Inverse Compton Scattering

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1 Thomson Scattering

经典电磁理论认为，当电磁辐射通过物质时，被散射的辐射应与入射辐射具有相同的波长。因为入射的电磁辐射使原子中的电子受到一个周期变化的力，迫使电子以入射波的频率振荡。

[1] Consider the scattering of an unpolarised parallel beam of radiation through an angle α by a stationary electron. It is assumed that the incident beam propagates in the positive z-direction and the geometry of the scattering is arranged to be such that the scattering angle α lies in the $x - z$ plane. The electric field strength of the unpolarised incident field is resolved into components of equal intensity with electric vectors in the orthogonal $\boldsymbol{i}_x$ and $\boldsymbol{i}_y$ directions. The electric fields experienced by the electron in the x and y directions, $E_x = E_{x0} \exp(i\omega t)$ and $E_y = E_{y0} \exp(i\omega t)$, respectively, cause the electron to oscillate and the accelerations in these directions are

$$\ddot{r}_x = \frac{eE_x}{m_\mathrm{e}} \ , \tag{1}$$

$$\ddot{r}_y = \frac{eE_y}{m_\mathrm{e}} \ . \tag{2}$$

The intensity of radiation scattered through angle θ into the solid angle $\mathrm{d}\Omega$ is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_x \mathrm{d}\Omega = \frac{e^2|\ddot{r}_x|^2 \sin^2\theta}{16\pi^2\epsilon_0 c^3}\mathrm{d}\Omega = \tag{3}$$

Thomson cross section

$$\begin{aligned}
\sigma_T &= \frac{e^4 E_0^2}{12\pi m_e^2 \epsilon_0 c^3}\frac{2\mu_0 c}{E_0^2} \\
&= \frac{\mu_0 \epsilon_0 e^4}{6\pi m_e^2 \epsilon_0^2 c^2} \\
&= \frac{e^4}{6\pi m_e^2 \epsilon_0^2 c^4} \\
&= \frac{8\pi}{3} r_0^2 \\
&= 6.6524 \times 10^{-25}\ \mathrm{cm}^2 = 6.6524 \times 10^{-29}\ \mathrm{m}^2
\end{aligned}$$

where r_0 is the classical electron radius. This is Thomson's famous result for the total cross-section for scattering of electromagnetic waves by stationary free electrons.

The scattering is symmetric with respect to the scattering angle α. Thus, as much radiation is scattered in the backward as in the forward direction.

The scattering cross-section for 100% polarised emission can be found by integrating the scattered intensity over all angles,

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_x = \frac{e^2|\ddot{r}_x|^2}{16\pi^2\epsilon_0 c^3}\int \sin^2\theta 2\pi \sin\theta \mathrm{d}\theta = \left(\frac{e^4}{6\pi\epsilon_0^2 m_\mathrm{e}^2 c^4}\right) S_x = \sigma_\mathrm{T} S_x \tag{4}$$

For incoherent radiation, the energy radiated is proportional to the sum of the incident intensities of the radiation field and so the only important quantity so far as the electron is concerned is the total intensity of radiation incident upon it. It does not matter how anisotropic the incident radiation field is. One convenient way of expressing this result is to write the formula for the scattered radiation in terms to the energy density of radiation

$u_{\rm rad}$ at the electron

$$u_{\rm rad} = \sum_i u_i = \sum_i \frac{S_i}{c} \tag{5}$$

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \sigma_{\rm T} c u_{\rm rad} \tag{6}$$

The scattered radiation is polarised, even if the incident beam of radiation is unpolarised. When the electron is observed precisely in the x—y plane, the scattered radiation is 100% polarised. On the other hand, if we look along the z-direction, we observe unpolarised radiation. The degree of polarisation is defined as

$$\Pi = \frac{I_{\rm max} - I_{\rm min}}{I_{\rm max} + I_{\rm min}} \,, \tag{7}$$

the fractional polarisation of the radiation is

$$\Pi = \frac{1 - \cos^2\alpha}{1 + \cos^2\alpha} \,. \tag{8}$$

This is therefore a means of producing polarised radiation from an initially unpolarised beam.

