The Orbits of Stars

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[1] Although galaxies are composed of stars, we shall neglect the forces from individual stars and consider only the large-scale forces from the overall mass distribution, which is made up of thousands of millions of stars. The gravitational fields of galaxies are smooth, neglecting small-scale irregularities due to individual stars or larger objects like globular clusters or molecular clouds. The gravitational fields of galaxies are sufficiently smooth that these irregularities can affect the orbits of stars only after many crossing times. Since only gravitational forces are dealt with, the trajectory of a star in a given field does not depend on its mass.

1   Orbits in static spherical potentials

Consider orbits in a static, spherically symmetric gravitational field. Such fields are appropriate for globular clusters, which are usually nearly spherical. $\boldsymbol{r} = r\hat{e}_r$ denotes the position vector of the star with respect to the center, and the radial acceleration is

$$\boldsymbol{g} = g(r)\hat{\boldsymbol{e}}_r \ , \tag{1}$$

the equation of motion of the star is

$$\frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} = g(r)\hat{\boldsymbol{e}}_r \ . \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \boldsymbol{r} \times \frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} = g(r)\boldsymbol{r} \times \hat{\boldsymbol{e}}_r = 0 \ . \tag{3}$$

$$\boldsymbol{L} = \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \tag{4}$$

$\boldsymbol{L}$ is the angular momentum per unit mass, a vector perpendicular to the plane defined by

the star's instantaneous position and velocity vectors. $\boldsymbol{L}$ is velocity vectors, i.e. the star

moves in a plane, the orbital plane. Establish the plane polar coordinates (r, ψ) in which

the center of attraction is at $r = 0$ and ψ is the azimuthal angle in the orbital plane. The

Lagrangian per unit mass is

$$\mathcal{L} = \frac{[\dot{r}^2 + (r\dot{\psi})^2]}{2} - \Phi(r) \ , \tag{5}$$

where ψ is the gravitational potential and $g(r) = -\dfrac{\mathrm{d}\psi}{\mathrm{d}r}$. The equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = \ddot{r} - r\dot{\psi}^2 + \frac{\mathrm{d}\Phi}{\mathrm{d}r} = 0 \ ,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \frac{\partial \mathcal{L}}{\partial \psi} = \frac{\mathrm{d}(r^2\dot{\psi})}{\mathrm{d}t} = 0 \ .$$

The second of these equations implies

$$L = r^2\dot{\psi} = \text{const.} \tag{6}$$

Geometrically, L is equal to twice the rate at which the radius vector sweeps out area.

1.1 Constants and integrals of the motion

Any stellar orbit traces a path in the six-dimensional space for which the coordinates are the position and velocity $\boldsymbol{x}, \boldsymbol{v}$. This space is called phase space. (In statistical mechanics phase space usually refers to position-momentum space rather than position-velocity space. Since all bodies have the same acceleration in a given gravitational field, mass is irrelevant.) A constant of motion in a given force field is any function $C(\boldsymbol{x}, \boldsymbol{v}; t)$ of the phase-space coordinates and time that is constant along stellar orbits. If the position and velocity along an orbit are given by $\boldsymbol{x}(t)$ and $\boldsymbol{v}(t) = \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}$,

$$C[\boldsymbol{x}(t_1), \boldsymbol{v}(t_1); t_1] = C[\boldsymbol{x}(t_2), \boldsymbol{v}(t_2); t_2] \tag{7}$$

for any t_1 and t_2.

An integral of motion $I(\boldsymbol{x}, \boldsymbol{v})$ is any function of the phase-space coordinates alone that is constant along an orbit:

$$I[\boldsymbol{x}(t_1), \boldsymbol{v}(t_1)] = I[\boldsymbol{x}(t_2), \boldsymbol{v}(t_2)] \tag{8}$$

While every integral is a constant of the motion, the converse is not true.

Any orbit in any force field always has six independent constants of motion. Indeed, since the initial phase-space coordinates $(\boldsymbol{x}_0, \boldsymbol{v}_0) \equiv [\boldsymbol{x}(0), \boldsymbol{v}(0)]$ can always be determined from $[\boldsymbol{x}(t), \boldsymbol{v}(t)]$ by integrating the equations of motion backward, $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ can be regarded as six constants of motion.

Orbits can have from zero to five integrals of motion. Two examples. In any static potential $\Phi(\boldsymbol{x})$, the Hamiltonian $H(\boldsymbol{x}, \boldsymbol{v}) = \dfrac{v^2}{2} + \Phi$ is an integral of motion. If a potential $\Phi(R, z, t)$ is axisymmetric about the z axis, the z-component of the angular momentum

is an integral, and in a spherical potential $\Phi(r,t)$, the three components of the angular-momentum vector $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{v}$ constitute three integrals of motion.

