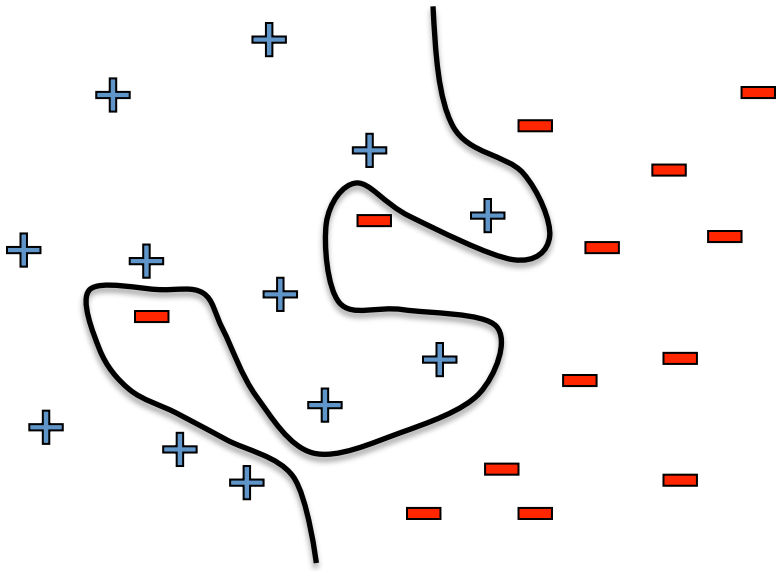
What if the data is not linearly separable?



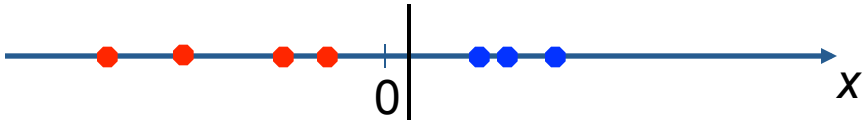
Use features of features of features of features….

$$\phi(x) = \begin{pmatrix} x_1 \\ \dots \\ x_n \\ x_1 x_2 \\ x_1 x_3 \\ \dots \\ e_{x_1} \\ \dots \end{pmatrix}$$

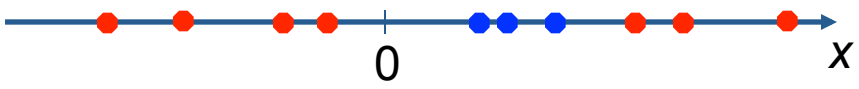
Feature space can get really large really quickly!

Non-linear features: 1D input

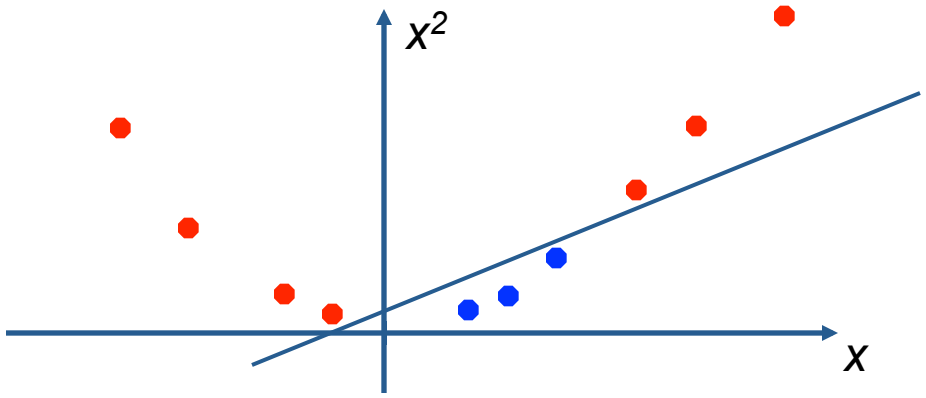
- Datasets that are linearly separable with some noise work out great:



- But what are we going to do if the dataset is just too hard?



