The Geometry of Vector Spaces

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1 Null Spaces, Column Spaces, and Linear Transformations

The subspaces of $\mathbb{R}^n$ usually arise in one of two ways: (1) as the set of all solutions to a system of homogeneous linear equations or (2) as the set of all linear combinations of certain specified vectors.

1.1 The Null Space of a Matrix

The set of $\boldsymbol{x}$ that satisfy $A\boldsymbol{x} = \mathbf{0}$ is called the null space of the matrix A.

Definition

The null space of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\boldsymbol{x} = \mathbf{0}$. In set notation,

Nul A $= \{\boldsymbol{x} : \boldsymbol{x} \text{ is in } \mathbb{R}^n \text{ and } A\boldsymbol{x} = \mathbf{0}\}$

A more dynamic description of Nul A is the set of all $\boldsymbol{x}$ in $\mathbb{R}^n$ that are mapped into the zero vector of $\mathbb{R}^m$ via the linear transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$.

Theorem

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\boldsymbol{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

1.2 An Explicit Description of Nul A

There is no obvious relation between vectors in Nul A and the entries in A. Nul A is defined implicitly, because it is defined by a condition that must be checked. No explicit list or description of the elements in Nul A is given. However, solving the equation $A\boldsymbol{x} = \mathbf{0}$ amounts to producing an explicit description of Nul A.

When Nul A contains nonzero vectors, the number of vectors in the spanning set for Nul A equals the number of free variables in the equation $A\boldsymbol{x} = \mathbf{0}$.

1.3 The Column Space of a Matrix

Definition

The column space of an $m \times n$ matrix A, written as Col A, is the set of all linear combinations of the columns of A. If $A = [\boldsymbol{a}_1 \cdots \boldsymbol{a}_n]$, then

Col $A =$ Span $\{\boldsymbol{a}_1 \cdots \boldsymbol{a}_n\}$

Theorem

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

1.4 The Contrast Between Nul A and Col A

1.5 Kernel and Range of a Linear Transformation

Definition

A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\boldsymbol{x}$ in V a unique vector $T(\boldsymbol{x})$ in W, such that

(i) $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} in \boldsymbol{V}$, and

(ii) $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$ for all $\boldsymbol{u}$ in V and all scalars c.

The kernel (or null space) of such a T is the set of all $\boldsymbol{u}$ in V such that $T(u) = 0$ (the zero vector in W). The range of T is the set of all vectors in W of the form $T(x)$ for some $\boldsymbol{x}$ in V. If T happens to arise as a matrix transformation-say, $T(\boldsymbol{x}) = A\boldsymbol{x}$ for some matrix A-then the kernel and the range of T are just the null space and the column space of A. The kernel of T is a subspace of V.

2 Linearly Independent Sets; Bases

3 Coordinate Systems

The Unique Representation Theorem

Let $\mathcal{B} = \{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ be a basis for a vector space V. Then for each $\boldsymbol{x}$ in V, there exists a unique set of scalars $c_1, \cdots, c_n$ such that

$$\boldsymbol{x} = c_1\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_n \ . \tag{1}$$

Definition

Suppose $\mathcal{B} = \{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ is a basis for V and $\boldsymbol{x}$ is in V. The coordinates of $\boldsymbol{x}$ relative to the basis $\mathcal{B}$ (or the $\mathcal{B}$-coordinates of $\boldsymbol{x}$) are the weights $c_1, \cdots, c_n$ such that $\boldsymbol{x} = c_1\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_n$.

3.1 A Graphical Interpretation of Coordinates

4 The Dimension of a Vector Space

5 Rank

6 Change of Basis

7 Applications to Difference Equations

8 Applications to Markov Chains