BOUNDARY EXTENSIONS FOR MAPPINGS BETWEEN METRIC SPACES

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Abstract. In this paper, we consider boundary extensions of two classes of mappings between metric measure spaces. These two mapping classes include in particular the well-studied geometric mappings such as quasiregular mappings and mappings with exponentially integrable distortion. Our main results extend the corresponding results of Äkinen and Guo [Ann. Mat. Pure. Appl. 2017] to the setting of metric measure spaces.

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1. Introduction

A mapping $f\colon \Omega \to \mathbb{R}^n$, on a domain $\Omega \subset \mathbb{R}^n$, is said to has finite distortion if the following three conditions are satisfied:

(1) $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$,
(2) $J_f = \det(Df) \in L^1_{\text{loc}}(\Omega)$,
(3) there exists a measurable $K_f\colon \Omega \to [1, \infty)$, so that for almost every $x \in \Omega$ we have

$$|Df(x)|^n \leq K_f(x) J_f(x),$$

where $|\cdot|$ is the operator norm. If $K_f \leq K < \infty$ almost everywhere, we say that f is K-quasiregular. If $n = 2$ and $K = 1$, we recover complex analytic functions. We say that f has expoentially integrable distortion if $\exp(\lambda K_f) \in L^1(\Omega)$ for some $\lambda > 0$.

The theory of quasiregular mappings, initiated by the works of Reshetnyak and Martio, Rickman and Väisälä, shows that they form, from the geometric function theoretic point of view, the correct generalization of the class of analytic functions to higher dimensions. In particular, Reshetnyak proved that non-constant quasiregular mappings are continuous, discrete and open, and that they preserve sets of measure zero; see [23], [22] for the theory of quasiregular mappings. This theory was further extended in a series of works, that were initiated by Iwaniec, Koskela and Onninen [16], to the setting of mappings of finite distortion with expoentially integrable distortion; see [13], [15].

In this article, we are mainly interested in the boundary behavior properties of this class of mappings in the setting of metric spaces. Recall that a classical theorem of Fatou states that a bounded analytic function defined on the unit disc has radial limits at almost every boundary point. It remains an open question whether this theorem continue to

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hold for quasiregular mappings in higher dimensions $n \geq 3$. It is even unknown whether bounded quasiregular maps have radial limits at any point on the boundary of the unit ball. A substantial progress on this problem was given by Rajala [21], who proved that quasiregular local homeomorphisms have radial limits at infinitely many boundary point of the unit ball. Adding certain growth assumptions on the multiplicity functions, Martio and Rickman [18] proved the existence of radial limits of quasiregular maps at almost every boundary point. The latter result was further extended by Äkinen [1] to mappings of finite distortion defined on the unit ball and by Äkinen and Guo [2] for mappings of finite distortion defined on general John domains. For homeomorphisms with finite distortion, an essentially sharp result along this direction was obtained earlier by Koskela and Nieminen [17]; see also [19] for examples of quasiregular maps without radial limits at any boundary point.

We now turn to the setting of metric spaces. The theory of quasiconformal maps in the setting of metric spaces was initiated by Heinonen and Koskela in [10] and was further developed for instance in [3, 24] (see also the references therein for more related works). The theory of quasiregular maps and mappings of finite distortion in singular metric spaces was initiated by Heinonen and Holopainen [9] and was later extended to more general setting in [12, 20, 6].

Let $X_1 = (X_1, d_1, \nu_1)$, $X_2 = (X_2, d_2, \nu_2)$ be two metric measure spaces and Ω be a bounded domain of X_1 . Through this paper, we shall assume that $\Omega \subset X_1$ is Ahlfors q-regular (see Section 2 below for precise definitions). For the statement of our main result, we introduce the following two classes of mappings.

Definition 1.1. Let $f : \Omega \to X_2$ be a mapping.

(1) We say that f belongs to class $\mathcal{A}_1$, if there exists $\alpha \in L^1(\Omega)$, $C \geqslant 1, \sigma \geq 1$, such that
$$\operatorname{diam} f(B) \leq C\bigg(\int_{\sigma B} \alpha(y) d\nu_1(y)\bigg)^{\frac{1}{q}}$$
for every $B = B(x, r)$ for which $\sigma B \subset\subset \Omega$.

(2) We say that f belongs to class $\mathcal{A}_2$, if there exists $\alpha \in L^1(\Omega)$, $C \geqslant 1, \sigma \geq 1$, such that
$$\operatorname{diam} f(B) \leq C\bigg(\int_{\sigma B} \alpha(y) d\nu_1(y)\bigg)^{\frac{1}{q}} \log\left(\frac{1}{\operatorname{diam} B}\right)^{\frac{1}{q}}$$
for every $B = B(x, r)$ for which $\sigma B \subset\subset \Omega$.

In the setting of Euclidean spaces, the class $\mathcal{A}_1$ contains all quasiregular maps between Euclidean domains, while $\mathcal{A}_2$ contains mappings of finite distortion with λ-exponentially integrable distortion. In both cases, α can be taken as the Jacobian determinant J_f of f; see for instance [17, 2]. We shall show in Example 2.7 below that this extends to the setting of sufficiently nice metric measure spaces as well.

Next, we introduce the class of φ-length John domains that were initially introduced in [7]. When $\varphi(t) = Ct$, it reduces to the well-known class of John domains (see [8] for more on John domains in the setting of metric spaces).

Definition 1.2. Let φ be a continuous, increasing function with $\varphi(0) = 0$ and $\varphi(t) \geq t$ for all $t > 0$. A bounded domain $\Omega \subset X_1$ is called a φ-length John domain with distinguished point $x_0 \in \Omega$ if there exists a constant $c \geq 1$ such that for each point $x \in \Omega$, there exists a curve $\gamma : [0, 1] \to \Omega$ satisfying $\gamma(0) = x, \gamma(1) = x_0$ and for all $t \in [0, 1]$,

$$\varphi(cd_1(\gamma(t), \partial\Omega)) \geq l(\gamma([0, t])).$$

The curve γ appearing in Definition 1.2 is called a φ-length John curve connecting x to the center x_0 and it needs not to be unique. Given $\xi \in \partial\Omega$, the set $I^{(\varphi,c)}(\xi, x_0)$ consists of all φ-length John curves in Ω connecting ξ to x_0 (see Lemma 3.1 below for the existence). When $\varphi(t) = t$, we simply write $I^c(\xi, x_0)$, instead of $I^{\varphi,c}(\xi, x_0)$, to denote the class of all John curves connecting ξ to x_0.

The purpose of this paper is to prove that the functions $f \in \mathcal{A}_i$, $i = 1, 2$, for a sufficiently large portion of the boundary, have limits along φ-length John curves.

Theorem 1.3. *Assume Ω is a φ-length John domain with center x_0 and Ahlfors q-regular. Given a map $f : \Omega \to X_2$, let E_f be the set of points $\omega \in \partial\Omega$ for which there exists a curve $\gamma \in I^{(\varphi,c)}(\omega, x_0)$ so that f does not have a limit along γ. Then we have the following:*

(1) *Let h be a doubling gauge function and $g(t) = \frac{\varphi(t)h(\varphi(t))^{\frac{1}{q-1}}}{t}$ be a continuous monotonic increasing (or monotonic decreasing) function, such that*
$$\int_0^1 \frac{g(t)}{t} dt < \infty.$$
If $f \in \mathcal{A}_1$, then $\mathcal{H}^h(E_f) = 0$.

(2) *Let h be a doubling gauge function and $g(t) = \frac{\varphi(t)h(\varphi(t))^{\frac{1}{q-1}}}{t} \left[\log \frac{1}{t}\right]^{\frac{1}{q-1}}$ be a continuous monotonic increasing (or monotonic decreasing) function, such that*
$$\int_0^1 \frac{g(t)}{t} dt < \infty.$$
If $f \in \mathcal{A}_2$, then $\mathcal{H}^h(E_f) = 0$.

