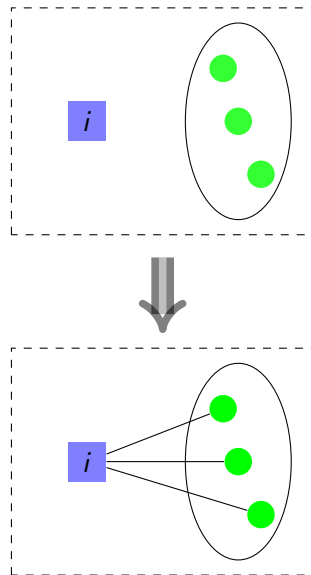
## Analysis

- **Fault-Tolerance:**  $\nu$  uses only facilities in  $N(\nu) \cup N_{\text{cls}}(\kappa)$
- **Cost:**  $\leq \gamma \cdot \text{LP}$  for  $\gamma = 1.575$ , because
  - Facility cost  $\leq \gamma \cdot F^*$
  - Connection cost  $\leq \gamma \cdot C^*$



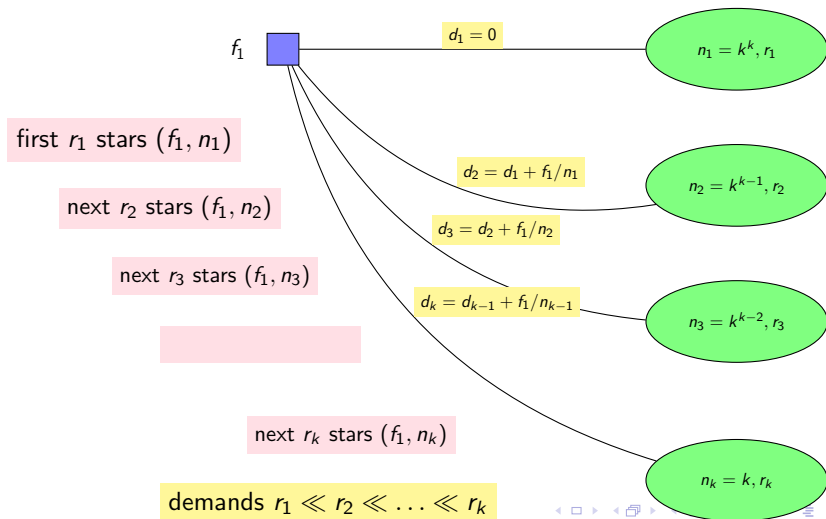
# Greedy and Dual-fitting

- Greedy in polynomial time
  - Best star can be found quickly
  - Best star remains best
- Ratio  $H_n$  (Wolsey's result): Greedy is  $H_n$ -approx for
  - Minimizing a linear function
  - Subject to submodular constraints
- Lower bound  $\Omega(\log n / \log \log n)$  for dual-fitting
  - Example has  $k$  groups,  $n = k^k$
  - Shrinking factor is  $k/2$



# Dual-fitting Example

Dual feasibility forces a ratio of  $k/2$ , number of clients  $n = k^k$



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# Summary

## Results

- 1.575-approximation algorithm for FTFP
- Technique for extending LP-rounding algorithms for UFL to FTFP

# Summary

## Results

- 1.575-approximation algorithm for FTFP
- Technique for extending LP-rounding algorithms for UFL to FTFP

## Open Problems

- Can FTFL be approximated with the same ratio?
- LP-free algorithms for FTFP or FTFL with constant ratio?
- Close the 1.463 – 1.488 gap for UFL!