Thomson scattering is one of the most important processes which impedes the escape of photons from any region. If the number density of photons of frequency ν is N, the rate at which energy is scattered out of the beam is

$$-\frac{\mathrm{d}(Nh\nu)}{\mathrm{d}t} = \sigma_{\rm T} c N h\nu \tag{9}$$

There is no change of energy of the photons in the scattering process and so, if there are $N_{\rm e}$ electrons per unit volume, the number density of photons decreases exponentially with distance

$$
\begin{aligned}
-\frac{\mathrm{d}N}{\mathrm{d}t} &= \sigma_{\mathrm{T}} c N_{\mathrm{e}} N \ , & (10) \\
-\frac{\mathrm{d}N}{\mathrm{d}x} &= \sigma_{\mathrm{T}} N_{\mathrm{e}} N \ , & (11) \\
N &= N_0 \exp(-\int \alpha_{\mathrm{T}} N_{\mathrm{e}} \mathrm{d}x) & (12)
\end{aligned}
$$

The optical depth T of the medium for Thomson scattering is

$$
\tau = \int \sigma_{\mathrm{T}} N_{\mathrm{e}} \mathrm{d}x \tag{13}
$$

In this process, the photons are scattered in random directions and so they perform a random walk, each step corresponding to the mean free path λ_{T} of the photon through the electron gas, where $\lambda_{\mathrm{T}} = (\sigma_{\mathrm{T}} N_{\mathrm{e}})^{-1}$.

In Thomson scattering, there is no change in the frequency of the radiation. This remains a good approximation provided the energy of the photon is much less than the rest mass energy of the electron, $\hbar\omega \ll m_{\mathrm{e}}c^2$. In general, as long as the energy of the photon is less than $m_{\mathrm{e}}c^2$ in the centre of momentum frame of reference, the scattering may be treated as Thomson scattering.

2 Compton Scattering

高能光子与低能电子相碰时，光子把一部分能量传递给电子，从而损失能量，能量降低，波长变长。

In the Compton scattering process, the incoming high energy photons collide with stationary electrons and transfer some of their energy and momentum to the electrons. Consequently, the scattered photons have less energies and momenta than before the collisions.

X 射线光子与自由电子发生碰撞；在被散射的 X 射线中，波长随散射角发生变化；证明了 X 射线的粒子性。

推导：

$$
\begin{aligned}
h\nu + E_0 &= h\nu' + E \ , \qquad (14) \\
\boldsymbol{p}_\lambda &= \boldsymbol{p}_{\lambda'} + \boldsymbol{p} \ , \qquad (15)
\end{aligned}
$$

$\boldsymbol{p}_\lambda = \dfrac{h}{\lambda}\hat{k}$ 和 $\boldsymbol{p}_{\lambda'} = \dfrac{h}{\lambda'}\hat{k}'$ 分别是光子碰撞前后的动量。

$$
\begin{aligned}
E^2 &= E_0^2 + p^2c^2 \ , \\
E_0 &= m_\mathrm{e}c^2 \ , \\
E &= \gamma m_\mathrm{e}c^2 \ , \\
\gamma &= \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} \ ,
\end{aligned}
$$

$$p_\lambda^2 + p_{\lambda'}^2 - 2p_\lambda p_{\lambda'}\cos\theta = p^2$$

Compton 散射公式

$$\lambda' - \lambda = \Delta\lambda = \frac{hc}{m_\mathrm{e}c^2}(1-\cos\theta) \qquad (16)$$

Compton 散射引起的最大位移

$$\Delta\lambda = \frac{2hc}{m_\mathrm{e}c^2} = 0.0049 \text{ nm}$$

散射光子的能量

$$h\nu' = \frac{h\nu}{1+\kappa(1-\cos\theta)} \ , \ \kappa = \frac{h\nu}{m_\mathrm{e}c^2}$$

