Feynman Rules

January 17, 2018

[1] Label each line with a momentum. If applicable, also label each line with an incoming and an outgoing Lorentz index (for a line describing a vector field), with an incoming and an outgoing internal index (for a line describing a field transforming under an internal symmetry), so on and so forth. Momentum is conserved at each vertex. Momenta associated with internal lines are to be integrated over with the measure $\int[\mathrm{d}^4p/(2\pi)^4]$. A factor of (-1) is to be associated with each closed fermion loop. External lines are to be amputated. For an incoming fermion line write $u(p,s)$ and for an outgoing fermion line $\overline{u}(p',s')$. For an incoming antifermion, write $\overline{v}(p,s)$, and for an outgoing antifermion, $v(p',s')$. If there are symmetry transformations leaving the diagram invariant, then we have to worry about the infamous symmetry factors.

1 Scalar field interacting with Dirac field

2 Vector field interacting with Dirac field

3 Nonabelian gauge theory

4 Cross sections and decay rates

Given the Feynman amplitude $\mathcal{M}$ for a process $p_1 + p_2 \to k_1 + k_2 + \cdots + k_n$, the differential cross section is given by

$$\mathrm{d}\sigma = \frac{1}{|\boldsymbol{v}_1 - \boldsymbol{v}_2|\mathcal{E}(p_1)\mathcal{E}(p_2)} \frac{\mathrm{d}^3k_1}{(2\pi)^3\mathcal{E}(k_1)} \cdots \frac{\mathrm{d}^3k_n}{(2\pi)^3\mathcal{E}(k_n)} (2\pi)^4\delta^{(4)}\left(p_1 + p_2 - \sum_{i=1}^{n} k_i\right) |\mathcal{M}|^2 \qquad (1)$$

Here $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ denote the velocities of the incoming particles. The energy factor $\mathcal{E}(p) = 2\sqrt{\boldsymbol{p}^2 + m^2}$ for bosons and $\mathcal{E}(p) = \sqrt{\boldsymbol{p}^2 + m^2}/m$ for fermions come from the different normalization of the creation and annihilation operators.

For a decay of a particle of mass M the differential decay rate in its rest frame is given by

$$\mathrm{d}\Gamma = \frac{1}{2M} \frac{\mathrm{d}^3k_1}{(2\pi)^3\mathcal{E}(k_1)} \cdots \frac{\mathrm{d}^3k_n}{(2\pi)^3\mathcal{E}(k_n)} (2\pi)^4\delta^{(4)}\left(P - \sum_{i=1}^{n} k_i\right) |\mathcal{M}|^2 \qquad (2)$$

[2] The scattering cross sections are naturally expressed in terms of time-ordered products of fields. The S-matrix has the form

$$\langle f|S|i\rangle \sim \langle\Omega|T\left\{\phi(x_1)\cdots\phi(x_n)\right\}|\Omega\rangle \ , \qquad (3)$$

where $|\Omega\rangle$ is the ground state/vacuum in the interacting theory. In this expression the fields $\phi(x)$ are not free but are the full interacting quantum fields. We also saw that in the free theory, the time-ordered product of two fields is given by the Feynman propagator:

$$D_F(x,y) \equiv \langle 0|T\left\{\phi_0(x)\phi_0(y)\right\}|0\rangle = \lim \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} e^{ik(x-y)} \ , \qquad (4)$$

where $|0\rangle$ is the ground state in the free theory.

We will develop a method of calculating time-ordered products in perturbation theory in terms of integrals over various Feynman propagators. There is a beautiful pictorial representation of the perturbation expansion using Feynman diagrams and an associated set of Feynman rules. There are position-space Feynman rules, for calculating time-ordered products, and also momentum-space Feynman rules, for calculating S-matrix elements. The momentum-space Feynman rules are by far the more important - they provide an extremely efficient way to set up calculations of physical results in quantum field theory.