Theorem 1.3 can be regarded as a natural extension of [17, Corollary 1.3] and [2, Theorem 3.5] from Euclidean domains to domains in general metric measure spaces, and from the linear John domains to general nonlinear φ-length John domains.

In case Ω is a uniform domain, we are able to prove the uniqueness of limits, which generalizes [2, Theorem 3.10].

Theorem 1.4. *Let $\Omega \subset X_1$ be a c-uniform domain with center x_0 and h be a doubling gauge function satisfying the condition (2) of Theorem 1.3. If $f \in \mathcal{A}_2$, then f has a unique limit along curves $\gamma \in I^c(w, x_0)$ for $\mathcal{H}^h$-almost every $\omega \in \partial\Omega$, that is, if $\gamma, \eta \in I^c(\omega, x_0)$ so that*

$$\lim_{t\to 0^+} f(\gamma(t)) = a \text{ and } \lim_{t\to 0^+} f(\eta(t)) = b.$$

Then $a = b$.

The idea for both Theorems 1.3, 1.4 is essentially the same as in [2] and thus many of the arguments are similar to the one used there. The main technical difficulty is to develop a suitable Dyadic-Whitney decomposition for proper domains in a general complete doubling metric space, which relies crucially the result of Hytönen and Kairema [14].

This paper is organized as follows. In Section 3.1, we will show that f can be extended to the boundary of a φ-length John domain along φ-length John curves with a small exceptional set with respect to Hausdorff gauges, and in Section 3.2, we prove our uniqueness theorem.

2. Preliminaries

A curve γ in Ω is a continuous mapping $\gamma : [0,1] \to \Omega$. A curve $\gamma : [0,1] \to \Omega$ is said to connect points $x, y \in \Omega$, if $\gamma(0) = y$ and $\gamma(1) = x$; similarly a curve $\gamma : (0,1] \to \Omega$ is said to connect points $x \in \Omega, y \in \partial\Omega$ if $\gamma(1) = x, \gamma((0,1]) \subset \Omega$ and

$$\lim_{t\to 0^+} \gamma(t) = y.$$

We use the notation $l(\gamma)$ to denote the (Euclidean) length of a curve γ.

Definition 2.1. (Ahlfors q-regular) Let $X_1 = (X_1, d_1, \nu_1)$ be a metric measure space. An open set $\Omega \subset X_1$ is said to be Ahlfors q-regular ($q > 1$) if there exists a contant C_q such that for any $x \in \Omega$ and for any r>0,

$$\frac{1}{C_q} r^q \le \nu_1\bigg(B(x,r) \cap \Omega\bigg) \le C_q r^q.$$

Definition 2.2. (Doubling metric space) A metric measure space (X, d, ν) is said to be a doubling metric space if there exists a positive integer $A \in \mathbb{N}$ such that for every $x \in X$ and for every $r > 0$, the ball $B(x,r) := \{y \in X, d(y,x) < r\}$ can be covered by at most A balls $B(x_i, \frac{r}{2})$.

The Dyadic-Whitney decomposition is based on the following theorem proved by Hytönen and Kairema [14].

Proposition 2.3. *Let $X = (X, \rho, \nu)$ be a doubling metric space and suppose that there are constants $0 < c_0 \le C_0 < \infty$, and $\delta \in (0,1)$ satisfying*

$$12 C_0 \delta \le c_0.$$

Given a set of points $\left\{z_\alpha^k\right\}_\alpha, \alpha \in \mathscr{A}_k$, $k \in \mathbb{Z}$, with the properties that

$$\rho\left(z_\alpha^k, z_\beta^k\right) \ge c_0 \delta^k \quad (\alpha \neq \beta), \quad \min_\alpha \rho\left(x, z_\alpha^k\right) < C_0 \delta^k \quad \text{for all } x \in X.$$

Then we can construct a family of sets Q_α^k, called open, half-open, and closed dyadic cubes, such that:

(1)$X = \bigcup_\alpha Q_\alpha^k$ (disjoint union) for all $k \in \mathbb{Z}$;
(2)if $\ell \ge k$, then either $Q_\beta^\ell \subseteq Q_\alpha^k$ or $Q_\alpha^k \cap Q_\beta^\ell = \varnothing$;
(3)$B\left(z_\alpha^k, c_0\delta^k\right) \subseteq Q_\alpha^k \subseteq B\left(z_\alpha^k, C_1\delta^k\right) =: B\left(Q_\alpha^k\right)$, where $c_1 := 3^{-1}c_0$ and $C_1 := 2C_0$.

The half-open cubes Q_α^k depend on z_β^ℓ for $\ell \ge \min(k, k_0)$, where $k_0 \in \mathbb{Z}$ is a preassigned number entering the construction.

Using the dydaic cubes satisfying Proposition 2.3, we can select a suitable subfamily of cubes to form a Whitney decomposition of a general open set $\Omega \subset X$. Recall that the Whitney decomposition states that any open subset $\Omega \subset X_1$ can be written as the union of cubes such that the diameter of each cube is comparable to its distance to the boundary. As a substitute for the Whitney decomposition in the Euclidian setting, we introduce Dyadic-Whitney decomposition in doubling metric spaces.

Definition 2.4. (Dyadic-Whitney decomposition). Let $X_1 = (X_1, d_1, \nu_1)$ be a doubling metric space. For an open subset $\Omega \subseteq X_1$, a dyadic-Whitney decomposition of Ω with data (δ, c_0, C_1, a), where $\delta \in (0,1)$, $C_1 > c_0 > 0$, and $a \geq 4$, is a collection $\mathscr{W}_\Omega$ of open subsets of X_1 satisfying the following properties:

(1) $\bigcup_{Q \in \mathscr{W}_\Omega} Q = \Omega$.

(2) $Q \bigcap Q' = \emptyset$ for all $Q, Q \in \mathscr{W}_\Omega$, where $Q \neq Q'$.

(3) For any $Q \in \mathscr{W}_\Omega$, there exists $x \in \Omega$ and $k \in \mathbb{Z}$ such that

$$B(x, c_0\delta^k) \subset Q \subset B(x, C_1\delta^k)$$

and

$$(a-2)C_1\delta^k \leq d_1(Q, \partial\Omega) \leq aC_1\delta^{k-1}.$$

The sets $Q \in \mathscr{W}_\Omega$ are referred to as (dyadic-Whitney) cubes. For each $k \in \mathbb{Z}$, we set

$$\mathscr{W}_\Omega^k = \{Q \in \mathscr{W}_\Omega : B(x, c_0\delta^k) \subset Q \subset B(x, C_1\delta^k)\}.$$

From the definition, if $Q \in W_\Omega$, then $B(c_0 \operatorname{diam} Q) \subset Q \subset B(C_1 \operatorname{diam} Q)$.We denote the lower ball by B_Q and the upper ball associated to Q by B^Q.

We first establish the folowing existence result.

