Green Function

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1 泊松方程

1.1 1-D

[1] The equation

$$\frac{\mathrm{d}^2 G(x,x')}{\mathrm{d}x^2} = \delta(x-x') \ , \tag{1}$$

It has no effect on our final observation if the source and observation points are interchanged. The observation (the measurable quantity) depends only on the spatial distance between source- and observation point. This symmetry is called Reciprocity, but now with respect to position. The Reciprocity condition

$$G(x,x') = G(x',x) \tag{2}$$

This condition is obviously fulfilled if

$$G(x,x') = F(|x-x'|) = F(u) \ , \tag{3}$$

is used as an ansatz with the so far unknown function F.

$$\frac{\mathrm{d}F(u)}{\mathrm{d}x} = F_x = F_u \cdot u_x = F_u \cdot [H(x-x') - H(x'-x)] \ , \tag{4}$$

and

$$F_{xx} = F_{uu} \cdot u_x^2 + 2 \cdot F_u \cdot \delta(x-x') = F_{uu} + 2 \cdot F_u \cdot \delta(x-x') = \delta(x-x') \ . \tag{5}$$

Require that the function F must be a solution of the homogeneous equation

$$F_{uu} = 0 \ , \tag{6}$$

$$F(u) = C_1 \cdot u + C_2 \ . \tag{7}$$

The additional condition is

$$[F_u]_{u=0} = \frac{1}{2} \ . \tag{8}$$

The Green's function is

$$\left[\frac{\mathrm{d}G(x,x')}{\mathrm{d}x} =\right]_{x=x'-\epsilon}^{x=x'+\epsilon} = 1 \ . \tag{9}$$

As known from potential theory, potentials are determined except for an arbitrary constant. Let us therefore choose $C_2 = 0$ and consider

$$G(x,x') = F(u) = \frac{1}{2}u = \frac{1}{2} \cdot |x - x'| \ , \tag{10}$$

as the free-space Green's function of the 1-dim. Poisson equation.

1.2 2-D

Assuming that the source point agrees with the origin of the polar coordinate system the Poisson equation for the Green' s function $G^{(2)}(R)$ reads

$$G^{(2)}_{RR}(R) + \frac{1}{R} \cdot G^{(2)}(R) = 2\delta_p(R) = \frac{1}{\pi R}\delta(R) \ . \tag{11}$$

Use

$$G^{(2)}(R) = F(R) \cdot H(R) \ , \tag{12}$$

as an appropriate ansatz. It seems as if the Heaviside function $H(R)$ can be omitted since $R \leqslant 0$. But according to our classical method we have to calculate the first and second derivative of the Green's function. In so doing $H(R)$ produces the required Dirac's delta function. However, $H(R)$ can be omitted in the final result, i.e., once we have determined $F(R)$.

$$\left[F_{RR}(R) + \frac{F_R(R)}{R}\right] \cdot H(R) + 2 \cdot F_R(R) \cdot \delta(R) = \frac{1}{\pi R}\delta(R) \ . \tag{13}$$

The unknown function $F(R)$ can be determined by looking for the general solution of the homogeneous equation in the square brackets, i.e., of the ordinary differential equation

$$F_{RR}(R) + \frac{F_R(R)}{R} = 0 \ . \qquad\qquad F(R) = C_1 \cdot \ln(R) + C_2 \ . \tag{14}$$

Constant C_1 can now be determined by applying condition

$$\left[R \cdot \frac{\mathrm{d}G^{(2)}(R)}{\mathrm{d}R}\right]_{R_\epsilon} = \frac{1}{2\pi} \ , \tag{15}$$

$$C_1 = \frac{1}{2\pi} \ . \tag{16}$$

If C_2 is again set to zero, as already done in the 1-dim. case, the free-space Green's function of the 2-dim. Poisson equation is given by

$$G^{(2)}(R) = \frac{1}{2\pi} \cdot \ln(R) \cdot H(R) \ . \tag{17}$$

But it becomes also clear that only the inhomogeneity $2\delta_p(R)$ produces a solution that is in correspondence with condition of the unit source. On the other hand, using the inhomogeneity $\delta_p(R)$ would result in

$$G^{(2)}(R) = \frac{1}{4\pi} \cdot \ln(R) \cdot H(R) \ . \tag{18}$$

Replace R by $|\boldsymbol{r}-\boldsymbol{r}_0|$ if the source point does not agrees with the origin of the coordinate system. Since

$$|\boldsymbol{r}-\boldsymbol{r}_0| = [(x-x')^2 + (y-y')^2]^{1/2} \ , \tag{19}$$

$$G^{(2)}(R) = \frac{1}{2\pi} \cdot \ln\left\{[R^2 + R'^2 - 2RR' \cdot \cos(\phi - \phi')]^{1/2}\right\} \tag{20}$$

for the free-space Green's function of the 2-dim. Poisson equation.