We will first derive the Feynman rules using a Lagrangian formulation of time evolution and quantization. This is the quickest way to connect Feynman diagrams to classical field theory. We will then derive the Feynman rules again using time-dependent perturbation theory, based on an expansion of the full interacting Hamiltonian around the free Hamiltonian. This calculation much more closely parallels the way we do perturbation theory in quantum mechanics. While the Hamiltonian-based calculation is significantly more involved, it has the distinct advantage of connecting time evolution directly to a Hermitian Hamiltonian, so time evolution is guaranteed to be unitary. The Feynman rules resulting from both approaches agree, confirming that the approaches are equivalent (at least in the case of the theory of a real scalar field, which is all we have so seen so far). As we progress in our understanding of field theory and encounter particles of different spin and more complicated interactions, unitarity and the requirement of a Hermitian Hamiltonian will play a more important role. A third independent way to derive the Feynman rules is through the path integral.

5 Hamiltonian derivation

[2] We reproduce the position-space Feynman rules using time-dependent perturbation theory. Instead of assuming that the quantum field satisfies the Euler-Lagrange equations, we instead assume its dynamics is determined by a Hamiltonian H by the Heisenberg equations of motion $i\partial_t\phi(x) = [\phi, H]$. The formal solution of this equation is

$$\phi(\boldsymbol{x}, t) = S(t, t_0)^\dagger \phi(\boldsymbol{x}) S(t, t_0) \ , \tag{5}$$

where $S(t, t_0)$ is the time-evolution operator (the S-matrix) that satisfies

$$i\partial_t S(t, t_0) = H(t) S(t, t_0) \ . \tag{6}$$

These are the dynamical equations in the Heisenberg picture where all the time dependence is in operators. States including the vacuum state $|\Omega\rangle$ in the Heisenberg picture are, by definition, time independent.

The Hamiltonian can either be defined at any given time as a functional of the fields $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ or equivalently as a functional of the creation and annihilation operators $a_p^\dagger$ and a_p. We will not need an explicit form of the Hamiltonian for this derivation so we just assume it is some time-dependent operator $H(t)$.

The first step in time-dependent perturbation theory is to write the Hamiltonian as

$$H(t) = H_0 + V(t) \ , \tag{7}$$

where the time evolution induced by H_0 can be solved exactly and V is small in some sense. For example, H_0 could be the free Hamiltonian, which is time independent, and V might be a ϕ^3 interaction:

$$V(t) = \int \mathrm{d}^3x \frac{g}{3!} \phi(\boldsymbol{x}, t)^3 \ . \tag{8}$$

The operators $\phi(\boldsymbol{x}, t), H, H_0$ and V are all in the Heisenberg picture.

Next, we need to change to the interaction picture. In the interaction picture the fields evolve only with H_0. The interaction picture fields are just what we had been calling (and will continue to call) the free fields:

$$\phi_0(\boldsymbol{x},t) = e^{iH_0(t-t_0)}\phi(\boldsymbol{x})e^{-iH_0(t-t_0)} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p e^{-ipx} + a_p^\dagger e^{ipx}) \ . \tag{9}$$

To be precise, $\phi(\boldsymbol{x})$ is the Schrödinger picture field, which does not change with time. The free fields are equal to the Schrödinger picture fields and also to the Heisenberg picture fields, by definition, at a single reference time, which we call t_0.

The Heisenberg picture fields are related to the free fields by

$$\begin{aligned}\phi(\boldsymbol{x},t) &= S^\dagger(t,t_0)e^{-iH_0(t-t_0)}\phi_0(\boldsymbol{x},t)e^{iH_0(t-t_0)}S(t,t_0) \\ &= U^\dagger(t,t_0)\phi_0(\boldsymbol{x},t)U(t,t_0) \ .\end{aligned} \tag{10}$$

The operator $U(t,t_0) \equiv e^{iH_0(t-t_0)}S(t,t_0)$ therefore relates the full Heisenberg picture fields to the free fields at the same time t. The evolution begins from the time t_0 where the fields in the two pictures (and the Schrödinger picture) are equal.

a differential equation for $U(t,t_0)$ is

$$\begin{aligned}i\partial_t U(t,t_0) &= i\left(\partial_t e^{iH_0(t-t_0)}\right)S(t,t_0) + e^{iH_0(t-t_0)}i\partial_t S(t,t_0) \\ &= -e^{iH_0(t-t_0)}H_0 S(t,t_0) + e^{iH_0(t-t_0)}H(t)S(t,t_0) \\ &= e^{iH_0(t-t_0)}[-H_0 + H(t)]e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}S(t,t_0) \\ &= V_I(t)U(t,t_0) \ ,\end{aligned} \tag{11}$$

where $V_I(t) \equiv e^{iH_0(t-t_0)}V(t)e^{-iH_0(t-t_0)}$ is the original Heisenberg picture potential $V(t)$, now expressed in the interaction picture.