Lemma 2.5. *Let X be a doubling metric space and let $\Omega \subsetneq X$ be open. For all $\delta, c_0, C_1, a > 0$ satisfying $a \geq 4, 0 < \delta + c_0 < 1/4$ and $C_1 > (1-\delta)^{-1}$, there exists a dyadic-Whitney decomposition of Ω with data (δ, c_0, C_1, a).*

Proof. Given a set of points $\left\{z_\alpha^k\right\}_\alpha, \alpha \in \mathscr{A}_k$, for every $k \in \mathbb{Z}$, with the properties that $\rho\left(z_\alpha^k, z_\beta^k\right) \geq c_0\delta^k(\alpha \neq \beta), \min_\alpha \rho\left(x, z_\alpha^k\right) < C_0\delta^k$ for all $x \in X$, we can construct families of sets $Q_\alpha^k \subset \Omega$ satisfying the properties in Proposition 2.2. Then, we divide Ω as

$$\Omega_k = \left\{x \in \Omega : aC_1\delta^k \leq d(x, X\backslash\Omega) \leq aC_1\delta^{k-1}\right\},$$

thus $\Omega = \bigcup_{k=-\infty}^\infty \Omega_k$. We make the first choice of sets,

$$\mathscr{W}_\Omega^0 = \bigcup_{k=-\infty}^\infty \left\{Q_\alpha^k : \alpha \in I_k, Q_\alpha^k \cap \Omega_k \neq \emptyset\right\}.$$

Note that if $Q_\alpha^k \in \mathscr{W}_\Omega^0$ then $d\left(Q_\alpha^k, X\backslash\Omega\right) \leq aC_1\delta^{k-1}$, and it follows from the triangle inequality that

$$d\left(Q_\alpha^k, X\backslash\Omega\right) \geq aC_1\delta^k - \operatorname{diam} Q_\alpha^k \geq (a-2)C_1\delta^k.$$

Hence any $Q \in \mathscr{W}_\Omega^0$ satisfies inequality for this value of k.

Since by Proposition 2.3 any two cubes in $\mathscr{W}_\Omega^0$ are either disjoint or one contains the other, we may choose the maximal subset of $\mathscr{W}_\Omega^0$ and notify it as $\mathscr{W}_\Omega$. Also property (2) and property (3) can be easily proved by the properties of the dyadic decomposition. $\square$

According to lemma 2.5, if (δ, c_0, C_1, a) satisfy $a \geq 4$, $0 < \delta + c_0 < 1/4$ and $C_1 > (1-\delta)^{-1}$, then the dyadic-Whitney decomposition exists.

Lemma 2.6. *Let λB^Q be the ball which has the same center as B^Q but is expanded by the factor λ . Under the condition of lemma 2.5, there exists $\lambda_0 \in \mathbb{R}$ such that for any $Q \in \mathscr{W}_\Omega$, $x \in \Omega$ and $1 \leq \lambda < \lambda_0$, there are at most N balls λB^{Q_i} containing x.*

Proof. We prove this lemma in three steps.
Step 1: We claim that if $Q_1, Q_2 \in \mathscr{W}$ and $Q_1 \cap Q_2 \neq \varnothing$, then

$$\frac{c_0\delta(a-2)}{C_1(2\delta+a)} \operatorname{diam} Q_1 \leq \operatorname{diam} Q_2 \leq \frac{C_1(2\delta+a)}{c_0\delta(a-2)} \operatorname{diam} Q_1.$$

By the definition 2.4, there exists $k_1, k_2 \in \mathbb{Z}$ such that,

$$\begin{aligned}
(a-2)C_1\delta^{k_1} &\leq d_1(Q_1, \partial\Omega) \leq aC_1\delta^{k_1-1}, \\
(a-2)C_1\delta^{k_2} &\leq d_1(Q_2, \partial\Omega) \leq aC_1\delta^{k_2-1}, \\
2c_0\delta^{k_1} &\leq \operatorname{diam} Q_1 \leq 2C_1\delta^{k_1}, \\
2c_0\delta^{k_2} &\leq \operatorname{diam} Q_2 \leq 2C_1\delta^{k_2}.
\end{aligned}$$

Since $Q_1 \cap Q_2 \neq \varnothing$,

$$d_1(Q_2, \partial\Omega) \leq \operatorname{diam} Q_1 + d_1(Q_1, \partial\Omega) \leq 2C_1\delta^{k_1} + aC_1\delta^{k_1-1} \leq \frac{C_1(2\delta+a)}{2c_0\delta} \operatorname{diam} Q_1.$$

On the other hand,

$$d_1(Q_2, \partial\Omega) \geq (a-2)C_1\delta^{k_2} \geq \frac{a-2}{2} \operatorname{diam} Q_2.$$

Therefore,

$$\operatorname{diam} Q_2 \leq \frac{C_1(2\delta+a)}{c_0\delta(a-2)} \operatorname{diam} Q_1.$$

Step 2: We claim that there are at most N dyadic-Whitney cubes touching Q. Suppose $Q \in W_\Omega$, then $B(c_0 \operatorname{diam} Q) \subset Q \subset B(C_1 \operatorname{diam} Q)$. From step 1, for any Q' touching Q, there exists constant C such that $\frac{1}{C} \operatorname{diam} Q \leq \operatorname{diam} Q' \leq C \operatorname{diam} Q$ and there exists $\lambda_0 = \frac{\delta(2C_1+ac_0-2c_0)+aC_1}{c_0\delta(a-2)} > 1$ such that the cubes touching Q can be covered by $\lambda_0 B^Q$. Since X_1 is a doubling metric space,there at most N cubes $B_{Q'}$ covering $\lambda_0 B^Q$. Therefore, there are at most $N\,(\delta, c_0, C_1, a)$ dyadic-Whitney cubes touching Q when $1 \leq \lambda < \lambda_0$.

Step 3: We claim that if $\lambda B^Q \cap \lambda B^{Q'} \neq \emptyset$, then there exists constant C such that $\frac{1}{C} \operatorname{diam} Q \leq \operatorname{diam} Q' \leq C \operatorname{diam} Q$. Since Q is dyadic-Whitney cube ,

$$(a-2)C_1 \operatorname{diam} Q \leq d_1(Q, \partial\Omega) \leq aC_1 \operatorname{diam} Q.$$

Suppose $x \in \lambda B^Q \cap \lambda B^{Q'}$, then

$$d_1(x, \partial\Omega) \geq d_1(\lambda B^{Q'}, \partial\Omega) \geq (a-3)C_1 \operatorname{diam} Q'.$$

On the other hand,

$$d_1(x,\partial\Omega) \leq d_1(x,x_{Q'})+d_1(x_{Q'},\partial\Omega) \leq \operatorname{diam} \lambda B^{Q'}+d_1(x_{Q'},\partial\Omega) \leq C' \operatorname{diam} Q'+aC_1 \operatorname{diam} Q'.$$

Since there exists constant c such that $\frac{1}{c}\operatorname{diam} Q \leq d_1(x,\partial\Omega) \leq c \operatorname{diam} Q$, $\operatorname{diam} Q'$ is comparable to $\operatorname{diam} Q$. Following by step 2,we get the conclusion.

$\square$

In the next example, we point out when Ω is sufficiently nice, the class $\mathcal{A}_1$ contains quasiregular maps and $\mathcal{A}_2$ contains mappings of finite distortion with exponentially integrable distortion.

Example 2.7. Let $f\colon \Omega \to X_2$ be a continuous $N^{1,p}(\Omega, X_2)$-map, where $\Omega \subset X_1$ is a bounded Ahlfors q-regular domain and supports an abstract $(1,p)$-Poincaré inequality (in the senes of Heinonen and Koskela [10]) with $p > q-1$. Assume Y is complete, unbounded, Ahlfors q-regular and supports a $(1,1)$-Poincaré inequality. Then

- $f \in \mathcal{A}_1$ whenever it is K-quasiregular;
- $f \in \mathcal{A}_2$ whenever it has λ-exponentially integrable distortion.