The motion in a spherically symmetric potential. In this case the Hamiltonian H and the three components of the angular momentum per unit mass $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{v}$ constitute four integrals. Use $|\boldsymbol{L}|$ and the two independent components of the unit vector $\hat{\boldsymbol{n}} = \boldsymbol{L}/|\boldsymbol{L}|$ as integrals in place of $\boldsymbol{L}$. $\hat{\boldsymbol{n}}$ defines the orbital plane within which the position vector $\boldsymbol{r}$ and the velocity vector $\boldsymbol{v}$ must lie. The two independent components of $\hat{\boldsymbol{n}}$ restrict the stars phase point to a four-dimensional region of phase space. The equations $H(\boldsymbol{x}, \boldsymbol{v}) = E$ and $|\boldsymbol{L}(\boldsymbol{x}, \boldsymbol{v})| = L$, where L is a constant, restrict the phase point to that two-dimensional surface in this four-dimensional region on which $v_r = \pm\sqrt{2[E - \Phi(r)] - L^2/r^2}$ and $v_\psi = L/r$. This surface is a torus and that the sign ambiguity in v_r is analogous to the sign ambiguity in the z-coordinate of a point on the sphere $r^2 = 1$ when one specifies the point through its x and y coordinates. Thus, given E, L, and $\hat{\boldsymbol{n}}$, the star's position and velocity (up to its sign) can be specified by two quantities, for example r and ψ.

Consider the motion in the potential

$$\Phi(r) = -GM \left(\frac{1}{r} + \frac{a}{r^2} \right) . \tag{9}$$

equation becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + \left(1 - \frac{2GMa}{L^2} \right) u = \frac{GM}{L^2} , \tag{10}$$

the general solution is

$$u = C \cos \left(\frac{\psi - \psi_0}{K} \right) + \frac{GMK^2}{L^2} , \tag{11}$$

where

$$K \equiv \left(1 - \frac{2GMa}{L^2} \right)^{-1/2} . \tag{12}$$

$$\psi_0 = \psi - K \arccos\left[\frac{1}{C}\left(\frac{1}{r} - \frac{2GMK^2}{L^2}\right)\right] \; , \tag{13}$$

where $t = \arccos x$ is the multiple-valued solution of $x = \cos t$, and C can be expressed in terms of E and L by

$$E = \frac{1}{2}\frac{C^2 L^2}{K^2} - \frac{1}{2}\left(\frac{GMK}{L}\right)^2 \; . \tag{14}$$

Replacing E by $H(\boldsymbol{x}, \boldsymbol{v})$ and L by $|L(\boldsymbol{x}, \boldsymbol{v})| = |\boldsymbol{x} \times \boldsymbol{v}|$, the quantity ψ_0 becomes a function of the phase-space coordinates which is constant as the particle moves along its orbit. Hence ψ_0 is a fifth integral of motion. (Since the function $\arccos x$ is multiple-valued, a judicious choice of solution is necessary to avoid discontinuous jumps in ψ_0.) Suppose we know the numerical values of E, L, ψ_0, and the radial coordinate r. Since we have four numbers—three integrals and one coordinate—it is natural to ask how to use these numbers to determine the azimuthal coordinate ψ. Rewrite the equation

$$\psi = \psi_0 \pm K \cos^{-1}\left[\frac{1}{C}\left(\frac{1}{r} - \frac{2GMK^2}{L^2}\right)\right] + 2nK\pi \; , \tag{15}$$

where $\cos^{-1}(x)$ is defined to be the value of $\arccos(x)$ that lies between 0 and π, and n is an arbitrary integer. If K is irrational—as nearly all real numbers are—then by a suitable choice of the integer n, we can make ψ modulo 2π approximate any given number as closely as we please. For any values of E and L, and any value of r between the pericenter and apocenter for the given E and L, an orbit that is known to have a given value of the integral ψ_0 can have an azimuthal angle as close as we please to any number between 0 and 2π.

If K is rational, these problems do not arise. The simplest and most important case is

that of the Kepler potential, when $a = 0$ and $K = 1$. The equation becomes

$$\psi = \psi_0 \pm \cos^{-1}\left[\frac{1}{C}\left(\frac{1}{r} - \frac{GM}{L^2}\right)\right] + 2n\pi \ , \tag{16}$$

which yields only two values of ψ modulo 2π for given E, L and r.

The phase space has six dimensions. The equation $H(\boldsymbol{x}, \boldsymbol{v}) = E$ confines the orbit to a five- dimensional subspace. The vector equation $L(\boldsymbol{x}, \boldsymbol{v}) = $ constant adds three further constraints, thereby restricting the orbit to a two-dimensional surface. Through the equation $\psi_0(\boldsymbol{x}, \boldsymbol{v}) = $ constant, the fifth integral confines the orbit to a one-dimensional curve on this surface.

Consider any volume of phase space, of any shape or size. Then if K is irrational, the fraction of the time that an orbit with given values of H and L spends in that volume does not depend on the value that ψ_0 takes on this orbit.

Integrals like ψ_0 for irrational K that do not affect the phase-space distribution of an orbit, are called non-isolating integrals. All other integrals are called isolating integrals. The examples of isolating integrals are H, $\boldsymbol{L}$, and the function ψ_0 when $K = 1$. All confine stars to a five-dimensional region in phase space. However, there can also be isolating integrals that restrict the orbit to a six-dimensional subspace of phase space.

References

[1] J. Binney and S. Tremaine. *Galactic Dynamics: Second Edition*. Princeton University Press, 2008.