# Vector Analysis

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[1]

# 1 Coordinate transformations

## 1.1 Rotations

$$\hat{\mathbf{e}}_x = \cos\varphi \hat{\mathbf{e}}'_x - \sin\varphi \hat{\mathbf{e}}'_y \ , \tag{1}$$

$$\hat{\mathbf{e}}_y = \sin\varphi \hat{\mathbf{e}}'_x - \cos\varphi \hat{\mathbf{e}}'_y \ , \tag{2}$$

the unchanged vector $\boldsymbol{A}$ now takes the changed form

$$\boldsymbol{A} = A_x \hat{\mathbf{e}}_x + A_y \hat{\mathbf{e}}_y \ , \tag{3}$$

$$= A_x(\cos\varphi \hat{\mathbf{e}}'_x - \sin\varphi \hat{\mathbf{e}}'_y) + A_y(\sin\varphi \hat{\mathbf{e}}'_x - \cos\varphi \hat{\mathbf{e}}'_y) \ , \tag{4}$$

$$= (A_x \cos\varphi + A_y \sin\varphi)\hat{\mathbf{e}}'_x + (-A_x \sin\varphi + A_y \cos\varphi)\hat{\mathbf{e}}'_y \ , \tag{5}$$

$$= A'_x \hat{\mathbf{e}}'_x + A'_y \hat{\mathbf{e}}'_y \ , \tag{6}$$

then

$$A'_x = A_x \cos\varphi + A_y \sin\varphi \ , \tag{7}$$

$$A'_y = -A_x \sin\varphi + A_y \cos\varphi \ , \tag{8}$$

i.e.

$$\boldsymbol{A}' = \begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix} \tag{9}$$

Suppose now starting from $\boldsymbol{A}$ as given by its components in the rotated system, $(A'_x, A'_y)$, and rotate the coordinate system back to its original orientation. This will entail a rotation in the amount $-\varphi$,

$$\boldsymbol{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos(-\varphi) & \sin(-\varphi) \\ -\sin(-\varphi) & \cos(-\varphi) \end{pmatrix} \begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} A'_x \\ A'_y \end{pmatrix} \tag{10}$$

Let

$$\boldsymbol{S} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \ , \quad \boldsymbol{S}' = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \ , \tag{11}$$

then $\boldsymbol{S}' = \boldsymbol{S}^{-1}$, $\boldsymbol{S}' = \boldsymbol{S}^T$ and $\boldsymbol{S}\boldsymbol{S}' = 1$. Since $\boldsymbol{S}$ is real, $\boldsymbol{S}^{-1} = \boldsymbol{S}^T$ means that it is orthogonal. The transformation connecting $\boldsymbol{A}$ and $\boldsymbol{A}'$ (the same vector, but represented in the rotated coordinate system) is

$$\boldsymbol{A}' = \boldsymbol{S}\boldsymbol{A} \ , \tag{12}$$

$$\boldsymbol{A} = \boldsymbol{S}'\boldsymbol{A}' \ , \tag{13}$$

$$\boldsymbol{A} = \boldsymbol{S}'\boldsymbol{S}\boldsymbol{A} \ . \tag{14}$$

with $\boldsymbol{S}$ an orthogonal matrix.

## 1.2 Orthogonal Transformations

$$\hat{\mathbf{e}}_x = (\hat{\mathbf{e}}'_x \cdot \hat{\mathbf{e}}_x)\hat{\mathbf{e}}'_x + (\hat{\mathbf{e}}'_y \cdot \hat{\mathbf{e}}_x)\hat{\mathbf{e}}'_y \ , \tag{15}$$

$$\hat{\mathbf{e}}_y = (\hat{\mathbf{e}}'_x \cdot \hat{\mathbf{e}}_y)\hat{\mathbf{e}}'_x + (\hat{\mathbf{e}}'_y \cdot \hat{\mathbf{e}}_y)\hat{\mathbf{e}}'_y \ , \tag{16}$$

$$\boldsymbol{S} = \begin{pmatrix} \hat{\mathbf{e}}'_x \cdot \hat{\mathbf{e}}_x & \hat{\mathbf{e}}'_x \cdot \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}'_y \cdot \hat{\mathbf{e}}_x & \hat{\mathbf{e}}'_y \cdot \hat{\mathbf{e}}_y \end{pmatrix} \tag{17}$$

The transformation from one orthogonal Cartesian coordinate system to another Cartesian system is described by an orthogonal matrix. An orthogonal matrix must have a determinant that is real and of magnitude unity, i.e., $\pm 1$. However, for rotations in ordinary space the value of the determinant will always be $+1$.

### 1.3 Reflections

$$\boldsymbol{S} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{18}$$

which results in $\det S = -1$.

$$\boldsymbol{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{19}$$

which results in $\det S = -1$.

### 1.4 Successive Operations

$$\boldsymbol{A}' = \boldsymbol{S}(R')\boldsymbol{S}(R)\boldsymbol{A} \tag{20}$$

## 2 Rotations in $\mathbb{R}^3$

In $\mathbb{R}^2$, all the elements of $\boldsymbol{S}$ depended on a single variable, the rotation angle. In $\mathbb{R}^3$, the number of independent variables needed to specify a general rotation is three: Two parameters (usually angles) are needed to specify the direction of $\hat{\mathbf{e}}_3'$; then one angle is needed to specify the direction of $\hat{\mathbf{e}}_1'$ in the plane perpendicular to $\hat{\mathbf{e}}_3'$; at this point the orientation of $\hat{\mathbf{e}}_2'$ is completely determined. Therefore, of the nine elements of $\boldsymbol{S}$, only three are independent. The usual parameters used to specify $\mathbb{R}^3$ rotations are the Euler angles.