We briefly indicate the proof of the first assertion, since the proof of the second one is similar (both are indeed similar to [17, Proof of Theorem 1.1]). Fix a ball $B = B(x,r)$ with $\sigma B \subset \Omega$, where σ is a constant to be determined later. By the abstract Sobolev embedding on spheres from [8, Theorem 7.1], we know

$$\operatorname{diam} f(B(x,r)) \leq Cr^{1-\frac{q-1}{p}}\Big(\frac{1}{r}\int_{5\mu B} g_f(y)^p d\nu(y)\Big)^{\frac{1}{p}} = r^{\frac{p-q}{p}}\Big(\int_{5\mu B} g_f(y)^p d\nu(y)\Big)^{\frac{1}{p}}, \quad (2.1)$$

where g_f is the minimal p-weak upper gradient of f; see [11]. Since f is K-quasiregular (see [6] for precise definition), $g_f \leq KJ_f$ and so by Hölder's inequality, we know

$$\Big(\int_{5\mu B} g_f(y)^p d\nu(y)\Big)^{\frac{1}{p}} \leq K^{\frac{p}{q-p}} r^{\frac{q-p}{p}}\Big(\int_{5\mu B} J_f(x) d\nu_1(x)\Big)^{1/q}.$$

On the other hand, by [6, Proof of Lemma 3.3], f is locally T-pseudomonotone, that is,

$$\operatorname{diam} f(B(x,r)) \leq T \operatorname{diam} f(\partial B(x,r))$$

for every ball $B = B(x,r)$ with $MB \subset \Omega$. Take $\sigma = \max\{5\mu, M\}$, which depends only on the data associated to the spaces. Then for each $B = B(x,r)$ with $\sigma B \subset \Omega$,

$$\operatorname{diam} f(B) \leq T \operatorname{diam} f(\partial B) \leq C\Big(\int_{\sigma B} J_f(x) d\nu_1(x)\Big)^{1/q}.$$

3. Proof of main results

3.1. **Existence of limits along curves.** There might be points on the boundary $\partial\Omega$ that are not accessible by a rectifiable curve inside Ω if we just think about general bounded domains; see for instance Figure 1 below.

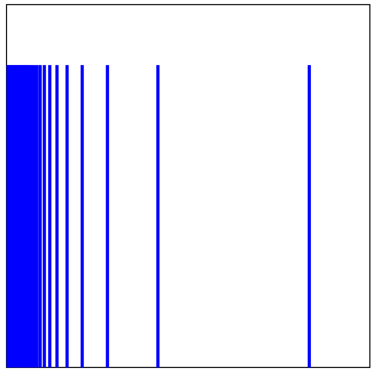

FIGURE 1. Topolpgist's comb

To avoid these situations, the authors considered the class of John domains [?]. Here, we consider the more general class of φ-length John domains.

Lemma 3.1. *Let $\Omega \subset X_1$ be a φ-length John domain with center x_0. Then each boundary point can be connected to x_0 by a φ-length John curve.*

Proof. For any $\omega \in \partial\Omega$, we can find a sequence $\omega_i \in \Omega$ such that $\lim_{i\to+\infty} \omega_i = \omega$. Since Ω is a φ-length John domain, every $\omega_i \in \Omega$ can be connected to x_0 by $\gamma_i : [0,1] \to \Omega$ such that for all $t \in [0,1]$,

$$l\left(\gamma_i|_{[0,t]}\right) \leq \varphi\left(cd_1\left(\gamma_i(t), \partial\Omega\right)\right).$$

Thus

$$\sup l_i = \sup l\left(\gamma_i\right) \leq \varphi(c\mathrm{diam}\,\Omega) < \infty.$$

If we parameterize γ_i by arc-length, then $\gamma_i : [0, l_i] \to \Omega$ is 1-Lipschitz. According to the Arzela-Ascoli theorem, there exists $\gamma : [0, L] \to \Omega$ such that γ_i converge to γ uniformly. Since the length functional is lower semicontinuous and φ, d_1 are continuous, we have for all $t \in [0, L]$,

$$l\left(\gamma|_{[0,t]}\right) \leq \varphi\left(cd_1(\gamma(t), \partial\Omega)\right).$$

Therefore, γ is a φ-length John curve connecting ω to x_0. □

We define $P^{(\varphi,c)}(\xi) = \{Q \in \mathscr{W}_\Omega : Q \cap \gamma \neq \emptyset \; for\ some\ \gamma \in I^{(\varphi,c)}(\xi, x_0)\}$. For $Q \in \mathscr{W}_\Omega$ and $E \subset \partial\Omega$, the definition of Q shadow on E is followed by

$$S_E^{(\varphi,c)}(Q) = \{\xi \in E : Q \in P^{(\varphi,c)}(\xi)\}.$$

If $E = \partial\Omega$, we write $S^{(\varphi,c)}(Q)$ instead of $S_{\partial\Omega}^{(\varphi,c)}(Q)$.

Notice that for a point $y \in \partial\Omega$ there might be infinitely many φ-length John curves connecting x_0 and y. For the rest of this section, the standing assumptions are: Ω is a φ-length John domain with center x_0 and $\mathscr{W}_\Omega$ is a dyadic decomposition of Ω.

We need the following two basic estimates for the shadow $S^{(\varphi,c)}$. The case of Euclidean setting is proved in [5].

Lemma 3.2. *Let $Q \in \mathscr{W}_\Omega$ and $\xi \in \partial\Omega$. Then $S^{(\varphi,c)}(Q)$ is closed and there exists a constant $C = C(c, c_0, C_1, \delta, a) > 0$ such that*

$$\mathrm{diam}\, S^{(\varphi,c)}(Q) \leq 3\varphi(C\,\mathrm{diam}\, Q).$$

Furthermore, for any $k \in \mathbb{Z}$,

$$\#\{Q \in \mathscr{W}_\Omega^k : \xi \in S^{(\varphi,c)}(Q)\} \leq C_q^2 \Big(\frac{\varphi(2C(c)C_1\delta^k)}{(c_0\delta^k)} \Big)^q.$$

Proof. Let $\xi \in S^{(\varphi,c)}(Q)$. We can find a φ-length John curve γ joining ξ to x_0 in Ω so that $\gamma(t_Q) \in Q$ for some $t_Q \in [0,1]$. Since φ is a continuous increasing function,

$$d_1(\xi, Q) \leq d_1(\xi, \gamma(t_Q)) \leq l(\gamma[0,t_Q])) \leq \varphi(cd_1(\gamma(t_Q), \partial\Omega)) \leq \varphi(c(\operatorname{diam} Q + d_1(Q, \partial\Omega))).$$

Note that,

$$d_1(Q, \partial\Omega) \leq aC_1\delta^{k-1} = 2bc_0\delta^k \leq b(\operatorname{diam} Q),$$

where $b = \frac{aC_1}{2c_0\delta}$. Thus,

$$d_1(\xi, Q) \leq \varphi(c(\operatorname{diam} Q + b(\operatorname{diam} Q))).$$

Then,

$$Q \subset B(\xi, 2\varphi((c(b+1)+1)\operatorname{diam} Q)).$$

Moreover,

$$\begin{aligned}
\operatorname{diam}(S^{(\varphi,c)}(Q)) &= \sup\{d_1(x,y) | x, y \in S^{(\varphi,c)}(Q)\} \\
&\leq \sup\{d_1(x,Q) + \operatorname{diam} Q + d_1(y,Q) | x, y \in S^{(\varphi,c)}(Q)\} \\
&\leq 2\varphi(c(\operatorname{diam} Q + b(\operatorname{diam} Q))) + \varphi(\operatorname{diam} Q) \\
&\leq 2\varphi(c(\operatorname{diam} Q + b(\operatorname{diam} Q)) + \operatorname{diam} Q) + \varphi(\operatorname{diam} Q) \\
&\leq 3\varphi((c(b+1)+1)\operatorname{diam} Q) \\
&= 3\varphi(C \operatorname{diam} Q).
\end{aligned}$$

Now, fix $\xi \in \partial\Omega$ and define

$$a_k = \#\{Q \in \mathscr{W}_\Omega^k : \xi \in S^{(\varphi,c)}(Q)\} = \#\{Q \in \mathscr{W}_\Omega^k : Q \in P^{(\varphi,c)}(\xi)\}, k \in \mathbb{Z}.$$

