Radiation Physics of Relativistic Flows

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RADIATION PRELIMINARIES

[1] The intensity I_ϵ is defined such that $I_\epsilon \mathrm{d}\epsilon \mathrm{d}A \mathrm{d}t \mathrm{d}\Omega$ is the infinitesimal energy $\mathrm{d}\mathcal{E}$ in photons with energy between ϵ and $\epsilon + \mathrm{d}\epsilon$ lying within solid angle element $\mathrm{d}\Omega$ that pass through area element $\mathrm{d}A$ oriented normal to the direction $\mathbf{\Omega}$ during differential time $\mathrm{d}t$. The intensity is a local quantity, $I_\epsilon = I_\epsilon(\boldsymbol{x}, t)$. Other than polarization, I_ϵ provides a complete description of the radiation field. The specific spectral energy density $u(\epsilon, \Omega) = \mathrm{d}\mathcal{E}/\mathrm{d}V \mathrm{d}\epsilon \mathrm{d}\Omega$. By following a pencil beam of rays contained within differential volume $\mathrm{d}V = c\mathrm{d}t\mathrm{d}A$, $\mathrm{d}\mathcal{E} = u(\epsilon, \Omega)\mathrm{d}V \mathrm{d}\epsilon \mathrm{d}\Omega = u(\epsilon, \Omega)c\mathrm{d}t\mathrm{d}A\mathrm{d}\epsilon\mathrm{d}\Omega = I_\epsilon \mathrm{d}A\mathrm{d}t\mathrm{d}\epsilon\mathrm{d}\Omega$, so that

$$I_\epsilon = cu(\epsilon, \Omega) \ . \tag{1}$$

Consider a pencil beam of radiation, passing through area elements $\mathrm{d}A_1 \cos\theta_1$ and $\mathrm{d}A_2 \cos\theta_2$ separated by distance d and oriented at angles θ_1 and θ_2, respectively, to the direction of the rays. If there is no absorption or emission of photons during propagation, then energy conservation of the radiation beam passing through $\mathrm{d}A_1$ and $\mathrm{d}A_2$ during time $\mathrm{d}t$

means that $\mathrm{d}\mathcal{E} = I_{\epsilon,1}\mathrm{d}A_1\cos\theta_1\mathrm{d}t\mathrm{d}\Omega_1\mathrm{d}\epsilon_1 = I_{\epsilon,2}\mathrm{d}A_2\cos\theta_2\mathrm{d}t\mathrm{d}\Omega_2\mathrm{d}\epsilon_2$. The bundle of rays passing through $\mathrm{d}A_2$ lies within solid angle element $\mathrm{d}\Omega_1 = \mathrm{d}A_2\cos\theta_2/d^2$ as seen from the location of $\mathrm{d}A_1$. The rays passing through $\mathrm{d}A_1$ lie within the solid angle element $\mathrm{d}\Omega_2 = \mathrm{d}A_1\cos\theta_1/d^2$ as seen from the location of $\mathrm{d}A_2$. For constant energy rays (no cosmological redshifting, which can be treated separately), $\epsilon_1 = \epsilon_2 = \epsilon$. $I_{\epsilon,1} = I_{\epsilon,2}$ or $\mathrm{d}I_\epsilon/\mathrm{d}s = 0$.

Effects of absorption or emission on the evolution of the intensity over differential path length $\mathrm{d}s$ are described by the equation of radiative transfer, given by

$$\frac{\mathrm{d}I_\epsilon}{\mathrm{d}s} = -\kappa_\epsilon I_\epsilon + j(\epsilon,\Omega)\ , \tag{2}$$

$$\frac{\mathrm{d}I_\epsilon}{\mathrm{d}\tau_\epsilon} = -I_\epsilon + \mathcal{S}_\epsilon\ . \tag{3}$$

$j(\epsilon,\Omega) = \mathrm{d}\mathcal{E}/\mathrm{d}V\mathrm{d}t\mathrm{d}\epsilon\mathrm{d}\Omega$ is the emissivity, and the differential optical depth $\tau_\epsilon = \kappa_\epsilon\mathrm{d}s$ is defined in terms of the spectral absorption coefficient κ_ϵ (units of inverse length). The source function

$$\mathcal{S}_\epsilon = \frac{j(\epsilon,\Omega)}{\kappa_\epsilon}\ . \tag{4}$$

INVARIANT QUANTITIES

[1]

BLACKBODY RADIATION FIELD

[1] The average energy in wave modes populated according to Boltzmann statistics is, letting $x = e^{-h\Delta U/k_B T}$,

$$\langle U \rangle = \frac{\sum\limits_{n=0}^{\infty} n\Delta U e^{-n\Delta U/k_B T}}{\sum\limits_{n=0}^{\infty} e^{-n\Delta U/k_B T}} = \Delta U \frac{\sum\limits_{n=0}^{\infty} n x^n}{\sum\limits_{n=0}^{\infty} x^n} = \Delta U (1-x) x \sum_{n=0}^{\infty} n x^{n-1} \tag{5}$$

$$= \Delta U x (1-x) \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{\Delta U}{x^{-1}-1} = \frac{h\nu}{e^{h\nu/k_B T}-1} \ , \tag{6}$$

where the energy $\Delta U = h\nu$ in a mode of frequency ν, according to Planck's quantum hypothesis.

The energy density of a blackbody radiation field is

$$u_{\mathrm{CMB}}(\Theta) = \frac{4\pi}{c}\int_0^\infty \mathrm{d}\epsilon I_\epsilon^{\mathrm{CMB}}(\Theta) = \frac{8\pi^5}{15}\frac{m_e c^2}{\lambda_C^3}\Theta^4$$
$$= 4.14 \times 10^{-13}(1+z)^4 \left(\frac{T}{2.72\mathrm{K}}\right)^4 \ \text{ergs cm}^{-3}$$
$$\simeq 0.26(1+z)^4\left(\frac{T}{2.72\mathrm{K}}\right)^4 \ \text{eV cm}^{-3} \tag{7}$$

$u_{\mathrm{CMB}}(\Theta) = m_e c^2 \langle \epsilon_{\mathrm{CMB}}(\Theta)\rangle n_{\mathrm{CMB}}(\Theta)$.

Because the blackbody radiation field is strongly peaked at photon energies $\epsilon \simeq \Theta$, a monochromatic δ-function approximation oftentimes provides sufficient accuracy for calculations. A convenient approximation for the CMBR field is

$$n_{\mathrm{CMB}}(\epsilon, z) = 407(1+z)^3 \delta[\epsilon - 1.24 \times 10^{-9}(1+z)] \ . \tag{8}$$

TRANSFORMED QUANTITIES

[1]

FLUXES OF RELATIVISTIC COSMOLOGICAL SOURCES

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References

[1] C. D. Dermer and G. Menon. *High Energy Radiation from Black Holes: Gamma Rays, Cosmic Rays, and Neutrinos*. 2009.