反冲电子动能

$$E_k = h\nu - h\nu' = h\nu\frac{\kappa(1-\cos\theta)}{1+\kappa(1-\cos\theta)}$$

反冲电子的最大能量 ($\theta = \pi$)

$$E_{k,\max} = h\nu \frac{2\kappa}{1+2\kappa}$$

相应光子的最小能量

$$h\nu'|_{\min} = \frac{h\nu}{1+2\kappa}$$

电子的 Compton 波长

$$\lambda = \frac{hc}{m_e c^2} = \frac{1.24\ \mathrm{nm}\cdot\mathrm{keV}}{511\ \mathrm{keV}} = 0.002426\ \mathrm{nm}$$

经典电子半径

$$\begin{aligned} m_e c^2 &= \frac{e^2}{4\pi\epsilon_0 r_e}\ , \\ r_e &= \frac{e^2}{4\pi\epsilon_0 m_e c^2} \approx 2.8\ \mathrm{fm} \end{aligned}$$

[1] Suppose the electron moves with velocity v through the laboratory frame of reference S. The momentum four-vectors of the electron and the photon before and after the collision are

$$\begin{aligned} \text{electron}\quad \boldsymbol{P} = [\gamma m_e c, \gamma m_e \boldsymbol{v}]\ ,\ \boldsymbol{P}' = [\gamma' m_e c, \gamma' m_e \boldsymbol{v}'] \\ \text{photon}\quad \boldsymbol{K} = \left[\frac{\hbar\omega}{c}, \frac{\hbar\omega}{c}\boldsymbol{i}_k\right]\ ,\ \boldsymbol{K}' = \left[\frac{\hbar\omega'}{c}, \frac{\hbar\omega'}{c}\boldsymbol{i}_{k'}\right] \end{aligned}$$

The collision conserves four-momentum

$$\boldsymbol{P} + \boldsymbol{K} = \boldsymbol{P}' + \boldsymbol{K}'$$

$$\boldsymbol{P}\cdot\boldsymbol{P} = \boldsymbol{P}'\cdot\boldsymbol{P}' = m_e c^2\ ,\ \boldsymbol{K}\cdot\boldsymbol{K} = \boldsymbol{K}'\cdot\boldsymbol{K}' = 0$$

$$\boldsymbol{P}\cdot\boldsymbol{K} = \boldsymbol{P}'\cdot\boldsymbol{K}'$$

$$\boldsymbol{P}\cdot\boldsymbol{K}' + \boldsymbol{K}\cdot\boldsymbol{K}' = \boldsymbol{P}\cdot\boldsymbol{K}$$

The scattering angle is given by $\boldsymbol{i}_k \cdot \boldsymbol{i}_{k'} = \cos\alpha$. The angle between the incoming photon and the velocity vector of the electron is θ and the angle between them after the collision is θ'. Then, $\cos\theta = \boldsymbol{i}_k \cdot \boldsymbol{v}/|\boldsymbol{v}|$ and $\cos\theta' = \boldsymbol{i}_{k'} \cdot \boldsymbol{v}/|\boldsymbol{v}|$.

$$\frac{\omega'}{\omega} = \frac{1 - (v/c)\cos\theta}{1 - (v/c)\cos\theta' + (\hbar\omega/\gamma m_e c^2)(1 - \cos\alpha)} \tag{17}$$

It shows how energy can be exchanged between the electron and the radiation field. In the limit $\omega \ll \gamma m_e c^2$, the change in frequency of the photon is

$$\frac{\omega' - \omega}{\omega} = \frac{\Delta\omega}{\omega} = \frac{v}{c}\frac{\cos\theta - \cos\theta'}{[1 - (v/c)\cos\theta']} \tag{18}$$

Thus, to first order, the frequency changes are $\sim v/c$. To first order, if the angles θ and θ' are randomly distributed, a photon is just as likely to decrease as increase its energy. It can be shown that there is no net increase in energy of the photons to first order in v/c and it is only in second order, that is, to order v^2/c^2, that there is a net energy change. If $v = 0$, $\gamma = 1$, i.e. the photon on scattering from a stationary electron,

$$\frac{\omega'}{\omega} = \frac{1}{1 + (\hbar\omega/m_e c^2)(1 - \cos\alpha)} \,, \tag{19}$$

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda' - \lambda}{\lambda} = \frac{\hbar\omega}{m_e c^2}(1 - \cos\alpha) \tag{20}$$

This effect of 'cooling' the radiation and transferring the energy to the electron is sometimes called the recoil effect.