Since the cubes $Q \in \mathscr{W}_\Omega^k$ are essentially disjoint, we have

$$\begin{aligned}
a_k \cdot \frac{1}{C_q}(c_0\delta^k)^q \leq a_k \cdot \nu_1(B(x, c_0\delta^k)) &\leq \sum_{Q \in \mathscr{W}_\Omega^k \bigcap P^{(\varphi,c)}(\xi)} \nu_1(Q) \\
&\leq \nu_1(B(\xi, 2\varphi((c(b+1)+1)\operatorname{diam} Q))) \\
&\leq C_q \Big(2\varphi((c(b+1)+1)\operatorname{diam} Q)\Big)^q \\
&\leq C_q \Big(2\varphi(2(c(b+1)+1)C_1\delta^k)\Big)^q.
\end{aligned}$$

Then,

$$a_k \leq C_q^2 \Big(\frac{2\varphi(2C(c)C_1\delta^k)}{(c_0\delta^k)} \Big)^q.$$

□

Lemma 3.3. *Assume that* μ *is a Borel measure on* $\partial\Omega$ *and* $E \subset \partial\Omega$ *measurable. Then for each* $k \in \mathbb{Z}$ *we have*

$$\sum_{Q \in \mathscr{W}_\Omega^k} \mu(S_E^{(\varphi,c)}(Q)) \leq C_q^2 \Big(\frac{2\varphi(C(c)C_1\delta^k)}{(c_0\delta^k)} \Big)^q \mu(E).$$

Proof. Lemma 3.2 implies that

$$\begin{aligned}\sum_{Q\in\mathscr{W}_\Omega^k}\mu(S_E^{(\varphi,c)}(Q)) &= \sum_{Q\in\mathscr{W}_\Omega^k}\int_E \chi_{S_E^{(\varphi,c)}(Q)}(\omega)d\mu(\omega)\\ &= \int_E \sum_{Q\in\mathscr{W}_\Omega^k}\chi_{S_E^{(\varphi,c)}(Q)}(\omega)d\mu(\omega)\\ &\le C_q^2\Big(\frac{2\varphi(C(c)C_1\delta^k)}{(c_0\delta^k)}\Big)^q\mu(E).\end{aligned}$$

□

Now we give the definition of discrete length of a curve.

Definition 3.4. Let $\Omega\subset X_1$ be a φ-length John domain with center x_0. Assume that $\xi\in\partial\Omega$ and $\gamma\in I^{(\varphi,c)}(\xi,x_0)$. Given a continuous mapping $f:\Omega\to X_2$, we define the discrete length of $f(\gamma)$ by

$$\ell_d[f(\gamma)] := \sum_{Q\in\mathscr{W}_\Omega, Q\cap\gamma\neq\emptyset}\operatorname{diam} f(Q).$$

Lemma 3.5. *If $\ell_d[f(\gamma)]<\infty$, then $\lim_{t\to0^+} f(\gamma(t))$ exists.*

Proof. By lemma 3.2, for fixed k, $\#\{Q\in\mathscr{W}_\Omega^k : \xi\in S^{(\varphi,c)}(Q)\}<\infty$. If $l_d[f(\gamma)] = \sum_{\substack{Q\in\mathscr{W}_\Omega\\ Q\cap\gamma\neq\emptyset}}\operatorname{diam} f(Q)<\infty$, then the general term of this series tends to zero. Since the cubes are Dyadic-Whitney cubes, for any two points $\gamma(t_1),\gamma(t_2)$ close to the boundary, we can find a cube Q_j which is also close to the boundary and includes $\gamma(t_1)$ and $\gamma(t_2)$. Then by definition 2.4 we have, for any $\varepsilon>0$, there exists $\delta' = 2C_1\delta^k>0$ such that when $|t_1-t_2|<\operatorname{diam} Q_j<\delta'$, $d(f(\gamma(t_1)),f(\gamma(t_2)))<\operatorname{diam} f(Q_j)<\varepsilon$. According to the Cauchy convergence criterion, the limit exists. □

Proof of Theorem 1.3. For simplicity,we write $S_E(Q)$ for $S_E^{(\varphi,c)}(Q)$. Our aim is to prove that $\mathcal{H}^h(A_\infty)=0$, where A_∞ is the set of points $\xi\in\partial\Omega$ for which there is a curve $\gamma\in I^{(\varphi,c)}(\xi,x_0)$ such that $\ell_d[f(\gamma)]=\infty$. Since $E_f\subset A_\infty$, $\mathcal{H}^h(E_f)=0$.

On the contrary, we assume that $\mathcal{H}^h(A_\infty)>0$. Then $\mathcal{H}^h(A_k)>0$, where A_k is the set of points $\xi\in\partial\Omega$ for which there exists $\gamma\in I^{(\varphi,c)}(\xi,x_0)$ so that $\ell_d[f(\gamma)]\ge k$. Then by Frostman's lemma there exists a Borel measure μ supported in A_k so that for every $B(x,r)\subset X_1$,

$$\mu(B(x,r))\le h(r),$$

and

$$\mu(E_k)\simeq\mathcal{H}^h_\infty(A_k)\ge\mathcal{H}^h_\infty(A_\infty)>0.$$

By the definition of A_k and the definition of the discrete length of $f(\gamma)$,

$$\begin{aligned}\mu(A_k)k &\le \int_{A_k}\ell_d[f(\gamma_\omega)]d\mu(\omega)\\ &\le \int_{A_k}\sum_{Q\in\mathscr{W}_\Omega, Q\cap\gamma_\omega\neq\emptyset}\operatorname{diam} f(Q)d\mu(\omega).\end{aligned}$$

From now on, we assume that $f \in \mathcal{A}_2$. Let $\lambda > 1$ be the constant in the lemma 2.6. If $Q \in \mathscr{W}_\Omega$, then set $B'_Q = B(x_Q, r_Q)$, where x_Q is the center of Q and $r_Q = \frac{\operatorname{diam} Q}{2}$. Then we have the following inequalities:

$$\begin{aligned}
\mu(A_k)k &\leq \int_{A_k} \sum_{Q \in \mathscr{W}_\Omega, Q \cap \gamma_\omega \neq \emptyset} \operatorname{diam} f(B'_Q) d\mu(\omega) \\
&\leq \sum_{Q \in \mathscr{W}_\Omega} \int_{A_k} \chi_{S(Q)}(\omega) d\mu(\omega) \operatorname{diam} f(B'_Q) \\
&\leq \sum_{Q \in \mathscr{W}_\Omega} \mu(S_{A_k}(Q)) \operatorname{diam} f(B'_Q).
\end{aligned}$$

Since $f \in \mathcal{A}_2$, for every $B = B(x, r)$ for which $\lambda B \subset\subset \Omega$,

$$\operatorname{diam} f(B) \leq C\Big(\int_{\lambda B} \alpha(x) d\nu_1(x) \Big)^{\frac{1}{q}} \log \Big(\frac{1}{\operatorname{diam} B} \Big)^{\frac{1}{q}}.$$

Then,

$$\begin{aligned}
\mu(A_k)k &\leq C \sum_{Q \in \mathscr{W}_\Omega} \Big(\mu(S_{A_k}(Q)) \log^{\frac{1}{q}} \Big(\frac{1}{\operatorname{diam} B'_Q} \Big) \Big) \times \Big(\int_{\lambda B'_Q} \alpha(x) d\nu_1(x) \Big)^{\frac{1}{q}} \\
&\leq C \Big(\sum_{Q \in \mathscr{W}_\Omega} \mu(S_{A_k}(Q))^{\frac{q}{q-1}} \log^{\frac{1}{q-1}} \Big(\frac{1}{\operatorname{diam} B'_Q} \Big) \Big)^{\frac{(q-1)}{q}} \times \Big(\sum_{Q \in \mathscr{W}_\Omega} \int_{\lambda B'_Q} \alpha(x) d\nu_1(x) \Big)^{\frac{1}{q}}.
\end{aligned}$$

