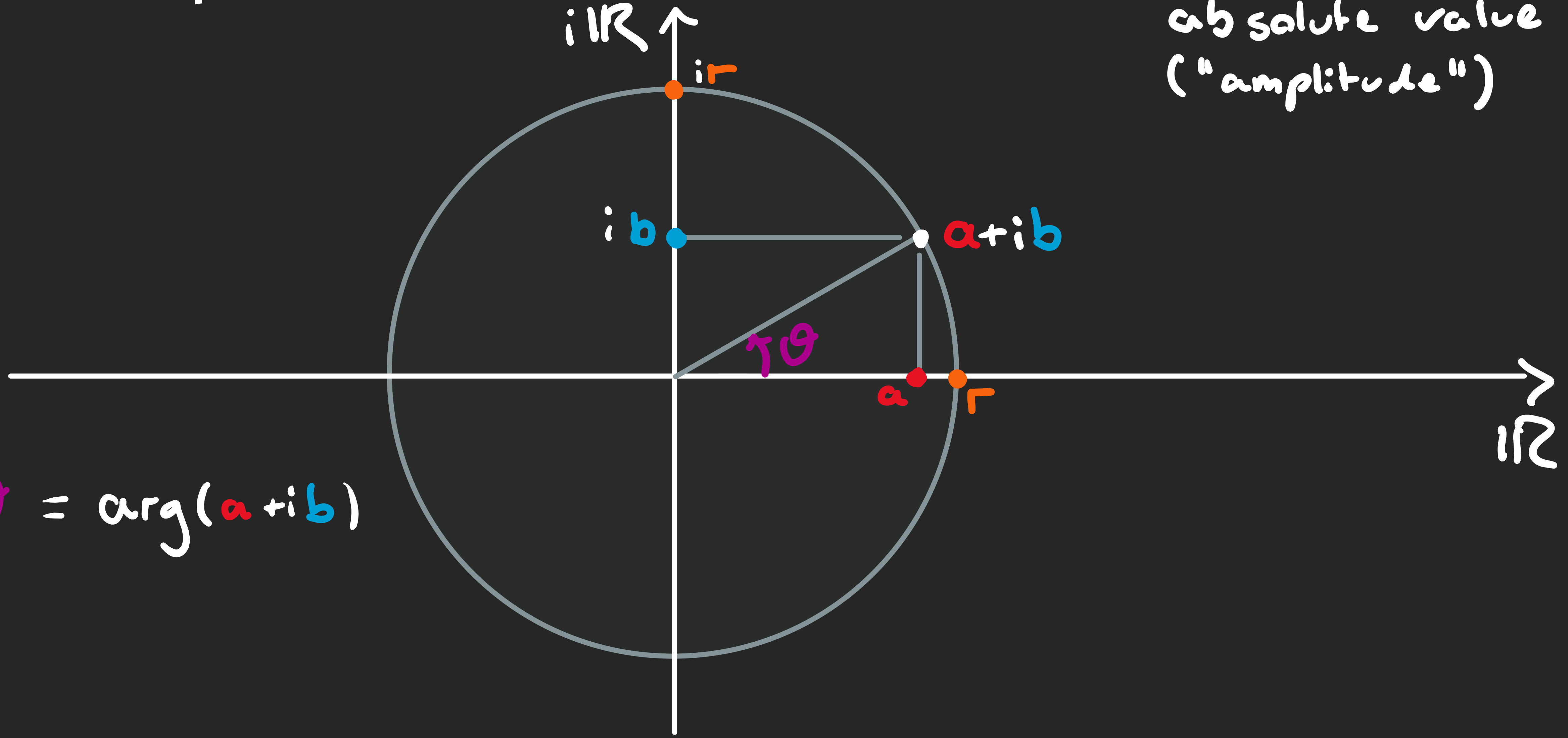
Single-qubit gates

Complex numbers $\mathbb{C}$

$r = |a+ib| := \sqrt{a^2+b^2}$
absolute value
("amplitude")



$\theta = \arg(a+ib)$

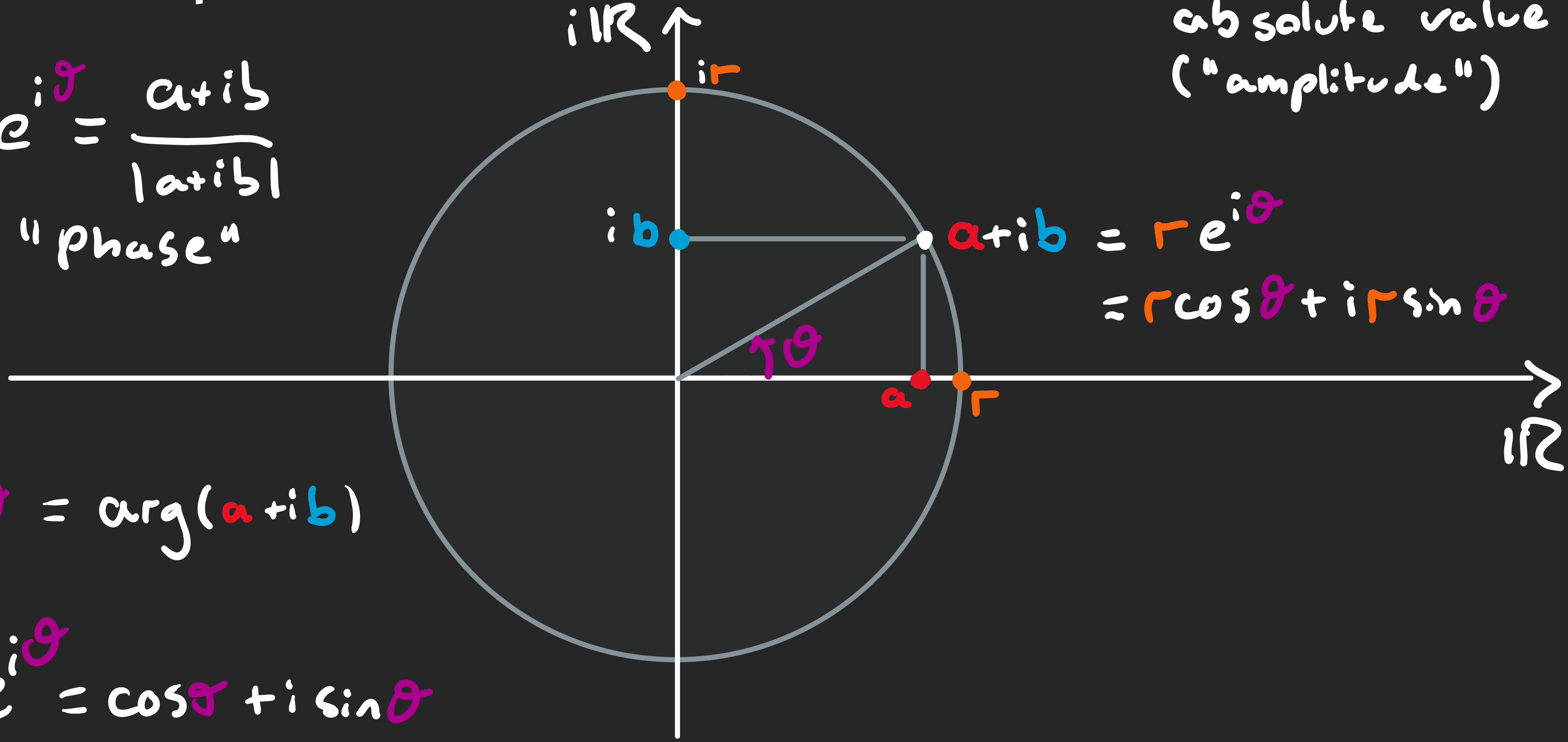
Complex numbers $\mathbb{C}$

$r = |a+ib| := \sqrt{a^2+b^2}$

absolute value ("amplitude")

$$e^{i\theta} = \frac{a+ib}{|a+ib|}$$

"phase"



$$a+ib = re^{i\theta} = r\cos\theta + ir\sin\theta$$

$$\theta = \arg(a+ib)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Complex numbers $\mathbb{C}$

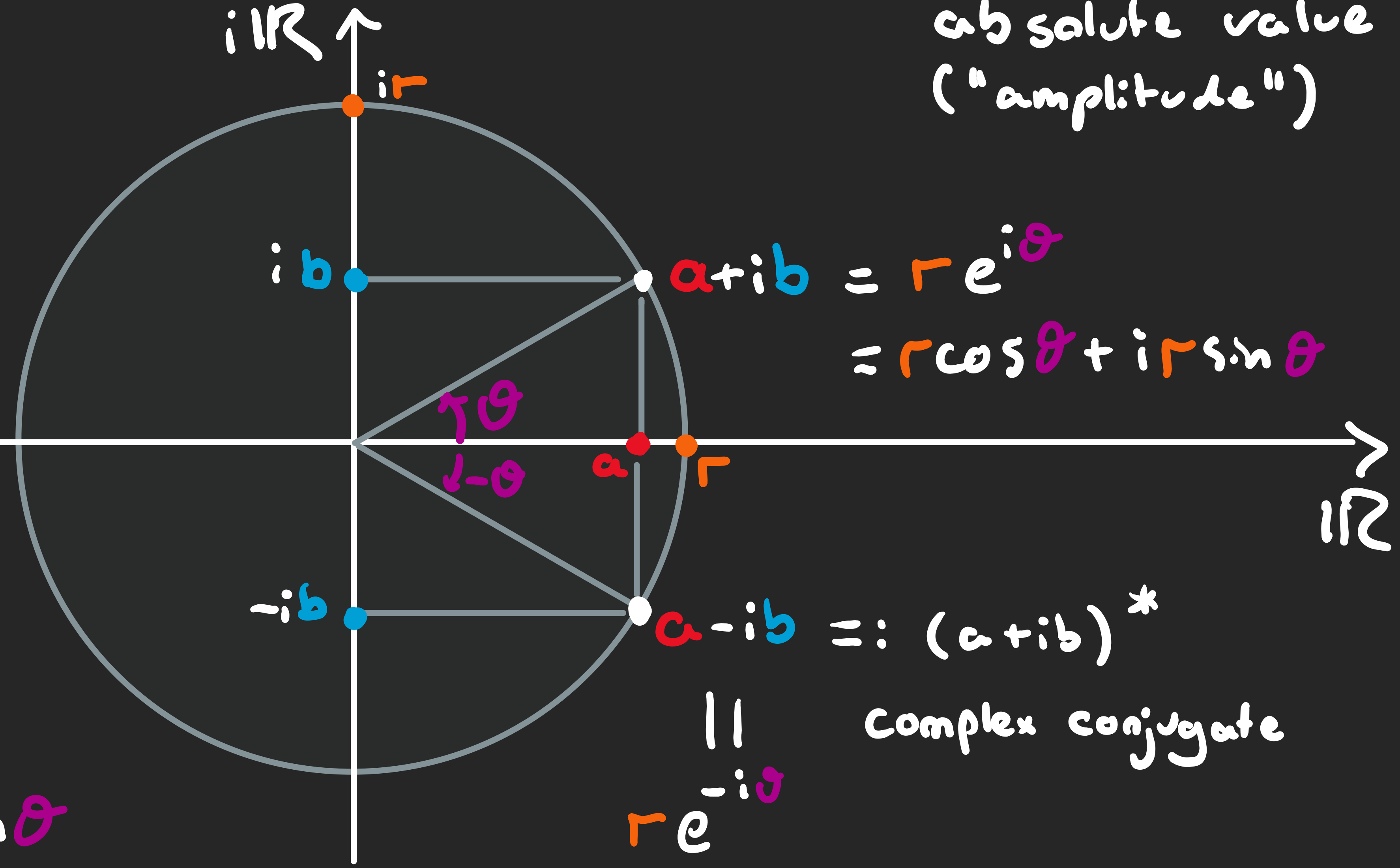
$r = |a+ib| := \sqrt{a^2+b^2}$

absolute value

("amplitude")

$$e^{i\vartheta} = \frac{a+ib}{|a+ib|}$$

"phase"



$$a+ib = re^{i\theta}$$
$$= r\cos\theta + ir\sin\theta$$

$$a-ib =: (a+ib)^*$$

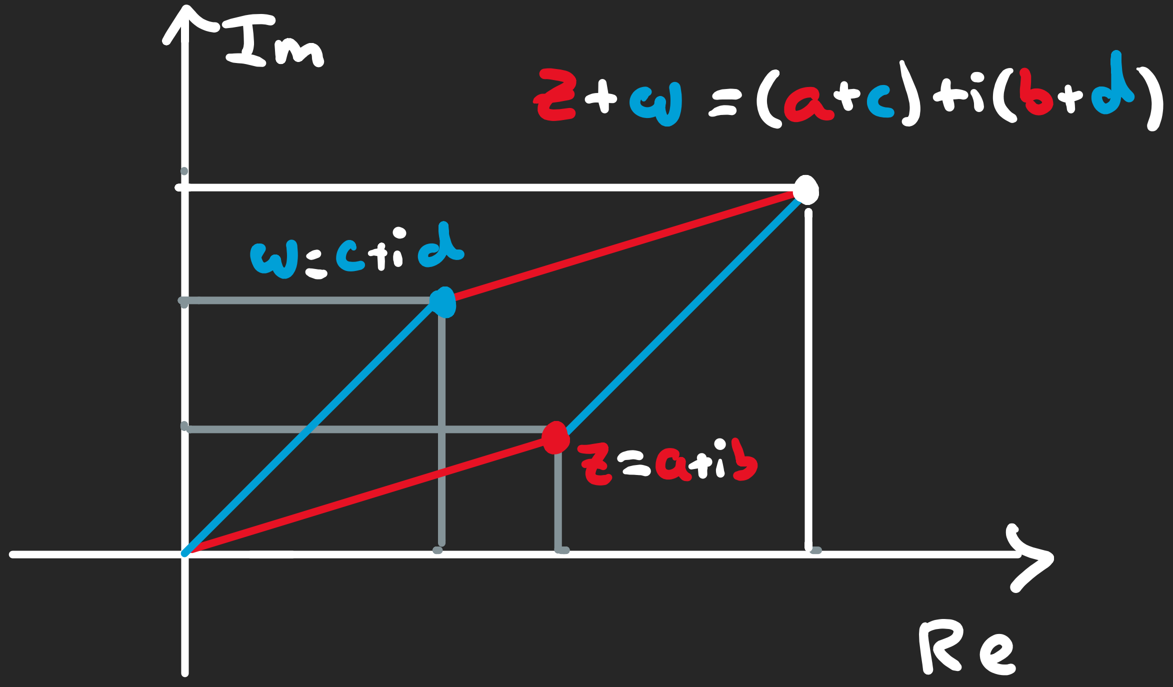
$=$ complex conjugate

$re^{-i\vartheta}$

$$\theta = \arg(a+ib)$$

$$e^{i\vartheta} = \cos\vartheta + i\sin\theta$$

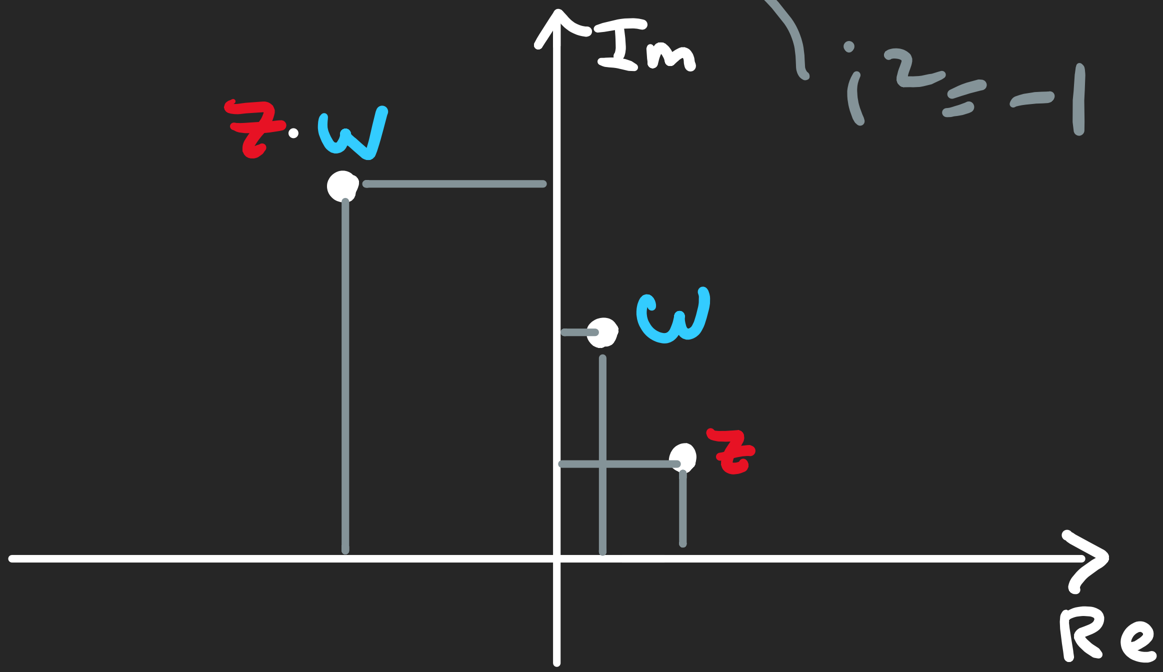
Complex numbers



Addition

$$z \cdot w = (a+ib)(c+id)$$
$$= (ac-bd)+i(ad+bc)$$

$i^2=-1$

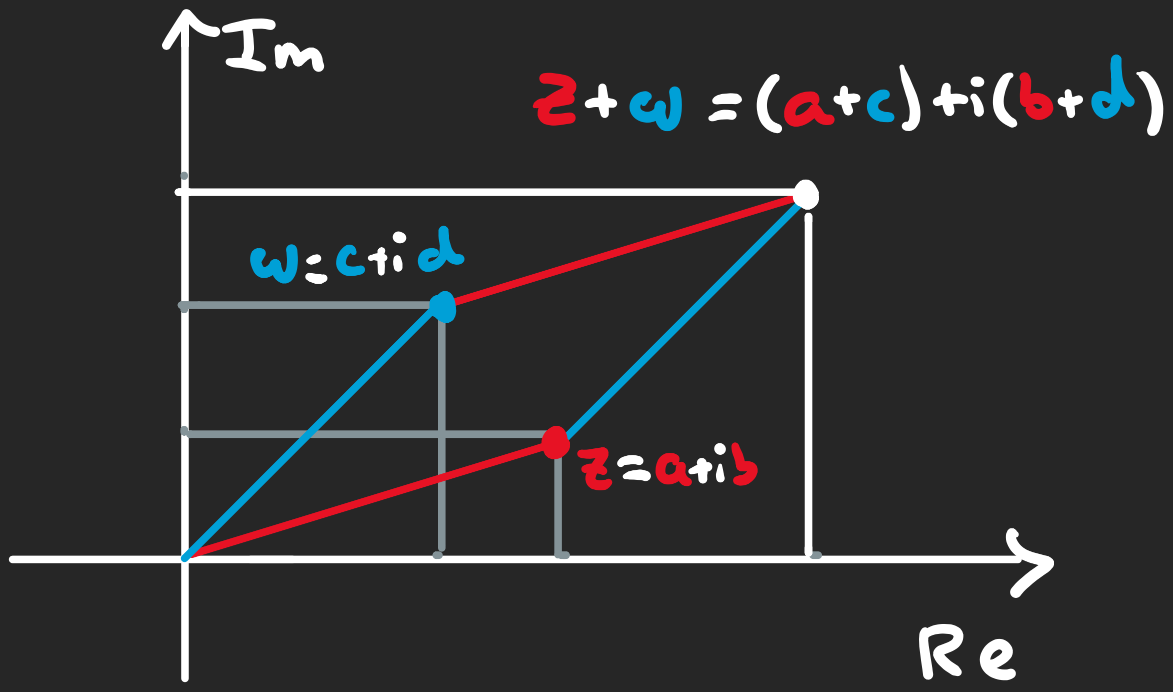


$$z z^* = (a+ib)(a-ib) =$$
$$= a^2+b^2+i(ab-ab) = |z|$$

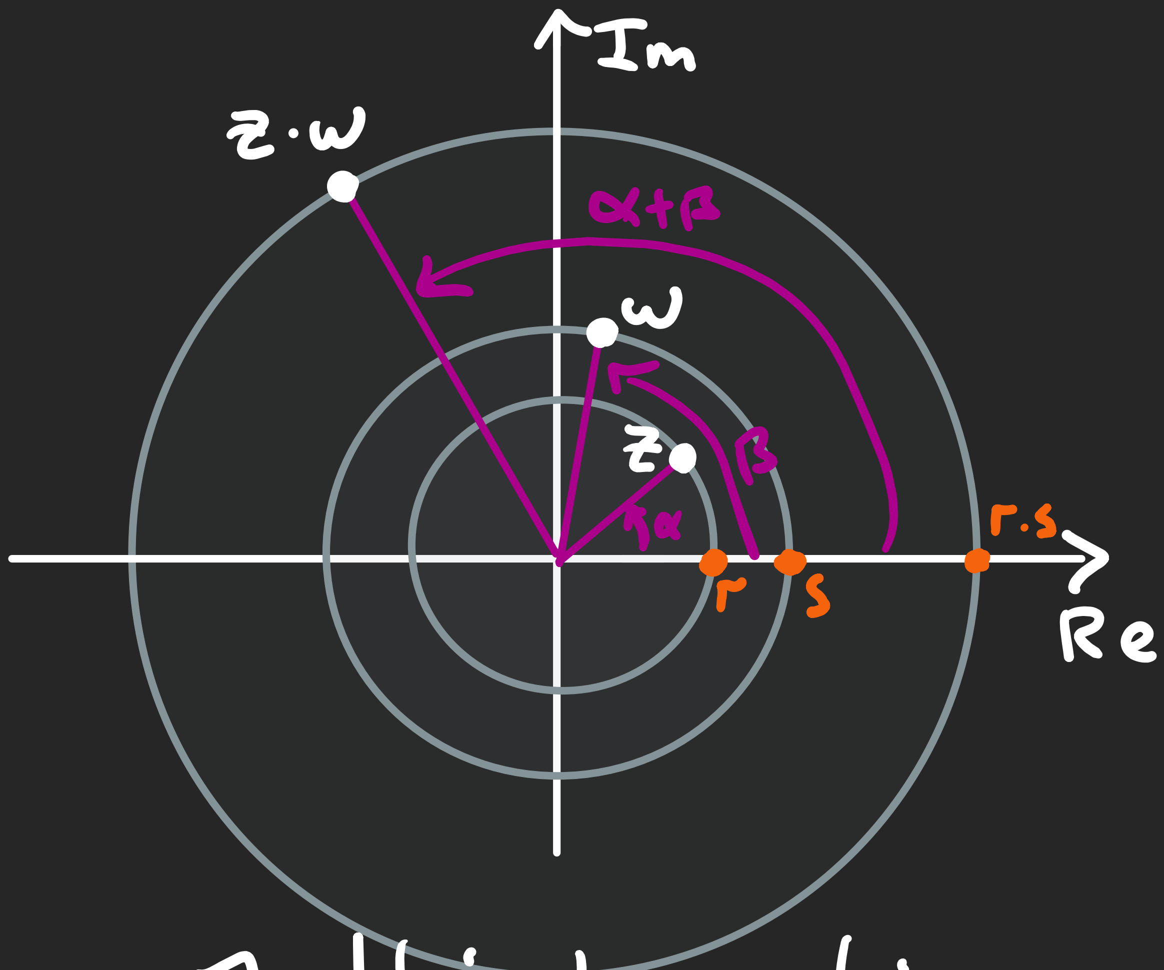
Multiplication

Complex numbers

$$z = r e^{i\alpha} \qquad z \cdot w = r s e^{i(\alpha+\beta)}$$
$$w = s e^{i\beta}$$

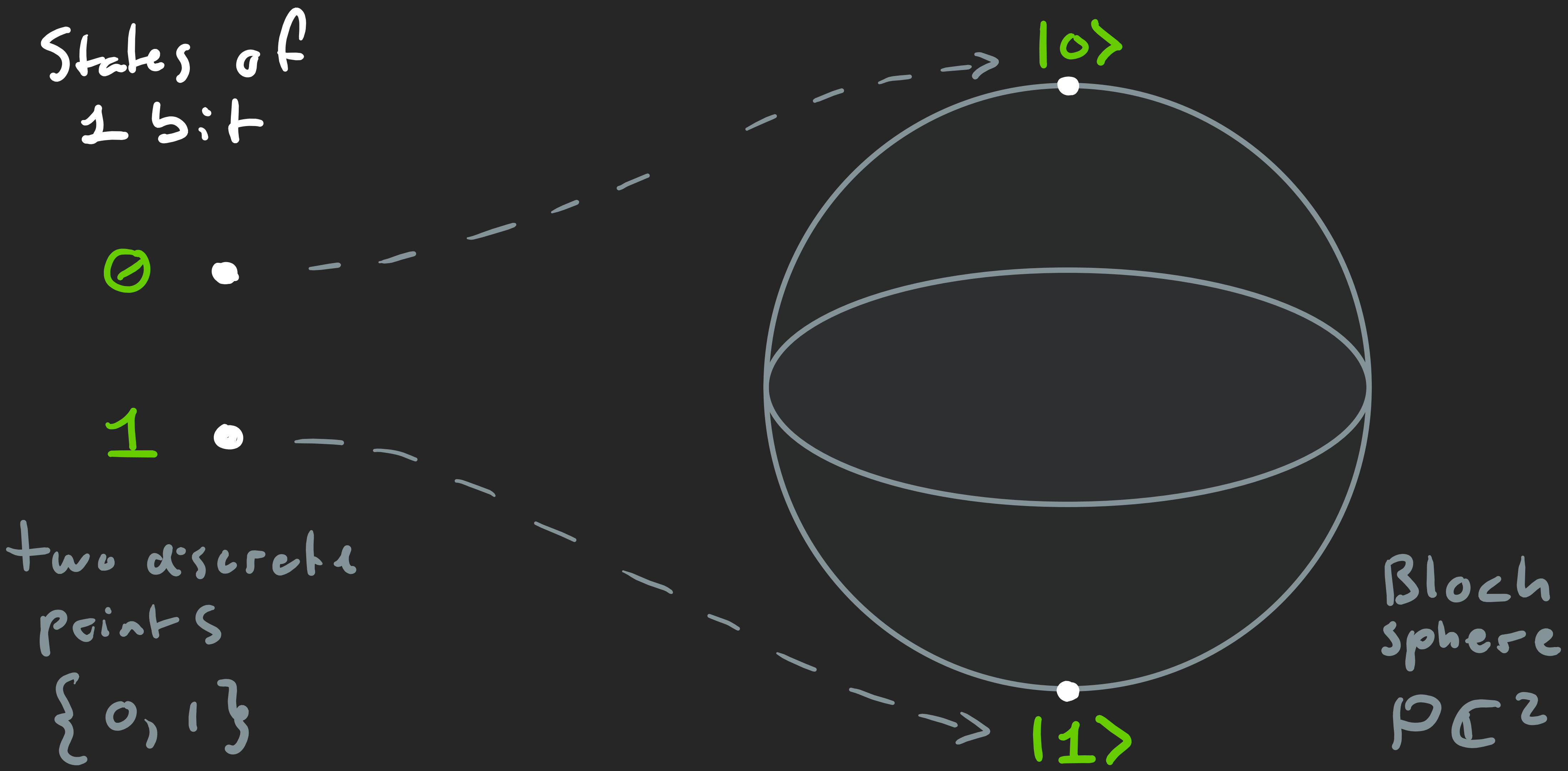


Addition



Multiplication

Bracket Notation



Bracket Notation

States of 1 qubit:

