On Reinhardt and its relationship with constructibility, inner models, and with I0

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Abstract

We show some properties of Reinhardt Cardinals and I0 models and their interaction with constructibility and inner models (particularly on constructible or inner models of ZF that are also Reinhardt/I0 models), and also definable embeddings. We also examine the general intersection between the properties and results of Reinhardt and I0, particularly on forcing notions that relate the two.

1 Introduction

This question was primarily inspired from a Discord server, about Large Cardinal Charts not showing Reinhardt cardinals. There, an anonymous user commented that the said chart only makes sense in ZFC, as Reinhardt is not consistent with ZFC, but the Rank-into-Rank axioms are consistent with ZFC. Another user commented on the possible implication of Reinhardt implies I0, particularly for inner models of ZF. This paper is largely inspired from the answer to this question. The "inner models of ZF part of the question is answered first, along with Reinhardt being stronger than I0 and on its implications and results.

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Another inspiration is from Goldberg's paper on Reinhardt cardinals and their connection with inner models.[6] Many of the theorems, especially Theorem 2.2, were meant to be somewhat tangential to the original problem (that Reinhardt gives I0 in an inner model, a.k.a. An inner model of ZF that also contains a Reinhardt cardinal can have an I0 model constructed from/inside it), but are expanded and critiqued on.

2 Reinhardt Cardinals

A Reinhardt Cardinal is a cardinal that is the critical point of an embedding from the set-theoretic universe V to itself. Kunen's Inconsistency Theorem implies that Reinhardt Cardinals are inconsistent with ZFC.

Theorem 2.1. *If there is a class of weakly Reinhardt cardinals, then there is an constructible model with a class of Reinhardt cardinals. (ZF)*[1]

Proof. Let C be a class of weakly Reinhardt cardinals[2]. Construct an structure-preserving non-trivial elementary embedding j_R from C that embeds into $L(V)$. Set the weakly Reinhardt cardinals as the critical points κ of such embeddings.[3] Take both classes in Theorem 2.1 to be class functions, but with "function" replaced by elementary embeddings; take the defining formulae of the classes Φ to be functions (injections) of sets (not sets of subsets; ZFC is first order) in V. $\square$

Open Question 1. *Can Theorem 2.1 be extended to Reinhardt cardinals or even stronger instead of weakly Reinhardt cardinals?*

The embedding from above has a critical point κ; κ a Reinhardt. **Note that such embedding j_R is of the form $M \models \phi(a_1, ..., a_n) \iff N \models \phi(L(a_1), ..., L(a_n))$, or $M \models \phi(a_1, ..., a_n) \iff N \models ((X, \in) \models \phi(a_1, ..., a_n))$**[4] Also, throughout this paper, this embedding j_R will extensively be used to prove properties of constructible and definable models of ZF + Reinhardt.

[1] I have removed "proper" from "proper class" in Goldberg's original formulation of the theorem because ZF(C) does not allow a strict definition of a formal class.

[2] As given by Corazza.

[3] This embedding also witnesses "regular" Reinhardt cardinals, along with a "shift-back" of levels in V from $\lambda + 1$.

[4] X is a class.

(**Definition 2.1**)

Lemma 2.1.1. *If there is a proper class of weakly Reinhardt cardinals, then there is an constructible model with a proper class of Reinhardt cardinals. (NBG, MK)*

This is an immediate result of the definition of proper class in NBG and MK.

Note that construction of an embedding j_R would be less annoying in MK than in NBG or ZF; MK is second-order. Then one can construct embeddings of sets of sets or classes of classes, particularly uncountable or inaccessible classes of classes, making it so that instead of individual functions or embeddings of classes (resulting in schemas), one can "stream-line" this.[5] Therefore, we now work in MK, without Choice.

Lemma 2.2. *j_R "enforces" upon formulae or models (say, of the form $\phi(a_1, ..., a_n)$) definability/constructability. That is, $j_R : M \to L(M)$.*

Theorem 2.3. *The existence of an elementary embedding j_R between Reinhardt Models is equivalent to there is an elementary embedding $L(V) \prec L(V)$.*

A Reinhardt cardinal in-between a weakly Reinhardt and a "regular" Reinhardt is nescessary for the proof, particularly to show that for some α and β, there is an elementary embedding $L_\alpha \prec L_\beta$ with a crit point less than α. This is to make sure the proof does not get "restrained" by specific levels of V or L. Such a cardinal will also be as strong as regular Reinhardt.