The Thomson cross-section is only adequate for cases in which the electron moves with velocity $v \ll c$ or if the photon has energy $\hbar\omega \ll m_e c^2$ in the centre of momentum frame of reference. If a photon of energy $\hbar\omega$ collides with a stationary electron, the centre of momentum frame moves at velocity

$$\frac{v}{c} = \frac{\hbar\omega}{m_e c^2 + \hbar\omega} \tag{21}$$

If the photons have energy $\hbar\omega \geq m_e c^2$, we must use the proper quantum relativistic cross-section for scattering. Another case which can often arise is if the photons are of low energy $\hbar\omega \ll m_e c^2$ but the electron moves ultra-relativistically with $\gamma \gg 1$. The centre of momentum frame moves with a velocity close to that of the electron and in this frame the energy of the photon is $\gamma\hbar\omega$. If $\gamma\hbar\omega \sim m_e c^2$, the quantum relativistic cross-section has to be used.

The total cross-section is the Klein–Nishina formula:

$$\sigma_{\mathrm{K-N}} = \pi r_{\mathrm{e}}^2 \frac{1}{x} \left\{ \left[1 - \frac{2(x+1)}{x^2} \right] \ln(2x+1) + \frac{1}{2} + \frac{4}{x} - \frac{1}{2(2x+1)^2} \right\} \tag{22}$$

where $x = \hbar\omega/m_e c^2$ and $r_{\mathrm{e}} = e^2/4\pi\epsilon_0 m_e c^2$ is the classical electron radius. For low energy photons, $x \ll 1$,

$$\sigma_{\mathrm{K-N}} = \frac{8\pi}{3} r_{\mathrm{e}}^2 (1-2x) = \sigma_{\mathrm{T}}(1-2x) \approx \sigma_{\mathrm{T}} \ . \tag{23}$$

In the ultra-relativistic limit, $\gamma \gg 1$, the Klein–Nishina cross-section becomes

$$\sigma_{\mathrm{K-N}} = \pi r_{\mathrm{e}}^2 \frac{1}{x} \left(\ln 2x + \frac{1}{2} \right) \ , \tag{24}$$

the cross-section decreases roughly as x^{-1} at the highest energies. If the atom has Z electrons, the total cross-section per atom is $Z\sigma_{\mathrm{K-N}}$. The scattering by nuclei can be neglected because they cause very much less scattering than electrons, roughly by a factor of $(m_e/m_{\mathrm{N}})^2$, where m_{N} is the mass of the nucleus.

3 Inverse Compton Scattering

[1] In inverse Compton scattering, ultra-relativistic electrons scatter low energy photons to high energies so that the photons gain energy at the expense of the kinetic energy of the electrons.

For an incident isotropic photon field at a single frequency ν_0, the spectral emissivity $I(\nu)$ is

$$I(\nu)\mathrm{d}\nu = \frac{3\sigma_{\mathrm{T}}c}{16\gamma^4}\frac{N(\nu_0)}{\nu_0^2}\nu\left[2\nu\ln\left(\frac{\nu}{4\gamma^2\nu_0}\right)+\nu+4\gamma^2\nu_0-\frac{\nu^2}{2\gamma^2\nu_0}\right]\mathrm{d}\nu \tag{25}$$

where the isotropic radiation field in the laboratory frame of reference S is assumed to be monochromatic with frequency ν_0; $N(\nu_0)$ is the number density of photons. At low frequencies, the term in square brackets is a constant and hence the scattered radiation has a spectrum of the form $I(\nu) \propto \nu$.

