Linear Equations in Linear Algebra

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Definition

An indexed set of vectors $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_p\boldsymbol{v}_p = \boldsymbol{0} \ , \tag{1}$$

has only the trivial solution. The set $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\}$ is said to be linearly dependent if there exist weights $c_1, \cdots, c_p$, not all zero, such that

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p = \boldsymbol{0} \ . \tag{2}$$

Equation (2) is called a linear dependence relation among $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\}$ when the weights are not all zero. An indexed set is linearly dependent if and only if it is not linearly independent. $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\}$ are linearly dependent when we mean that $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\}$ is a linearly dependent set.

7.1 Linear Independence of Matrix Columns

The matrix equation $A\boldsymbol{x} = \mathbf{0}$ can be written as

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}\ . \tag{3}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\boldsymbol{x} = \mathbf{0}$.

The columns of a matrix A are linearly independent if and only if the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

7.2 Sets of One or Two Vectors

A set of two vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

7.3 Sets of Two or More Vectors

Characterization of Linearly Dependent Sets

An indexed set $S = \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\boldsymbol{v}_1 \neq \mathbf{0}$, then some $\boldsymbol{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{j-1}$.

Theorem

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem

If a set $S = \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

8 Introduction to Linear Transformations

Solving the equation $A\boldsymbol{x} = \boldsymbol{b}$ amounts to finding all vectors $\boldsymbol{x}$ in $\mathbb{R}^4$ that are transformed into the vector $\boldsymbol{b}$ in $\mathbb{R}^2$ under the "action" of multiplication by A.

A transformation (or function or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\boldsymbol{x}$ in $\mathbb{R}^n$ a vector $T(\boldsymbol{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the domain of T, and $\mathbb{R}^m$ is called the codomain of T. The notation $T : \mathbb{R}^n \to \mathbb{R}^m$ indicates that the domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$. For $\boldsymbol{x}$ in $\mathbb{R}^n$, the vector $T(\boldsymbol{x})$ in $\mathbb{R}^m$ is called the image of $\boldsymbol{x}$ (under the action of T). The set of all images $T(\boldsymbol{x})$ is called the range of T.

8.1 Matrix Transformations

9 The Matrix of a Linear Transformation

10 Linear Models in Business, Science, and Engineering