Definition 2.4. *A **moderately Reinhardt cardinal** is the critical point κ of an elementary embedding $j : V_{k+n} \to V_{k+m}$, $n \leq m$, such that $V_n \preceq V_m$ for $k < m$.*

Proof. We prove Theorem 2.3 for moderate Reinhardts first. Let M_{MR} be a model of MK that satisfies moderate Reinhardt-ness. Let $j_R : M_{MR\star m} \to M_{MR\star m_1}$, m and m_1 in this context representing the level m of V that the

[5]This is the advantage with most second-order theories. A "second-order" ZFC could be used for the rest of this paper.

embedding in the moderately Reinhardt cardinal maps to. Lemma 2.2 is used to show that the embedding j_R enforces upon $M_{MR\star m}$ to become contructible as the form $M_{MR\star m_1}$. Therefore, $M_{MR\star m_1}$ is constructible and of the form $L_{MR\star m_1}$. $M_{MR\star m-1}$ embeds into $M_{MR\star m}$, therefore making it constructible and of the form $L_{MR\star m}$. We could keep going, recursively "pushing down" the m to keep the constructability aspect of the models. Specifically, start at the base m, and name this level 0 of the "constructible push-down". Level η of the push-down is $m - \eta$ for any ordinal η. Requiring that V_0, the maximum "push-down" be constructible is not nescessary. Therefore, if we "shift" the m's, it is possible for the statement α and β, there is an elementary embedding $L_\alpha \prec L_\beta$ with a crit point less than α to be satisfied. This case can be generalized to Reinhardt cardinals in general in that the non-levels of the cardinals could be treated as a special case of the "push-down hierarchy"; simply set $k = m = 0$. $\square$

As a side remark, the "push-down" method can be thought of as essentially a reverse reflection; we go bottom-up instead of top-bottom.

Open Question 2. *Can this system of proof be used for other Large Cardinals?*

Open Question 3. *Where do moderate Reinhardt cardinals fall on the Large Cardinal Hierarchy?*

Moderately Reinhardt Cardinals are ammunition for another paper.

2.1 Reinhardt and Inner Models

Theorem 2.5. *If there is a proper class of weakly Reinhardt cardinals, then there is a inner model with a proper class of Reinhardt cardinals.*

$V \neq L$ per the Jensen covering lemma.[6] The ordinals for the proper class of weakly Reinhardt cardinals can be defined "as usual", only with the hierarchy going up to weak Reinhardt-ness (same thing applies for classes of Reinhardt cardinals), and with cardinals being defined as an ordinal number

[6] Because there is an embedding from L to L, at least when constructed from Reinhardt Models.[8] **Ord** in L is simply the ordinals in V.

that is not in bijection with a smaller cardinal (Goldstein). Jensen also implies that for all ordinals α, $|P(\alpha) \cap L| = |\alpha|$, along with 0 sharp. Let $L(j)$ represent the class of constructible sets relative[7] to an elementary embedding $L \to L$.

There are instances in which L is not inner without AC. We work in KP set theory for this proof. L is constructed in the usual way, but most ordinals in L are not admissible in KP because the "def" relation is Σ_1^{KP}. As an additional remark, absoluteness of L in an inner model W implies V = L, which implies GCH, which implies AC. We can avoid V = L by "loosening" the notion of absoluteness in def; X or ϕ could be more "variable". This also protects κ being measurable, and therefore L being an inner model. Therefore, def is newly defined as $Def^v(X) = \{\{\forall y : y \in X \text{ and } (X, \in) \models \phi(z_1, ..., z_n)\} \mid \phi \text{ is first order and } z_1, ..., z_n, y \in X\}$.[8,9] (**Defintion 2.2.**)

L^v[10] constructed from $Def^v(X)$ still satisfies all the axioms of ZF, but not AC. L^v is still transitive, but with the additional component of y, therefore extensionality. Foundation is trivial. Comprehension goes like this: show that $\forall z_1, ..., z_n \in L^v(\{y \in X : \phi(y, z_1, ..., z_n)\}X \in L^v)$.[11] Proceed via reflection in L^v. Pairing, Union, Replacement, and Infinity are all trivial. Power Set is of the form $\forall z_1, ..., z_n \in L^v(\{y \in X : \phi(y, z_1, ..., z_n)\ X \in L^v\}) \iff \forall z_1, ..., z_n \in L^v\ \exists y\ \forall z[z \in y \iff \forall w \in (z_1, ..., z_n), w \in z \implies w \in \phi(z_1, ..., z_n)]$.