2-dim complex vector $|\psi\rangle \in \mathbb{C}^2$

$$\underbrace{|\psi\rangle}_{\text{a "ket"}} = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$$

(i) normalised $\quad |\psi_0|^2 + |\psi_1|^2 = 1$

(ii) up to phase $\quad |\psi\rangle = e^{i\theta} |\psi\rangle$

Bracket Notation

complex conjugate

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \qquad \langle\psi| = \begin{pmatrix} \psi_0^* & \psi_1^* \end{pmatrix}$$

ket = column vector   bra = row vector

$$\langle\varphi|\psi\rangle = \varphi_0^*\psi_0 + \varphi_1^*\psi_1$$

braket = inner product

$|\psi\rangle$ normalised $\iff$ $\langle\psi|\psi\rangle = \psi_0^*\psi_0 + \psi_1^*\psi_1 = |\psi_0|^2 + |\psi_1|^2 = 1$

Bracket Notation

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

computational / Z basis

$$|\psi\rangle = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \psi_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \psi_0 |0\rangle + \psi_1 |1\rangle$$

$$\langle 0 | \psi \rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \psi_0 \qquad \langle 1 | \psi \rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \psi_1$$

Bracket Notation

Z basis:

$|0\rangle \qquad |1\rangle$

X basis:

$|+\rangle := \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |-\rangle := \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Y basis:

$|R\rangle := \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \qquad |L\rangle := \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$

Bracket Notation

$$\left.\begin{aligned} \psi_0 &= c\, e^{i\alpha} \\ \psi_1 &= s\, e^{i\beta} \end{aligned}\right\} \underset{\underbrace{|\psi_0|^2}_{c^2} + \underbrace{|\psi_1|^2}_{s^2} = 1}{\Longrightarrow} \exists\, \theta \text{ s.t. } \left\{\begin{aligned} \psi_0 &= \cos\theta\, e^{i\alpha} \\ \psi_1 &= \sin\theta\, e^{i\beta} \end{aligned}\right.$$

Setting $\varphi := \beta - \alpha$ we get:

relative phase

$$|\psi\rangle = e^{i\alpha}\left(\cos\theta\, |0\rangle + \sin\theta\, e^{i\varphi}\, |1\rangle\right)$$

irrelevant global phase

Bracket Notation

Z basis:

$$|0\rangle = \overbrace{1}^{\cos\theta}\cdot|0\rangle + \overbrace{0}^{\sin\theta}\cdot e^{\overbrace{i\varphi}^{\text{irrelevant}}}|1\rangle \qquad |1\rangle = \overbrace{0}^{\cos\pi}\cdot|0\rangle + \overbrace{1}^{\sin\pi}\cdot e^{\overbrace{i\varphi}^{\text{irrelevant}}}|1\rangle$$

X basis:

$$|+\rangle := \overbrace{\frac{1}{\sqrt{2}}}^{\cos\frac{\pi}{2}}|0\rangle + \overbrace{\frac{1}{\sqrt{2}}}^{\sin\frac{\pi}{2}}\underbrace{e^{i\theta}}_{1}|1\rangle \qquad |-\rangle := \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\underbrace{e^{i\pi}}_{-1}|1\rangle$$

Y basis:

$$|R\rangle := \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\underbrace{e^{i\frac{\pi}{2}}}_{i}|1\rangle \qquad |L\rangle := \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\underbrace{e^{i\frac{3\pi}{2}}}_{-i}|1\rangle$$

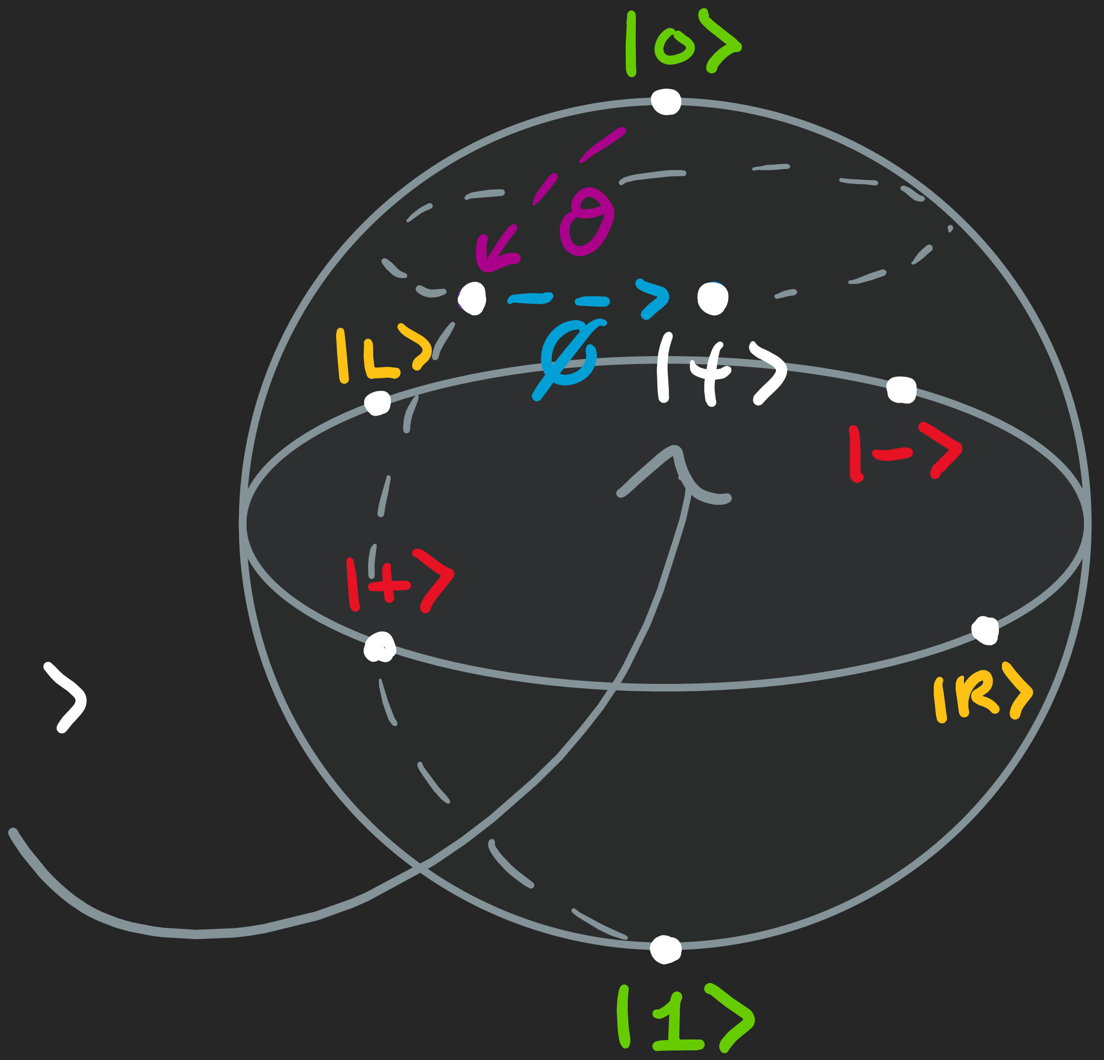
Qubits States

Spherical Coordinates

(ϕ, θ)

$\cos\theta\,|0\rangle + \sin\theta\, e^{i\phi}\,|1\rangle$

Bloch Sphere



Adjoints

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1c} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rc} \end{pmatrix}$$

$$A^\dagger := \begin{pmatrix} A_{11}^* & \cdots & A_{r1}^* \\ \vdots & \ddots & \vdots \\ A_{1c}^* & \cdots & A_{rc}^* \end{pmatrix}$$

adjoint of A

= conjugate transpose

(i) $(A^\dagger)^\dagger = A$

(ii) $(AB)^\dagger = B^\dagger A^\dagger$

Adjoints

$$\begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix}^\dagger = \begin{pmatrix} v_1^* & \cdots & v_r^* \end{pmatrix}$$

$$\begin{pmatrix} u_1 & \cdots & u_c \end{pmatrix}^\dagger = \begin{pmatrix} u_1^* \\ \vdots \\ u_c^* \end{pmatrix}$$

$$|\psi\rangle^\dagger = \langle\psi|$$

$$\langle\varphi|^\dagger = |\varphi\rangle$$

Qubit Matrices

Inner product:

$$\langle \varphi | \psi \rangle = \begin{pmatrix} \varphi_0^* & \varphi_1^* \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \varphi_0^* \psi_0 + \varphi_1^* \psi_1$$

Outer product:

$$|\psi\rangle\langle\varphi| = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \begin{pmatrix} \varphi_0^* & \varphi_1^* \end{pmatrix} = \begin{pmatrix} \psi_0 \varphi_0^* & \psi_0 \varphi_1^* \\ \psi_1 \varphi_0^* & \psi_1 \varphi_1^* \end{pmatrix}$$

Qubit Matrices

$|\psi\rangle, |\phi\rangle$ orthogonal $\overset{\text{def}}{\iff} \langle\psi|\phi\rangle = 0$

Inner products:

$\langle 0|0\rangle = 1 = \langle 1|1\rangle$ $\qquad$ $\langle 0|1\rangle = 0 = \langle 1|0\rangle$

Outer products:

$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\qquad$ $|1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$|0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\qquad$ $|1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Qubit Matrices

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \qquad \text{any matrix } A \in \mathbb{C}^{2\times 2}$$

$$= A_{00} \underbrace{|0\rangle\langle 0|}_{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} + A_{01} \underbrace{|0\rangle\langle 1|}_{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} + A_{10} \underbrace{|1\rangle\langle 0|}_{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} + A_{11} \underbrace{|1\rangle\langle 1|}_{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

Qubit Matrices

$$A|\psi\rangle = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} A_{00}\psi_0 + A_{01}\psi_1 \\ A_{10}\psi_0 + A_{11}\psi_1 \end{pmatrix}$$

$$A|\psi\rangle = \underbrace{|0\rangle}_{\text{vectors}} \underbrace{\left(A_{00}\langle 0| + A_{01}\langle 1|\right)|\psi\rangle}_{\text{complex numbers}}$$

$$+ \underbrace{|1\rangle}_{\text{vectors}} \underbrace{\left(A_{10}\langle 0| + A_{11}\langle 1|\right)|\psi\rangle}_{\text{complex numbers}}$$

Qubit Matrices

$$A|\psi\rangle = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} A_{00}\psi_0 + A_{01}\psi_1 \\ A_{10}\psi_0 + A_{11}\psi_1 \end{pmatrix}$$

$$A|\psi\rangle = |0\rangle\left(A_{00}\langle 0|\psi\rangle + A_{01}\langle 1|\psi\rangle\right)$$

$$+\ |1\rangle\left(A_{10}\langle 0|\psi\rangle + A_{11}\langle 1|\psi\rangle\right)$$

Qubit Matrices

$$A|\psi\rangle = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} A_{00}\psi_0 + A_{01}\psi_1 \\ A_{10}\psi_0 + A_{11}\psi_1 \end{pmatrix}$$

$$\begin{aligned} A|\psi\rangle = \; & |0\rangle \left(A_{00} \cancelto{\psi_0}{\langle 0|\psi\rangle} + A_{01} \cancelto{\psi_1}{\langle 1|\psi\rangle} \right) \\ & + |1\rangle \left(A_{10} \cancelto{\psi_0}{\langle 0|\psi\rangle} + A_{11} \cancelto{\psi_1}{\langle 1|\psi\rangle} \right) \end{aligned}$$

Qubit Matrices

$$A|\psi\rangle = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} A_{00}\psi_0 + A_{01}\psi_1 \\ A_{10}\psi_0 + A_{11}\psi_1 \end{pmatrix}$$

$$A|\psi\rangle = \underbrace{(A_{00}\psi_0 + A_{01}\psi_1)}_{\text{complex number}}|0\rangle + \underbrace{(A_{10}\psi_0 + A_{11}\psi_1)}_{\text{complex number}}|1\rangle$$

Unitary Matrices

Also up to phase!

$U = e^{i\theta} U$

Transformations of 1 qubit:

$$U = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} \in \mathbb{C}^{2\times 2}$$

which are unitary: $U^\dagger U = I = U U^\dagger$

i.e. $U^\dagger = U^{-1}$

Unitary Matrices

In particular:

$|\psi\rangle$ normalised $\Rightarrow U|\psi\rangle$ normalised

Unitary $\Rightarrow$ preserve inner products

$|\psi'\rangle := U|\psi\rangle$ $\qquad$ $|\varphi'\rangle := U|\varphi\rangle$

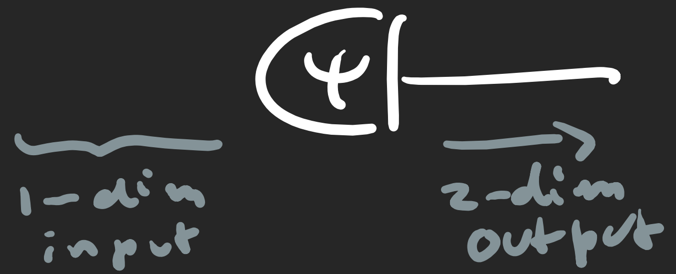
$$\langle\varphi'|\psi'\rangle = (|\varphi'\rangle)^\dagger|\psi'\rangle = (U|\varphi\rangle)^\dagger U|\psi\rangle = \langle\varphi|U^\dagger U|\psi\rangle$$

$$= \langle\varphi|I|\psi\rangle = \langle\varphi|\psi\rangle$$

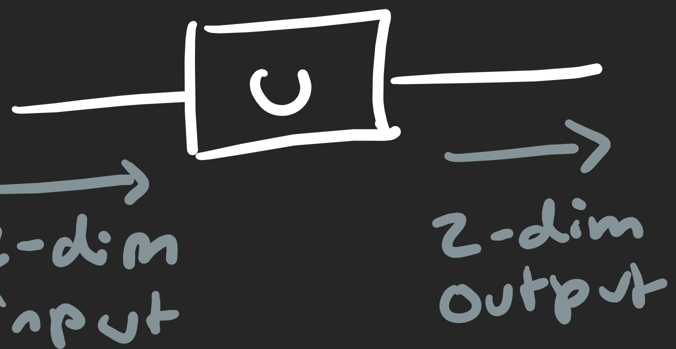
no-wire = numbers
wire = vectors

Graphical Notation

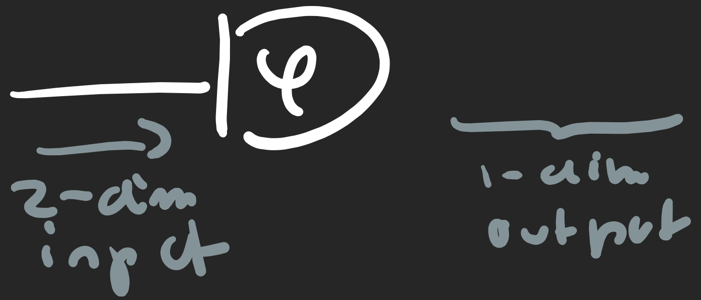
column vector $\begin{pmatrix}\psi_0 \\ \psi_1\end{pmatrix}$



matrix $\begin{pmatrix}U_{00} & U_{01} \\ U_{10} & U_{11}\end{pmatrix}$



row vector $(\psi_0^* \ \ \psi_1^*)$



Graphical Notation

right-to-left by convention

left-to-right by convention

Matrix multiplication = Sequential composition

$$\begin{pmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{pmatrix} \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix}$$

=

intermediate vector space is "contracted", i.e. summed over

Graphical Notation

right-to-left by convention

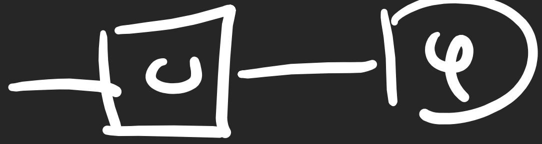
left-to-right by convention

Matrix multiplication = Sequential composition

$V \cdot U$

$U|\psi\rangle$

$\langle\psi|U$



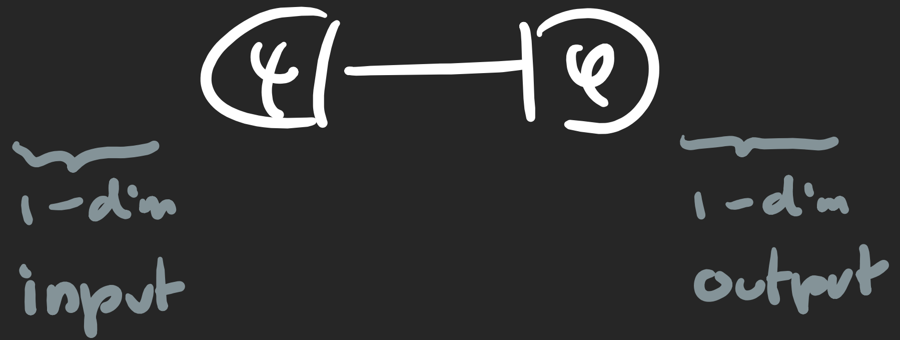
Graphical Notation

Numbers are a little weird:

Complex number $\in \mathbb{C}$

$\langle \varphi | \psi \rangle$

We could do this, but it is confusing later on!



$$\underbrace{\begin{pmatrix}\varphi_0^* & \varphi_1^*\end{pmatrix}}_{\substack{1\times 2\\ \text{matrix}}} \cdot \underbrace{\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}}_{\substack{2\times 1\\ \text{matrix}}} = \underbrace{\begin{pmatrix}\varphi_0^*\psi_0 + \varphi_1^*\psi_1\end{pmatrix}}_{\substack{1\times 1\\ \text{matrix}}} = \underbrace{\varphi_0^*\psi_0 + \varphi_1^*\psi_1}_{\text{number}}$$

Graphical Notation

Numbers are a little weird:

✓ $|\langle\varphi|\psi\rangle|^2$

$\uparrow$

non-negative
real number
$\mathbb{R}^+$

1-dim
output

Instead, we work with $\underbrace{\text{real numbers}}_{\text{(probabilities!)}}$ from the start

Graphical Notation

$\mathbb{R}^+$ numbers $\longleftrightarrow$ Diagrams with no open legs

$|\langle \varphi | \psi \rangle|^2$

$|\langle \varphi | U | \psi \rangle|^2$

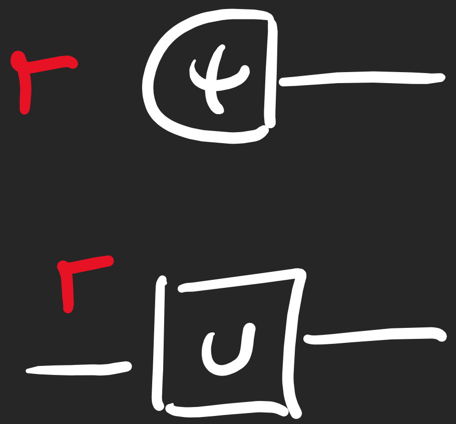
$|\psi_0|^2 = |\langle 0 | \psi \rangle|^2$

Graphical Notation

We can write numbers in diagrams:

$r \in \mathbb{R}^+$ $\quad \sqrt{r}\,|\psi\rangle$

$\sqrt{r}\,U$



(We write numbers in red, to see them better.)

Graphical Notation

Multiple numbers in the same diagram are free to move around, and are multiplied together:

r  s $=$ rs 

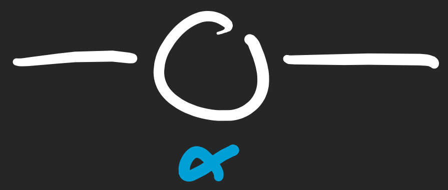
$$\left|\sqrt{s}\langle\varphi|\psi\rangle\sqrt{r}\right|^2 = rs\,|\langle\varphi|\psi\rangle|^2$$

(a.k.a. gates)

Qubit Rotations

$|0\rangle\langle 0| + e^{i\alpha}|1\rangle\langle 1|$

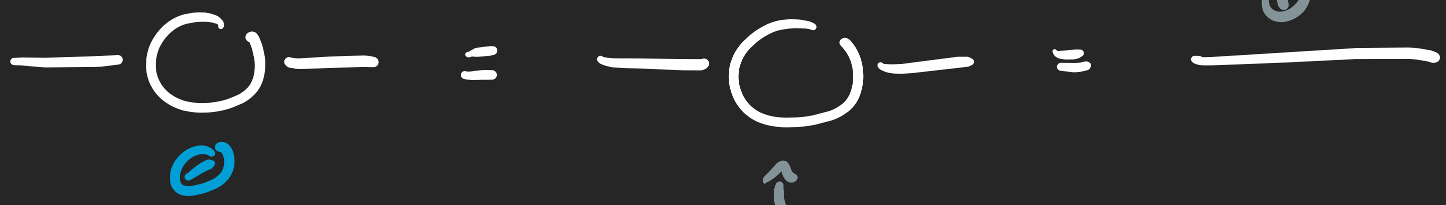
$$R_z(\alpha) := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \longleftrightarrow$$



Angles and other parameters are blue in diagrams, to help distinguish them from probabilities.

