Elliptic Equations

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1 Residual Correction Methods

To solve equation

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \ , \tag{1}$$

where the variables are ordered in lexicographical order,

$$\boldsymbol{u} = [u_{11}u_{21}\cdots u_{M_x-11}u_{12}\cdots u_{M_x-1}u_{M_y-1}]^T$$

Let $\boldsymbol{w}$ denote an approximation to $\boldsymbol{u}$, the solution of equation, and denote the algebraic error by $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{w}$ and the residual error by $\boldsymbol{r} = \boldsymbol{f} - \boldsymbol{A}\boldsymbol{w}$. At different times, measure both the algebraic and residual errors with respect to either the sup-norm or the ℓ_2 norm defined on R^L, where $L = (M_x - 1)(M_y - 1)$.

$$\boldsymbol{A}\boldsymbol{e} = \boldsymbol{A}(\boldsymbol{u} - \boldsymbol{w}) = \boldsymbol{f} - \boldsymbol{A}\boldsymbol{w} = \boldsymbol{r} \ . \tag{2}$$

The correction equation is

$$\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{e} = \boldsymbol{w} + \boldsymbol{A}^{-1}\boldsymbol{r} \ . \tag{3}$$

The residual correction method is to approximate $\boldsymbol{A}^{-1}$ and define the iteration

$$\boldsymbol{w}_{k+1} = \boldsymbol{w}_k + \boldsymbol{B}\boldsymbol{r}_k \ , \tag{4}$$

where $\boldsymbol{w}_k = \boldsymbol{f} - \boldsymbol{A}\boldsymbol{w}_k$ and B is some approximation to A^{-1}.

If $\boldsymbol{B} = \boldsymbol{I}$, it is the Richardson iterative scheme,

$$\boldsymbol{w}_{k+1} = \boldsymbol{w}_k + \boldsymbol{r}_k \ . \tag{5}$$

Decompose the matrix $\boldsymbol{A}$ into $\boldsymbol{A} = \boldsymbol{L} + \boldsymbol{D} + \boldsymbol{U}$ where $\boldsymbol{L}$ is the lower triangular matrix consisting of the elements of $\boldsymbol{A}$ below the diagonal, $\boldsymbol{D}$ is a diagonal matrix consisting of the diagonal of $\boldsymbol{A}$, and $\boldsymbol{U}$ is an upper triangular matrix consisting of the elements of $\boldsymbol{A}$ above the diagonal. If $\boldsymbol{B} = \boldsymbol{D}^{-1}$, it is Jacobi relaxation scheme; if $\boldsymbol{B} = (\boldsymbol{L} + \boldsymbol{D})^{-1}$, it is Gauss-Seidel relaxation scheme; if $\boldsymbol{B} = \omega(\boldsymbol{I} + \omega\boldsymbol{D}^{-1}\boldsymbol{L})^{-1}\boldsymbol{D}^{-1} = \omega(\boldsymbol{D} + \omega\boldsymbol{L})^{-1}$, it is successive overrelaxation scheme, where ω is a free parameter; if $\boldsymbol{B} = \omega(2-\omega)(\boldsymbol{D} + \omega\boldsymbol{U})^{-1}\boldsymbol{D}(\boldsymbol{D} + \omega\boldsymbol{L})^{-1}$, it is symmetric successive overrelaxation scheme, where ω is a free parameter.

1.1 Analysis of Residual Correction Schemes

The approach that is used often to obtain an approximate solution to equation (1) is to choose an initial guess, $\boldsymbol{w}_0$, and use the iterative scheme (4). The sequence $\{\boldsymbol{w}_k\}$ will not converge to the solution of equation (1) for all choices of $\boldsymbol{B}$ and $\boldsymbol{w}_0$.

2 Elliptic Difference Equations: Neumann Boundary Conditions

3 Multigrid

The method of multigrid is to eliminate the high frequency components of the error quickly on a fine grid. To accomplish this, the high frequency components of the error will have to correspond to the smallest eigenvalues of the iteration matrix. Then transfer the problem to a coarser grid where high frequency components of the error correspond to some of the lower frequency errors on the previous grid. We can then eliminate these high frequency components of the error on this coarse grid quickly.