Proof. (MK) Denote C_{wR} for the proper class of weak Reinhardts, and C_R for the proper class of weak Reinhardts. Use j_R from Definition 2.1.[12] Let $j_R : C_{wR} \to C_R$, and then set the critical points κ of j_R as weak Reinhardt, while it still witnessing regular Reinhardt cardinals. Like in the proof of Theorem 2.1, take both classes to be defined in terms of subsets of j_R.[13] Def^v is Π_1, therefore most ordinals in C_{wR} are admissible into C_R. Both models

[7] "Relative" means constructible in the sense of an embedding similar to in Definition 2.1.

[8] y is defined separately to "indivdualize" it and the classes.

[9] "v" represents variability. We also do this in order to make Def^v a non Σ_1-formula.

[10] L^v is L but from Def^v.

[11] Inspired from Kunen (1980). This is also a schema; there is a separate statement for each ϕ.

[12] First form of j_R; def as in Definition 2.2.

[13] As in, an ordered pair of sets from C_{wR} and C_R.

are transitive. Also, a "loosened", but still strong notion of absoluteness avoids V = L. If not, this would contradict the Jensen covering lemma and absoluteness of L in an inner model W.
(ZF) Overall the same, but a schema of functions/formulae would be needed in place of embeddings. □

3 Rank-into-Rank Cardinals

3.1 Introduction to this section

This paper grew out of the following question:

(Solved) Question 4. *Reinhardt gives I0 in an inner model of ZF.*

With "gives" meaning that another model containing I0 can be constructed from the Reinhardt-ZF-inner model. Essentially, given an inner model of ZF containing Reinhardt, a model that is inner and is also I0 can be constructed from said model.

The original proof was of forcing, namely a forcing notion called "Skibidi", which was inspired from shooting a fast club. Essentially, for S a stationary set $\subseteq \omega_1$, P is the set of closed and constructible sequences, and HOD sequences from S in M_R. Then G will consist of nontrivial embeddings, which satisfy the above P. The construction of the new embedding makes Skibidi forcing redundant, but would be very useful in first-order theories.

In particular, both Reinhardt and I0 share that (ZF) models containing both Reinhardt and I0, respectively, can be made constructible (from Skibidi forcing), and also an embedding of the form j_R (Theorem 3.1). They also both involve critical points, and nontrivial elementary embeddings.

First, we prove some constructability theorems. Next, those theorems and embeddings will be related to Reinhardt. Then, the notion of Skibidi forcing will be explored, along with the above theorems.

3.2 Inner Models; Constructability

Lemma 3.2. *A constructible model of ZF containing an I0 cardinal exists.*

Proof. Rather easy. Under I0, transitive proper class obtained by starting with $V_{\lambda+1}$ and forming the constructible hierarchy over $V_{\lambda+1}$ in the usual fashion i.e. usual construction of the constructible hierarchy. $\square$

Remark. *An embedding j_R as in Definition 2.1 might need not be changed much to suit I0, but only that its critical point $< \lambda$. (j_{I0}) We could also use ultrapowers and model extenders, inspired from Gabriel Goldberg's other paper on Rank-to-Rank embeddings.[5] Ultrapowers and model extenders are beneficial in that I0 is much less dependent on embeddings and crit. points and much more on ultrapowers; they can even be wholly formulated via ultrapowers.*

Let U_α be an model-specific ultrafilter over a class X_α over the said model. Define a function $f_{\beta,\alpha}$ from X_β to X_α. We still use j_{I0}. Also, let λ be an ordinal in X_α. Denote $E = \langle U_\alpha, X_\alpha, f_{b,a} : a \subseteq b \in [\lambda]^{<\omega}\rangle$, which is an X_α-extender.[14]

Theorem 3.3. *An inner model containing I0 exists.*

Proof Sketch. We work in ZF. A second-order formulation of set theory is not necessary,and we could just use extenders (Ult(M,E), M is a model, and E is an M-extender, then Ult(M,E) is a def. inner model of M) or j_{I0}. Such a model of I0 is already transitive; we just need to prove that it contains all ordinals, which can again be done using extenders. $\square$

As a somewhat immediate corollary, the addition of I0 to certain classes can induce inner models. For example, take the class V_λ (or $V_{\lambda+1}$). Then, we can construct an embedding from it to a constructible version, and from there we can use an extender Ult((L(M),E)), in which E is an M-extender, and E $= \langle U_\lambda, V_\lambda, f_{\lambda,L(\lambda)} : L(\lambda) \subseteq L(\lambda) \in [\Lambda]^{<\omega}\rangle$.[15,16]

[14] This will be repeatedly referenced as an M-extender, given that the respective model-specific ultrafilter is over a class M.

[15] A remark on the influence of I0 on extenders: U from I0 typically(?) has most(?) of the image of the embedding included within it.

