Synchrotron

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1 Non-relativistic gyroradiation and cyclotron radiation

critical angular frequency $\omega_{\mathrm{c}} = 3c\gamma^3/2a$, $x = \omega/\omega_{\mathrm{c}} = \nu/\nu_{\mathrm{c}}$. a is the radius of curvature of the electron's spiral orbit. At any instant, the plane of the electron's orbit is inclined at a pitch angle α to the magnetic field. With respect to the guiding centre of the electron' s trajectory, the radius of curvature is $a = v/(\omega_{\mathrm{r}} \sin\alpha)$ and

$$\omega_{\mathrm{c}} = 2\pi\nu_{\mathrm{c}} = \frac{3}{2}\left(\frac{c}{v}\right)\gamma^3\omega_{\mathrm{r}}\sin\alpha \tag{1}$$

when $v \to c$ and rewriting the expression in terms of the non-relativistic gyrofrequency $\nu_{\mathrm{g}} = eB/2\pi m_{\mathrm{e}} = 28$ GHz T^{-1}

$$\nu_{\mathrm{c}} = \frac{3}{2}\gamma^2\nu_{\mathrm{g}}\sin\alpha \tag{2}$$

The emissivities of the electron in the two polarisations are

$$j_{\perp}(\omega) = \frac{I_{\perp}(\omega)}{T_{\mathrm{r}}} = \frac{\sqrt{3}e^3 B\sin\alpha}{16\pi^2\epsilon_0 c m_{\mathrm{e}}}[F(x) + G(x)], \tag{3}$$

$$j_{\parallel}(\omega) = \frac{I_{\parallel}(\omega)}{T_{\mathrm{r}}} = \frac{\sqrt{3}e^3 B\sin\alpha}{16\pi^2\epsilon_0 c m_{\mathrm{e}}}[F(x) - G(x)] \tag{4}$$

The total emissivity of a single electron by synchrotron radiation is the sum of $j_{\perp}(\omega)$ and $j_{\parallel}(\omega)$:

$$j(\omega) = j_{\perp}(\omega) + j_{\parallel}(\omega) = \frac{\sqrt{3}e^3 B \sin\alpha}{8\pi^2 \epsilon_0 c m_{\rm e}} F(x), \tag{5}$$

the spectral emissivity of a single electron by synchrotron radiation in the ultrarelativistic limit. The spectrum has a broad maximum, $\Delta\nu/\nu \sim 1$, centred roughly at the frequency $\nu \approx \nu_{\rm c}$ – the maximum of the emission spectrum in fact has value $\nu_{\rm max} = 0.29\nu_{\rm c}$. The spectrum is smooth and continuous.

2 The synchrotron radiation of a power-law distribution of electron energies

2.1 physical arguments

The spectrum of synchrotron radiation is quite sharply peaked near the critical frequency ν_c, much narrower than the breadth of the power-law electron energy spectrum. Assumed that an electron of energy E radiates away its energy at the critical frequency ν_c,

$$\nu \approx \nu_c \approx \gamma^2 \nu_{\rm g} = \left(\frac{E}{m_{\rm e}c^2}\right)^2 \nu_{\rm g}; \quad \nu_{\rm g} = \frac{eB}{2\pi m_{\rm e}} \tag{6}$$

Then the energy radiated in the frequency range ν to $\nu + {\rm d}\nu$ can be attributed to electrons with energies in the range E to $E + {\rm d}E$

$$J(\nu){\rm d}\nu = \left(-\frac{{\rm d}E}{{\rm d}t}\right) N(E){\rm d}E \tag{7}$$

where

$$\begin{aligned}
E &= \gamma m_{\mathrm{e}} c^2 = \left(\frac{\nu}{\nu_{\mathrm{g}}}\right)^{1/2} m_{\mathrm{e}} c^2, \\
\mathrm{d}E &= \frac{m_{\mathrm{e}} c^2}{2\nu_{\mathrm{g}}^{1/2}} \nu^{-1/2} \mathrm{d}\nu, \\
-\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{4}{3}\sigma_{\mathrm{T}} c \left(\frac{E}{m_{\mathrm{e}} c^2}\right)^2 \frac{B^2}{2\mu_0}, \\
N(E)\mathrm{d}E &= \kappa E^{-p} \mathrm{d}E
\end{aligned}$$

the emissivity is expressed in terms of κ, B, ν and fundamental constants:

$$J(\nu) = (\text{constants})\ \kappa B^{(p+1)/2} \nu^{-(p-1)/2} \tag{8}$$

The emitted spectrum, written as $J(\nu) \propto \nu^{-a}$, where $a = (p-1)/2$ is known as the spectral index, is determined by the slope of the electron energy spectrum p, rather than by the shape of the emission spectrum of a single electron. The emissivity also depends upon $\kappa B^{(p+1)/2} \propto \kappa B^{a+1}$.