wire = identity

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longleftrightarrow$$

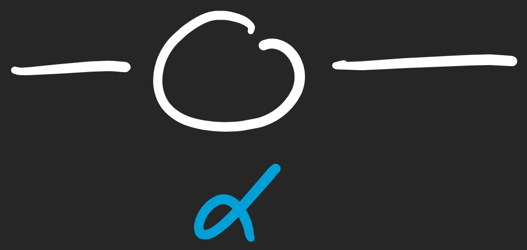


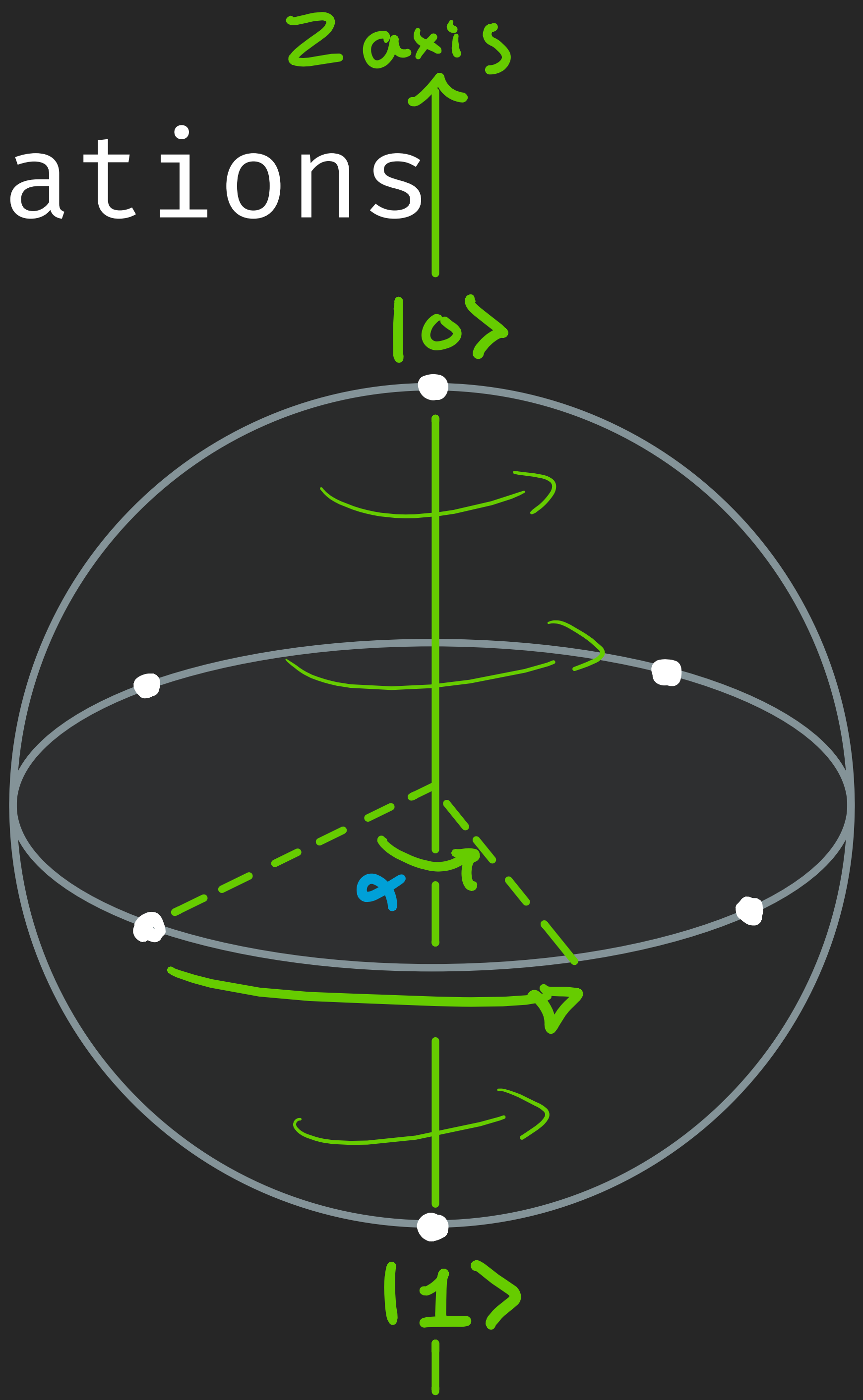
identity matrix

angle 0 omitted

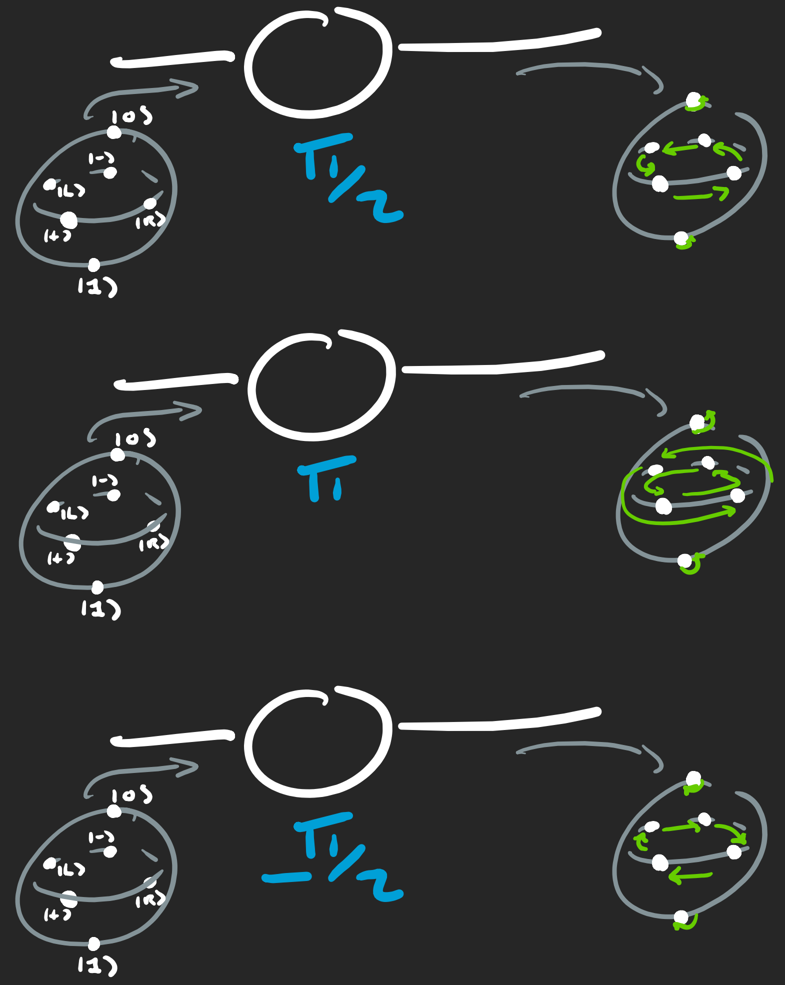
Qubit Rotations

Z rotation by angle α



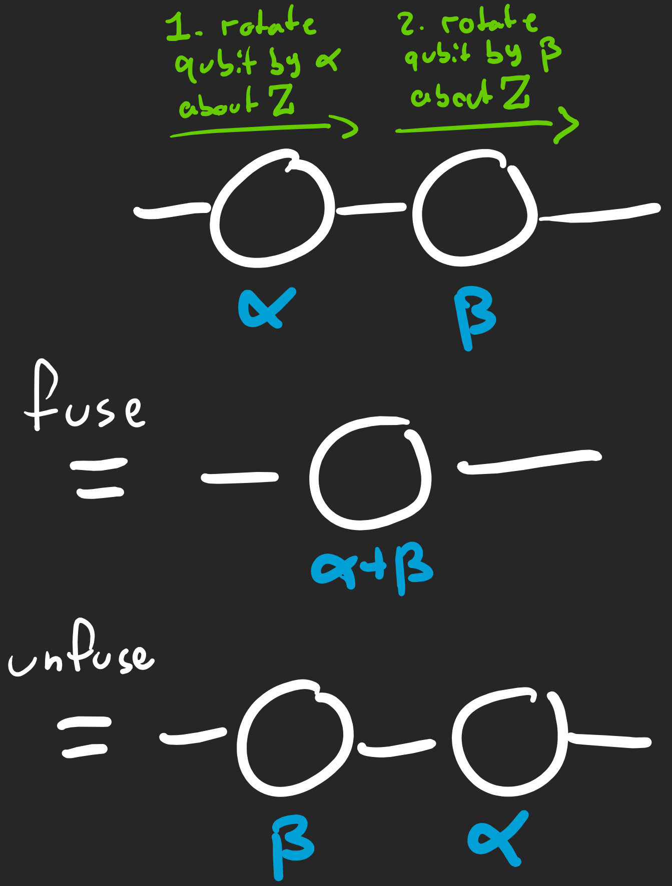


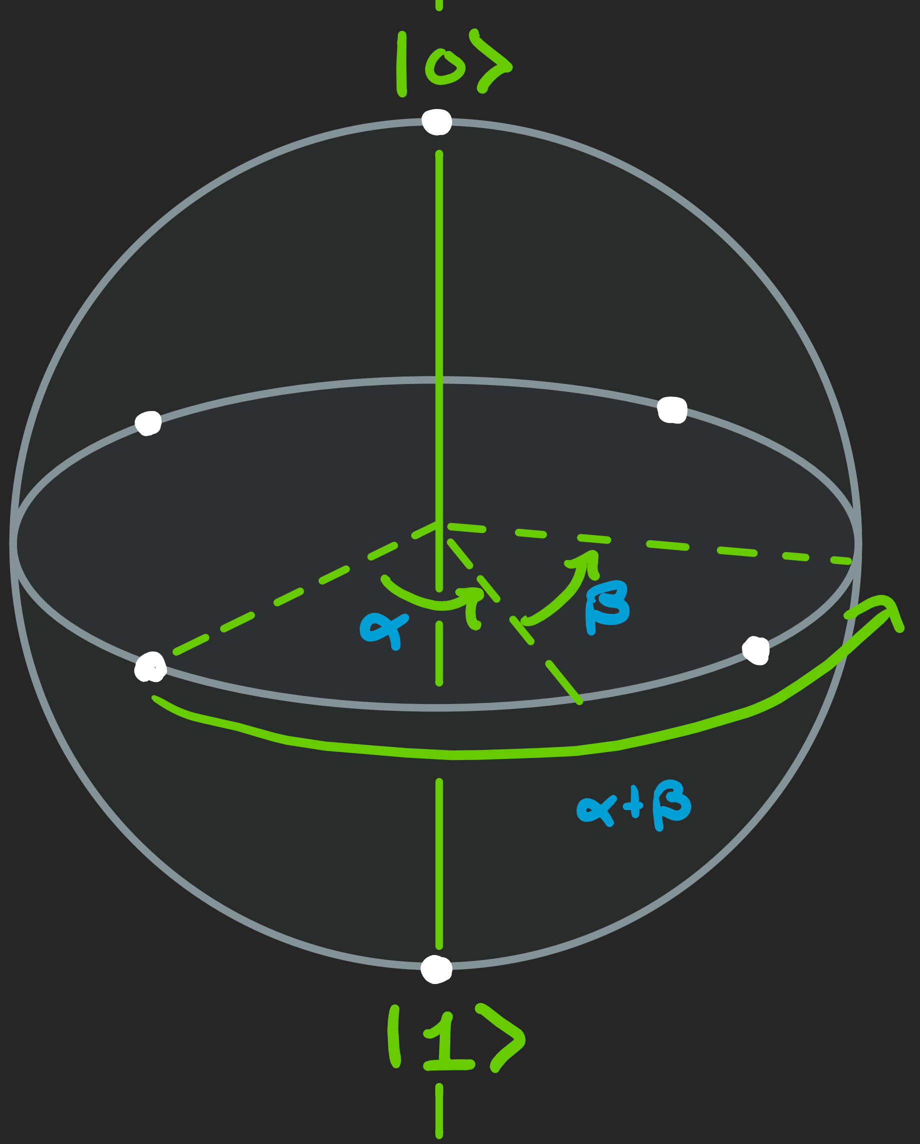
Examples



Qubit Rotations

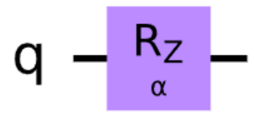
All rotation angles in this course follow the right-hand rule unless stated otherwise.



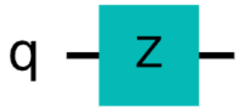


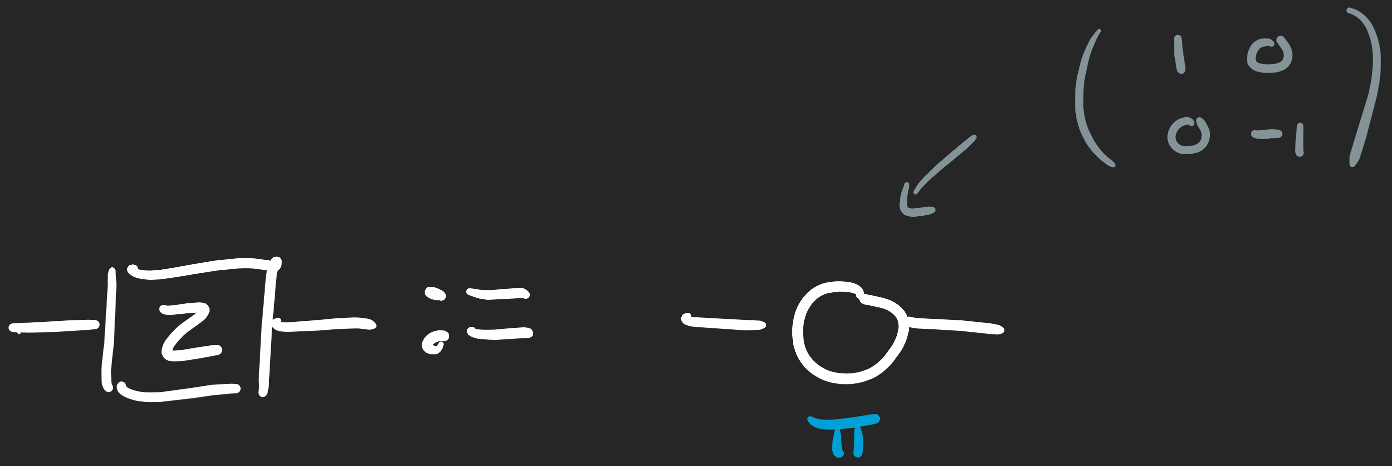
Z rotations

```python
circ = QuantumCircuit(1)
circ.rz(Parameter("α"), 0)
circ.draw("mpl")
```



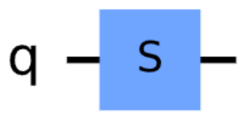

```python
circ = QuantumCircuit(1)
circ.z(0)
circ.draw("mpl")
```



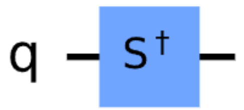


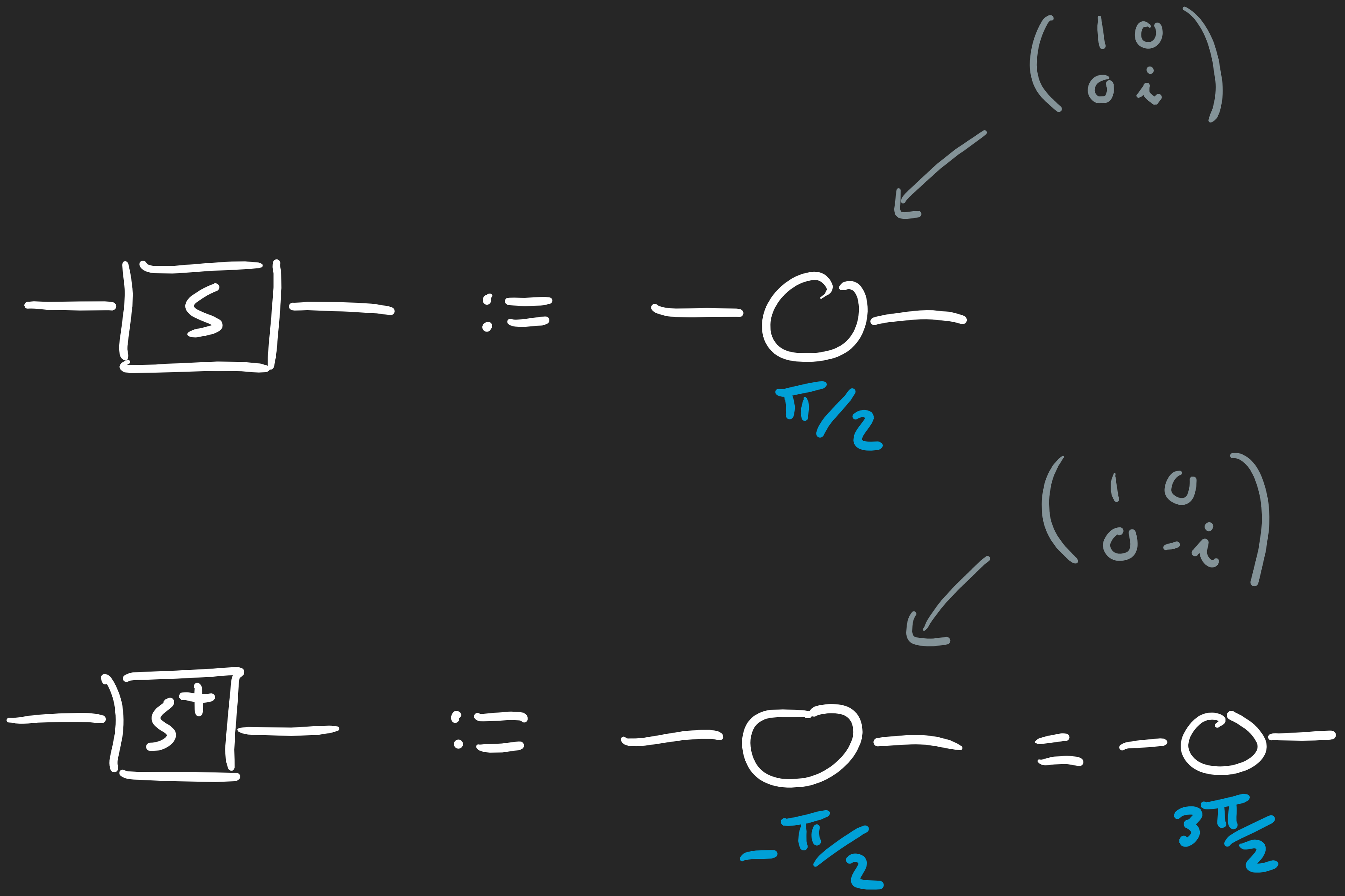
The S gate

```python
circ = QuantumCircuit(1)
circ.s(0)
circ.draw("mpl")
```




```python
circ = QuantumCircuit(1)
circ.sdg(0)
circ.draw("mpl")
```

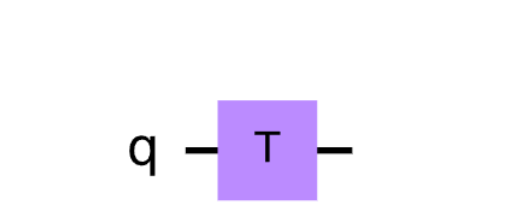


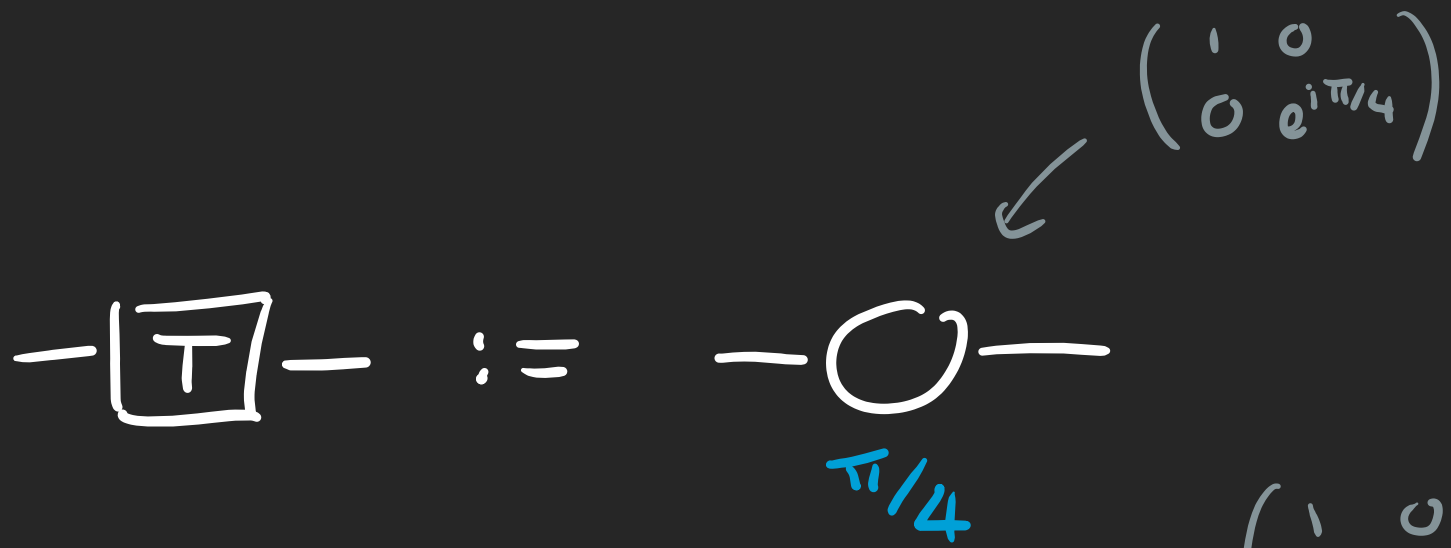


(For quantum gates, the dagger superscript means inverse)

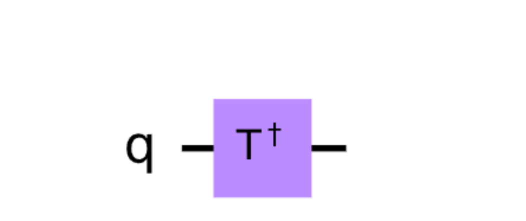
The T gate

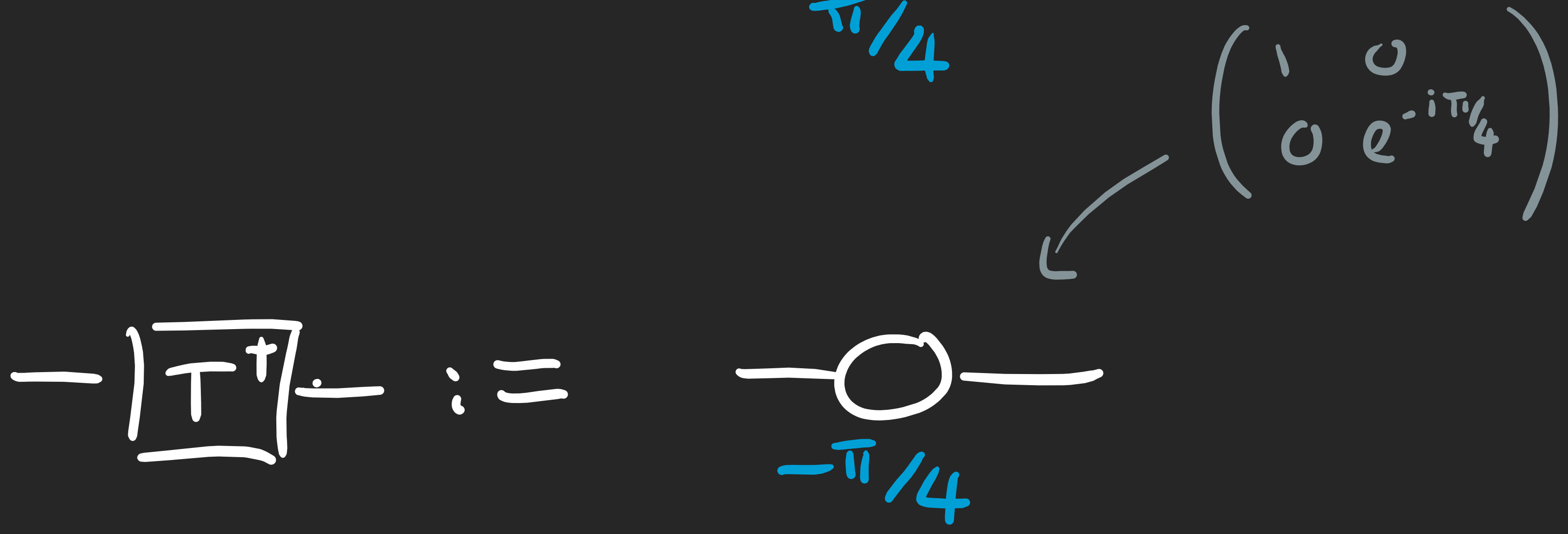
```
circ = QuantumCircuit(1)
circ.t(0)
circ.draw("mpl")
```





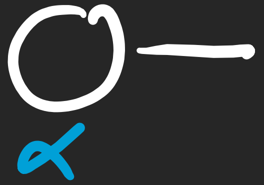

```
circ = QuantumCircuit(1)
circ.tdg(0)
circ.draw("mpl")
```



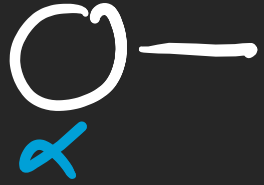


Some qubit states

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix}$

$|0\rangle + e^{i\alpha}|1\rangle \longleftrightarrow$ 

(not normalised)

$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\alpha}|1\rangle\right) \longleftrightarrow \frac{1}{2}$ 

(normalised)

angle θ is omitted

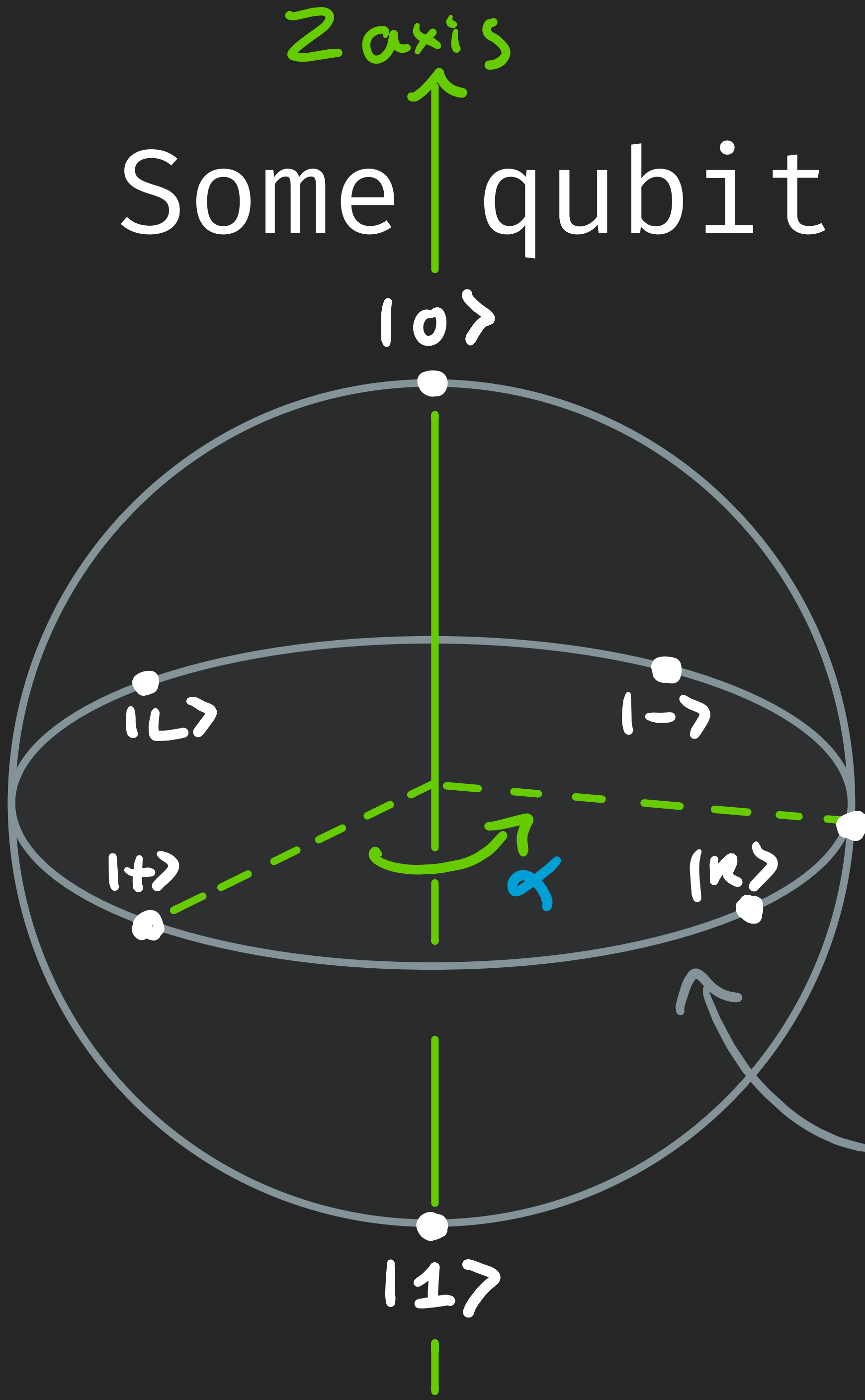
$|+\rangle \longleftrightarrow \frac{1}{2}$ 

$|-\rangle \longleftrightarrow \frac{1}{2}$ 

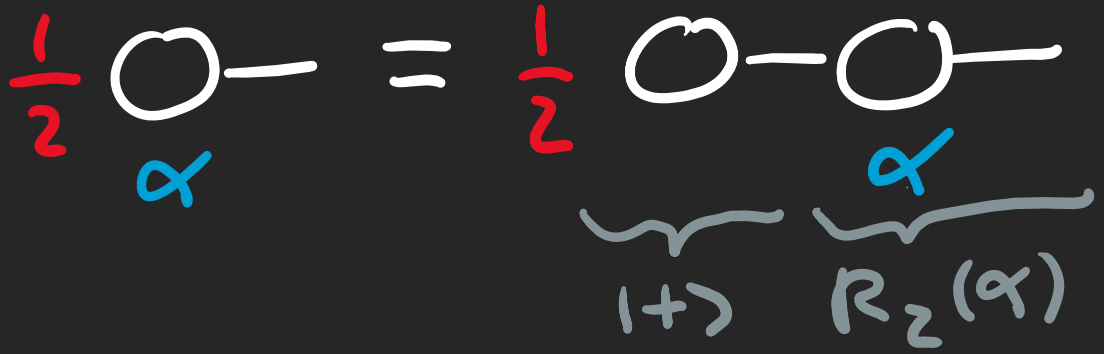
$|R\rangle \longleftrightarrow \frac{1}{2}$ 

$|L\rangle \longleftrightarrow \frac{1}{2}$ 

Some qubit states



Generic "Z-equatorial" state:

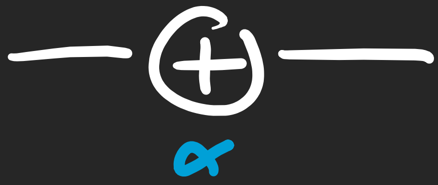


This circle is the equator for the Z axis (|0⟩ and |1⟩ are poles)

Qubit Rotations

$|+\rangle\langle+| + e^{i\alpha}|-\rangle\langle-|$

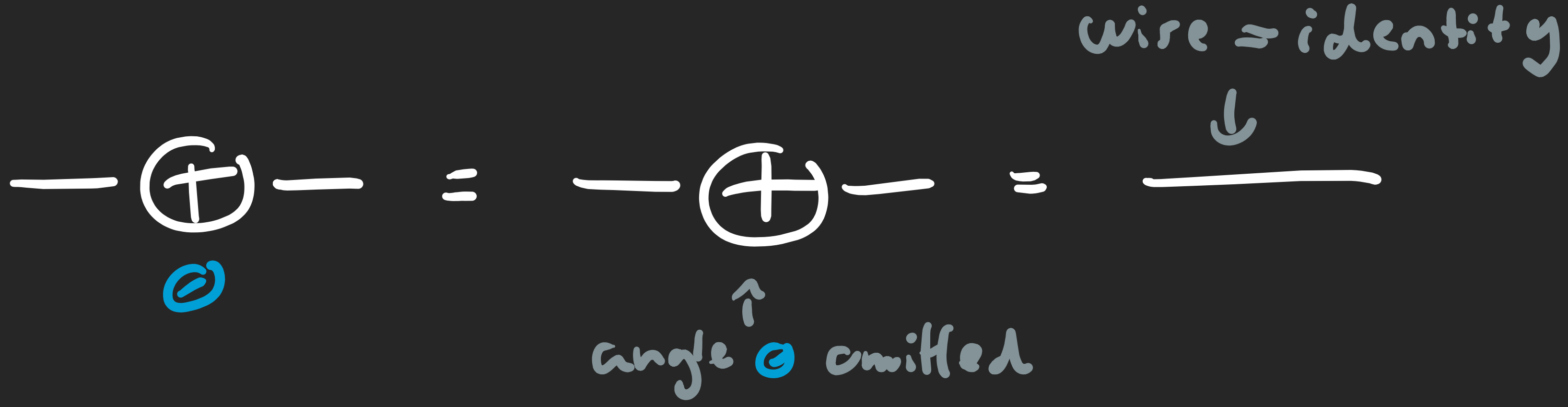
$$R_x(\alpha) := \begin{pmatrix} \cos\alpha/2 & -i\sin\alpha/2 \\ -i\sin\alpha/2 & \cos\alpha/2 \end{pmatrix} \longleftrightarrow$$



Angles and other parameters are blue in diagrams, to help distinguish them from probabilities.