[16] The modified Def relation is not nescessary for rank-into-rank.

Open Question 5. *Can the addition of I0 to classes induce inner models in all classes?*

3.2.1 Axiom of Choice

Theorem 3.4. *The Axiom of Choice is not inconsistent with I0.*

If it were, then, particularly in ZF, an infinite schema of functions would not be able to be constructed and then ordered by their rank.

Corollary 3.5. *I0 $\to$ AC.*

The converse does not apply; if AC implies I0, then the elements of the ultrafilter U over a class X over a model containing I0 can be well ordered. But then

3.3 Ultrafilters of I0

3.3.1 Los's Theorem

Los's Theorem implies that if M_{I0} is a model containing I0, a class X_α over said model, an ultrafilter U over X_α, and an assignment of $i \in I$ to $\mathcal{M}_i$, in which $\mathcal{M}$ is a λ-structure[17], then for $\Pi_U \mathcal{M}_\bullet$, given $a_1, .., a_n \in \Pi_{i \in I} \mathcal{M}$, and for a λ-function f[18], $\Pi_U \mathcal{M}_\bullet \models f(a_1, ..., a_n)$, in which $a_1, .., a_n \in U \iff \{i \in I : \mathcal{M}_i \models f[a_1, ..., a_n]$ (relativized to an i $\in$ I)$\} \in U$.

4 Interaction of Reinhardt and Rank-into-Rank, particularly I0

A motivation for this is that they both "edge" consistency, and show very elegant properties for constructibility together. With Reinhardt, this shows up as results regarding embeddings, and with I0, this shows up as results regarding ultrafilters and ultraproducts. For this section, primarily MK will be used.

[17] λ is an ordinal from the class.

[18] Similar to the one used to define E in p. 7.

4.1 Skibidi Forcing; original proof of Question 4

Here is the unabridged proof of Question 4.

Proof. Let M_R be a model of ZF containing a Reinhardt cardinal. It must be shown that $I0$ be can "contained" or proven in (or using) the model, in that a constructible version of the embedding $j : V \to V$ exists in M_R. Construct a forcing notion based off of shooting a fast club; more specifically for S stationary in $\subseteq \omega_1$, P is the set of closed and constructible sequences, and HOD sequences from S in M_R. Then G will consist of nontrivial embeddings, which satisfy the above P. Clearly $p \models Def(u_1, .., u_n)$, especially for embeddings. Therefore, $I0$ holds in $M_R[G]$, showing the existence of a constructible version of $j : V \to V$ in a model which satisfies Reinhardt-ness. □

Additional Remarks:

1. A condition $p \in P$ if and only if $p \in L(S) \cap M_R$ and p is a closed subset of S.

2. The forcing poset (Skibidi) forces constructibility; take $p < q$ iff $p \subseteq q$. (Both are sequences of elements, which is very important in ZF) Let $p \in P$ and (it) assume[s] $p \models \dot{s}$ to be constructible, s a sequence or embedding.

3. S in condition 1 is allowed to be non-ω_1.

Skibidi Forcing can be used to force constructibility for models without using MK or any other second-order theory, because we do not have to add embeddings "directly" on the model; names for "functions" (or embeddings) of functions, or first-order sequences that add on to the Generic Set of the model will be used instead. As a Reinhardt $\to$ I0 relation is inherently a relation regarding constructibility (as one deals with embeddings of constructible models), Skibidi forcings can be very useful in order to show relations between Reinhardts and Rank-into-Rank, and axioms that involve constructible models or classes, such as an embedding from L to L.

Theorem 4.1. *Skbidi Forcing adds an embedding from L to L.*

Lemma 4.1.1. *There is a generic set which satisfies the existence of at least one isomorphism between two constructible sets.*

Note that for Theorem 4.1 to work, modify (1) such that M_R is simply M, or $L(M)$[19].

Proof Sketch. Represent isomorphisms between two constructible sets in G as $I = \{p \in P \mid p : S \in L \to S_1 \in L\}$, and $I \in G$, in which p is a function. □

Lemma 4.1.2. *Such isomorphisms comprise an embedding.*

Proof. Take p and q from the forcing poset. Define $I_{p_0} = \{p_0 \in P \mid p_0 : C \in L \to C_1 \in L\}$, $I_{p_1} = \{p_1 \in P \mid p_1 : C \in L \to C_1 \in L\}$, with C representing classes of sets, and $I_p = \{p \in P \mid p : C \in L \to C_1 \in L\}$. Then p is an embedding from $C \in L$ to $C_1 \in L$. □

Lemma 4.1.3. *Propositions 4.1.1 and 4.1.2 apply to all sets and classes in L.*[20]

Proof Sketch. Suppose that there exists a set or class in which 4.1.1 or 4.1.2 does not apply. If 4.1.1 does not apply, this is trivial. If 4.1.2 does not apply, then let $p_1 < p$; $p \subset p_1$, and proceed via recursion. □

4.2 Some general constructibility/inner model properties

Theorem 4.2. *A constructible model L_R implies the existence of a constructible model L_{I0}.*

Let the order of the elements of the forcing poset be $p < q$ if $p \subset q$.