- How about… mapping data to a higher-dimensional space:

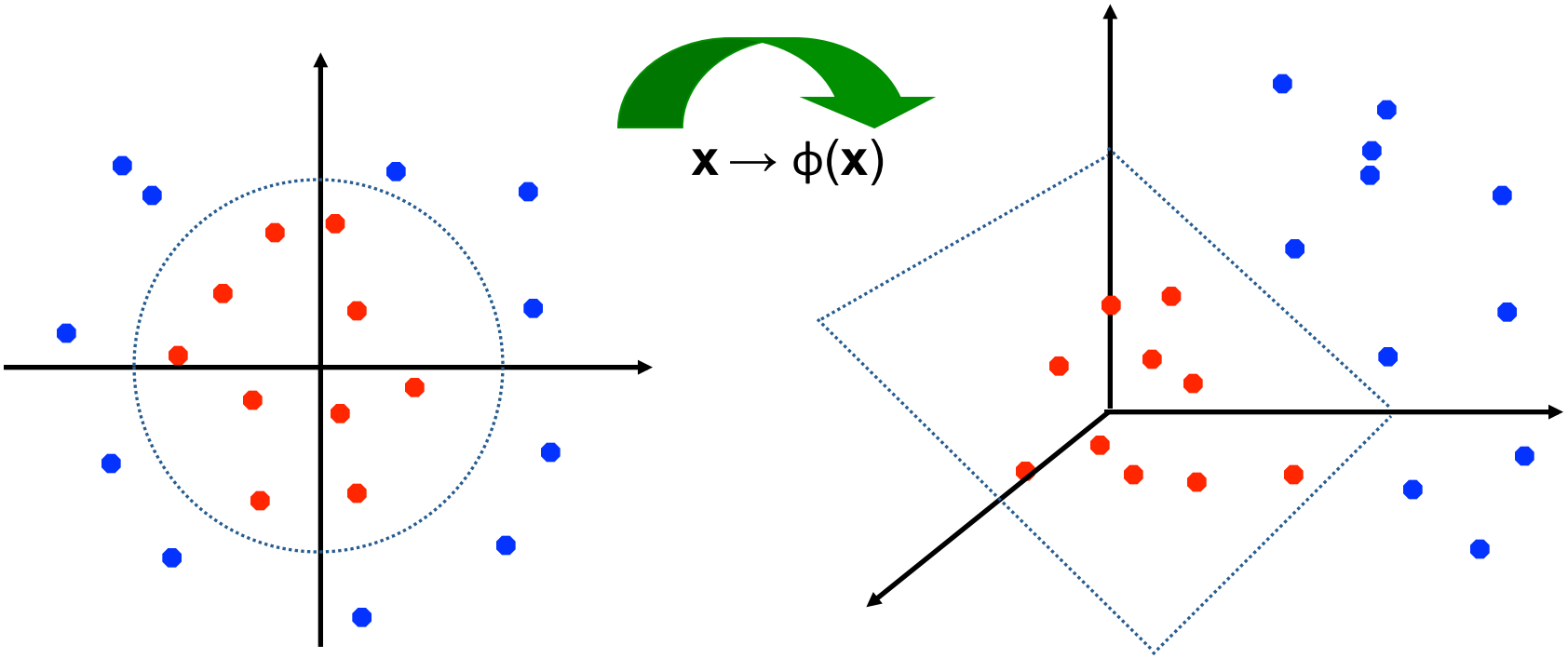


Feature spaces

- General idea: map to higher dimensional space
  - if **x** is in R^n, then $\phi(\mathbf{x})$ is in R^m for m>n
  - Can now learn feature weights **w** in R^m and predict:

