Fourier Transform

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1 Integral transforms

The pairs of functions are related by an expression of the form

$$g(x) = \int_a^b f(t)K(x,t)\mathrm{d}t = \mathcal{L}f(t) \ , \tag{1}$$

$$\mathcal{L}^{-1}g(x) = f(t) \ , \tag{2}$$

where a, b, and $K(x,t)$ (called the kernel) will be the same for all function pairs f and g. Eq. (1) can be interpreted as an operator equation. The function $g(x)$ is called the integral transform of $f(t)$ by the operator $\mathcal{L}$, with the specific transform determined by the choice of a, b, and $K(x,t)$. The operator defined by Eq. (1) will be linear:

$$\int_a^b [f_1(t) + f_2(t)]K(x,t)\mathrm{d}t = \int_a^b f_1(t)K(x,t)\mathrm{d}t + \int_a^b f_2(t)K(x,t)\mathrm{d}t \ , \tag{3}$$

$$\int_a^b cf(t)K(x,t)\mathrm{d}t = c\int_a^b f(t)K(x,t)\mathrm{d}t = \tag{4}$$

1.1 Some Important Transforms

The Fourier transform

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \mathrm{d}t \ , \tag{5}$$

The Laplace transform,

$$F(s) = \int_{0}^{\infty} e^{-st} f(t) \mathrm{d}t \ , \tag{6}$$

The Hankel transform,

$$g(\alpha) = \int_{0}^{\infty} f(t) t J_n(\alpha t) \mathrm{d}t \ , \tag{7}$$

represents the continuum limit of the Bessel series.

The Mellin transform,

$$g(\alpha) = \int_{0}^{\infty} f(t) t^{\alpha-1} \mathrm{d}t \ , \tag{8}$$

2 Fourier transform

The Fourier and inverse Fourier transforms are

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \mathrm{d}t \ , \tag{9}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} \mathrm{d}\omega \ . \tag{10}$$

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos \omega t \mathrm{d}t \ , \tag{11}$$

$$f_c(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(\omega) \cos \omega t \mathrm{d}\omega \ . \tag{12}$$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t \mathrm{d}t \ , \tag{13}$$

$$f_s(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\omega) \sin \omega t \mathrm{d}\omega \ . \tag{14}$$

$$g(\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^\infty f(\boldsymbol{r}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 r \ , \tag{15}$$

$$f(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^\infty g(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \mathrm{d}^3 k \ . \tag{16}$$

3 Fourier convolution theorem

3.1 Parseval Relation

4 Discrete Fourier transform