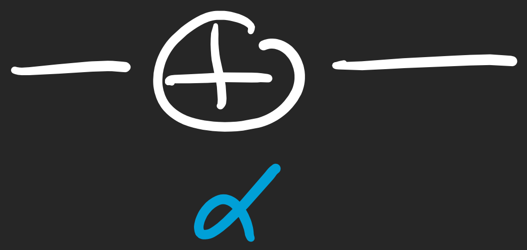
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longleftrightarrow$$

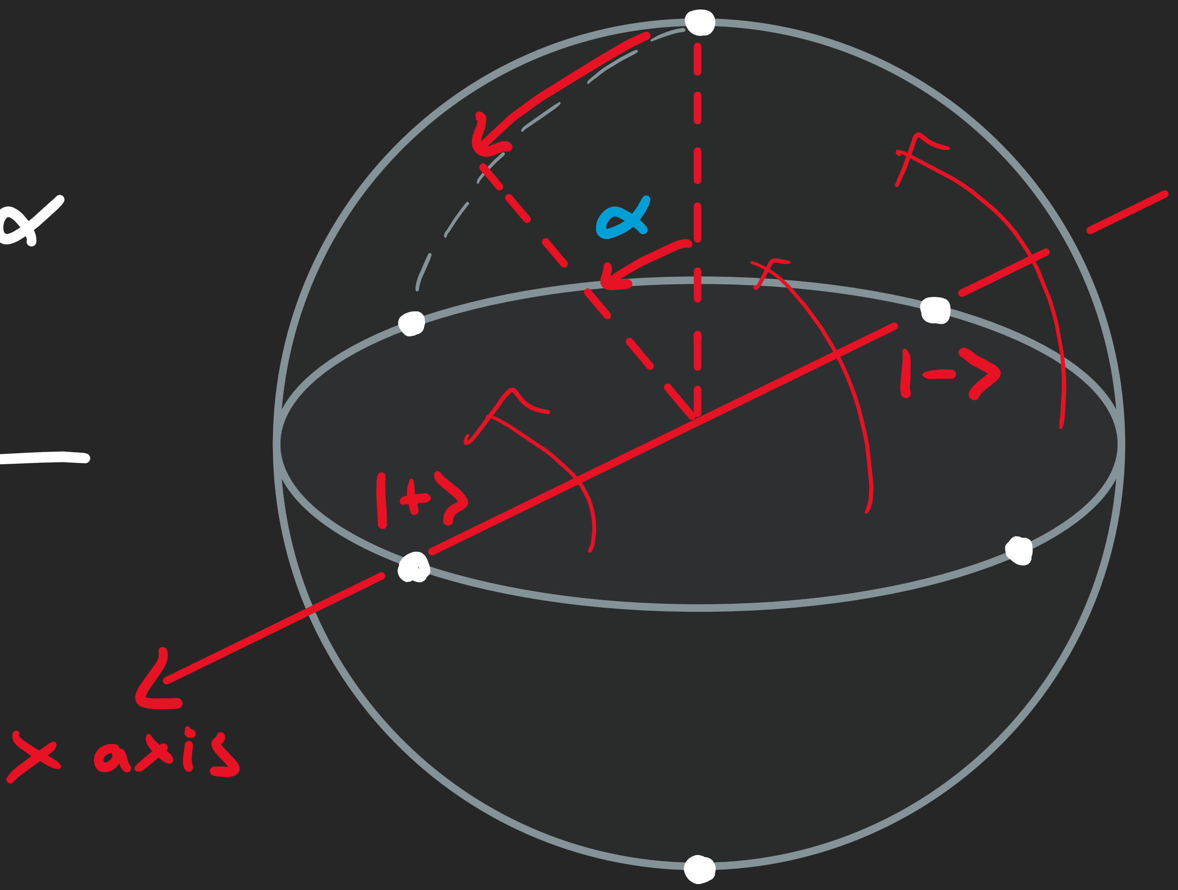
identity matrix



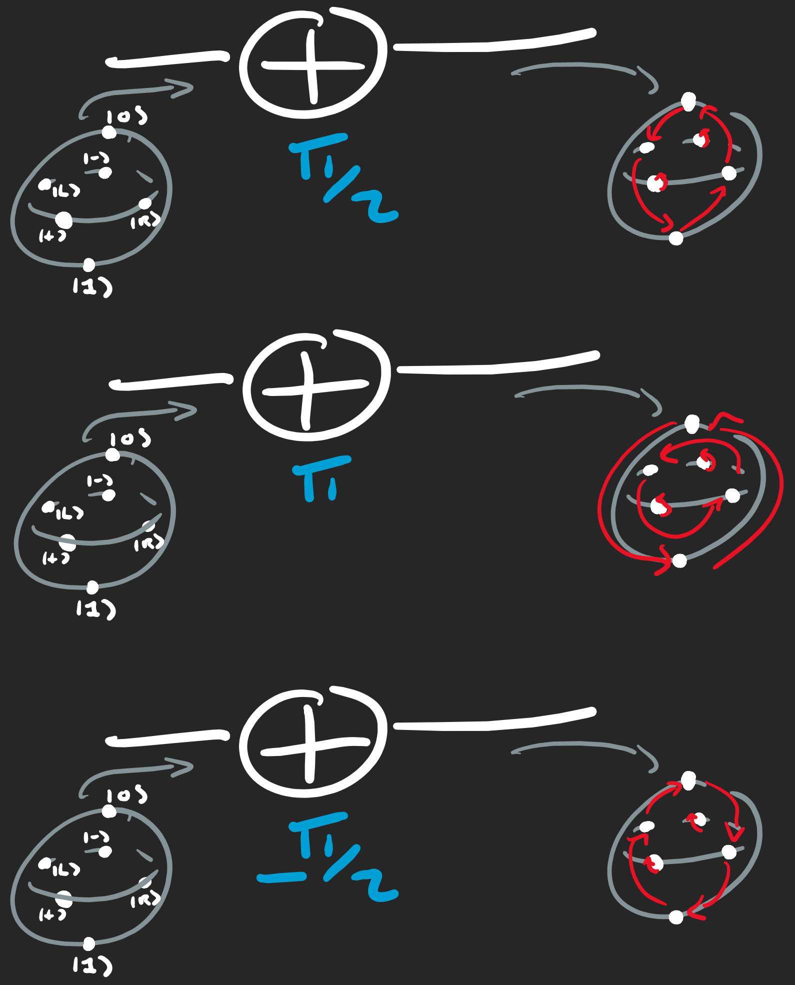
Qubit Rotations

X rotation
by angle α

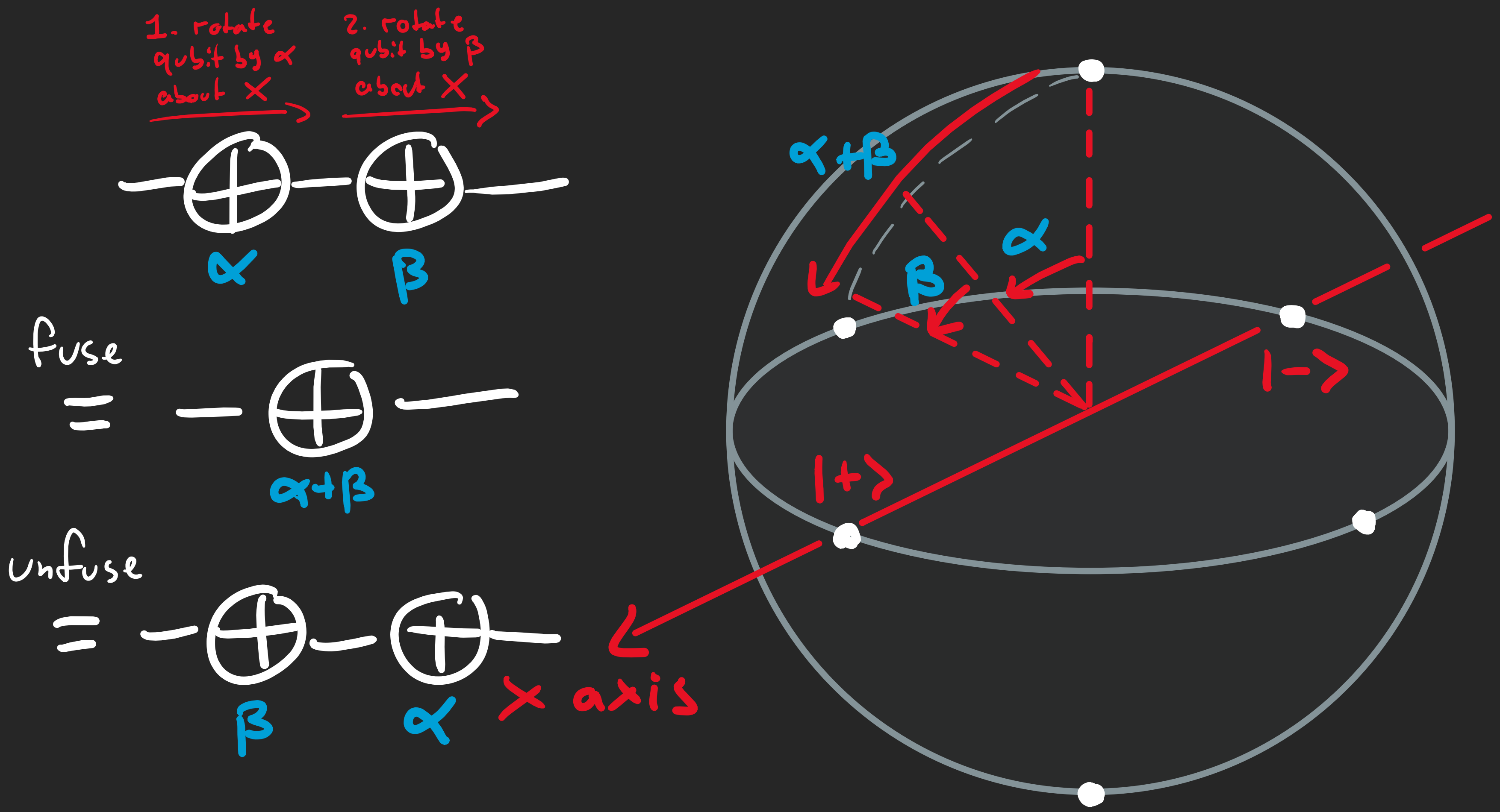




Examples

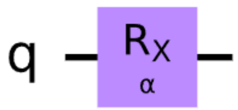


Qubit Rotations

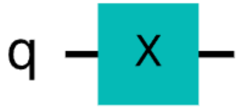


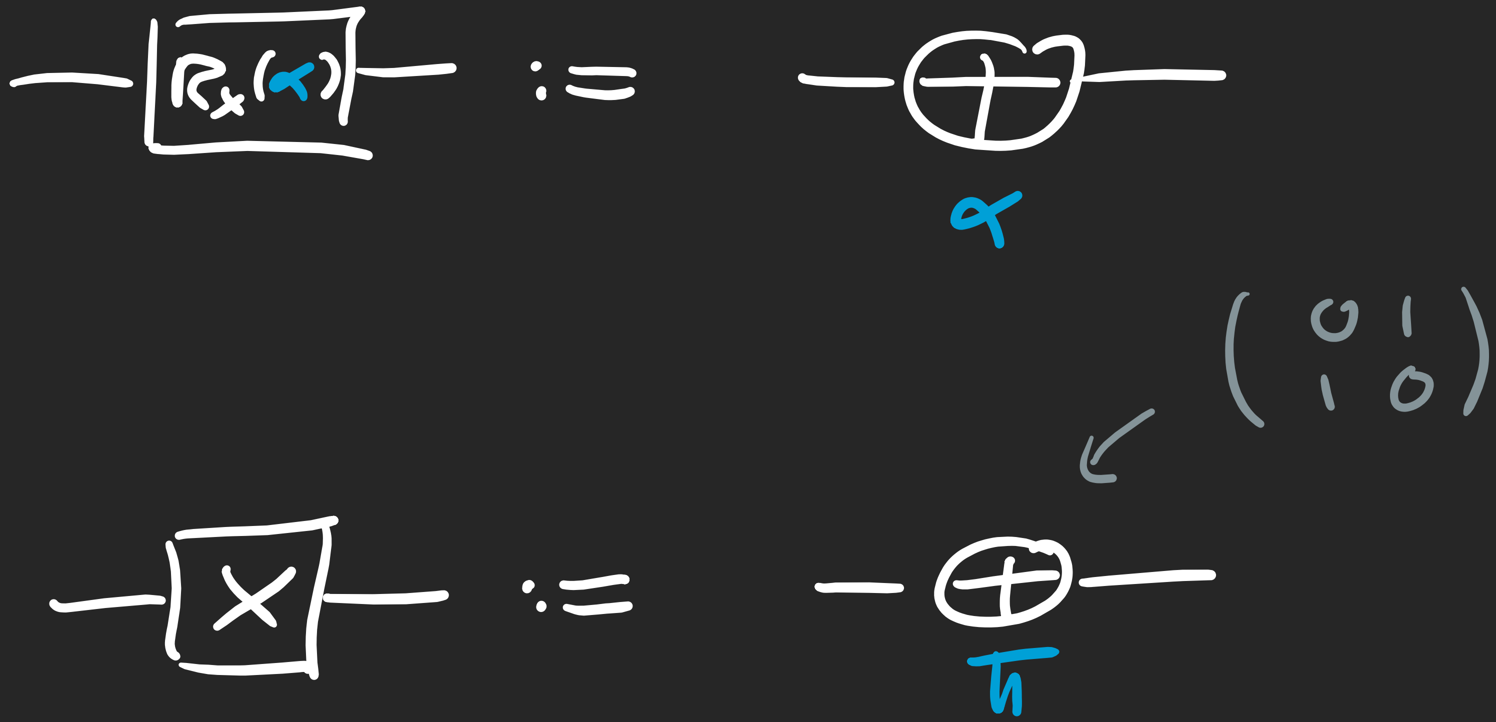
X rotations

```python
circ = QuantumCircuit(1)
circ.rx(Parameter("α"), 0)
circ.draw("mpl")
```




```python
circ = QuantumCircuit(1)
circ.x(0)
circ.draw("mpl")
```

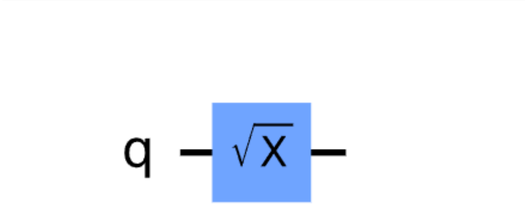


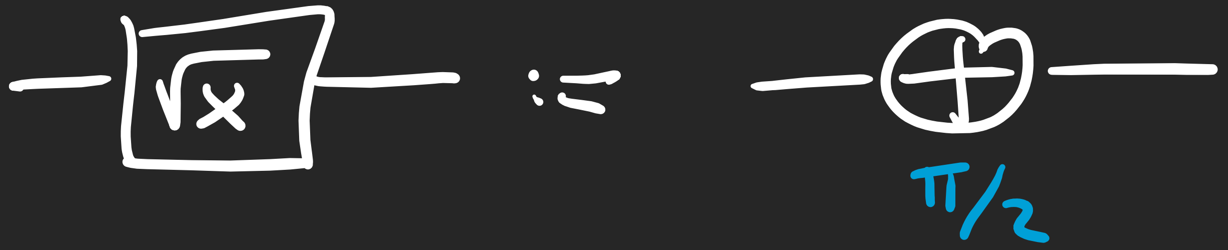


The $\sqrt{X}$ gate

as matrices: $\sqrt{X} \cdot \sqrt{X} = X$

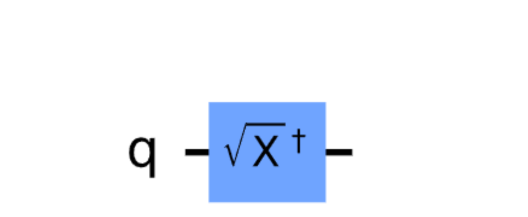
```python
circ = QuantumCircuit(1)
circ.sx(0)
circ.draw("mpl")
```





$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

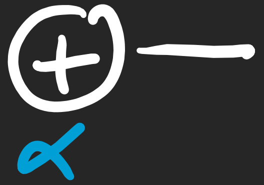
```python
circ = QuantumCircuit(1)
circ.sxdg(0)
circ.draw("mpl")
```



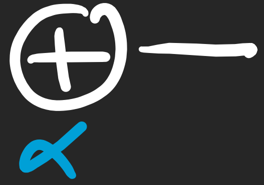
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

Some qubit states

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1+e^{i\alpha}\\1-e^{i\alpha}\end{pmatrix}$$

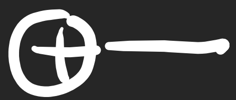
$|+\rangle + e^{i\alpha}|-\rangle \longleftrightarrow$ 

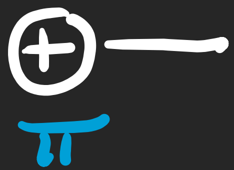
(not normalised)

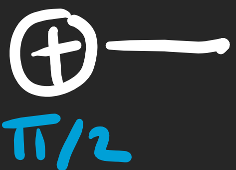
$\frac{1}{\sqrt{2}}\left(|+\rangle + e^{i\alpha}|-\rangle\right) \longleftrightarrow \frac{1}{2}$ 

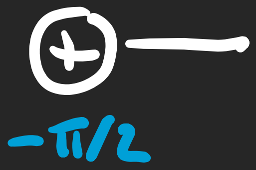
(normalised)

angle θ is omitted

$|0\rangle \longleftrightarrow \frac{1}{2}$ 

$|1\rangle \longleftrightarrow \frac{1}{2}$ 

$|L\rangle \longleftrightarrow \frac{1}{2}$ 

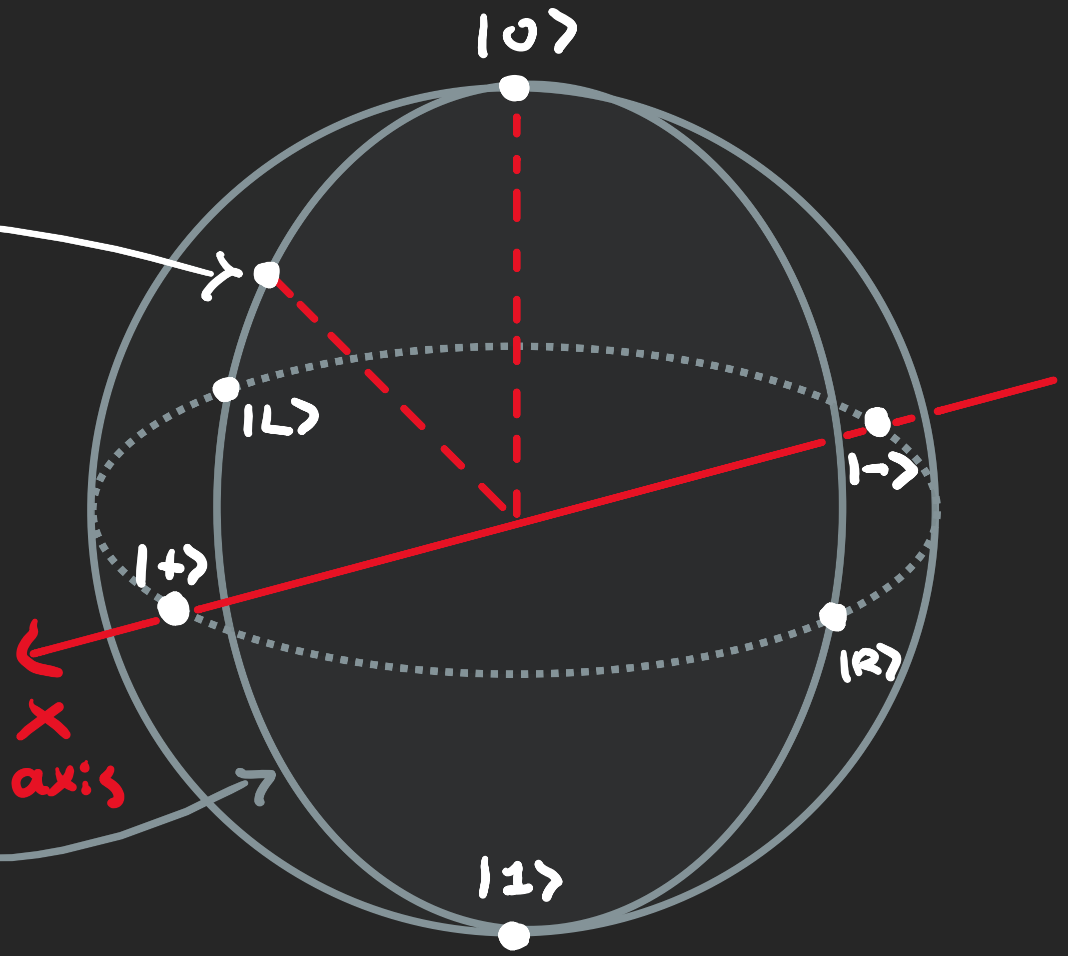
$|R\rangle \longleftrightarrow \frac{1}{2}$ 

Some qubit states

$\oplus\!-\; := |0\rangle$

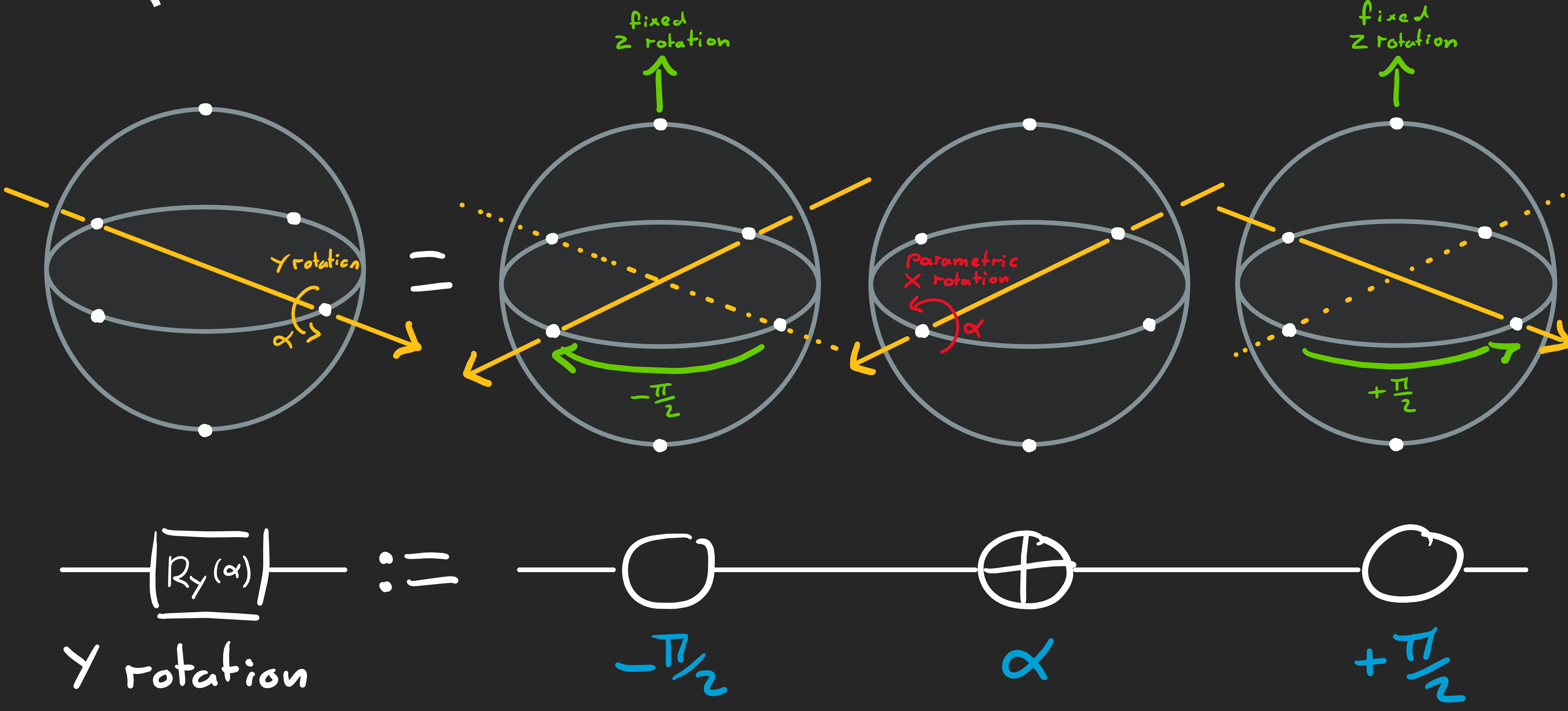
Generic "X-equatorial" state:

$$\frac{1}{2}\underset{\alpha}{\oplus}\!- \;=\; \frac{1}{2}\underbrace{\oplus\!-}_{|0\rangle}\underbrace{\underset{\alpha}{\oplus}\!-}_{R_x(\alpha)}$$



This circle is the equator for the X axis (|+⟩ and |−⟩ are poles)

Qubit Rotations



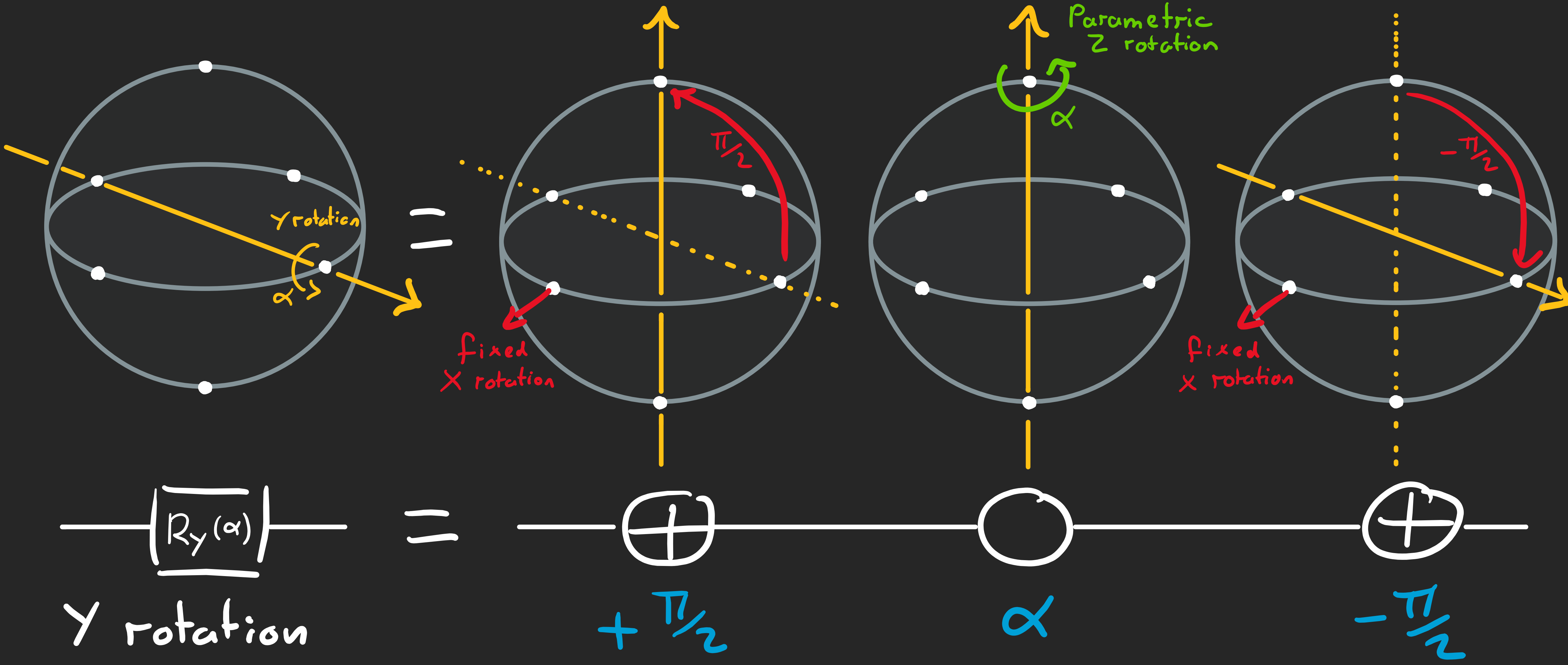
Qubit Rotations

$$R_y(\alpha) := \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2} \\ \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \underbrace{e^{i\pi/2}}_{i} \end{pmatrix} \begin{pmatrix} \cos\alpha/2 & -i\sin\alpha/2 \\ -i\sin\alpha/2 & \cos\alpha/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \underbrace{e^{i\frac{3\pi}{2}}}_{-i} \end{pmatrix}$$

