Special Relativity

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1 Invariance of Electric Charge

Lorentz force equation for a particle of charge q,

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \tag{1}$$

$\boldsymbol{p}$ transforms as the space part of the 4-vector of energy and momentum,

$$p^\alpha = (p_0, \boldsymbol{p}) = m(U_0, \boldsymbol{U})$$

where $p_0 = E/c$ and U^α is the 4-velocity.

2 Covariance of Electrodaynamics

the electric and magnetic fields are the elements of a second-rank, antisymmetric field-strength tensor,

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{2}$$

Explicitly, the field-strength tensor is

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

the field-strength tensor with two covariant indices

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

The elements of $F_{\alpha\beta}$ are obtained from $F^{\alpha\beta}$ by putting $\boldsymbol{E} \to -\boldsymbol{E}$.

The dual field-strength tensor $\mathscr{F}^{\alpha\beta}$

$$\mathscr{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

The elements of the dual tensor $\mathscr{F}^{\alpha\beta}$ are obtained from $F^{\alpha\beta}$ by putting $\boldsymbol{E} \to \boldsymbol{B}$ and $\boldsymbol{B} \to -\boldsymbol{E}$. $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric fourth-rank tensor:

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{for} \alpha = 0, \beta = 1, \gamma = 2, \delta = 3, \text{and any even permutation} \\ -1 & \text{for any odd permutation} \\ 0 & \text{if any two indices are equal} \end{cases} \tag{3}$$

The nonvanishing elements all have one time and three(different) space indices and that $\epsilon_{\alpha\beta\gamma\delta} = -\epsilon^{\alpha\beta\gamma\delta}$. The tensor $\epsilon^{\alpha\beta\gamma\delta}$ is a pseudotensor under spatial inversions.

Write Maxwell equations in an explicitly covariant form:

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 4\pi\rho \\ \nabla \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} &= \frac{4\pi}{c}\boldsymbol{J} \end{aligned}$$

In terms of $F^{\alpha\beta}$ and the 4-current J^α,

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \tag{4}$$

$$\begin{aligned} \nabla \cdot \boldsymbol{B} &= 0 \\ \nabla \times \boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} &= 0 \end{aligned}$$

$$\partial_\alpha \mathscr{F}^{\alpha\beta} = 0 \tag{5}$$

or in terms of $F^{\alpha\beta}$, rather than $\mathscr{F}^{\alpha\beta}$,

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{6}$$

3 transformation of electromagnetic fields

The fields $\boldsymbol{E}'$ and $\boldsymbol{B}'$ in one inertial frame K' can be expressed in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ in another inertial frame K according to

$$F^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta} \tag{7}$$

i.e.

$$F' = AF\tilde{A} \tag{8}$$

A is the Lorentz transformation matrix.

A boost along the x_1 axis with speed $c\beta$ from the unprinted frame to the primed frame, the explicit equations of transformation are

$$E'_1 = E_1 \qquad\qquad B'_1 = B_1$$

$$E'_2 = \gamma(E_2 - \beta B_3) \qquad B'_2 = \gamma(B_2 + \beta E_3)$$

$$E'_3 = \gamma(E_3 + \beta B_2) \qquad B'_3 = \gamma(B_3 - \beta E_2)$$

For a general Lorentz transformation from K to a system K' moving with velocity $\boldsymbol{v}$ relative to K, the transformation of the fields is

$$\boldsymbol{E}' = \gamma(\boldsymbol{E} + \boldsymbol{\beta} \times \boldsymbol{B}) - \frac{\gamma^2}{\gamma + 1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \boldsymbol{E}) \ , \tag{10}$$

$$\boldsymbol{B}' = \gamma(\boldsymbol{B} - \boldsymbol{\beta} \times \boldsymbol{E}) - \frac{\gamma^2}{\gamma + 1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \boldsymbol{B}) \tag{11}$$

$\boldsymbol{E}$ and $\boldsymbol{B}$ have no independent existence. A purely electric or magnetic field in one coordinate system will appear as a mixture of electric and magnetic fields in another coordinate frame.