Bessel Functions

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1 Bessel Functions of the First kind

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2 Modified Bessel Functions

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3 Spherical Bessel functions

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3.1 Rayleigh Formula

[1] A plane wave can be expanded in a series of spherical waves by the Rayleigh equation

$$\mathrm{e}^{ikr\cos\theta} = \sum_{n=0}^{\infty} a_n j_n(kr) P_n(\cos\theta) \ , \tag{1}$$

where $j_n(kr)$ is the spherical Bessel function of oder n. $a_n = i^n(2n+1)$.

Multiply both sides of Eq. (1) by $P_l(\cos\theta)\sin\theta \mathrm{d}\theta$, $l \neq n$ and integrate with respect to θ.

$$\begin{aligned}\int_0^\pi \mathrm{e}^{ikr\cos\theta} P_l(\cos\theta)\sin\theta \mathrm{d}\theta &= \sum_{n=0}^{\infty} a_n j_n(kr) \int_0^\pi P_n(\cos\theta) P_l(\cos\theta)\sin\theta \mathrm{d}\theta \\ &= \sum_{n=0}^{\infty} a_n j_n(kr) \frac{2}{2n+1}\delta_{n,l} = a_l j_l(kr)\frac{2}{2l+1} \ .\end{aligned}$$

Differentiate equation with respect to (kr) l times, then take the limit of $(kr) \to 0$. For the spherical Bessel part,

$$\frac{2a_l}{2l+1}\lim_{kr\to 0}\left(\frac{\mathrm{d}^l}{\mathrm{d}(kr)^l} j_l(kr)\right) = \frac{a_l}{2l+1}\frac{2l!}{(2l+1)!!} \ . \tag{2}$$

$$\begin{aligned}\lim_{(kr)\to 0}\left(\frac{\mathrm{d}^l}{\mathrm{d}(kr)^l}\int_0^\pi \mathrm{e}^{ikr\cos\theta} P_l(\cos\theta)\sin\theta \mathrm{d}\theta\right) &= \int_0^\pi (i\cos\theta)^l P_l(\cos\theta)\sin\theta \mathrm{d}\theta \\ &= i^l \frac{2l!}{(2l+1)!!} \ ,\end{aligned}$$

where

$$\int_{-1}^{1} x^n P_n(x)\mathrm{d}x = \frac{2n!}{(2n+1)!!} \ . \tag{3}$$

$$a_n = i^n(2n+1) \ . \tag{4}$$

References

[1] George B. Arfken and Hans J. Weber. *Mathematical Methods for Physicists, Seventh Edition: A Comprehensive Guide*. Academic Press, 7 edition, January 2012.