[5] Consider

$$\nabla^2 u = f \ , \quad \text{in} \ \ D, \tag{21}$$

$$u = g \ , \quad \text{on} \ \ C, \tag{22}$$

The Green 's Function satisfies the differential equation and homogeneous boundary conditions. The associated problem is given by

$$\nabla^2 G = \delta(\xi - x, \eta - y) \ , \quad \text{in} \ \ D, \tag{23}$$

$$G \equiv 0 \ , \quad \text{on} \ \ C. \tag{24}$$

Green's Function is symmetric in its arguments. However, this is not always the case and depends on things such as the self-adjointedness of the problem. we will assume that the Green's Function satisfies

$$\nabla^2_{r'} G = \delta(\xi - x, \eta - y) \ , \tag{25}$$

where the notation $\nabla_{r'}$ means differentiation with respect to the variables ξ and η.

$$\int_D (u\nabla^2_{r'}G - G\nabla^2_{r'}u)\mathrm{d}A' = \int_C (u\nabla_{r'}G - G\nabla_{r'}u) \cdot \mathrm{d}\boldsymbol{s}' \ .$$

$$\begin{aligned}\int_D (u\nabla^2_{r'}G - G\nabla^2_{r'}u)\mathrm{d}A' &= \int_D (u(\xi,\eta)\delta(\xi - x, \eta - y) - G(x,y;\xi,\eta)f(\xi,\eta))\mathrm{d}\xi\mathrm{d}\eta \\ &= u(x,y) - \int_D G(x,y;\xi,\eta)f(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta \ .\end{aligned}$$

Using the boundary conditions, $u(\xi,\eta)=g(\xi,\eta)$ on C and $G(x,y;\xi,\eta)=0$ on C,

$$\int_C\int_C (u\nabla_{r'}G - G\nabla_{r'}u)\cdot \mathrm{d}\boldsymbol{s'} = \int_C g(\xi,\eta)\nabla_{r'}G\cdot \mathrm{d}\boldsymbol{s'}\ . \tag{26}$$

$$u(x,y)=\int_D G(x,y;\xi,\eta)f(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta + \int_C g(\xi,\eta)\nabla_{r'}G\cdot \mathrm{d}\boldsymbol{s'} \tag{27}$$

For the Laplacian in polar coordinates, the Green's function is

$$v_{rr}+\frac{1}{r}v_r=\delta(r)\ . \tag{28}$$

For $r\neq 0$, this is a Cauchy-Euler type of differential equation. The general solution is $v(r)=A\ln r+B$.

Due to the singularity at $r=0$, we integrate over a domain in which a small circle of radius e is cut form the plane and apply the two-dimensional Divergence Theorem.

$$\begin{aligned}1 &= \int_{D_\epsilon}\delta(r)\mathrm{d}A\\ &= \int_{D_\epsilon}\nabla^2 v\mathrm{d}A\\ &= \int_{C_\epsilon}\nabla v\cdot \mathrm{d}\boldsymbol{s}\\ &= \int_{C_\epsilon}\frac{\partial v}{\partial r}\mathrm{d}S=2\pi A\ .\end{aligned} \tag{29}$$

$A=1/2\pi$. B is arbitrary, so we will take $B=0$.