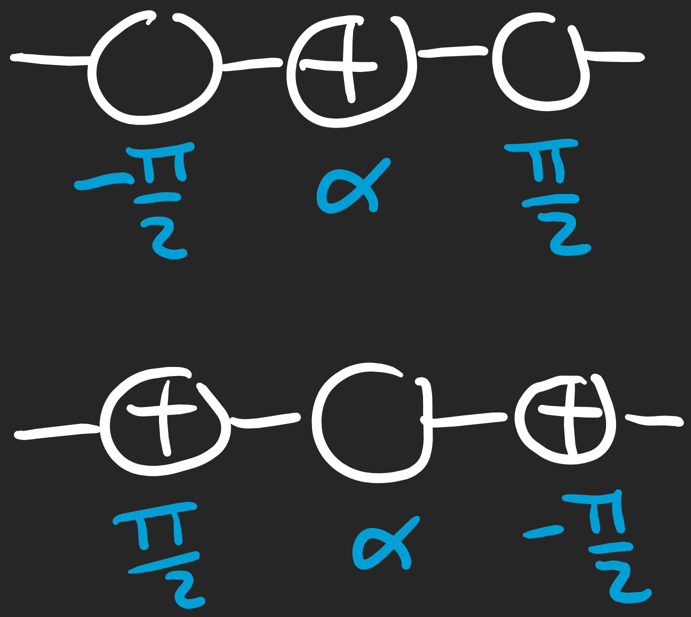
$$R_z\left(\frac{\pi}{2}\right) \qquad R_x(\alpha) \qquad R_z\left(-\frac{\pi}{2}\right)$$

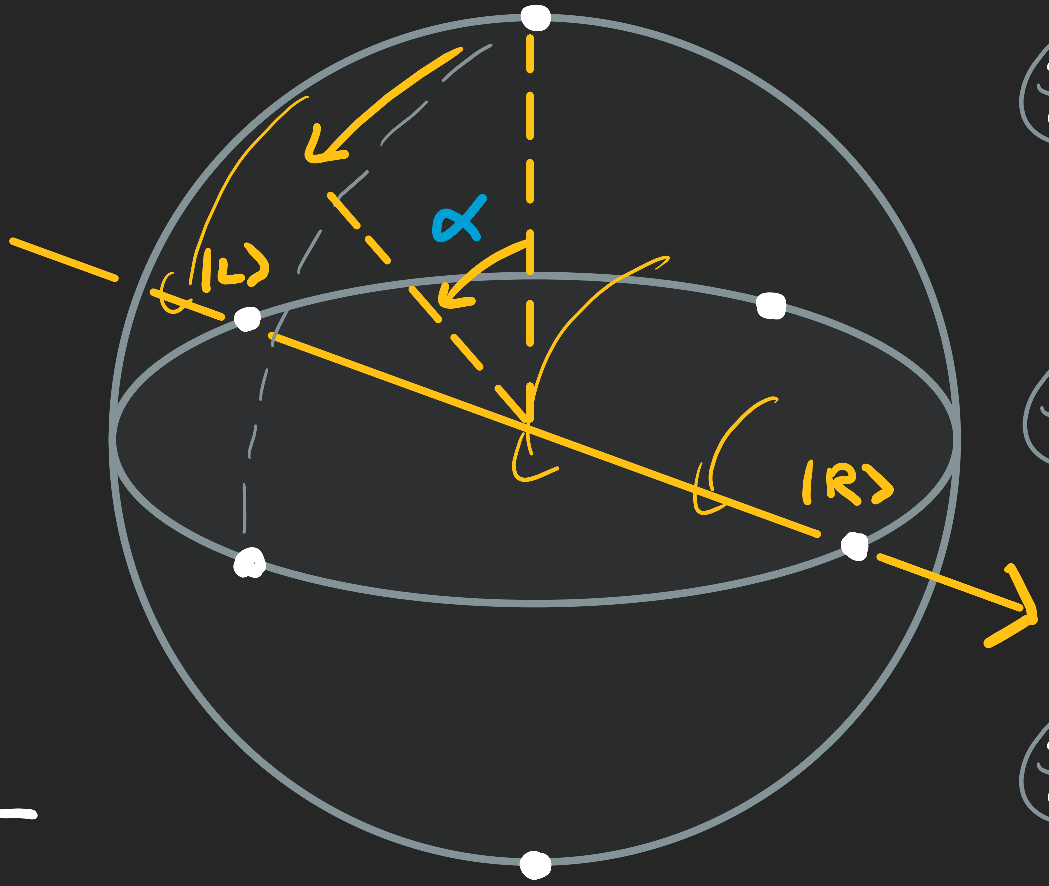
Qubit Rotations



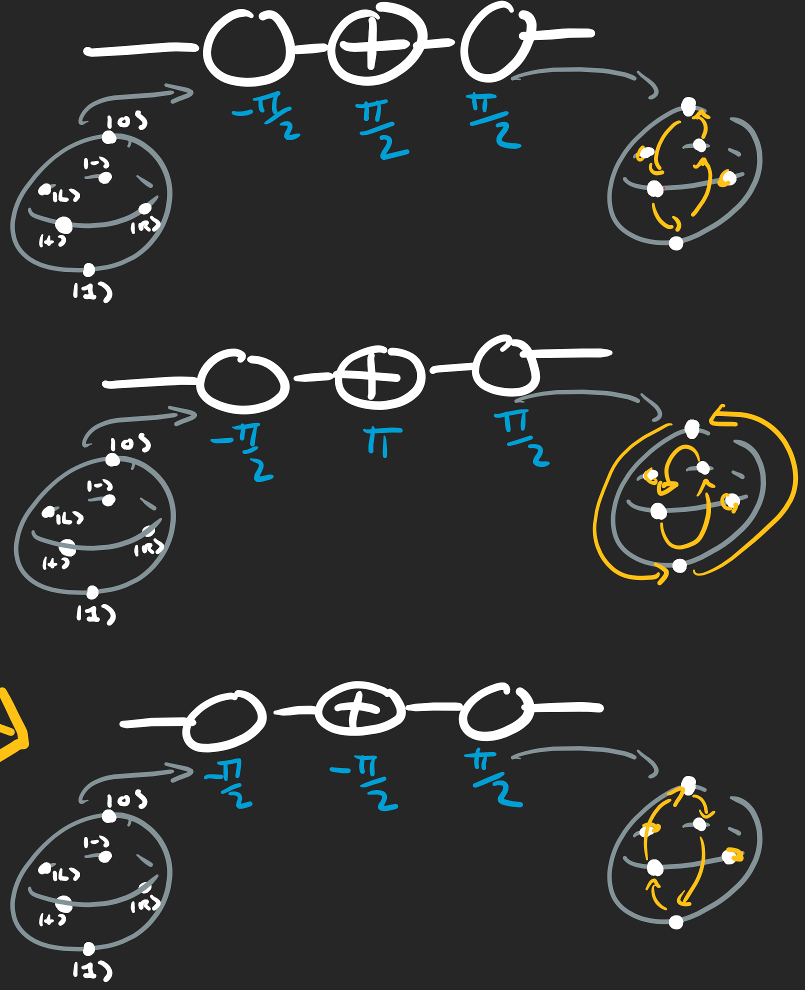
Qubit Rotations

Y rotation by angle α



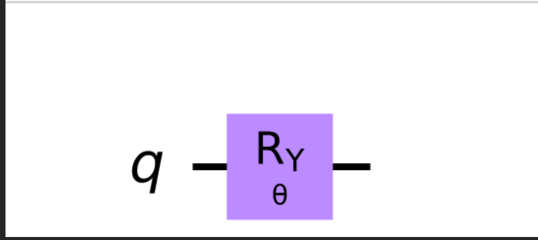


Examples

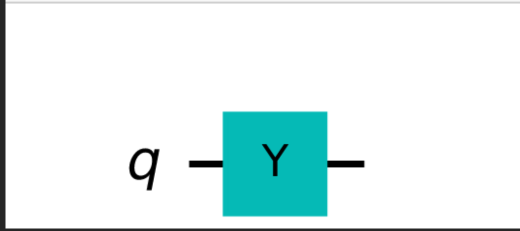


Y rotations

```python
circ = QuantumCircuit(1)
circ.ry(theta, 0)
circ.draw("mpl")
```

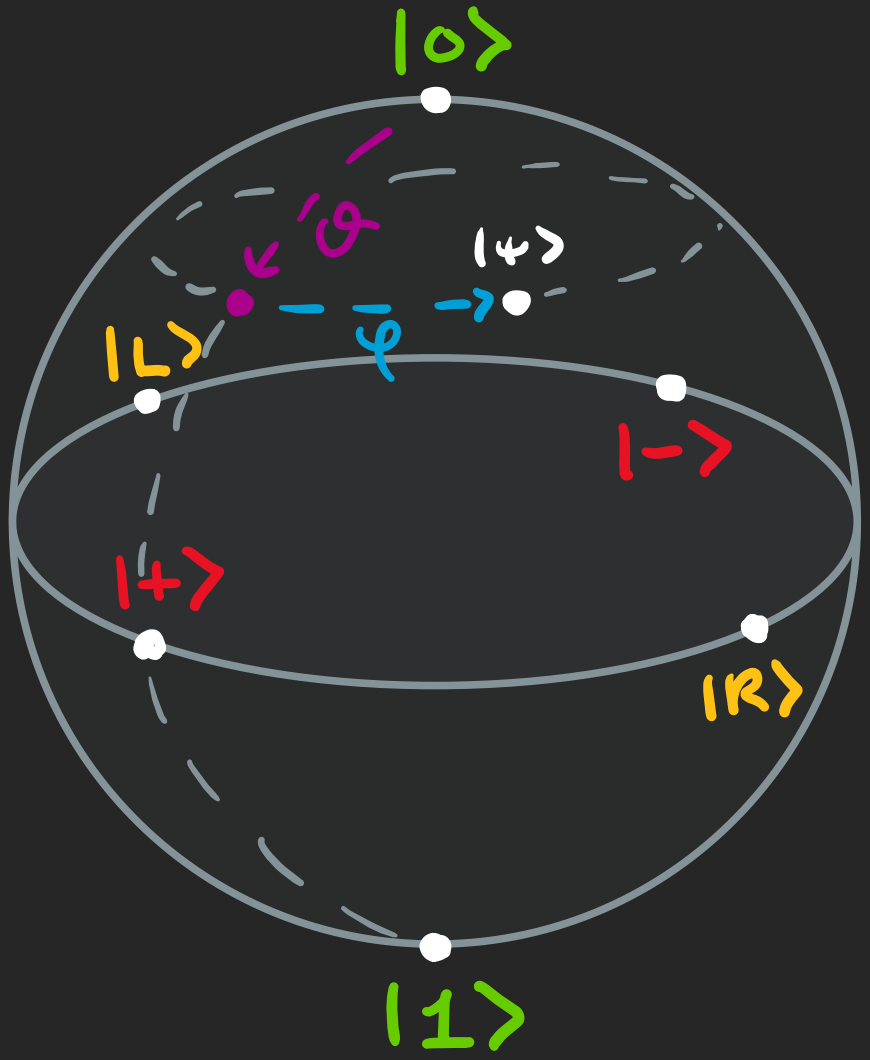


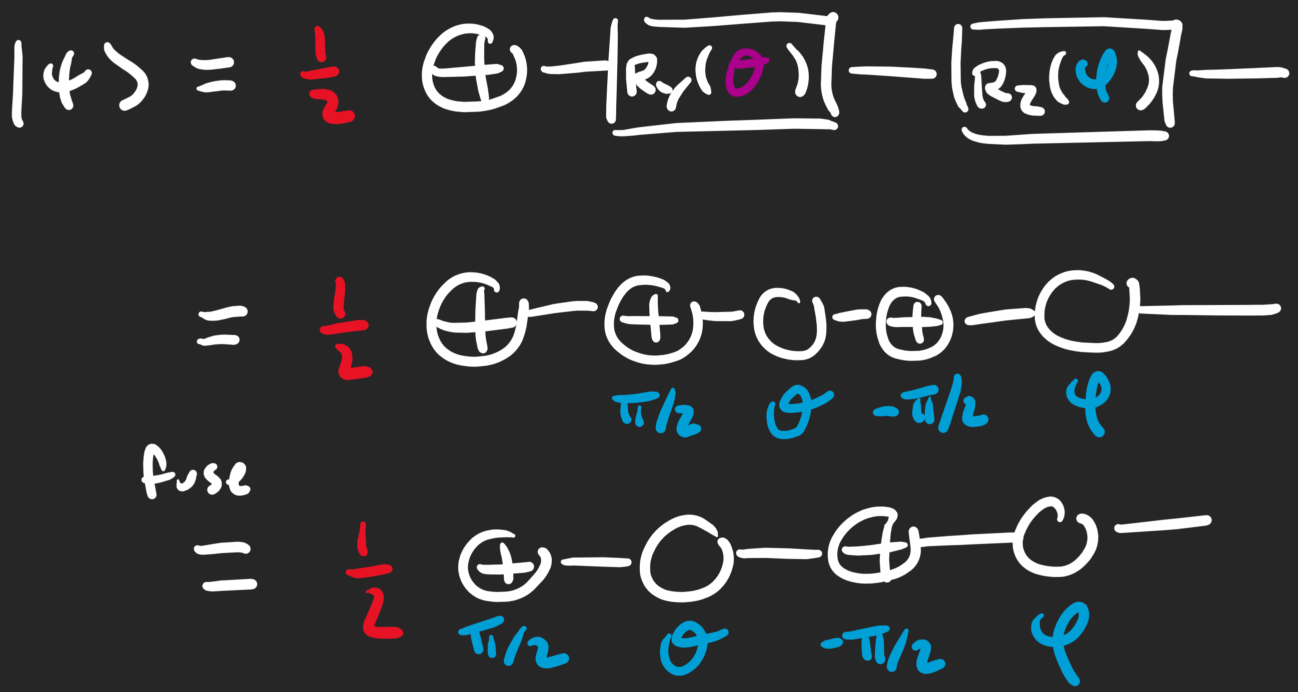

```python
circ = QuantumCircuit(1)
circ.y(0)
circ.draw("mpl")
```



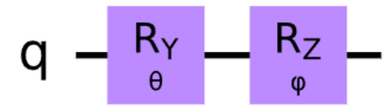
$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Qubit States



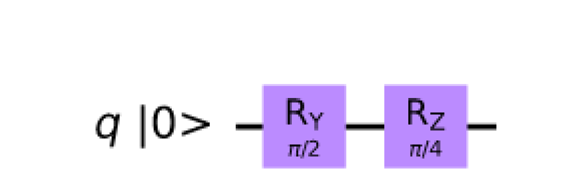


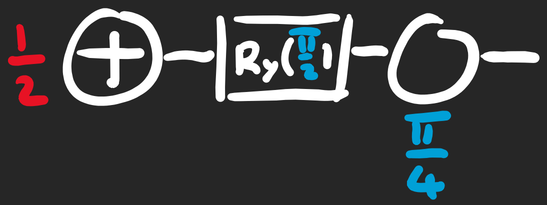

```
circ = QuantumCircuit(1)
circ.ry(Parameter("θ"), 0)
circ.rz(Parameter("φ"), 0)
circ.draw("mpl")
```



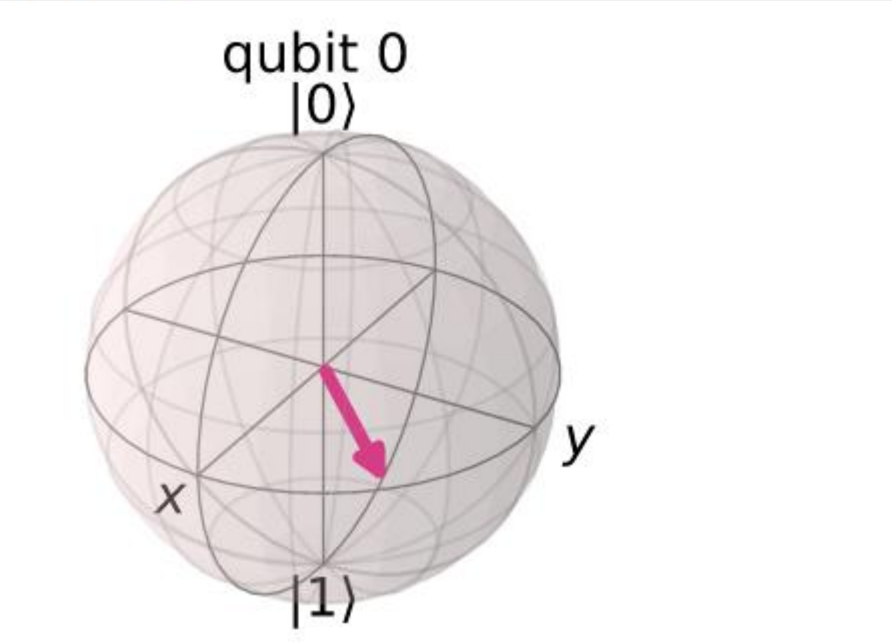
Visualisation in Qiskit

```
from qiskit import QuantumCircuit
circ = QuantumCircuit(1)
circ.ry(pi/2, 0)
circ.rz(pi/4, 0)
circ.draw("mpl", initial_state=True)
```

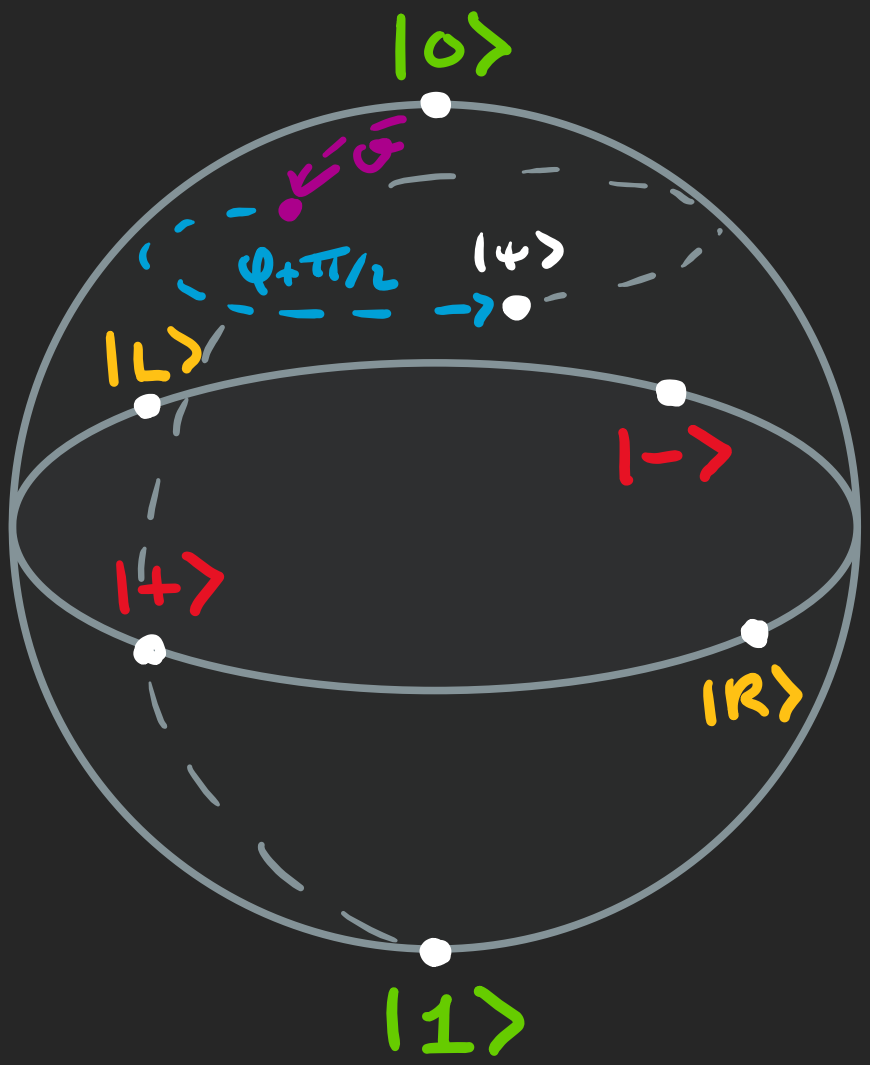


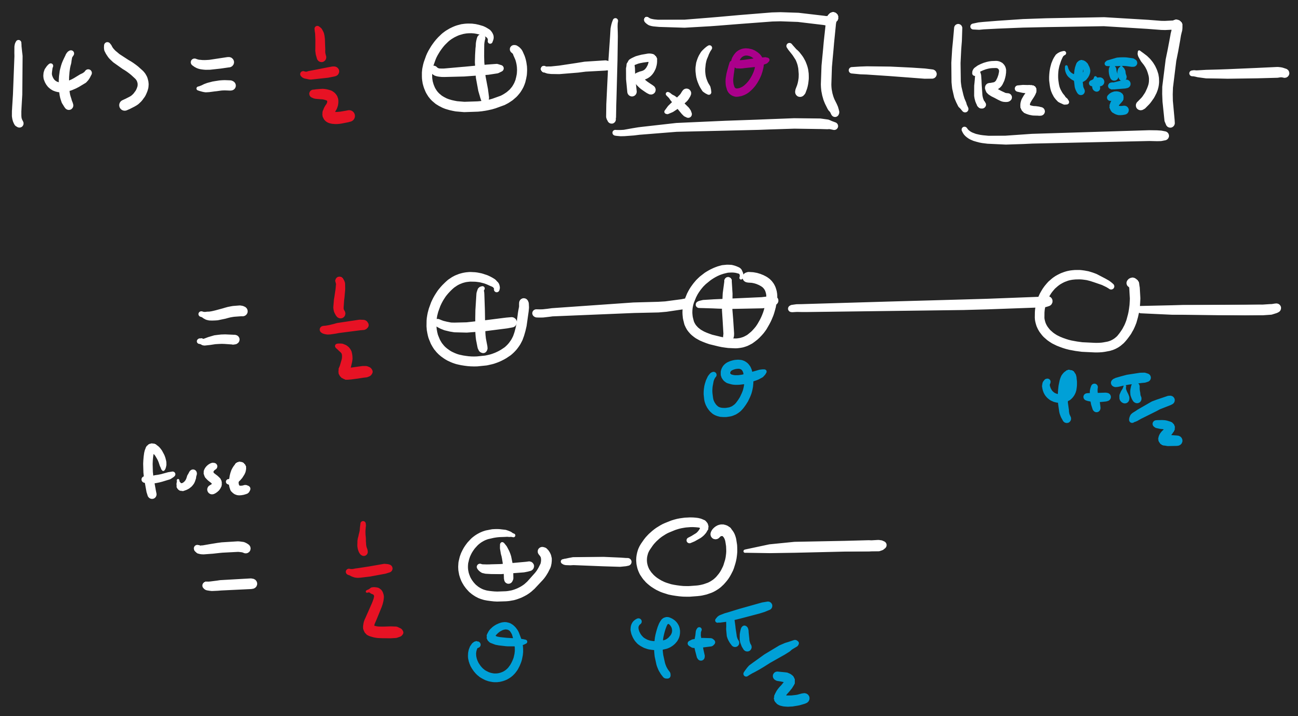



```
from qiskit.visualization import plot_bloch_multivector
circ = QuantumCircuit(1)
circ.ry(pi/2, 0)
circ.rz(pi/4, 0)
plot_bloch_multivector(circ)
```

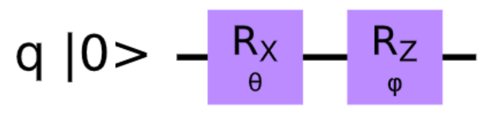


Qubit States

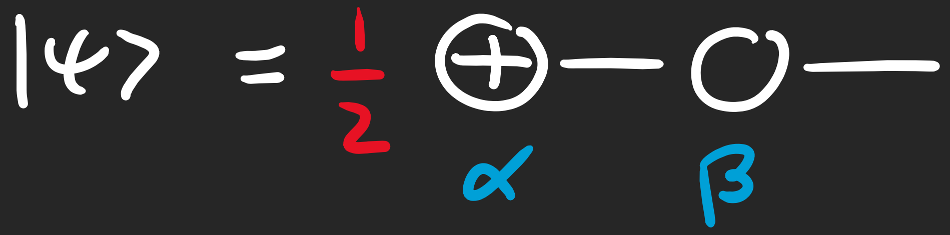




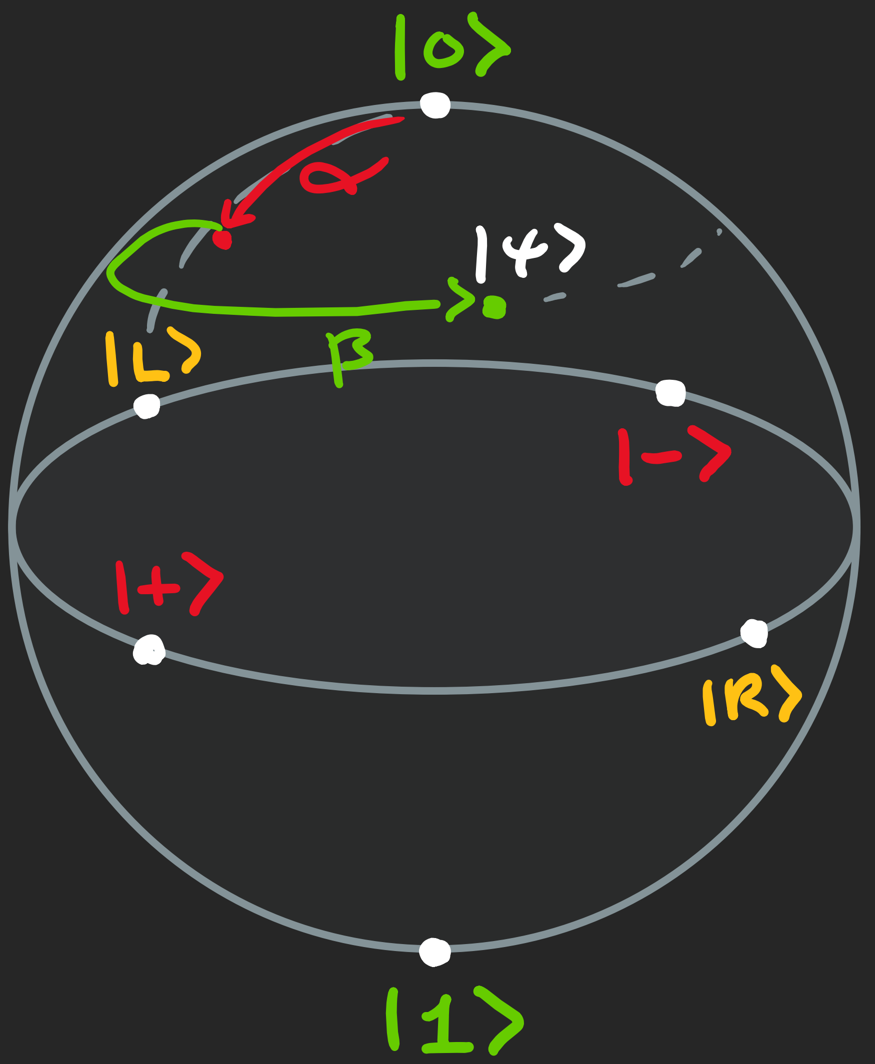

```
circ = QuantumCircuit(1)
circ.rx(Parameter("θ"), 0)
circ.rz(Parameter("φ"), 0)
circ.draw("mpl", initial_state=True)
```



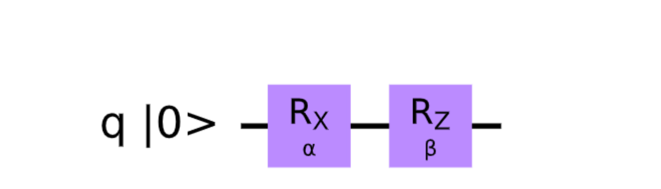
Qubit States



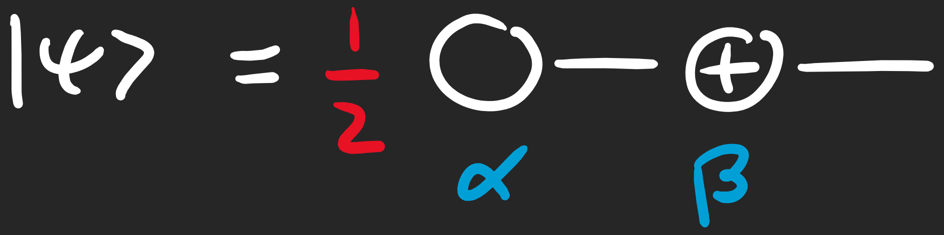
All qubit states can be obtained by Z rotation of a suitable X-equatorial state.