2.2 full analysis

Consider a power-law distribution of electron energies at a fixed pitch angle α. To integrate the contributions of electrons of different energies to the intensity at angular frequency ω, or equivalently, at fixed $x = \omega/\omega_{\mathrm{c}}$.

3 Energy flux

For synchrotron radiation the energy flux at an energy is given by [1]

$$\Phi(\epsilon) = \frac{\sqrt{3}Be^3}{hmc^2} \int p^2 F(p) K(\epsilon/\epsilon_c) \mathrm{d}p \tag{9}$$

where $\epsilon_c = h\nu_c$ is the energy of critical frequency $\nu_c = 3eBp^2/(4\pi m^3 c^3)$ and $K(\epsilon/\epsilon_c)$ is the emission produced by the single electron of momentum p, charge e and mass m. For the exact expression for the kernel function $K(\epsilon/\epsilon_c)$ in the case of a turbulent magnetic field one can refers to [2]; or it can be approximated by the analytical expression in [3] to several percent accuracy, i.e.

$$K(\epsilon/\epsilon_c) = \frac{1.81\ \mathrm{e}^{-\epsilon/\epsilon_c}}{\sqrt{(\epsilon/\epsilon_c)^{-2/3} + (3.62/\pi)^2}} \tag{10}$$

4 Energy loss rate

In a uniform magnetic field, a high energy electron moves in a spiral path at a constant pitch angle α. Its velocity along the field lines is constant whilst it gyrates about the magnetic field direction at the relativistic gyrofrequency $\nu_{\mathrm{g}} = eB/2\pi\gamma m_{\mathrm{e}} = 28\gamma^{-1}$ GHz T^{-1}, where γ is the Lorentz factor of the electron $\gamma = (1 - v^2/c^2)^{-1/2}$. The radiation loss rate of a charged particle q with accelerations $a_\perp$ and $a_\parallel$ as measured in the laboratory frame of reference is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} = \frac{q^2\gamma^4}{6\pi\epsilon_0 c^3}[|a_\perp|^2 + \gamma^2|a_\parallel|^2] \tag{11}$$

The acceleration is always perpendicular to the velocity vector of the particle, i.e. $a_\perp = evB\sin\alpha/\gamma m_{\mathrm{e}}$ and $a_\parallel = 0$. The total radiation loss rate of the electron is

$$\begin{aligned} -\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} &= \frac{\gamma^4 e^2}{6\pi\epsilon_0 c^3}|a_\perp|^2 \\ &= \frac{\gamma^4 e^2}{6\pi\epsilon_0 c^3}\frac{e^2 v^2 B^2 \sin^2\alpha}{\gamma^2 m_{\mathrm{e}}^2} \\ &= \frac{e^4 B^2}{6\pi\epsilon_0 m_{\mathrm{e}}^2 c}\frac{v^2}{c^2}\gamma^2\sin^2\alpha \end{aligned} \tag{12}$$

4.1 derivation 2

The pitch angle distribution is likely to be randomised either by irregularities in the magnetic field distribution or by streaming instabilities. As a result, the distribution of pitch angles for a population of high energy electrons is expected to be isotropic. During its lifetime, any high energy electron is also randomly scattered in pitch angle. Averaging over an isotropic distribution of pitch angles $p(\alpha)\mathrm{d}\alpha = 1/2\sin\alpha\mathrm{d}\alpha$, the average energy loss rate,

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} = 2\sigma_{\mathrm{T}}cU_{\mathrm{mag}}\gamma^2\frac{v^2}{c^2}\frac{1}{2}\int_0^\pi \sin^3\alpha\mathrm{d}\alpha = \frac{4}{3}\sigma_{\mathrm{T}}cU_{\mathrm{mag}}\frac{v^2}{c^2}\gamma^2 \tag{13}$$

The synchrotron energy loss rate of a single electron in a large-scale random magnetic field of constant strength B is

$$|\dot{\gamma}|_{\mathrm{S}} = \frac{4\sigma_{\mathrm{T}}c}{3m_{\mathrm{e}}c^2}U_{\mathrm{B}}\gamma^2 \tag{14}$$

$U_{\mathrm{B}} = B^2/8\pi = 0.22\ b_3^2$ eV cm^{-3}, where $B = 3b_3\ \mu$G.

References

[1] G. Vannoni, S. Gabici, and F. A. Aharonian. Diffusive shock acceleration in radiation-dominated environments. A&A, 497:17–26, April 2009.

[2] A. Crusius and R. Schlickeiser. Synchrotron radiation in random magnetic fields. A&A, 164:L16–L18, August 1986.

[3] V. N. Zirakashvili and F. Aharonian. Analytical solutions for energy spectra of electrons accelerated by nonrelativistic shock-waves in shell type supernova remnants. A&A, 465:695–702, April 2007.