Lemma 4.2.1. *There are generic extensions of L_R which satisfy the existence of an embedding between Reinhardt models and I0 models.*

Proof. (Skibidi Forcing) Set S in (1) in 4.1 to be essentially unrestricted in terms of ordinality. Let G be defined such that for embeddings from V to V, such sequences "enforce" the embedding to become constructible (for $L(V) \to L(V)$), and then another forcing (we use iterated forcing) to "push it" to $L(V_{\lambda+1}) \to L(V_{\lambda+1})$.

[19] $L(M_{I0})$ too.

[20] In order to make Theorem 4.1 a function.

For G, we can basically proceed as how Lemma 4.1.1 goes, but with V and non-constructible sets, and represent p (functions implied by the original 4.1.1) with isomorphisms between sets of V into $L(V)$ (such that $p : v \to l, v \in V$ and $l \in L(V)$), and to avoid breaking first-order-ness, represent embeddings as schemata of isomorphisms. $V \to V$ becomes $l(f) : L(V) \to L(V)$ for a particular $f : V \to V$, and $l(f)$ is the "constructibly lifted" version of f. The fact that $L(V) \to L(V)$ exists comes from Theorem 4.1. Define P-names for V_α and $L(V_\alpha)$ recursively; order P-names of rank α of V or $L(V)$ by their individual rank, i.e., $\rho(\dot{V_\alpha}) = \alpha$, and the same goes for $L(V)$.

$$\begin{tikzcd}
L(V) \arrow[r, "l(f)"] & L(V) \\
V \arrow[u, "p"] \arrow[r, "f"] & V \arrow[u, "p"]
\end{tikzcd}$$

ω_1 is not collapsed under the original Skibidi forcing notion because given an element $p < \omega_1$, then there is an element q so that $p < q$ (slightly handwave-y). For the generic set $\dot{H}$, let it be constructed using the forcing notion Q which is $L_R[G]$-generic; $Q \cap E = \emptyset, \forall E \in L_R[G]$ implies that $r \in Q$ if $r \in L(S)^C$ (is complement, with rest of universe being the rest of L) $\cap$ M_R and r is a closed subset of S. We can still force constructibility using this new notion; let $r_0 < r_1$ iff $r_0 \subsetneq r_1$, with both being sequences. $r \models_{\dot{L_R},Q} \dot{\dot{s}}$ is equivalent to $r \models_{\dot{L_R},Q} s \dot{\in} G$. Also, add another criterion for admission into Q; $r \in Q$ if $r : \rho(s \in G) \to r+1 : \rho + 1(s \in G)$. Then, given P-names for $L(V)$, we have that r "moves" $L(\dot{V})$ by a rank ($r \models_{\dot{L_R},Q} \dot{L_R}\ \rho(L(V)) + 1$). We can extend this into $\lambda + 1$ by recursively defining r_1; $r : \rho(s \in G) \to r + 2 : \rho + 2(s \in G)$, and $r_\lambda : \rho(s \in G) \to r + n : \rho + n(s \in G), n < \lambda$. Therefore, we have successfully been able to "push" $L(V)$ to $L(V_{\lambda+1})$.

$$\begin{tikzcd}
L(V_{\lambda+1}) \arrow[r, "r(\dot{f})"] & L(V_{\lambda+1}) \\
L(V) \arrow[u, "r"] \arrow[r, "l(f)"] & L(V) \arrow[u, "r"]
\end{tikzcd}$$

□

Open Question 5. *What are some uses for Skibidi Forcing on Large Cardinals and Constructibility besides Reinhardts and I0? Can Skibidi Forcing be applied to study properties about constructibility for, say, other Rank-into-rank cardinals?*

5 Some acknowledgments

I would like to thank the people of Discord for helping and ultimately inspiring the background of this paper. Although I have contacted mathematicians not via Discord, this community was extremely helpful, with it not being uncommon for those mathematicians not in Discord to simply just "ghost" me. For privacy, those users will be kept anonymous.

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