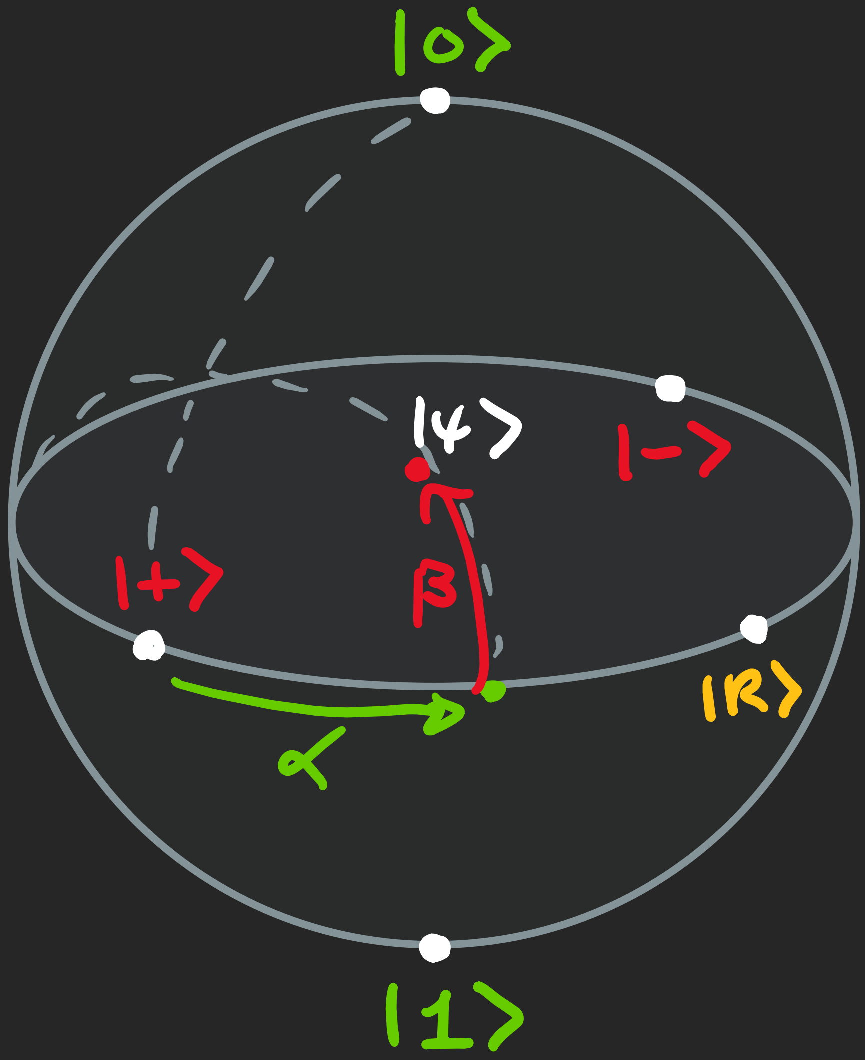

```
circ = QuantumCircuit(1)
circ.rx(Parameter("α"), 0)
circ.rz(Parameter("β"), 0)
circ.draw("mpl", initial_state=True)
```

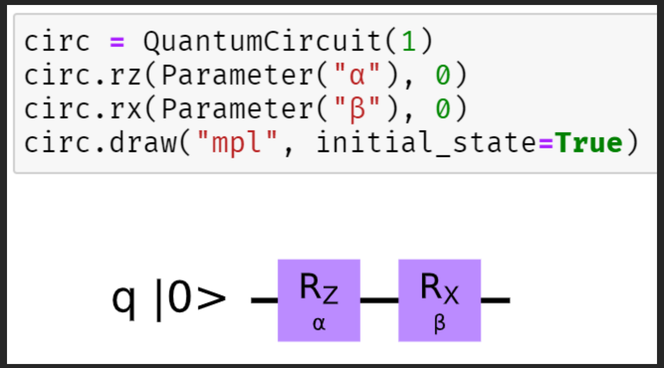


Qubit States



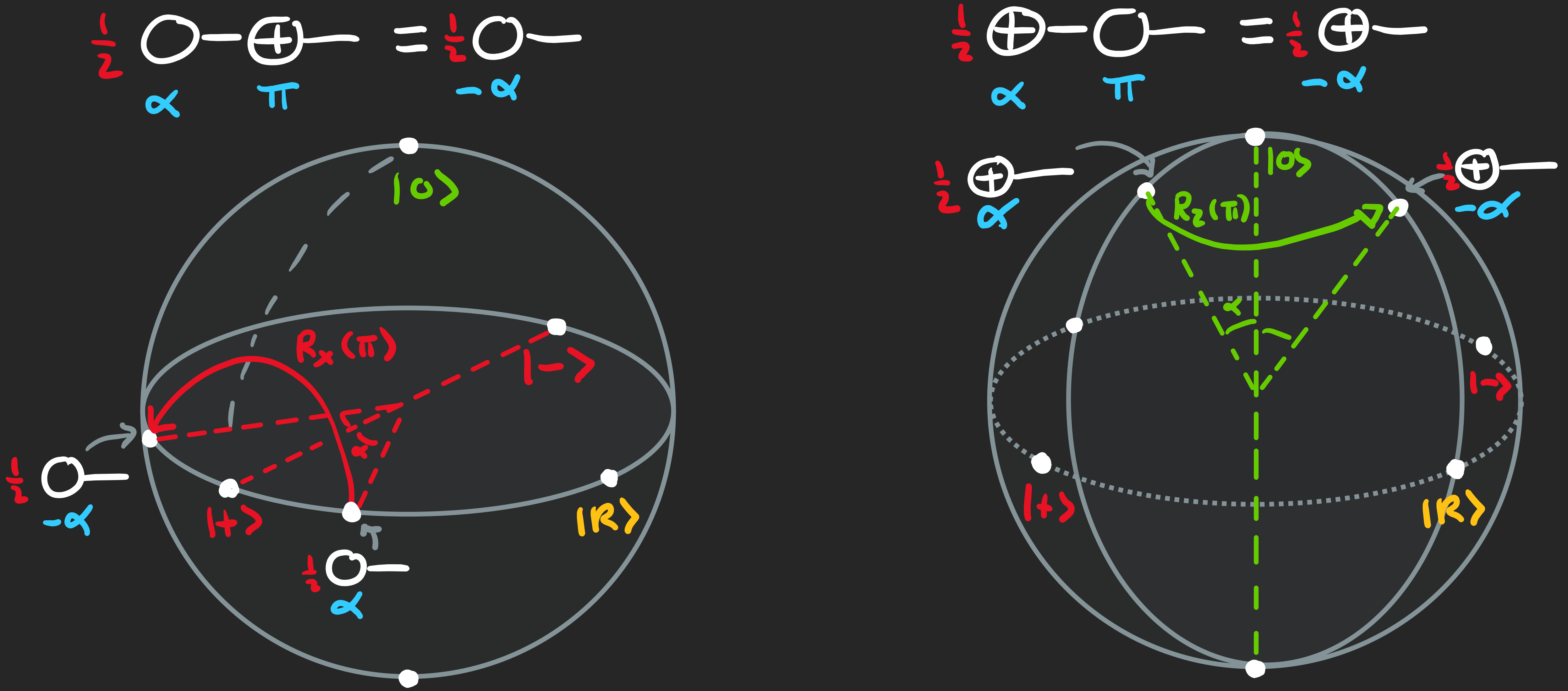
All qubit states can be obtained by X rotation of a suitable Z-equatorial state.



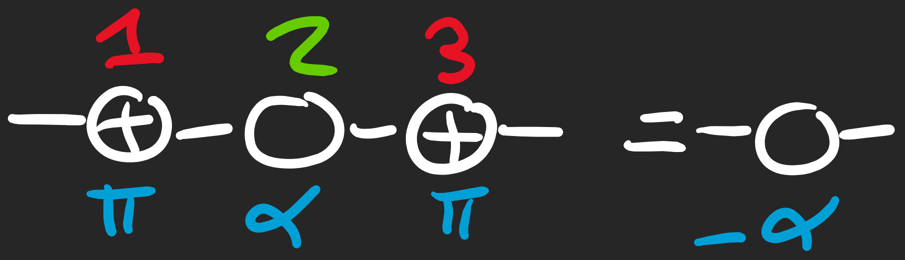


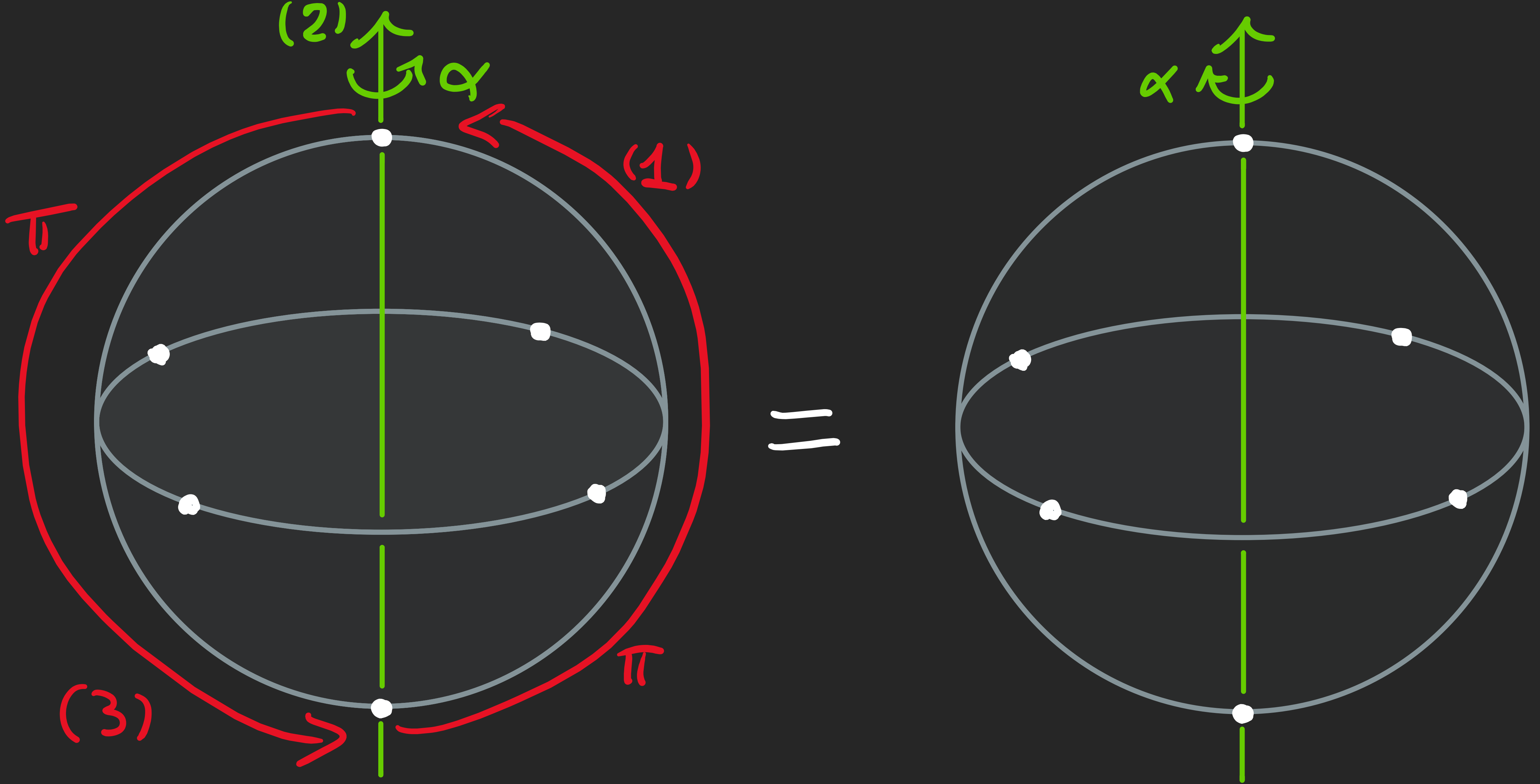
```
circ = QuantumCircuit(1)
circ.rz(Parameter("α"), 0)
circ.rx(Parameter("β"), 0)
circ.draw("mpl", initial_state=True)
```

Effect of π Rotations

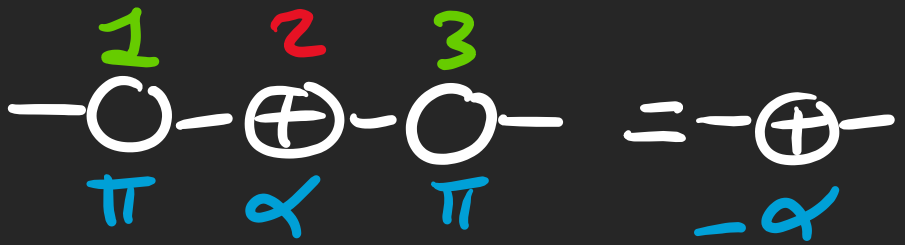


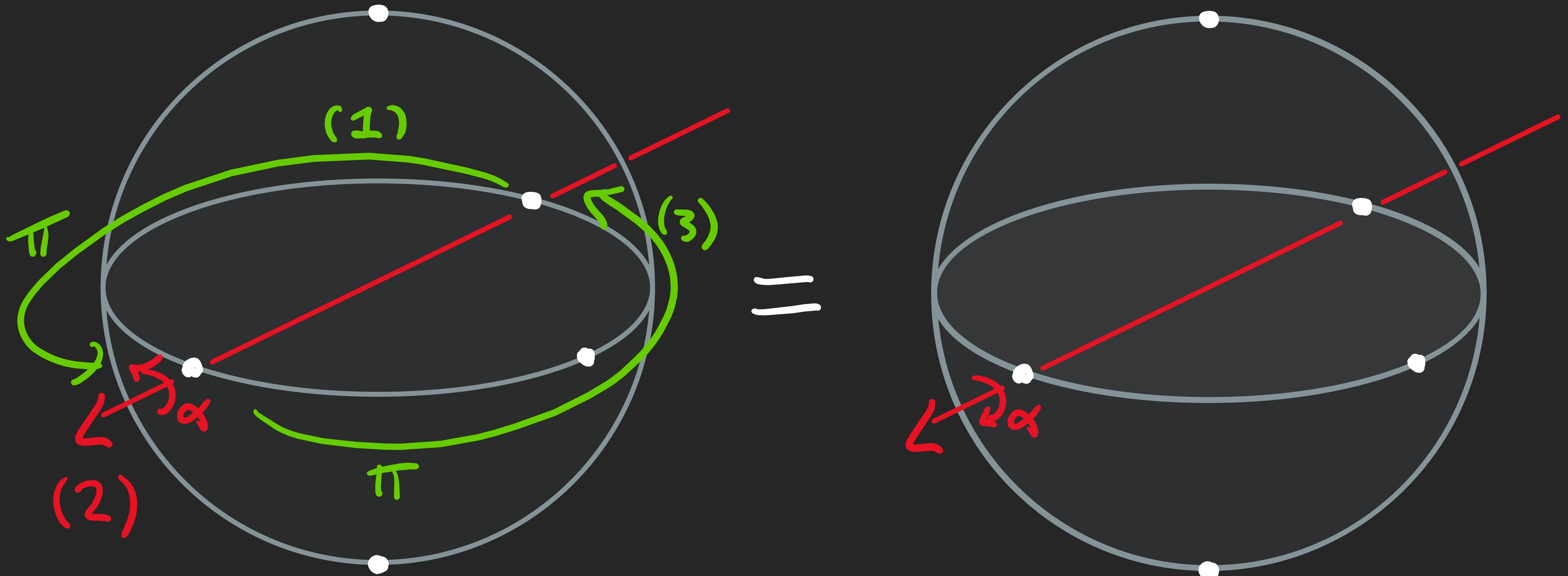
Effect of π Rot's



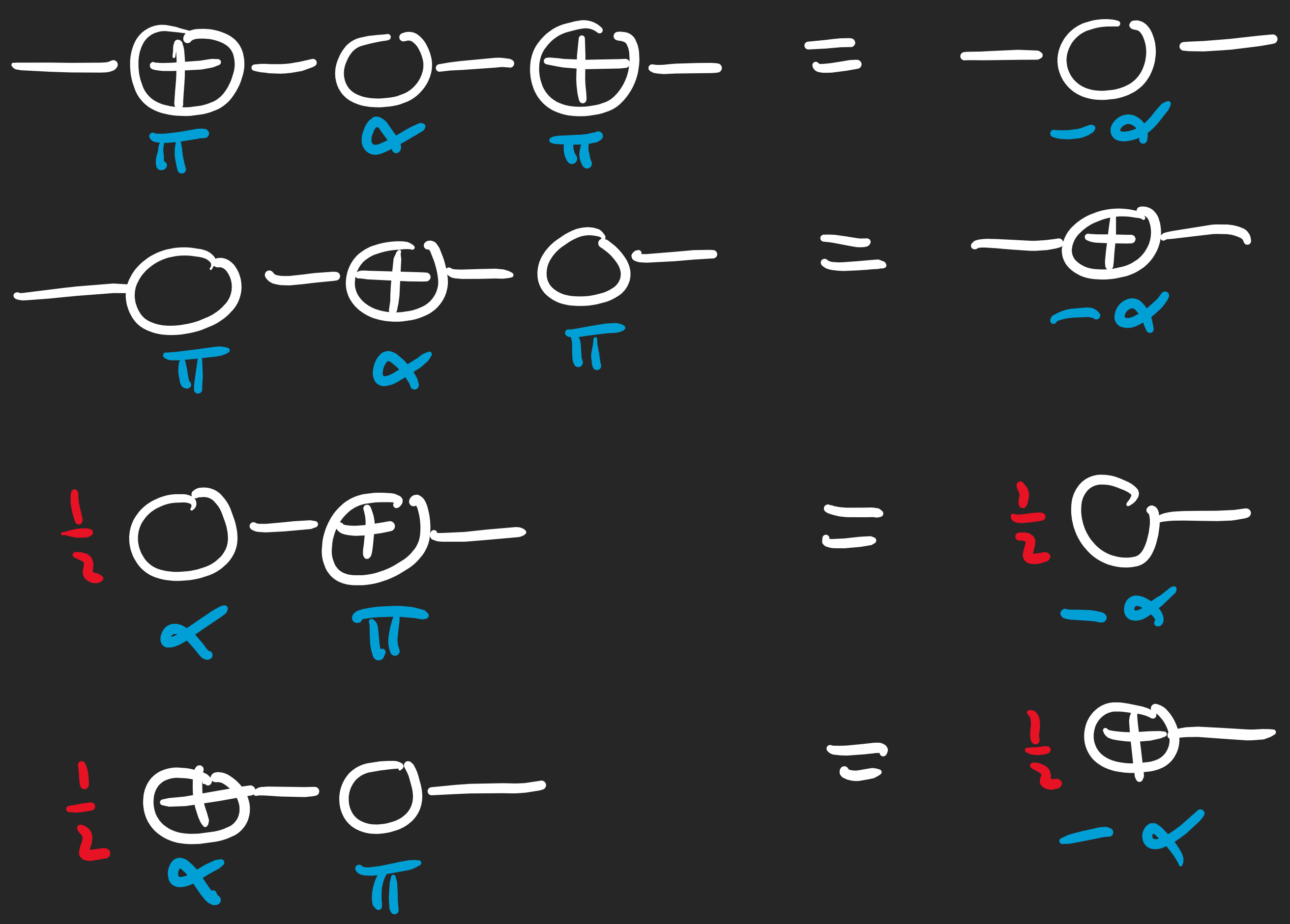


Effect of π Rot's





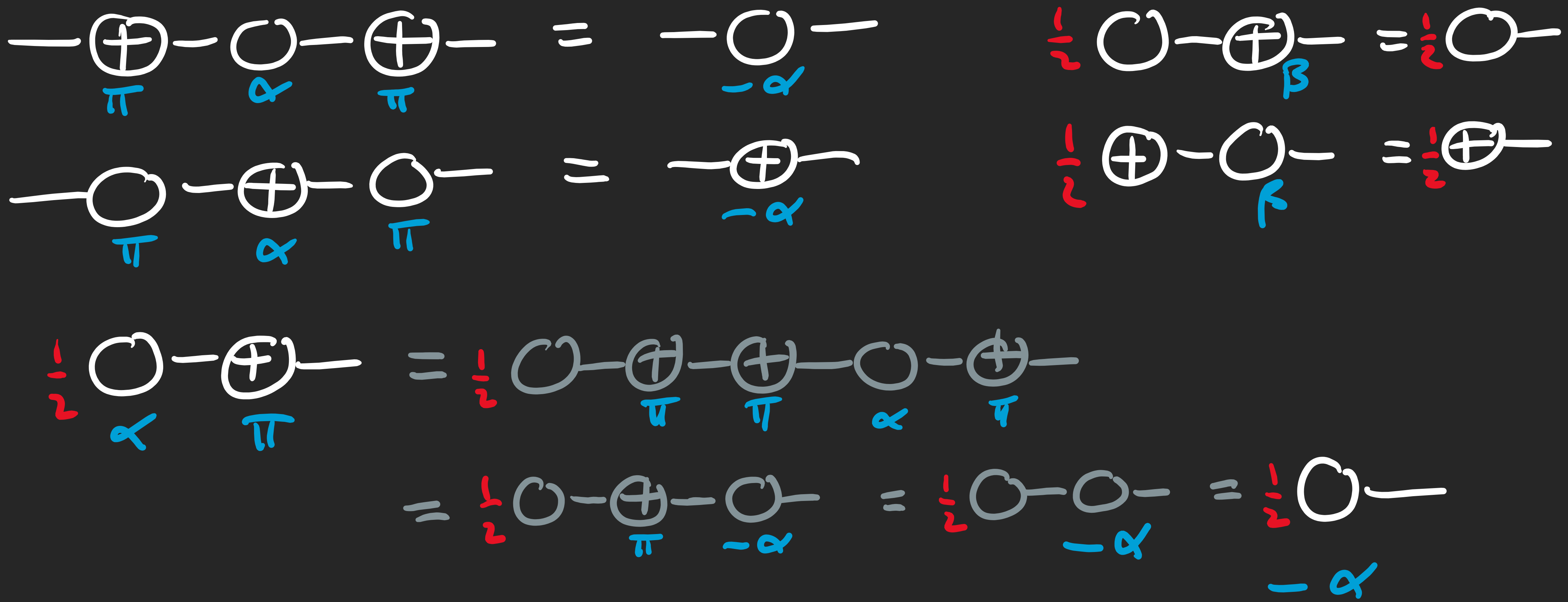
Effect of π Rotations



on rotations

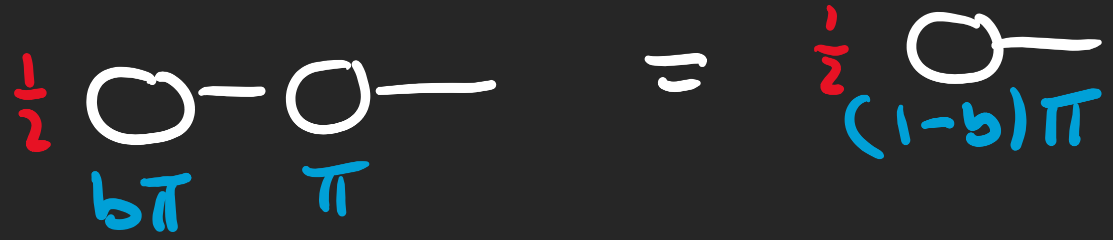
on states

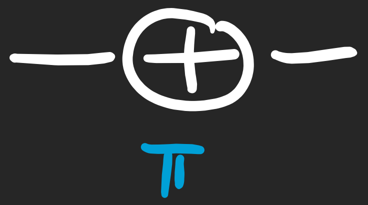
Effect of π Rotations



Effect of π Rotations

for $b \in \{0,1\}$



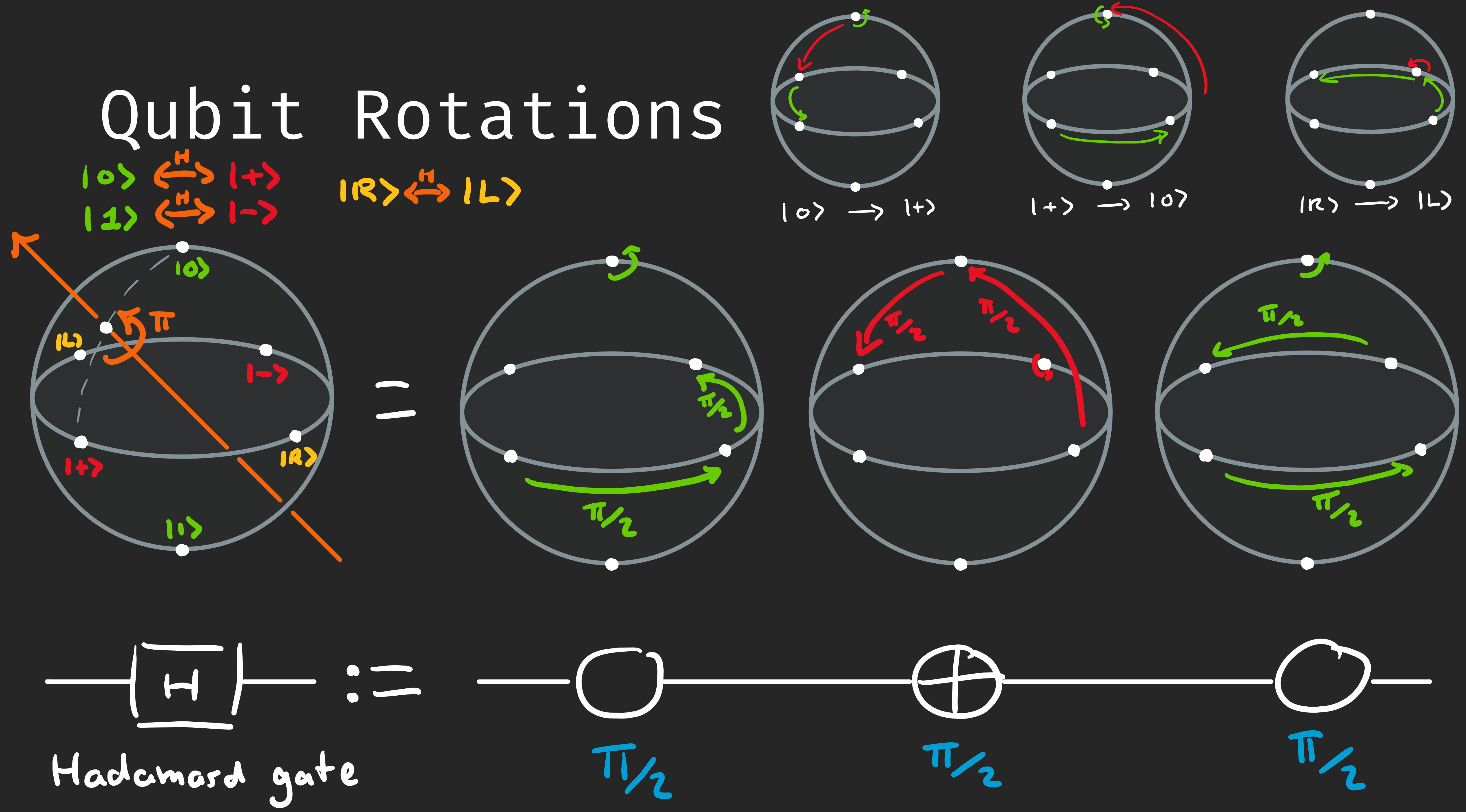


$$\begin{cases} |0\rangle \mapsto |1\rangle \\ |1\rangle \mapsto |0\rangle \end{cases}$$

acts as boolean negation on Z basis

$$\begin{cases} |+\rangle \mapsto |-\rangle \\ |-\rangle \mapsto |+\rangle \end{cases}$$

acts as the flipping of sign on X basis


Qubit Rotations
|0⟩ ⟷H |+⟩
|1⟩ ⟷H |−⟩
|R⟩ ⟷H |L⟩
|0⟩ → |+⟩
|+⟩ → |0⟩
|R⟩ → |L⟩
|0⟩
|L⟩
π
|−⟩
|+⟩
|R⟩
|1⟩
=
π/2
π/2
π/2
π/2
π/2
π/2
H
:=
Hadamard gate
π/2
π/2
π/2