The maximum energy which the photon can acquire corresponds to a head-on collision in which the photon is sent back along its original path. The maximum energy of the photon is

$$\hbar\omega|_{\max} = \hbar\omega\gamma^2\left(1+\frac{v}{c}\right)^2 \approx 4\gamma^2\hbar\omega_0 \tag{26}$$

The number of photons scattered per unit time is $\sigma_{\mathrm{T}}cu_{\mathrm{rad}}/\hbar\omega_0$ and hence the average energy of the scattered photons is

$$\hbar\bar{\omega} = \frac{4}{3}\gamma^2\left(\frac{v}{c}\right)^2\hbar\omega_0 \approx \frac{4}{3}\gamma^2\hbar\omega_0 \tag{27}$$

The photon gains typically one factor of γ in transforming into S' and then gains another on transforming back into S. The frequency of photons scattered by ultra-relativistic electrons is $\nu \sim \gamma^2\nu_0$.

3.1 Energy flux

The gamma-ray flux from the ICS process for an isotropic distribution of soft photons $n(\epsilon')$ upscattered by a population of electrons with spectrum $F(p)$ is [2, 3]

$$\Phi(\epsilon) = \frac{2\pi e^4 \epsilon}{c} \int \mathrm{d}p F(p) \int \frac{n(\epsilon')\mathrm{d}\epsilon'}{\epsilon'} \times \left[2q \ln q + (1+2q)(1-q) + \frac{1}{2} \frac{\epsilon^2}{pc(pc-\epsilon)}(1-q) \right] \tag{28}$$

where

$$q = \frac{\epsilon}{\frac{4\epsilon' pc}{(mc^2)^2}(pc - \epsilon)} \tag{29}$$

The gamma-ray flux from the ICS process is given by

$$\left.\frac{\mathrm{d}N}{\mathrm{d}E}\right|_{\mathrm{IC}} = c \int \mathrm{d}\epsilon\, n(\epsilon) \int \mathrm{d}E_e \frac{\mathrm{d}n}{\mathrm{d}E_e} \times F_{\mathrm{KN}}(\epsilon, E_e, E), \tag{30}$$

The differential Klein-Nishina cross section $F_{\mathrm{KN}}(\epsilon, E_e, E)$ is adopted as the following form [4, 2]

$$F_{\mathrm{KN}}(\epsilon, E_e, E) = \frac{3\sigma_T}{4\gamma^2 \epsilon} \left[2q \ln q + (1+2q)(1-q) + \frac{(\Gamma q)^2 (1-q)}{2(1+\Gamma q)} \right], \tag{31}$$

where σ_T is the Thomson cross section, γ is the Lorentz factor of electron, $\Gamma = 4\epsilon\gamma/m_e$, and $q = E/\Gamma(E_e - E)$. On a separate note, when $q < 1/4\gamma^2$ or $q > 1$, $F_{\mathrm{KN}}(\epsilon, E_e, E) = 0$.

3.2 Energy loss rate

The inverse Compton energy loss rate of a single electron in one graybody photon field is

$$|\dot{\gamma}|_{\mathrm{C}} \simeq \frac{4\sigma_{\mathrm{T}} c W}{3 m_{\mathrm{e}} c^2} \frac{\gamma_K^2 \gamma^2}{\gamma_K^2 + \gamma^2} \tag{32}$$

The critical Klein-Nishina Lorentz factor is

$$\gamma_K \equiv \frac{3\sqrt{5}}{8\pi} \frac{m_e c^2}{k_B T} = \frac{0.27 m_e c^2}{k_B T} \tag{33}$$

When $\gamma \ll \gamma_K$, reduces to the Thomson limit

$$|\dot{\gamma}|_{\rm C}(\gamma \ll \gamma_K) \simeq \frac{4\sigma_{\rm T} c W}{3 m_{\rm e} c^2}\gamma^2, \tag{34}$$

When $\gamma \gg \gamma_K$, obtain the energy-independent extreme Klein-Nishina limit

$$|\dot{\gamma}|_{\rm C}(\gamma \gg \gamma_K) \simeq \frac{4\sigma_{\rm T} c W}{3 m_{\rm e} c^2}\gamma_K^2, \tag{35}$$

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