To simplify the inequalities, we use the fact that $2c_0\delta^j \leq \operatorname{diam} B'_Q \leq 2C_1\delta^j$ for $Q \in \mathscr{W}_\Omega^j$ and the property that $\lambda B'_Q$ have uniformly bounded overlap. By lemma 3.2, $\lambda B'_Q \subset\subset \Omega$ and $\alpha \in L^1(\Omega)$ we have,

$$\Big(\sum_{Q \in \mathscr{W}_\Omega} \int_{\lambda B'_Q} \alpha(x) d\nu_1(x) \Big)^{\frac{1}{q}} \in L^1(\Omega).$$

Therefore,

$$\mu(A_k)k \leq C' \Big(\sum_{j=1}^{\infty} \sum_{Q \in \mathscr{W}_\Omega^j} \mu(S_{A_k}(Q))^{\frac{q}{q-1}} j^{\frac{1}{q-1}} \Big)^{\frac{(q-1)}{q}},$$

where $C' = C\big(\sum_{Q \in \mathscr{W}_\Omega} \int_{\lambda B'_Q} \alpha(x) d\nu_1(x), C_1, c_0 \big)$. Using lemma 3.3 and the doubling property of gauge function h,

$$\sum_{Q \in \mathscr{W}_\Omega^j} \mu(S_{A_k}(Q)) \leq C_q^2 \Big(\frac{2\varphi(C(c)C_1\delta^j)}{(c_0\delta^j)} \Big)^q \mu(A_k),$$

$$\mu(B(x, r)) \leq h(r),$$

then we have the following estimates:

$$\begin{aligned}\sum_{Q\in\mathscr{W}_\Omega^j}\mu(S_{A_k}(Q))^{\frac{q}{q-1}} &\le \max_{Q\in\mathscr{W}_\Omega^j}\mu(S_{A_k}(Q))^{\frac{1}{q-1}}\sum_{Q\in\mathscr{W}_\Omega^j}\mu(S_{A_k}(Q))\\ &\le C_q^2\Big(\frac{2\varphi(2C'(c)C_1\delta^j)}{(c_0\delta^j)}\Big)^q\max_{Q\in\mathscr{W}_\Omega^j}\mu(S(Q))^{\frac{1}{q-1}}\mu(A_k)\\ &\le C_q^2\Big(\frac{2\varphi(2C'(c)C_1\delta^j)}{(c_0\delta^j)}\Big)^q h\big(\varphi(C_1\delta^j)\big)^{\frac{1}{q-1}}\mu(A_k).\end{aligned}$$

Putting these estimates together gives

$$\mu(A_k)k \le C\mu(A_k)^{\frac{(q-1)}{q}}\Big(\sum_{j=1}^{\infty}C_q^2[\frac{2\varphi(2C'(c)C_1\delta^j)}{(c_0\delta^j)}]^q h\big(\varphi(C_1\delta^j)\big)^{\frac{1}{q-1}}j^{\frac{1}{q-1}}\Big)^{\frac{(q-1)}{q}}.$$

By the condition of h, we know that the term on the right side is finite(and independent of k). Indeed, by the definition of $f\in\mathcal{A}_2$ we have

$$\begin{aligned}\int_0^1\frac{g(t)}{t}dt &= \int_0^1\frac{\varphi(t)h(\varphi(t))^{\frac{1}{q-1}}}{t^2}\left[\log\frac{1}{t}\right]^{\frac{1}{q-1}}dt\\ &=\sum_j\int_{\delta^j}^{\delta^{j-1}}\frac{\varphi(t)h(\varphi(t))^{\frac{1}{q-1}}}{t^2}\left[\log\frac{1}{t}\right]^{\frac{1}{q-1}}dt\\ &\sim\sum_j\frac{\varphi\left(\delta^j\right)h\left(\varphi(\delta^j)\right)^{\frac{1}{q-1}}j^{\frac{1}{q-1}}\left(\delta^{j-1}-\delta^j\right)}{\left(\delta^j\right)^2}\\ &=\left(\frac{1}{\delta}-1\right)\sum_j\frac{\varphi\left(\delta^j\right)h\left(\varphi(\delta^j)\right)^{\frac{1}{q-1}}j^{\frac{1}{q-1}}}{\delta^j\delta^j\left(\frac{1}{\delta}-1\right)}\left(\delta^{j-1}-\delta^j\right)\\ &=\left(\frac{1}{\delta}-1\right)\sum_j\frac{\varphi\left(\delta^j\right)h\left(\varphi(\delta^j)\right)^{\frac{1}{q-1}}j^{\frac{1}{q-1}}}{\delta^j}<\infty.\end{aligned}$$

Then

$$\sum_{j=1}^{\infty}C_q[\frac{2\varphi(2C'(c)C_1\delta^j)}{(c_0\delta^j)}]^q h(\varphi(C_1\delta^j))^{\frac{1}{q-1}}j^{\frac{1}{q-1}}<\infty.$$

By the certification process as before we get $\mu^{\frac{1}{q}}(A_k)k\le C$ for every $k\in\mathbb{N}$. Therefore, $\mu(A_k)\to 0$ as $k\to\infty$. We claim the conclusion. For $f\in\mathcal{A}_1$ we do not get the term $j^{\frac{1}{(q-1)}}$ in the last inequality and by an obvious modification of the above proof we can prove the similar conclusion. □

3.2. **Uniqueness of limits along John curves.** From the Section 3.1, Ω is a φ-length John domain.Then for the extension of f defined by definition 1.1 to $\partial\Omega$, we have $\tilde{f}:\partial\Omega\to X_2$ that satisfies

$$\tilde{f}(\omega)=\lim_{t\to 0^+}f(\gamma_\omega(t)),$$

where γ_ω is a φ-length John curve connecting x_0 and ω. The problem with this extension is that $(f:H^2\to H^2)$; see for instance Figure 2 below.

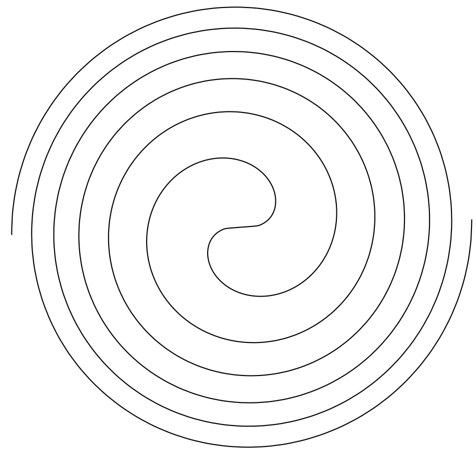

FIGURE 2.

So we must make more assumptions on Ω to avoid such a situation. In this section, we assume $\varphi(t) = ct$.

Definition 3.6. Let $X_1 = (X_1, d_1, \nu_1)$ be a metric measure space. A bounded domain $\Omega \subset X_1$ is a c-John domain ($c \geq 1$) with a distinguished point $x_0 \in \Omega$ if all points $x \in \Omega$ can be connected to x_0 by a c-John curve . A curve $\gamma : [0,1] \to \Omega$ is called a c-John curve if it satisfies $\gamma(0) = x, \gamma(1) = x_0$ and for all $t \in [0,1]$,

$$cd_1(\gamma(t), \partial\Omega) \geq l(\gamma([0,t])).$$

Definition 3.7. A bounded domain $\Omega \subset X_1$ is a c-uniform domain if there exists a constant c>0 such that each pair of points $x_1, x_2 \in \Omega$ can be joined by a uniform curve γ in Ω, namely a curve γ with the following two properties:

$$l(\gamma) \leq cd_1(x_1, x_2), \tag{3.1}$$

$$\min_{i=1,2} l(\gamma(x_i, x)) \leq cd_1(x, \partial\Omega) \; for \; all \; x \in \gamma. \tag{3.2}$$

These domains can be referred to [5]. It is necessary to introduce the definition below.