If everything commuted, the solution to Eq. (11) would be $U(t,t_0) = \exp\left(-i\int_{t_0}^t V_I(t')\mathrm{d}t'\right)$. But $V_I(t_1)$ does not necessarily commute with $V_I(t_2)$, so this is not the right answer. It turns out

that the right answer is very similar:

$$U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} \mathrm{d}t' V_I(t')\right]\right\} \ , \tag{12}$$

where $T\{\}$ is the time-ordering operator. This solution works because time ordering effectively makes everything inside commute:

$$T\{A\cdots B\cdots\} = T\{B\cdots A\cdots\} \ . \tag{13}$$

Since it has the right boundary conditions, namely $U(t,t) = 1$, this solution is unique.

Time ordering of an exponential is defined in the obvious way through its expansion:

$$U(t,t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' V_I(t') - \frac{1}{2}\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t}\mathrm{d}t'' T\left\{V_I(t')V_I(t'')\right\} + \cdots \tag{14}$$

This is known as a Dyson series. Dyson defined the time-ordered product and this series in his classic paper. In that paper he showed the equivalence of old-fashioned perturbation theory or, more exactly, the interaction picture method developed by Schwinger and Tomonaga based on time-dependent perturbation theory, and Feynman's method, involving space-time diagrams, which we are about to get to.

5.0.1 Perturbative solution for the Dyson series

Removing the subscript on V for simplicity, the differential equation we want to solve is

$$i\partial_t U(t,t_0) = V(t)U(t,t_0) \ . \tag{15}$$

Integrating this equation lets us write it in an equivalent form:

$$U(t,t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' V(t')U(t',t_0) \ , \tag{16}$$

where 1 is the appropriate integration constant so that $U(t_0,t_0) = 1$.

Solve the integral equation order-by-order in V. At zeroth order in V,

$$U(t,t_0) = 1 \ . \tag{17}$$

To first order in V,

$$U(t,t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' V(t') + \cdots \ . \tag{18}$$

To second order,

$$\begin{aligned} U(t,t_0) &= 1 - i\int_{t_0}^{t} \mathrm{d}t' V(t') \left[1 - i\int_{t_0}^{t'} \mathrm{d}t'' V(t'') + \cdots\right] \\ &= 1 - i\int_{t_0}^{t} \mathrm{d}t' V(t') + (-i)^2 \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' V(t')V(t'') + \cdots \ . \end{aligned} \tag{19}$$

The second integral has $t_0 < t'' < t' < t$, which is the same as $t_0 < t'' < t$ and $t'' < t' < t$. So it can also be written as

$$\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' V(t')V(t'') = \int_{t_0}^{t} \mathrm{d}t'' \int_{t''}^{t} \mathrm{d}t' V(t')V(t'') = \int_{t'}^{t} \mathrm{d}t'' \int_{t_0}^{t} \mathrm{d}t' V(t'')V(t') \tag{20}$$

where we have relabeled $t'' \leftrightarrow t'$ and swapped the order of the integrals to form. Averaging the first and third form gives

$$\begin{aligned} \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' V(t')V(t'') &= \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t' \left[\int_{t_0}^{t'} \mathrm{d}t'' V(t')V(t'') + \int_{t'}^{t} \mathrm{d}t'' V(t'')V(t')\right] \\ &= \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t} \mathrm{d}t'' T\left\{V(t')V(t'')\right\} \ . \end{aligned} \tag{21}$$

Thus

$$U(t,t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' V(t') + \frac{(-i)^2}{2}\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t} \mathrm{d}t'' T\left\{V(t')V(t'')\right\} + \cdots \ . \tag{22}$$

Continuing this way, we find, restoring the subscript on V, that

$$U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} \mathrm{d}t' V_I(t')\right]\right\} \ . \tag{23}$$

References

[1] A. Zee. *Quantum Field Theory in a Nutshell: Second Edition.* 2010.

[2] M. D. Schwartz. *Quantum Field Theory and the Standard Model.* March 2014.