Qubit Rotations

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

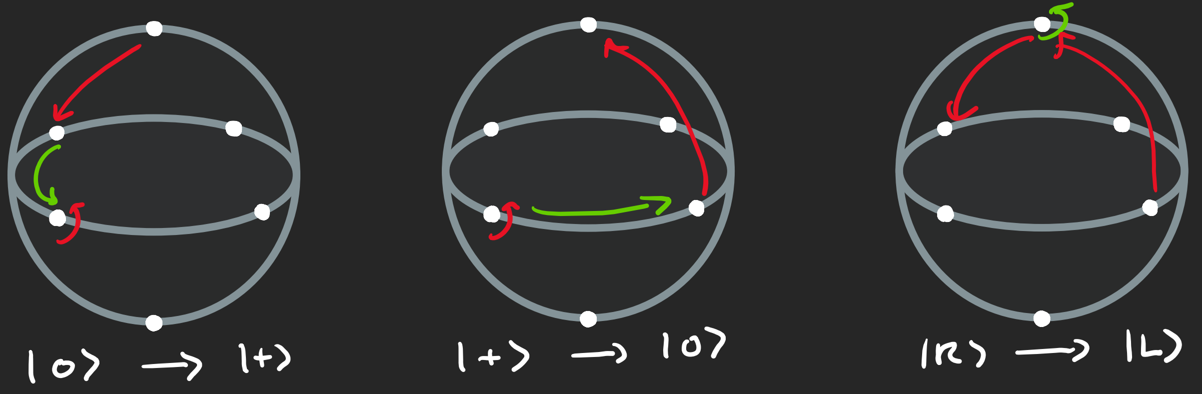
Hadamard gate

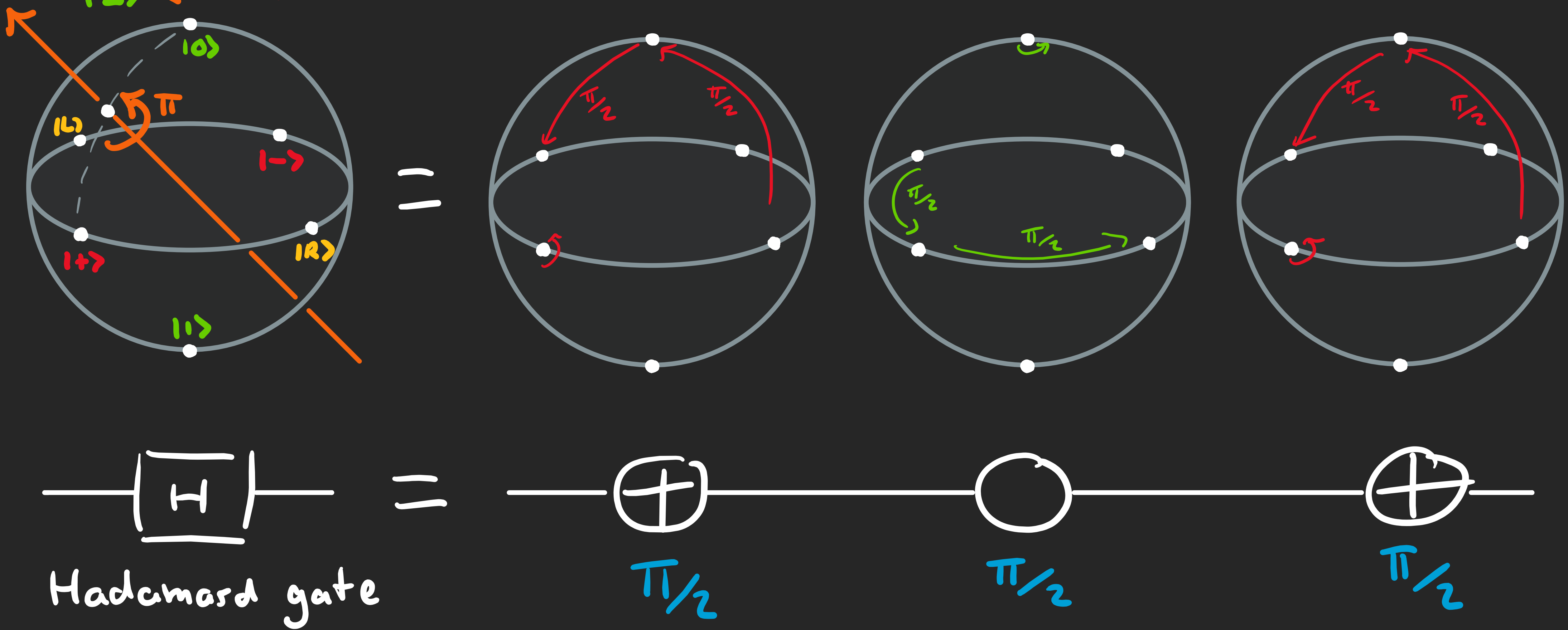
Qubit Rotations

$|0\rangle \overset{H}{\longleftrightarrow} |+\rangle$

$|1\rangle \overset{H}{\longleftrightarrow} |-\rangle$

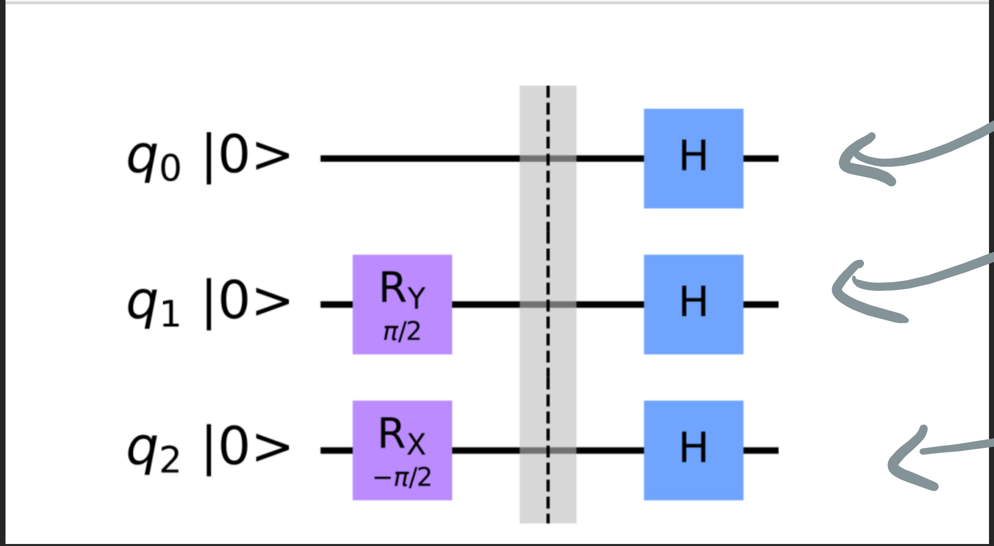
$|R\rangle \overset{H}{\longleftrightarrow} |L\rangle$





Visualisation in Qiskit

```python
from qiskit import QuantumCircuit
circ = QuantumCircuit(3)
# do nothing, q0 remains |0>
circ.ry(+pi/2, 1)# q1 in |+>
circ.rx(-pi/2, 2)# q2 in |R>
circ.barrier() # separate init states from gate
circ.h(range(3)) # apply H to q0, q1 and q2
circ.draw("mpl", initial_state=True)
```



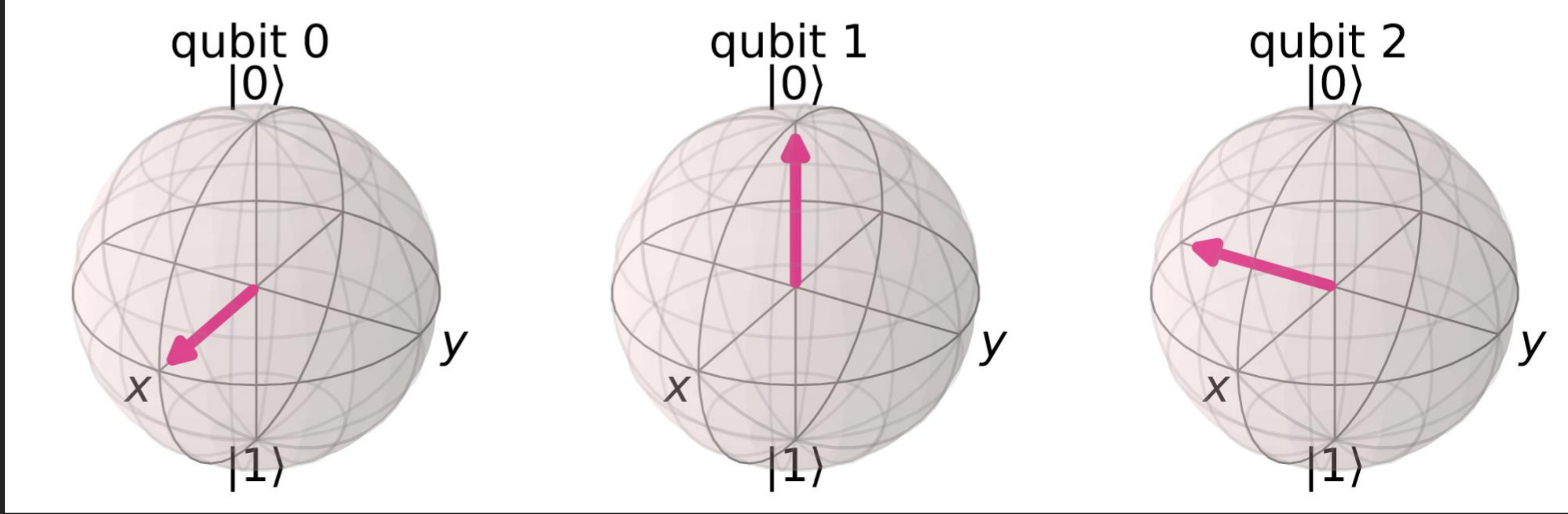
$H|0\rangle$

$H|+\rangle$

$H|R\rangle$

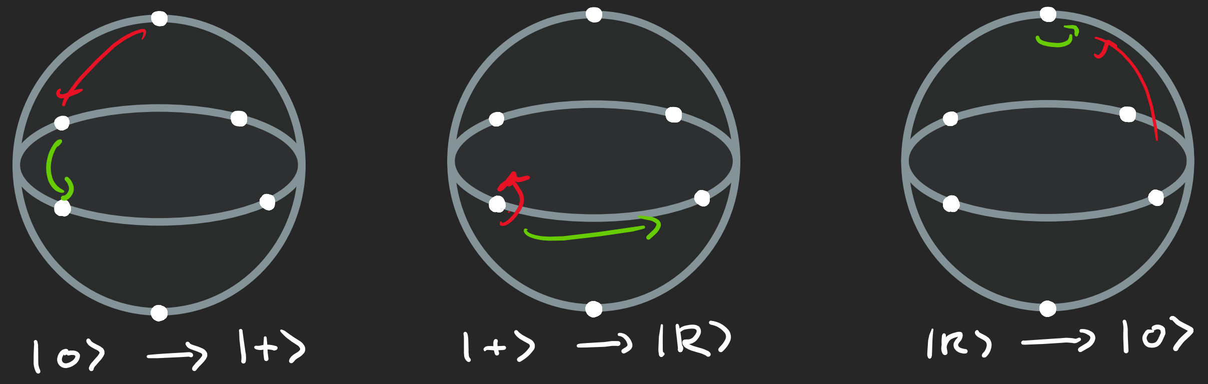
Visualisation in Qiskit

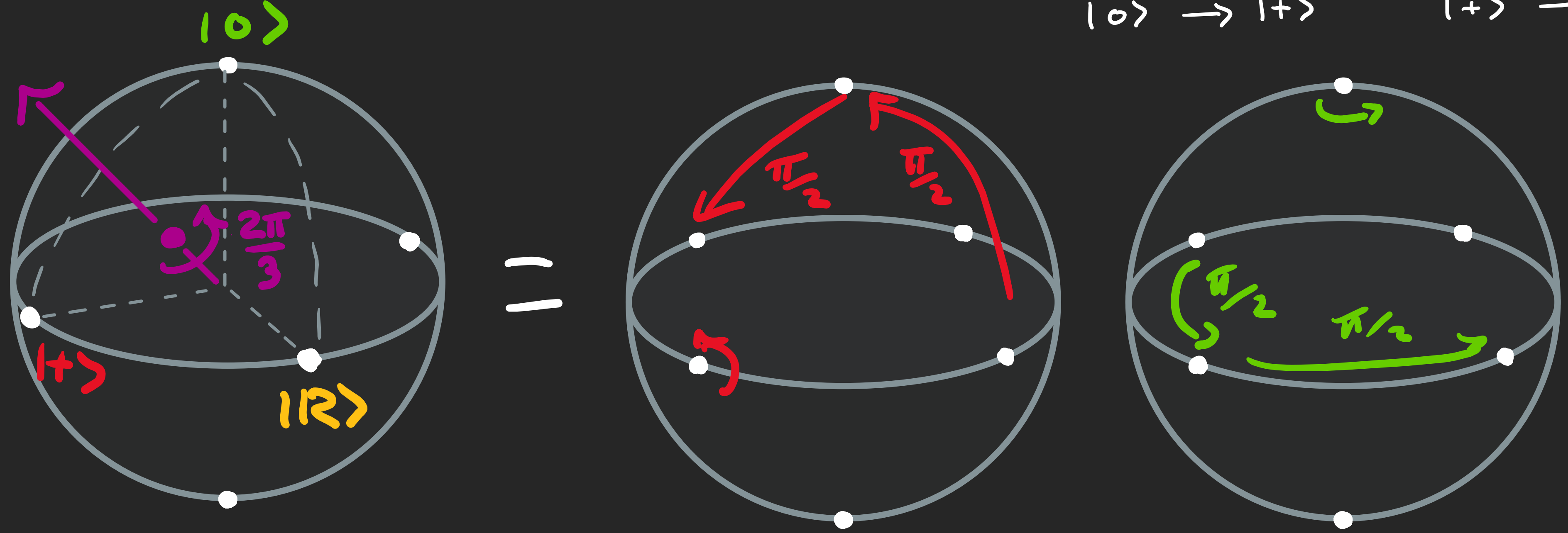
```
from qiskit.visualization import plot_bloch_multivector
circ = QuantumCircuit(3)
# do nothing, q0 remains |0>
circ.ry(+pi/2, 1)# q1 in |+>
circ.rx(-pi/2, 2)# q2 in |R>
circ.barrier() # separate init states from gate
circ.h(range(3)) # apply H to q0, q1 and q2
plot_bloch_multivector(circ)
```

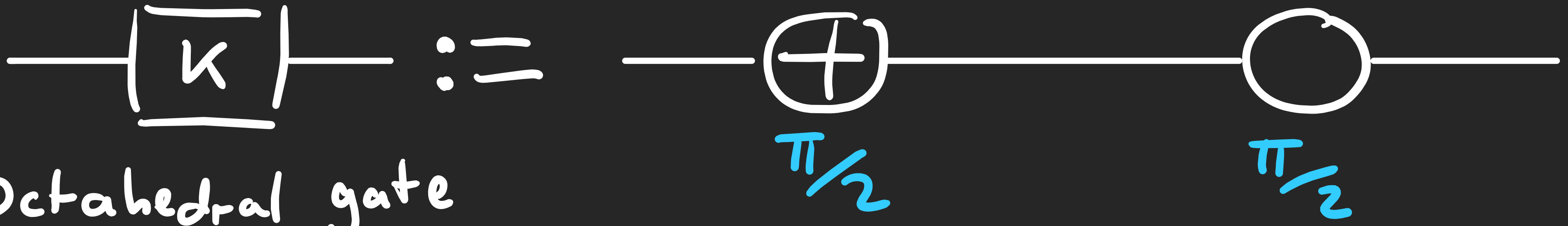


$H|0\rangle = |+\rangle$ $\quad$ $H|+\rangle = |0\rangle$ $\quad$ $H|R\rangle = |L\rangle$

Qubit Rotations







Octahedral gate

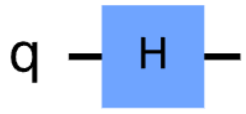
Qubit Rotations

$$K = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

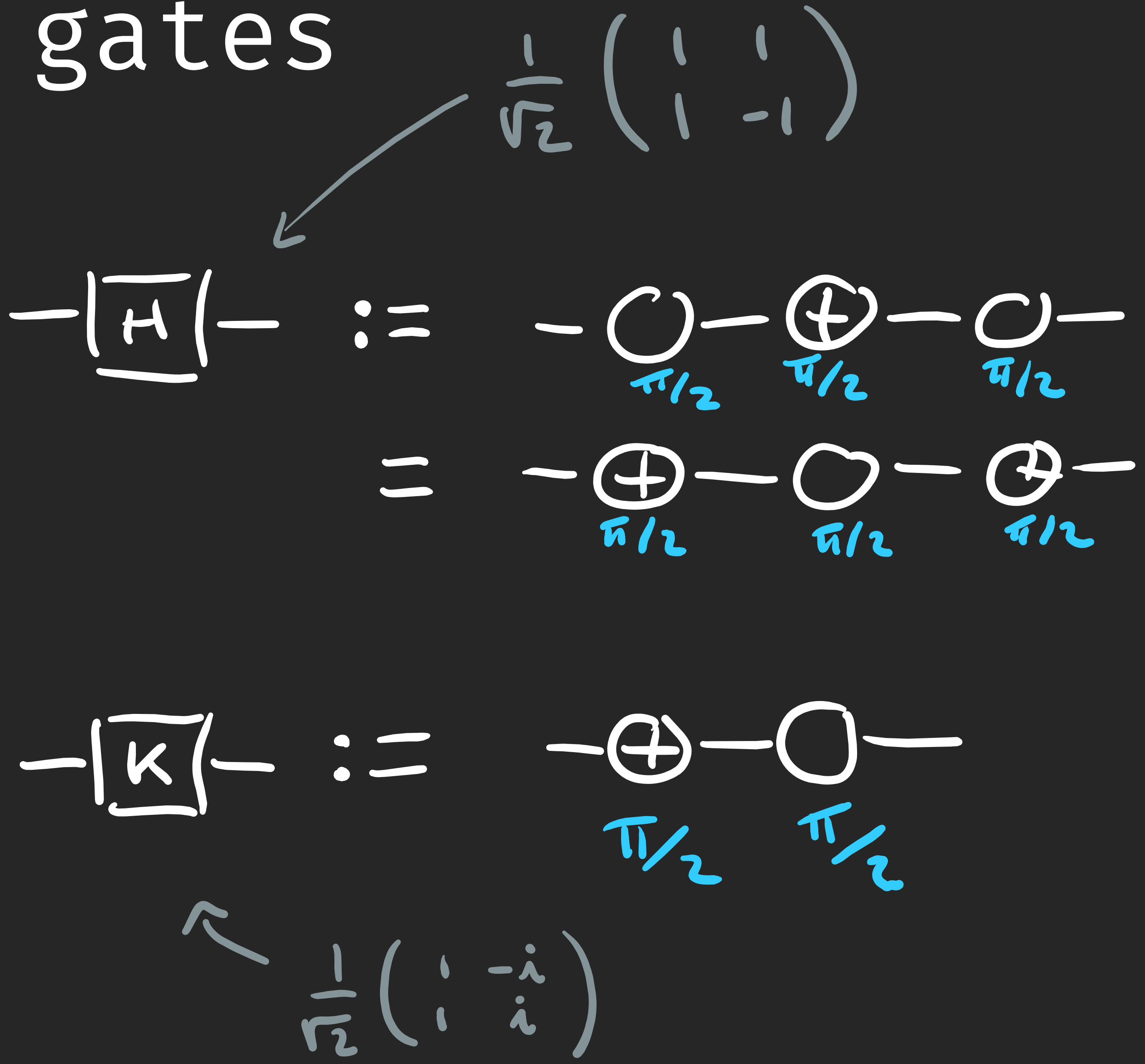
Octahedral gate

The H and K gates

```python
circ = QuantumCircuit(1)
circ.h(0)
circ.draw("mpl")
```



The K gate is currently not available in Qiskit.



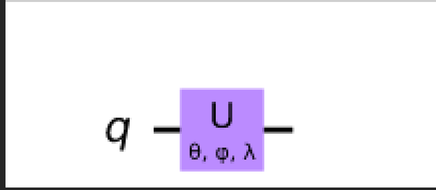
Qubit Rotations = Rotations of the Bloch sphere

$$U(\theta, \varphi, \lambda) := R_z(\varphi) R_y(\theta) R_z(\lambda)$$

"U gate"

$$= e^{-\frac{\varphi+\lambda}{2}} \begin{pmatrix} \cos\frac{\theta}{2} & -e^{i\lambda}\sin\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} & e^{i(\varphi+\lambda)}\cos\frac{\theta}{2} \end{pmatrix}$$

```
theta = Parameter("θ")
phi = Parameter("φ")
lam = Parameter("λ")
circ = QuantumCircuit(1)
circ.u(theta, phi, lam, 0)
circ.draw("mpl")
```



Qubit Rotations = Rotations of the Bloch sphere

$$U(\vartheta, \varphi, \lambda) := R_z(\varphi) R_y(\vartheta) R_z(\lambda)$$

Generic unitary matrix:

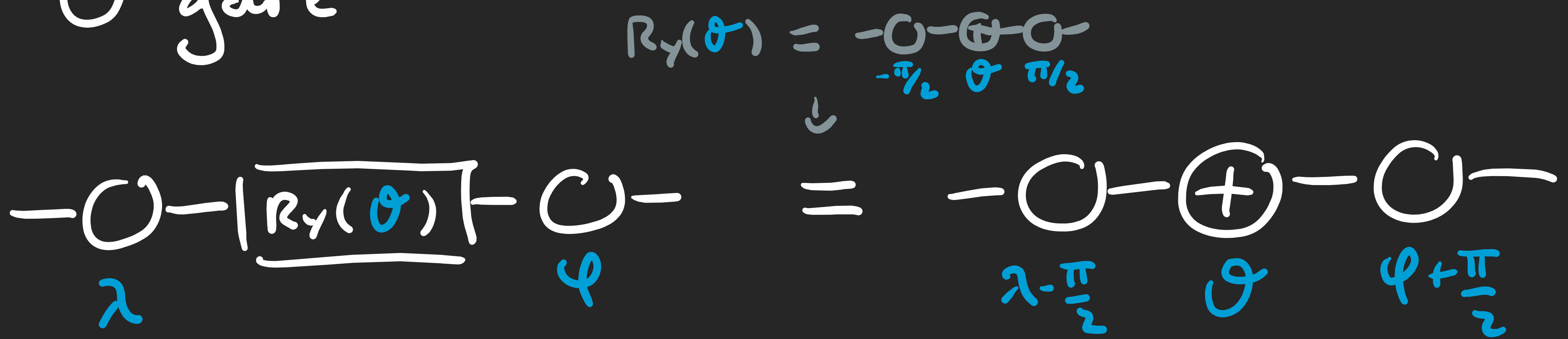
$$\cos\vartheta = 2\cos^2\frac{\vartheta}{2} - 1$$

$$\begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} \Rightarrow \begin{cases} \vartheta = \cos^{-1}(2U_{11}U_{22} - 1) \\ \lambda + \varphi = \arg(U_{12}/U_{21}) \\ \lambda + \varphi = \arg(U_{22}/U_{11}) \end{cases}$$

Qubit Rotations = Rotations of the Bloch sphere

$$U(\vartheta, \varphi, \lambda) := R_z(\varphi)\, R_y(\vartheta)\, R_z(\lambda)$$

"U gate"

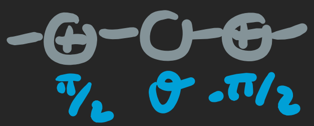


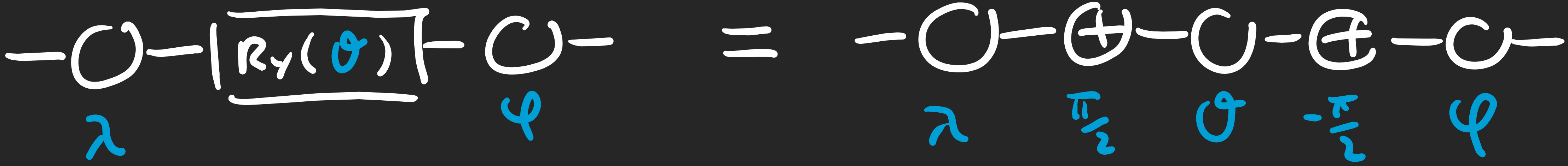
Qubit Rotations = Rotations of the Bloch sphere

$$U(\theta, \varphi, \lambda) := R_z(\varphi)\, R_y(\theta)\, R_z(\lambda)$$

"U gate"