Definition 3.8. (Uniform domain with center) We say that Ω is a c-uniform domain with center x_0 if Ω is a uniform domain and as a John domain, it is c-John with center x_0.

As was shown in Section 3.1, there might be several John curves connecting $\xi - x_0$ for a given point $\xi \in \partial\Omega$. Moreover, $f \in \mathcal{A}_2$ may have different limits along different John curves. In the rest of this section, we write $I^c(\xi, x_0)$ to denote the class of all c-John curves connecting ξ to x_0 and

$$S^c(Q) = \{\xi \in \partial\Omega : Q \in P^c(\xi)\}$$

where $P^c(\xi) = \{Q \in \mathscr{W}_\Omega : Q \cap \gamma \neq \emptyset \; for \; some \; \gamma \in I^c(\xi, x_0)\}$. We will show that if Ω is a c-uniform domain with center x_0, then f can be extended to $\partial\Omega$ in a unique way along c-John curves. We need the following lemmas to prove Theorem 1.4.

Lemma 3.9. *Let Ω be a c_o-John domain with center x_0 and $c \geq c_0$. If $f \in \mathcal{A}_2$ and F_f is the set of points $\xi \in \partial\Omega$ for which there exists a curve $\gamma \in I^c(\xi, x_0)$ so that*

$$\sum_{i=1}^{\infty} a_i = \infty,$$

then $\mathcal{H}^h(F_f) = 0$, where h is as definition in Theorem 1.3. Above, a_i is defined in the following way: for a fixed $\gamma \in I^c(\xi, x_0)$ connecting $x_0 - \xi \in \partial\Omega$, we denote by $Q_i \in \mathscr{W}_\Omega$ those cubes that intersect γ and labeled in the order from x_0 to ξ, and set

$$a_i = \left(\int_{\lambda B_{Q_i}} \alpha(x) d\nu_1(x) \right)^{\frac{1}{q}} \log \left(\frac{1}{\operatorname{diam} B_{Q_i}} \right)^{\frac{1}{q}}.$$

Proof. Let A_k be the collection of points $\xi \in \partial\Omega$ for which there exists $\gamma \in I^c(\xi, x_0)$ such that

$$\sum_{i=1}^{\infty} \left(\int_{\lambda B_{Q_i}} \alpha(x) d\nu_1(x) \right)^{\frac{1}{q}} \log \left(\frac{1}{\operatorname{diam} B_{Q_i}} \right)^{\frac{1}{q}} \geq k.$$

Notice that if $\xi \in F_f$, then

$$\sum_{i=1}^{\infty} \left(\int_{\lambda B_{Q_i}} \alpha(x) d\nu_1(x) \right)^{\frac{1}{q}} \log \left(\frac{1}{\operatorname{diam} B_{Q_i}} \right)^{\frac{1}{q}} = \infty.$$

As the proof of Theorem 1.3 we can show that $\mathcal{H}^h(A_\infty) = 0$ then $\mathcal{H}^h(F_f) = 0$ since $F_f \subset A_\infty$. Indeed, we assume that $\mathcal{H}^h(A_\infty) > 0$ then $\mathcal{H}^h(A_k) > 0$. And denote by μ the Frostman measure whose support lies in A_k, which satisfies $\mu(A_k) \simeq \mathcal{H}^h_\infty(A_k) \geq \mathcal{H}^h_\infty(A_\infty)$ and

$$\mu(B(x,r)) \leq Ch(r)$$

for every ball $B(x,r)$. Then we can use the same method in theorem 1.3 and get

$$\mu(A_k)\, k \leq C$$

where C is a positive constant.So we get the contraction and the lemma is proved. □

Then we need the following lemma to prove lemma 3.11 that the number of cubes is uniformly bounded, which is compared to the work in [5, lemma 5.2].

Lemma 3.10. *Let $\Omega \subset X_1$ be a uniform domain. Then there exists a positive constant C, depending only on the data, such that*

$$k_\Omega(x,y) \leq C \int_{\min\{d_1(x,\partial\Omega), d_1(y,\partial\Omega)\}}^{C d_1(x,y)} \frac{1}{s} ds + 2$$

for each pair x, y of points in Ω.

Proof. Let $x, y \in \Omega$. Since Ω is a uniform domain, there exists a uniform curve γ from x to y. We suppose $\gamma : [0, l(\gamma)] \to \Omega$ is parameterized by arc length and $\gamma(0) = x, \gamma(l(\gamma)) = y$ then

$$k_\Omega(x,y) \leq \int_0^{l(\gamma)} \frac{dt}{d_1(z,\partial\Omega)} \leq c \int_0^{l(\gamma)} \frac{dt}{\min\{t, l(\gamma) - t\}}.$$

If $t \leq \frac{1}{2}l(\gamma)$ and suppose first that,

$$l(\gamma(x,z)) \leq \frac{1}{2}d_1(x,\partial\Omega).$$

Then for any $z_0 \in \gamma(x,z)$, we have

$$d_1\left(z_0,\partial\Omega\right) \geq d_1(x,\partial\Omega) - l\left(\gamma\left(z_0,x\right)\right) \geq \frac{1}{2}d_1(x,\partial\Omega).$$

Thus,

$$k_\Omega\left(x,y\right) \leq \int_\gamma \frac{dt}{d_1(z,\partial\Omega)} \leq 2\int_0^{l(\gamma(x,y))} \frac{dt}{d_1(x,\partial\Omega)} \leq 2\frac{l\left(\gamma\left(x,y\right)\right)}{d_1(x,\partial\Omega)} \leq 1.$$

Next, we consider the case that,

$$l(\gamma(x,z)) > \frac{1}{2}d_1(x,\partial\Omega),$$

then

$$\begin{aligned}
k_\Omega\left(x,y\right) &\leq \int_0^{\frac{1}{2}d_1(x,\partial\Omega)} \frac{dt}{d_1(x,\partial\Omega)-t} + C\int_{\frac{1}{2}d_1(x,\partial\Omega)}^{l(\gamma)} \frac{1}{min\{t,l(\gamma)-t\}}dt \\
&= \log\left(\frac{d_1(x,\partial\Omega)}{d_1(x,\partial\Omega)-\frac{1}{2}d_1(x,\partial\Omega)}\right) + C\int_{\frac{1}{2}d_1(x,\partial\Omega)}^{l(\gamma)} \frac{1}{t}dt \\
&\leq \log 2 + C\int_{d_1(x,\partial\Omega)}^{Cd_1(x,y)} \frac{1}{t}dt.
\end{aligned}$$

If $t > \frac{1}{2}l(\gamma)$, we may denote a new curve $\hat{\gamma}$ parameterized by arc length from y to x i.e. $\hat{\gamma} : [0, l(\hat{\gamma})] \to \Omega$ and $\hat{\gamma}(0) = y, \hat{\gamma}(l(\hat{\gamma})) = x$. Then

$$\int_0^{l(\gamma)} \frac{dt}{\min\{t,l(\gamma)-t\}} = \int_0^{l(\gamma)} \frac{dt}{l(\gamma)-t} = \int_0^{l(\hat{\gamma})} \frac{dt}{t}.$$

According to the above steps, we have

$$k_\Omega(x,y) \leq \log 2 + C\int_{d_1(y,\partial\Omega)}^{Cd_1(x,y)} \frac{dt}{t}.$$

Therefore,

$$k_\Omega(x,y) \leq C\int_{\min\{d_1(x,\partial\Omega),d_1(y,\partial\Omega)\}}^{Cd_1(x,y)} \frac{1}{t}dt + 2$$

for each pair x, y of points in Ω. □

Lemma 3.11. *Let $\Omega \subset X_1$ be a c_0 uniform domain with center x_0 and $\mathscr{W}_\Omega$ be a Dyadic-Whitney decomposition of Ω. Then there exists a constant* C *such that for any* s > 0 *and pair $x_1, x_2 \in \Omega$ with $d_1(x_2,\partial\Omega) \geq s, d_1(x_1,\partial\Omega) \geq s$ and $d_1(x_1,x_2) \leq c_1 s$ there exists a chain of dyadic cubes $\{Q_k\}_{k=1}^N$ connecting points x_1 and x_2 such that the number of cubes is uniformly bounded with respect to s, i.e. $N \leq C$, where C depends only on q, c_0, c_1 but not on s. Moreover if $\xi \in S^c(Q)$ for some $Q \in \cup_{i=1}^N Q_i$ then $\xi \in S^{c'}(Q')$ for any Q' from this collection with* c$' = c'\left(C, a, c_0\right) \geq c_0$.