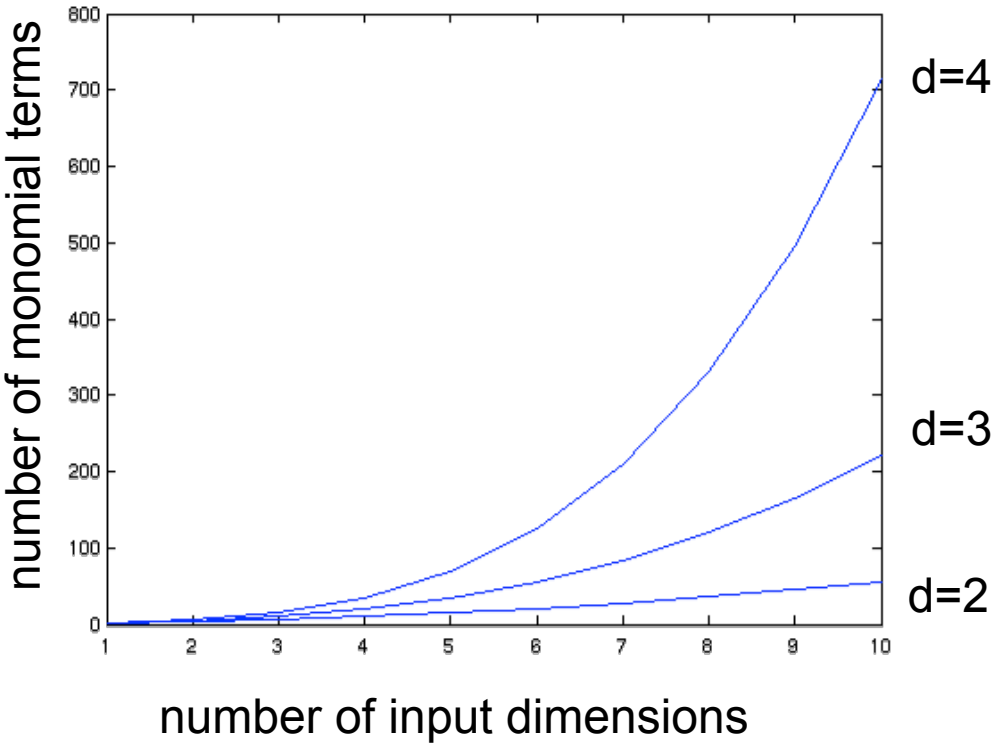
$$y = sign(\mathbf{w} \cdot \phi(\mathbf{x}))$$

  - Linear function in the higher dimensional space will be non-linear in the original space



Higher order polynomials

$$\text{num. terms} = \begin{pmatrix} d+m-1 \\ d \end{pmatrix} = \frac{(d+m-1)!}{d!(m-1)!}$$



m – input features
d – degree of polynomial

grows fast!
d = 6, m = 100
about 1.6 billion terms

Efficient dot-product of polynomials

Polynomials of degree exactly *d*

d=1

$$\phi(u).\phi(v) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} . \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1v_1 + u_2v_2 = u.v$$

d=2

$$\phi(u).\phi(v) = \begin{pmatrix} u_1^2 \\ u_1u_2 \\ u_2u_1 \\ u_2^2 \end{pmatrix} . \begin{pmatrix} v_1^2 \\ v_1v_2 \\ v_2v_1 \\ v_2^2 \end{pmatrix} = u_1^2v_1^2 + 2u_1v_1u_2v_2 + u_2^2v_2^2$$
$$= (u_1v_1 + u_2v_2)^2$$
$$= (u.v)^2$$

For any *d (we will skip proof):*

$$K(u, v) = \phi(u).\phi(v) = (u.v)^d$$

- Cool! Taking a dot product and an exponential gives same results as mapping into high dimensional space and then taking dot product

The "Kernel Trick"

- A *kernel function* defines a dot product in some feature space.

$$K(\mathbf{u},\mathbf{v}) = \boldsymbol{\phi}(\mathbf{u}) \bullet \boldsymbol{\phi}(\mathbf{v})$$

- Example:

  2-dimensional vectors $\mathbf{u}=[u_1 \;\; u_2]$ and $\mathbf{v}=[v_1 \;\; v_2]$; let $K(\mathbf{u},\mathbf{v})=(1 + \mathbf{u}\bullet\mathbf{v})^2$,