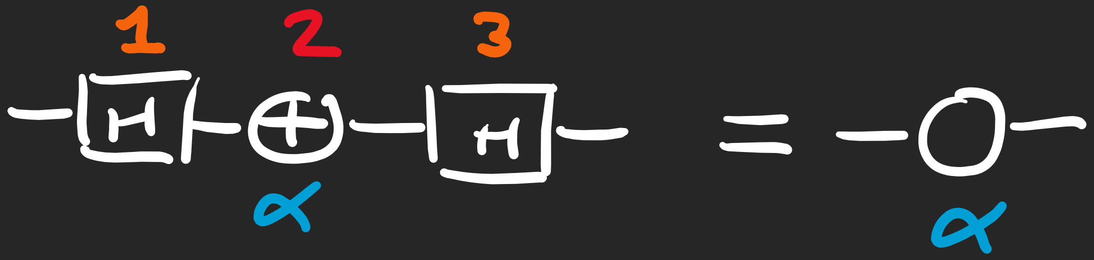
$R_y(\theta) =$

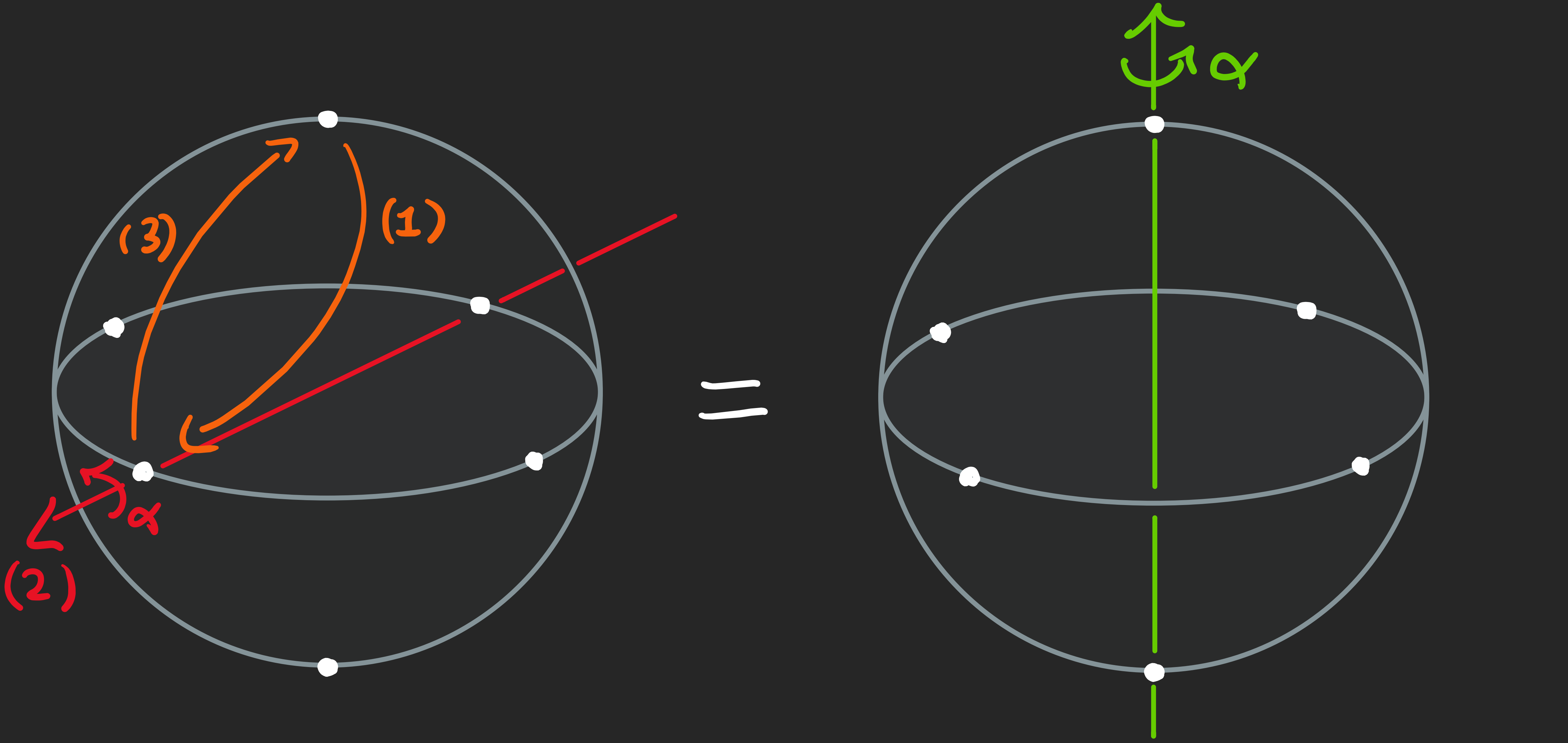




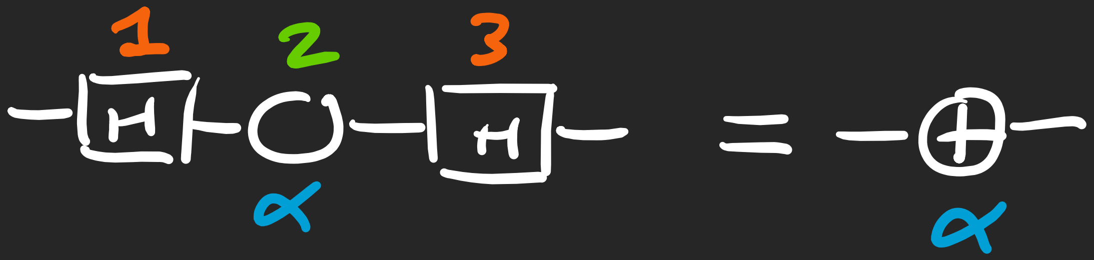
(rotations native to IBM Q)

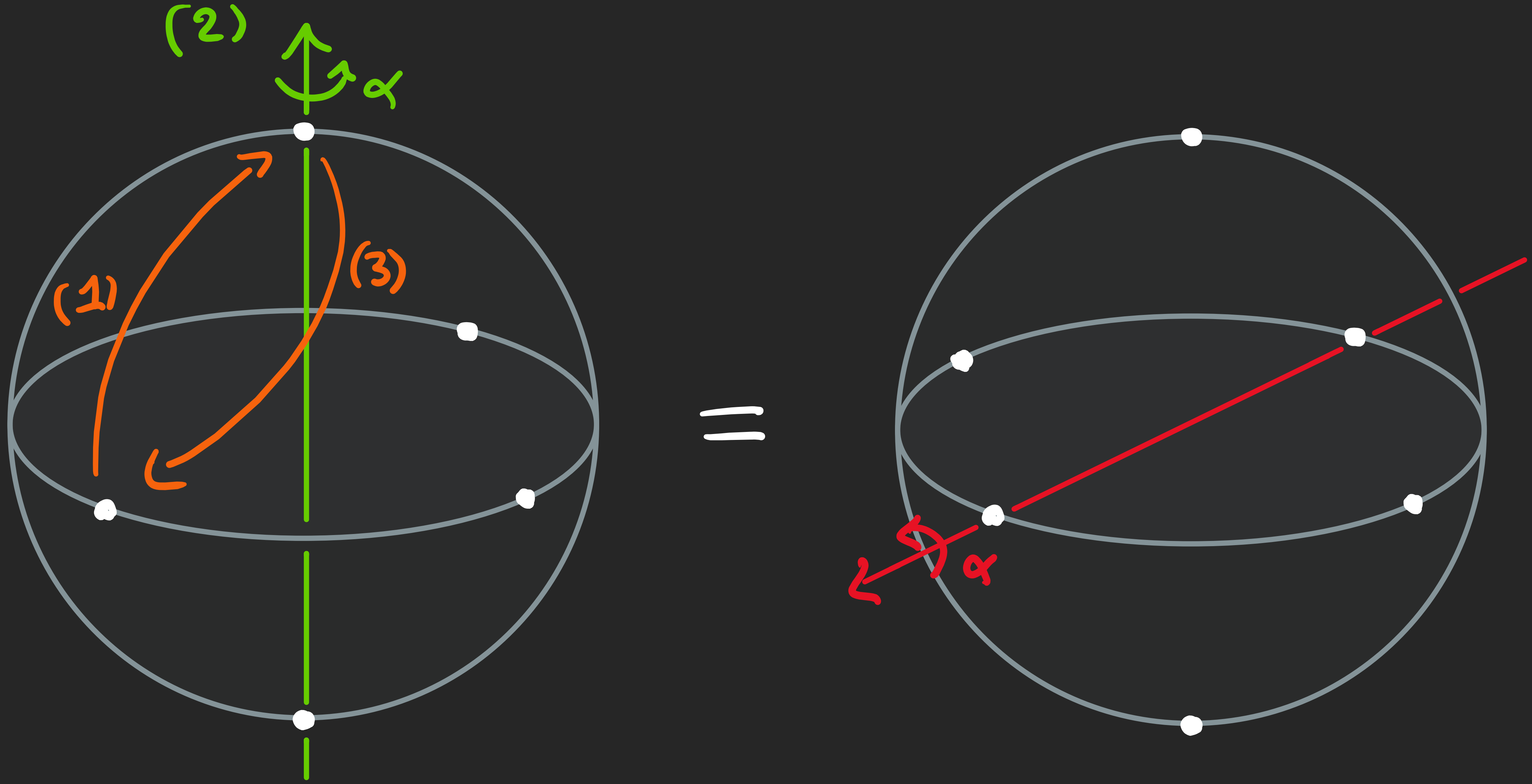
Change of basis



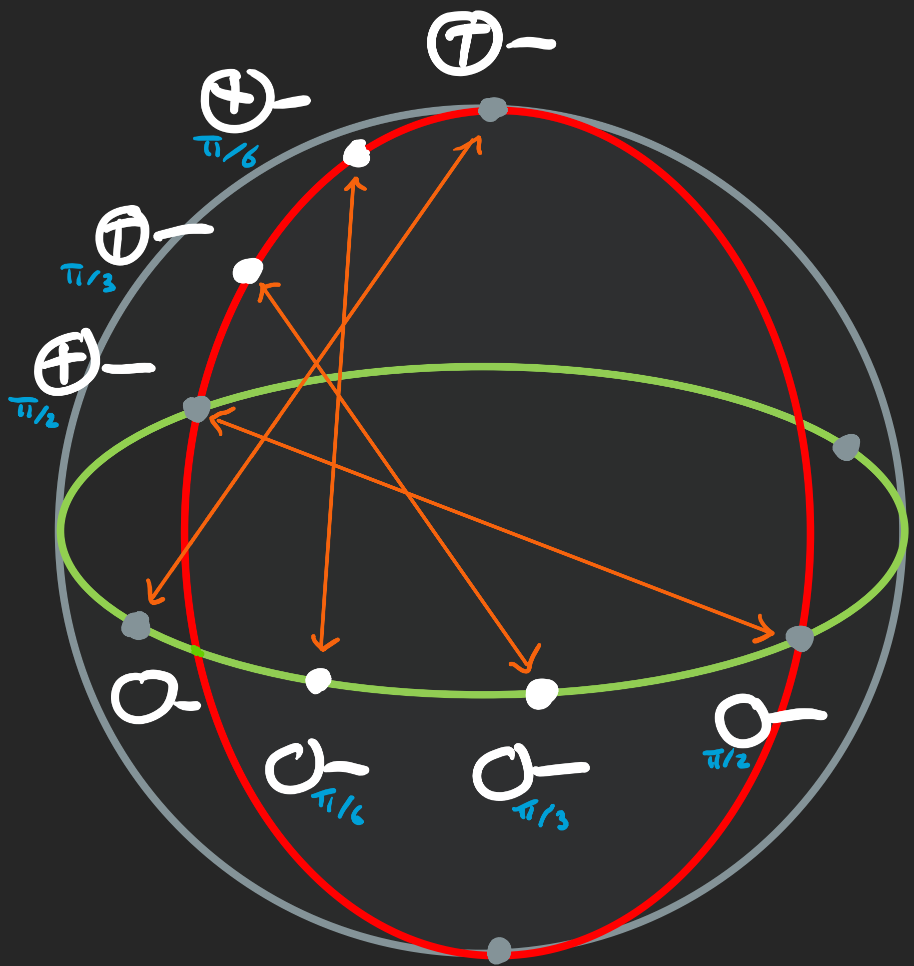


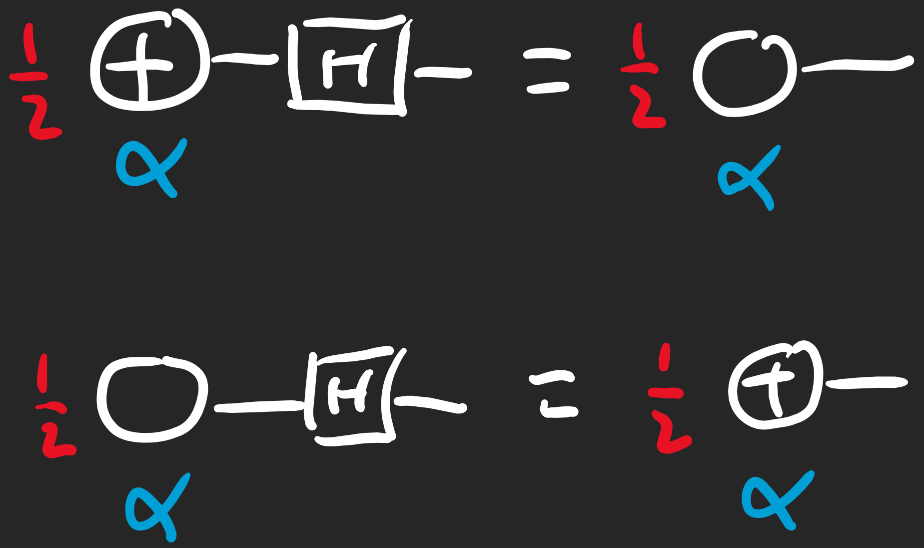
Change of basis



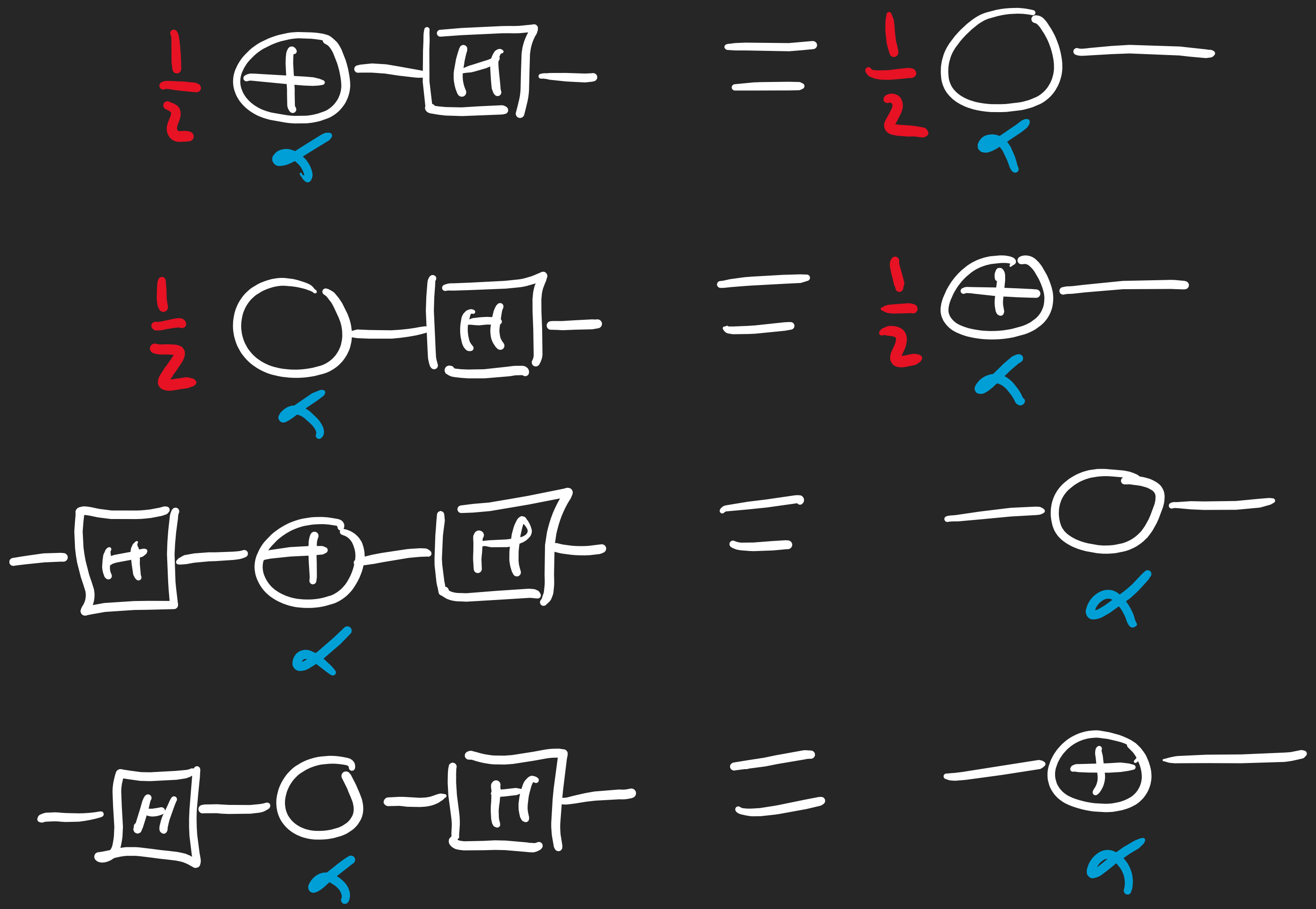


Change of basis

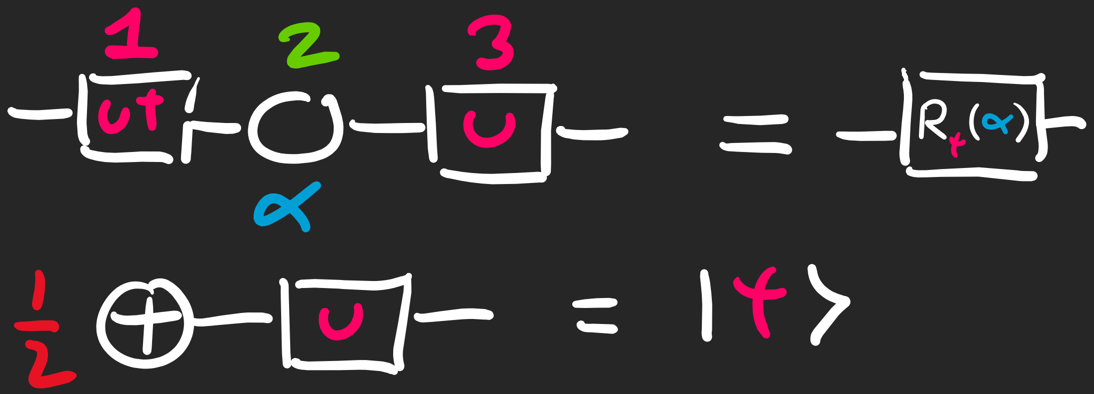


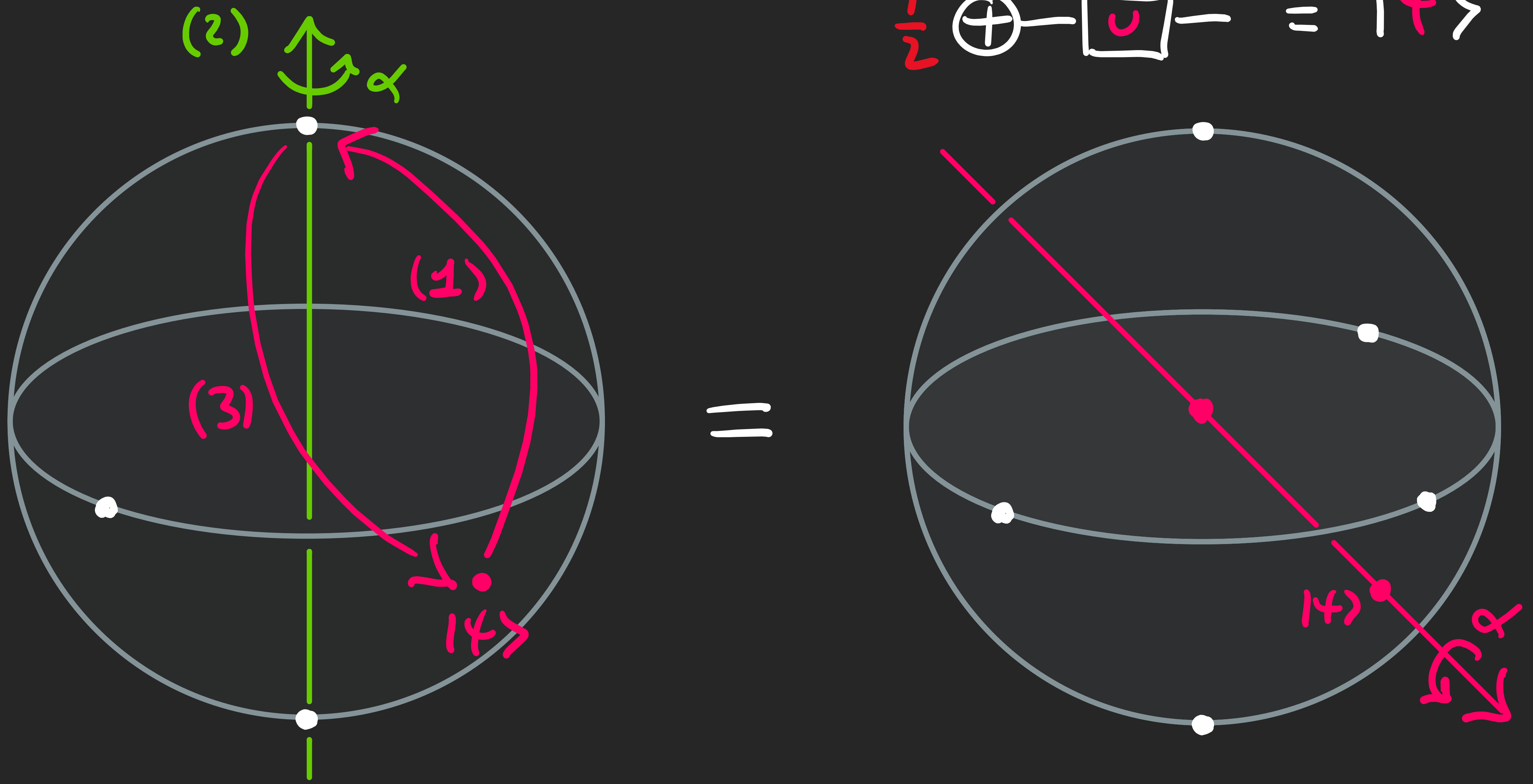


Change of basis

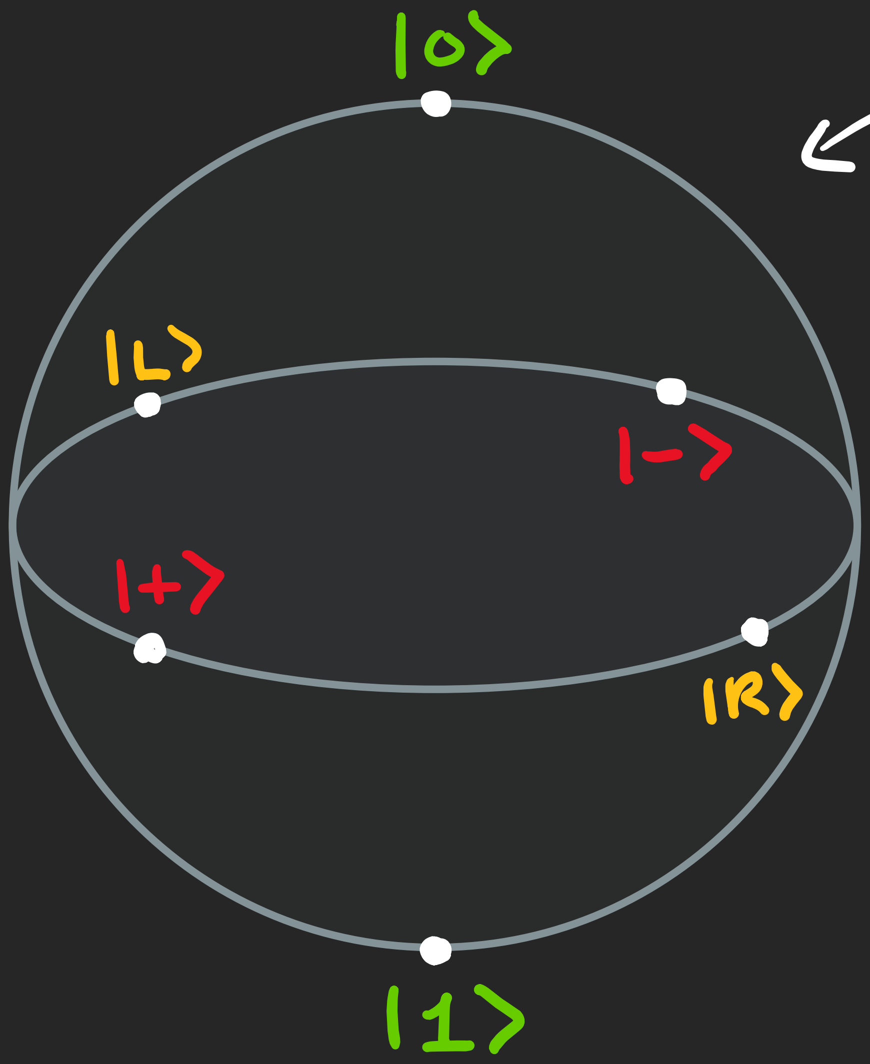


Change of basis





Clifford Rotations



The 6 stabilizer states

Rotations by $\frac{\pi}{2}$ about X, Y, Z permute the stabilizer states, as do all of their combinations:

Clifford Rotations

Clifford Rotations

Qubit rotations are characterised by where they send any two non-antipodal points of the sphere, e.g. $|0\rangle$ and $|+\rangle$.

This gives an easy way to count the Clifford rotations

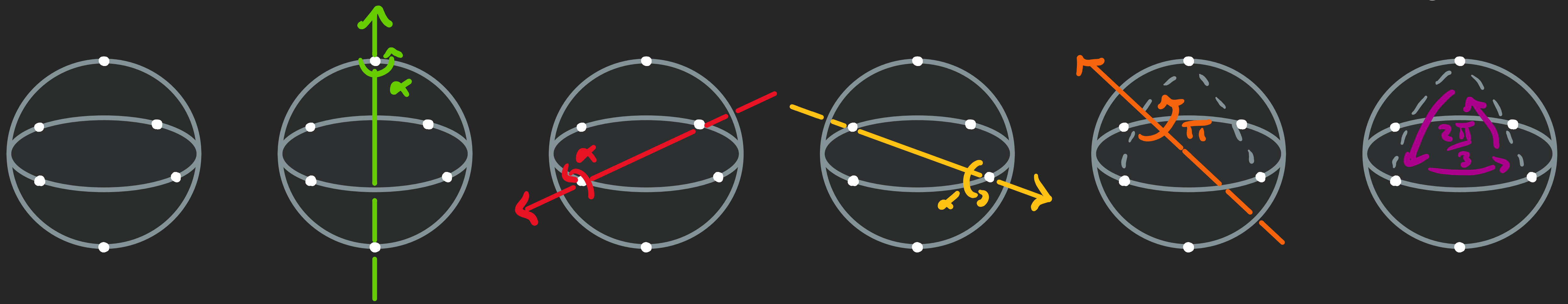
1. Send $|0\rangle$ to any one of the 6 stabiliser states.
2. Then $|1\rangle$ is automatically sent to the antipodal point (because rotations are "rigid").
3. Send $|+\rangle$ to any one of the 4 remaining stabiliser states.

There are thus $6 \cdot 4 = 24$ Clifford rotations.

Clifford Rotations

There are 24 Clifford Rotations in total:

1x	3x	3x	3x	6x	8x



Identity	Z rotations	X rotations	Y rotations	Hadamard	Octahedral
	$\frac{\pi}{2}, \pi, -\frac{\pi}{2}$	$\frac{\pi}{2}, \pi, -\frac{\pi}{2}$	$\frac{\pi}{2}, \pi, -\frac{\pi}{2}$	(+5 similar)	(+7 similar)