Proof. Fix $x_1, x_2 \in \Omega$ satisfying the conditions in the lemma. Then we can connect x_1, x_2 by a quasihyperbolic geodesic $[x_1, x_2]_k$ and get a chain of cubes $Q \in \mathscr{W}_\Omega$ that intersect

$[x_1, x_2]_k$ in a such way that $x_1 \in Q_1, x_2 \in Q_N$ and $Q_i \cap Q_{i+1} \neq \varnothing$ for every i. By [?] we know that the number of cubes in this chain is comparable to $k_\Omega(x_1, x_2)$, joining x to y. Since Ω is uniform, by lemma 3.10, there exists a constant $C > 0$ which does not depend on s such that

$$k_\Omega(x, y) \leq C \int_{\min\{d_1(x,\partial\Omega), d_1(y,\partial\Omega)\}}^{Cd_1(x,y)} \frac{1}{s} ds + 2$$

Next we introduce a method to prove the second part which can be referred to [?]. Let $\xi \in S(Q)$ for some $Q \in \bigcup_{i=1}^N Q_i$ then there exists a curve $\gamma \in I^{c_0}(\xi, x_0)$ and $\gamma \cap Q \neq \emptyset$. Let $x_3 \in Q \cap \gamma$ and Q' be any other dyadic cubes from $\bigcup_{i=1}^N Q_i$ and fix $x_4 \in Q'$. Let $\tilde{\gamma}$ denote the piecewise linear curve from x_3 to x_4 such that $|\tilde{\gamma}| \subset \bigcup_{i=1}^N Q_i$ Now we define a new curve $\gamma' = \gamma_1 \circ \gamma_2 \circ \gamma_3 \circ \gamma_4$ connecting ξ and x_0 where γ_1: follow γ from ξ to x_3; γ_2: follow $\tilde{\gamma}$ from x_3 to x_4; γ_3: follow $\tilde{\gamma}$ from x_4 to $x3$; γ_4: follow γ from x_3 to x_0. We suppose that $\gamma' : [0, l'] \to \Omega$ is parameterized by arc length and we will prove that γ' is a c'-John curve with $c' = c'(C, a, c_0) \geq c_0$.

(1) For $t \in [0, l(\gamma_1)]$, since $\gamma_1 \subset \gamma \in I^{c_0}(\xi, x_0)$,
$$t \leq c_0 d_1(\gamma(t), \partial\Omega).$$

(2) For $t \in [l(\gamma_1), l(\gamma_1) + 2l(\gamma_2)]$, by the lemma 3.2, the number of cubes in $\bigcup_{i=1}^N Q_i$ is uniformly bounded from above by C then the cubes have uniformly comparable size. Then we have the following inequalities:
$$l(\gamma_2) \leq c d_1(\gamma'(t), \partial\Omega) \text{ and } d_1(x_3, \partial\Omega) \leq c\ d_1(\gamma'(t), \partial\Omega),$$
where c is constant of a, C. Since γ is a c_0-John curve,
$$l(\gamma_1) \leq c_0 d_1(x_3, \partial\Omega).$$
Putting the estimates together we have:
$$\begin{aligned} t &\leq l(\gamma_1) + 2l(\gamma_2) \\ &\leq c_0 d_1(\gamma'(t), \partial\Omega) + 2c d_1(\gamma'(t), \partial\Omega) \\ &\leq c'(c, c_0)\, d_1(\gamma'(t), \partial\Omega). \end{aligned}$$

(3) For $t \in [l(\gamma) + 2l(\gamma_2), l(\gamma')]$, by the definition of uniform domain we have
$$d_1(x_3, \gamma'(t)) \leq c_0 d_1(\gamma'(t), \partial\Omega)$$
which means that,
$$l(\gamma_2) \leq c d_1(x_3, \partial\Omega) \leq c(c, c_0)\, d_1(\gamma'(t), \partial\Omega).$$
Therefore,
$$t \leq c_0 d_1(\gamma'(t), \partial\Omega) + 2l(\gamma_2) \leq c'(c, c_0)\, d_1(\gamma'(t), \partial\Omega).$$

□

Proof of Theorem 1.4. By lemma 3.9, We know that $\mathcal{H}^h(F_f) = 0$, where F_f is defined in the lemma. Let $\xi \in \partial\Omega \backslash F_f$ and $\gamma \in I^c(\xi, x_0)$ such that
$$\lim_{t \to 0^+} f(\gamma(t)) = a.$$

We will prove that f has the same limit along any curve $\eta \in I^c(\xi, x_0)$. Given a constant $s > 0$, let $t_1, t_2 \in [0,1]$ be such that $l(\gamma(t_1), w) = s$ and $l(\eta(t_2), w) = s$. Since γ, η are both c-John curves, this implies that

$$cd_1(\gamma(t_1), \partial\Omega) \geq s \text{ and } cd_1(\eta(t_2), \partial\Omega) \geq s.$$

Then we can find a chain of Dyadic-Whitney cubes $\{Q_i\}$ and the number of the cubes is finite by lemma 3.11. Choose $y_i \in Q_i$ so that $y_1 = \gamma(t_1)$ and $y_N = \eta(t_2)$. Since $f \in \mathcal{A}_2$,

$$\begin{aligned}
d_2(f(\gamma(t_1)), f(\eta(t_2))) &\leq \sum_{i=1}^{N} d_2(f(y_{i+1}), f(y_i)) \\
&\leq C_1 \sum_{i=1}^{N} \left(\int_{\lambda B_{Q_i}} \alpha(y) \mathrm{d}\nu_1(y) \right)^{1/q} \log \left(\frac{1}{\operatorname{diam} Q_i} \right)^{1/q} \\
&\leq c(C, C_1) \max_{1 \leq i \leq N} \left(\int_{\lambda B_{Q_i}} \alpha(y) \mathrm{d}\nu_1(y) \right)^{1/q} \log \left(\frac{1}{\operatorname{diam} Q_i} \right)^{1/q}.
\end{aligned}$$

By lemma 3.11 for every $i = 1, \ldots, N$ there exists a constant $c' \geq c$ such that $\xi \in S^{c'}(Q_i)$. Since N does not depend on s, by lemma 3.9 we can choose $s > 0$ so small such that,

$$\max_{1 \leq i \leq N} \left(\int_{\lambda B_{O_i}} \alpha(y) \mathrm{d}\nu_1(y) \right)^{1/q} \log \left(\frac{1}{\operatorname{diam} Q_i} \right)^{1/q} < \varepsilon.$$

This together with the above inequality shows that $d_2(f(\gamma(t_1)), f(\eta(t_2))) \to 0$ as $s \to 0$ and the proof is complete. □

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