TENSOR AND DIFFERENTIAL FORMS

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An isotropic distribution of mass would be a spherical distribution. No matter what direction one looks, the distribution looks the same. We had been discussed in the Friedmann's cosmological model of the universe. A simpler model is a spherical ball. However, an ellipsoidal distribution is slightly non-isotropic. In general, the distribution of matter can be described using the inertia tensor, a 3×3 matrix

$$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

The entries are the moments of inertia about the origin for a continuous distribution of mass:

$$\begin{aligned} I_{xx} &= \int y^2 + z^2 \mathrm{d}m \ , \quad I_{xy} = I_{yx} = -\int xy \mathrm{d}m \ , \\ I_{yy} &= \int x^2 + z^2 \mathrm{d}m \ , \quad I_{yz} = I_{zy} = -\int yz \mathrm{d}m \ , \\ I_{zz} &= \int x^2 + y^2 \mathrm{d}m \ , \quad I_{zx} = I_{xz} = -\int xz \mathrm{d}m \ , \end{aligned}$$

The total angular momentum is $\boldsymbol{L} = I\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is a column vector of the components of angular velocity. The components of the moment of inertia can be written more compactly using index notation as

$$I_{ij} = \int (r^2 \delta_{ij} - x_i x_j) \rho \mathrm{d}V \ .$$

Under a rotation of the coordinate system, the angular momentum equation becomes $\boldsymbol{L}' = I'\boldsymbol{\Omega}'$. The relation to the original quantities is

$$\boldsymbol{L}' = I'\boldsymbol{\Omega}' \ ,$$

$$\hat{R}_\theta \boldsymbol{L} = I'\hat{R}_\theta \boldsymbol{\Omega} \ ,$$

$$\boldsymbol{L} = \hat{R}_\theta^{-1} I'\hat{R}_\theta \boldsymbol{\Omega} \ .$$

The moment of inertia changes under a transformation,

$$I = \hat{R}_\theta^{-1} I'\hat{R}_\theta \ .$$

If the resulting matrix is diagonal, the moment of inertia tensor have been diagonalized,

$$I = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

The diagonal entries are called the principal moments of inertia.

1 Tensor Analysis

1.1 Covariant and Contravariant Tensors

[1] The rotational transformation of a vector $\boldsymbol{A} = A_1\hat{\boldsymbol{e}}_1 + A_2\hat{\boldsymbol{e}}_2 + A_3\hat{\boldsymbol{e}}_3$ from the Cartesian system defined by $\hat{\boldsymbol{e}}_i(i = 1, 2, 3)$ into a rotated coordinate system defined by $\hat{\boldsymbol{e}}'_i$, with the same vector $\boldsymbol{A}$ then represented as $\boldsymbol{A}' = A'_1\hat{\boldsymbol{e}}'_1 + A'_2\hat{\boldsymbol{e}}'_2 + A'_3\hat{\boldsymbol{e}}'_3$. The components of $\boldsymbol{A}$ and $\boldsymbol{A}'$ are related by

$$A'_i = \sum_j (\hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}'_j) A_j \ , \tag{1}$$

where the coefficients $(\hat{\boldsymbol{e}}'_i \cdot \hat{\boldsymbol{e}}'_j)$ are the projections of $\hat{\boldsymbol{e}}'_i$ in the $\hat{\boldsymbol{e}}'_j$ directions. Because the $\hat{\boldsymbol{e}}_i$ and the $\hat{\boldsymbol{e}}_j$ are linearly related,

$$A'_i = \sum_j \frac{\partial x'_i}{\partial x_j} A_j \ . \tag{2}$$

The gradient of a scalar φ has in the unrotated Cartesian coordinates the components $(\nabla\varphi)_j = \frac{\partial \varphi}{\partial x_j}\hat{\boldsymbol{e}}_j$, i.e. in a rotated system

$$(\nabla\varphi)'_i \equiv \frac{\partial \varphi}{\partial x'_i} = \sum_j \frac{\partial x_j}{\partial x'_i}\frac{\partial \varphi}{\partial x_j} \ , \tag{3}$$

Quantities transforming according to Eq. (2) are called contravariant vectors, while those transforming according to Eq. (3) are termed covariant vectors.

$$(A')^i = \sum_j \frac{\partial (x')^i}{\partial x^j} A^j \quad , \boldsymbol{A} \text{ a contravariant vector} \tag{4}$$

$$A'_i = \sum_j \frac{\partial x^j}{\partial (x')^i} A^j \quad , \boldsymbol{A} \text{ a covariant vector} \tag{5}$$

1.2 Tensors of Rank 2

Define contravariant, mixed, and covariant tensors of rank 2 by the following equations for their components under coordinate transformations:

$$(A')^{ij} = \sum_{kl} \frac{\partial (x')^i}{\partial x^k}\frac{\partial (x')^j}{\partial x^l} A^{kl} \ , \tag{6}$$

$$(B')^i_j = \sum_{kl} \frac{\partial (x')^i}{\partial x^k}\frac{\partial x^l}{\partial (x')^j} B^k_l \ , \tag{7}$$

$$(C')_{ij} = \sum_{kl} \frac{\partial x^k}{\partial (x')^i}\frac{\partial x^l}{\partial (x')^j} C_{kl} \ , \tag{8}$$

The second-rank tensor A (with components A^{kl}) may be represented by writing out its components in a square array

$$A = \begin{pmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{pmatrix} \tag{9}$$

For A, it takes the form

$$(A')^{ij} = \sum_{kl} S_{ik} A^{kl} (S^T)_{lj}, \tag{10}$$

$$A' = SAS^T \ , \tag{11}$$

which is known as a similarity transformation.

1.3 Isotropic Tensors

1.4 Contraction

1.5 Direct Product

1.6 Inverse Transformation

1.7 Quotient Rule

References

[1] George B. Arfken and Hans J. Weber. *Mathematical Methods for Physicists, Seventh Edition: A Comprehensive Guide*. Academic Press, 7 edition, January 2012.