The Green's function for Poisson's Equation is

$$G(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{2\pi}\ln|\boldsymbol{r}-\boldsymbol{r}'|\ . \tag{30}$$

1.3 3-D

[2, 3, 4] 确定了 G，就能利用积分表示式求得泊松方程边值问题的解。虽然求格林函数的问题本身也是边值问题，但这是特殊的边值问题。无界区域的格林函数称为相应方程的基本解。将一个一般边值问题的格林函数 G 分成两部分

$$G=G_0+G_1\ . \tag{31}$$

其中 G_0 是基本解。对于三维泊松方程，G_0 满足

$$\Delta G_0=\delta(\boldsymbol{r}-\boldsymbol{r}_0)\ . \tag{32}$$

G_1 满足相应的齐次方程 (Laplace 方程)

$$\Delta G_1 = 0 \tag{33}$$

及相应的边界条件。方程31描述的是位于点 $\boldsymbol{r}_0$，电量 $-\varepsilon_0$ 的点电荷在无界空间中所产生电场在 $\boldsymbol{r}$ 的电势，即 $G_0 = -1/4\pi|\boldsymbol{r}-\boldsymbol{r}_0|$。

假设点源位于坐标原点，由于区域是无界的，点源产生的场与方向无关，选取球坐标 (r,θ,φ)，则 G_0 只是 r 的函数，方程变为常微分方程，当 $r \neq 0$ 时，G_0 满足 Laplace 方程

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}G_0}{\mathrm{d}r}\right) = 0 \ , \tag{34}$$

解为

$$G_0 = -\frac{C_1}{r} + C_2 \ . \tag{35}$$

令无穷远处 $G_0 = 0$，$C_2 = 0$。将方程31在包含 $r_0 = 0$ 的区域作体积分，区域可取为以 $r_0 = 0$ 为球心，半径为 ε 的小球 K_ε，其边界面为 Σ_ε，

$$\iiint\limits_{K_\varepsilon} \Delta G_0 \mathrm{d}V = 1 \ . \tag{36}$$

$$\iiint\limits_{K_\varepsilon} \Delta G_0 \mathrm{d}V = \iint\limits_{\Sigma_\varepsilon} \frac{\partial G_0}{\partial r}\mathrm{d}S = \int_0^{2\pi}\int_0^{\pi}\frac{\partial}{\partial r}\left(-\frac{C_1}{r}\right) r^2 \sin\theta \mathrm{d}\theta \mathrm{d}\varphi = 4\pi C_1 \ . \tag{37}$$

则 $C_1 = \dfrac{1}{4\pi}$，

$$G_0(r) = -\frac{1}{4\pi r} \ . \tag{38}$$

若电荷位于任意点 $\boldsymbol{r}_0$，则

$$G_0(r) = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|} \ . \tag{39}$$

[5] Poisson's Equation for the electric potential is

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \ . \tag{40}$$

The Poisson's Equation also arises in Newton's Theory of Gravitation for the gravitational potential in the form $\nabla^2\phi = -4\pi G\rho$, where ρ is the matter density.

Consider Poisson's Equation

$$\nabla^2\phi(\boldsymbol{r}) = -4\pi f(\boldsymbol{r}) \ , \tag{41}$$

for $\boldsymbol{r}$ defined throughout all space. The Fourier transform can be generalized to three dimensions as

$$\hat{\phi}(\boldsymbol{k}) = \int_V \phi(\boldsymbol{r})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3 r \ , \tag{42}$$

where the integration is over all space, V. The inverse Fourier transform is

$$\phi(\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int_{V_k}\hat{\phi}(\boldsymbol{k})\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}^3k \ . \tag{43}$$

The Fourier transform of the Laplacian follows from computing Fourier transforms of any derivatives that are present. Assuming that ϕ and its gradient vanish for large distances, then

$$\mathcal{F}[\nabla^2\phi] = -(k_x^2 + k_y^2 + k_z^2)\hat{\phi}(\boldsymbol{k}) \ . \tag{44}$$

Poisson's Equation becomes

$$k^2\hat{\phi}(\boldsymbol{k}) = 4\pi\hat{f}(\boldsymbol{k}) \ . \tag{45}$$

$$\hat{\phi}(\boldsymbol{k}) = \frac{4\pi}{k^2}\hat{f}(\boldsymbol{k}) \ . \tag{46}$$

The solution to Poisson's Equation is then determined from the inverse Fourier transform,

$$\phi(\boldsymbol{r}) = \frac{4\pi}{(2\pi)^3}\int_{V_k}\hat{\phi}(\boldsymbol{k})\frac{\mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2}\mathrm{d}^3k \ . \tag{47}$$

Set $f(\boldsymbol{r}) = f_0\delta^3(\boldsymbol{r})$ in order to represent a point source. For a unit point charge, $f_0 = 1/4\pi\epsilon_0$.

2 Helmholtz Equation

3 Modified Helmholtz Equation

References

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