The Euler angles describe an $\mathbb{R}^3$ rotation in three steps, the first two of which have the effect of fixing the orientation of the new $\hat{\mathbf{e}}'_3$ axis (the polar direction in spherical coordinates), while the third Euler angle indicates the amount of subsequent rotation about that axis. The first two steps do more than identify a new polar direction; they describe rotations that cause the realignment.

1. The coordinates are rotated about the $\hat{\mathbf{e}}_3$ axis counterclockwise (as viewed from positive $\hat{\mathbf{e}}_3$) through an angle $\alpha$ in the range $0 \leqslant \alpha < 2\pi$, into new axes denoted $\hat{\mathbf{e}}'_1$, $\hat{\mathbf{e}}'_2$, $\hat{\mathbf{e}}'_3$. (The polar direction is not changed; the $\hat{\mathbf{e}}'_3$ and $\hat{\mathbf{e}}_3$ axes coincide.)

2. The coordinates are rotated about the $\hat{\mathbf{e}}'_2$ axis counterclockwise (as viewed from positive $\hat{\mathbf{e}}'_2$) through an angle $\beta$ in the range $0 \leqslant \beta < 2\pi$, into new axes denoted $\hat{\mathbf{e}}''_1$, $\hat{\mathbf{e}}''_2$, $\hat{\mathbf{e}}''_3$. (This tilts the polar direction toward the $\hat{\mathbf{e}}'_1$ direction; but leaves $\hat{\mathbf{e}}'_2$ unchanged.)

3. The coordinates are now rotated about the $\hat{\mathbf{e}}''_3$ axis counterclockwise (as viewed from positive $\hat{\mathbf{e}}''_3$) through an angle $\gamma$ in the range $0 \leqslant \gamma < 2\pi$, into the final axes, denoted $\hat{\mathbf{e}}'''_1$, $\hat{\mathbf{e}}'''_2$, $\hat{\mathbf{e}}'''_3$. (This rotation leaves the polar direction, $\hat{\mathbf{e}}''_3$, unchanged.)

In terms of the usual spherical polar coordinates $(r, \theta, \varphi)$, the final polar axis is at the orientation $\theta = \beta, \varphi = \alpha$. The final orientations of the other axes depend on all three Euler angles.

$$\boldsymbol{S}_1(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{21}$$

$$\boldsymbol{S}_2(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \tag{22}$$

$$\boldsymbol{S}_3(\gamma) = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{23}$$

The total rotation is described by the triple matrix product

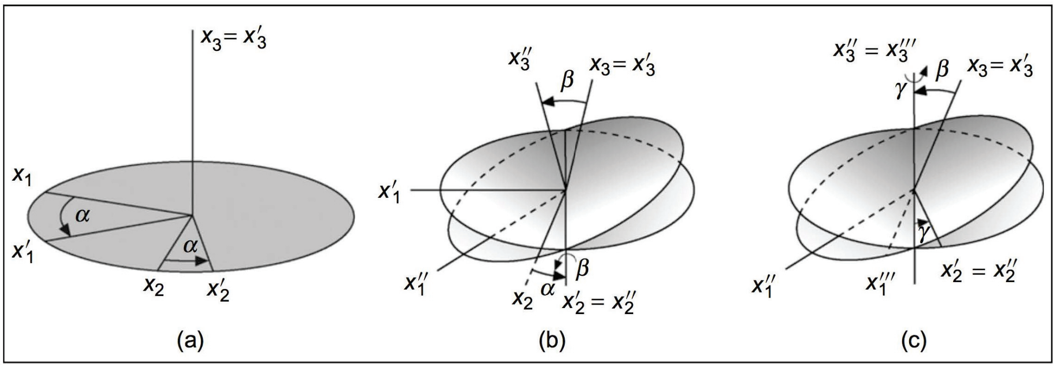


Fig. 1: Euler angle rotations : (a) about $\hat{\mathbf{e}}_3$ through angle $\alpha$; (b) about $\hat{\mathbf{e}}'_2$ through angle $\beta$; (c) about $\hat{\mathbf{e}}''_3$ through angle $\gamma$.

$$\boldsymbol{S}(\alpha, \beta, \gamma) = \boldsymbol{S}_3(\gamma)\boldsymbol{S}_2(\beta)\boldsymbol{S}_1(\alpha) \ , \tag{24}$$

Note the order: $\boldsymbol{S}_1(\alpha)$ operates first, then $\boldsymbol{S}_2(\beta)$, and finally $\boldsymbol{S}_3(\gamma)$, i.e.

$$\boldsymbol{S}(\alpha, \beta, \gamma) = \begin{pmatrix} \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & -\cos\gamma\sin\beta \\ -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha & -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta \\ \sin\beta\cos\alpha & \sin\beta\sin\alpha & \cos\beta \end{pmatrix} \tag{25}$$

Each of $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, and $\boldsymbol{S}_3$ are orthogonal, with determinant $+1$, so that the overall $\boldsymbol{S}$ will also be orthogonal with determinant $+1$.

## 3 Curvilinear coordinates

## References

[1] George B. Arfken and Hans J. Weber. *Mathematical Methods for Physicists, Seventh Edition: A Comprehensive Guide.* Academic Press, 7 edition, January 2012.