  Need to show that $K(\mathbf{x}_i,\mathbf{x}_j)= \boldsymbol{\phi}(\mathbf{x}_i) \bullet \boldsymbol{\phi}(\mathbf{x}_j)$:

  $K(\mathbf{u},\mathbf{v})=(1 + \mathbf{u}\bullet\mathbf{v})^2 = 1+ u_1^2v_1^2 + 2\, u_1v_1\, u_2v_2 + u_2^2v_2^2 + 2u_1v_1 + 2u_2v_2 =$

  $= [1,\ u_1^2,\ \sqrt{2}\, u_1u_2,\ u_2^2,\ \sqrt{2}u_1,\ \sqrt{2}u_2] \bullet [1,\ v_1^2,\ \sqrt{2}v_1v_2,\ v_2^2,\ \sqrt{2}v_1,\ \sqrt{2}v_2] =$

  $= \boldsymbol{\phi}(\mathbf{u}) \bullet \boldsymbol{\phi}(\mathbf{v})$, where $\boldsymbol{\phi}(\mathbf{x}) = [1,\ x_1^2,\ \sqrt{2}\, x_1x_2,\ x_2^2,\ \sqrt{2}x_1,\ \sqrt{2}x_2]$

- Thus, a kernel function *implicitly* maps data to a high-dimensional space (without the need to compute each $\boldsymbol{\phi}(\mathbf{x})$ explicitly).
- But, it isn't obvious yet how we will incorporate it into actual learning algorithms…

"Kernel trick" for The Perceptron!

- Never compute features explicitly!!!
  - Compute dot products in closed form K(u,v) = Φ(u) • Φ(v)
- Standard Perceptron:
  - set w_i=0 for each feature i
  - set a^i=0 for each example i
  - For t=1..T, i=1..n:
    - $y = sign(w \cdot \phi(x^i))$
    - if y ≠ y^i
      - $w = w + y^i \phi(x^i)$
      - a^i += y^i
  - At all times during learning:

$$w = \sum_k a^k \phi(x^k)$$

- Kernelized Perceptron:
  - set a^i=0 for each example i
  - For t=1..T, i=1..n:
    - $y = sign((\sum_k a^k \phi(x^k)) \cdot \phi(x^i))$

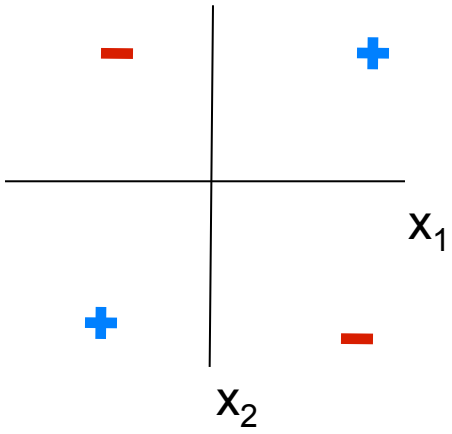
$$= sign(\sum_k a^k K(x^k, x^i))$$

    - if y ≠ y^i
      - a^i += y^i

Exactly the same computations, but can use K(u,v) to avoid enumerating the features!!!

- set a^i=0 for each example i
- For t=1..T, i=1..n:
  - $y = sign(\sum_k a^k K(x^k, x^i))$
  - if y ≠ y^i
    - a^i += y^i

x_1	x_2	y
1	1	1
-1	1	-1
-1	-1	1
1	-1	-1



$K(u,v) = (u \bullet v)^2$

e.g.,

$K(x^1,x^2)$

$= K([1,1],[-1,1])$

$= (1x-1+1x1)^2$

$= 0$

K	x^1	x^2	x^3	x^4
x^1	4	0	4	0
x^2	0	4	0	4
x^3	4	0	4	0
x^4	0	4	0	4

Initial:

- a = [a^1, a^2, a^3, a^4] = [0,0,0,0]

t=1,i=1

- $\Sigma_k a^k K(x^k,x^1)$ = 0x4+0x0+0x4+0x0 = 0, sign(0)=-1
- a^1 += y^1 → a^1+=1, new a= [1,0,0,0]

t=1,i=2

- $\Sigma_k a^k K(x^k,x^2)$ = 1x0+0x4+0x0+0x4 = 0, sign(0)=-1

t=1,i=3

- $\Sigma_k a^k K(x^k,x^3)$ = 1x4+0x0+0x4+0x0 = 4, sign(4)=1

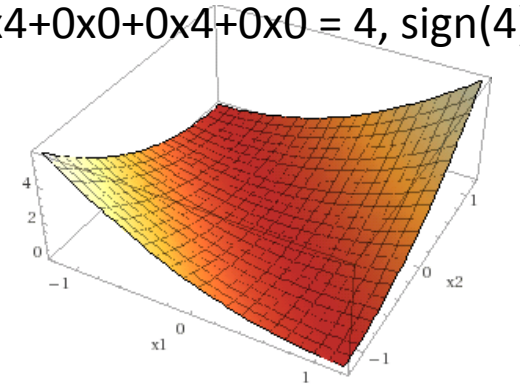
t=1,i=4

- $\Sigma_k a^k K(x^k,x^4)$ = 1x0+0x4+0x0+0x4 = 0, sign(0)=-1

t=2,i=1

- $\Sigma_k a^k K(x^k,x^1)$ = 1x4+0x0+0x4+0x0 = 4, sign(4)=1

…



Converged!!!

- $y=\Sigma_k a^k K(x^k,x)$
  $= 1\times K(x^1,x)+0\times K(x^2,x)+0\times K(x^3,x)+0\times K(x^4,x)$
  $= K(x^1,x)$
  $= K([1,1],x)$ (because x^1=[1,1])
  $= (x_1+x_2)^2$ (because $K(u,v) = (u \bullet v)^2$)

Common kernels

- Polynomials of degree exactly *d*

$$K(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \cdot \mathbf{v})^d$$

- Polynomials of degree up to *d*

$$K(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \cdot \mathbf{v} + 1)^d$$

- Gaussian kernels

$$K(\mathbf{u}, \mathbf{v}) = \exp\left(-\frac{||\mathbf{u} - \mathbf{v}||}{2\sigma^2}\right)$$

- Sigmoid

$$K(\mathbf{u}, \mathbf{v}) = \tanh(\eta \mathbf{u} \cdot \mathbf{v} + \nu)$$

- And many others: very active area of research!

Overfitting?

- Huge feature space with kernels, what about overfitting???
  - Often robust to overfitting, e.g. if you don't make too many Perceptron updates
  - SVMs (which we will see next) will have a clearer story for avoiding overfitting
  - But everything overfits sometimes!!!
    - Can control by:
      - Choosing a better Kernel
      - Varying parameters of the Kernel (width of Gaussian, etc.)

Kernels in logistic regression

$$P(Y = 0|\mathbf{X} = \mathbf{x}, \mathbf{w}, w_0) = \frac{1}{1 + exp(w_0 + \mathbf{w} \cdot \mathbf{x})}$$

- Define weights in terms of data points:

$$\mathbf{w} = \sum_j \alpha^j \phi(\mathbf{x}^j)$$

$$P(Y = 0|\mathbf{X} = \mathbf{x}, \mathbf{w}, w_0) = \frac{1}{1 + exp(w_0 + \sum_j \alpha^j \phi(\mathbf{x}^j) \cdot \phi(\mathbf{x}))}$$

$$= \frac{1}{1 + exp(w_0 + \sum_j \alpha^j K(\mathbf{x}^j, \mathbf{x}))}$$

- Derive gradient descent rule on α^j, w_0
- Similar tricks for all linear models: SVMs, etc

What you need to know

- The kernel trick
- Derive polynomial kernel
- Common kernels
- Kernelized perceptron