Qualitative Theory

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1 Qualitative Properties of Solutions

2 Nonlinear Equations

2.1 Autonomous Systems. The Phase Plane and Its Phenomena

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = f\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) \tag{1}$$

The values of x (position) and $\mathrm{d}x/\mathrm{d}t$ (velocity), which at each instant characterize the state of the system, are called its phases, and the plane of the variables x and $\mathrm{d}x/\mathrm{d}t$ is called the phase plane. Introduce the variable $y = \mathrm{d}x/\mathrm{d}t$,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = f(x, y) \ . \end{cases} \tag{2}$$

When t is regarded as a parameter, then in general a solution of (2) is a pair of functions

$x(t)$ and $y(t)$ defining a curve in the xy-plane,

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = F(x,y) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = G(x,y) \ . \end{cases} \tag{3}$$

where F and G are continuous and have continuous first partial derivatives throughout the plane. A system of this kind, in which the independent variable t does not appear in the functions F and G on the right, is said to be autonomous.

If t_0 is any number and (x_0, y_0) is any point in the phase plane, then there exists a unique solution

$$\begin{cases} x = x(t) \\ y = y(t) \ . \end{cases} \tag{4}$$

of (3) such that $x(t_0) = x_0$ and $y(t_0) = y_0$. If $x(t)$ and $y(t)$ are not both constant functions, then (4) defines a curve in the phase plane called a path of the system. If (4) is a solution of (3), then

$$\begin{cases} x = x(t+c) \\ y = y(t+c) \ . \end{cases} \tag{5}$$

is also a solution for any constant c. Each path is represented by many solutions, which differ from one another only by a translation of the parameter. Any path through the point (x_0, y_0) must correspond to a solution of the form (5). At most one path passes through each point of the phase plane. The direction of increasing t along a given path is the same for all solutions representing the path. We shall use arrows to indicate the direction in which the path is traced out as t increases.

In general the path of (3) cover the entire phase plane and do not intersect one another. The only exceptions to this statement occur at points (x_0, y_0) where both F and G vanish

:

$$F(x_0, y_0) = 0 \ , \quad \text{and} \quad G(x_0, y_0) = 0 \tag{6}$$

These points are called critical points.

2.2 Types of Critical Points. Stability

2.3 Critical Points and Stability for Linear Systems

2.4 Stability by Liapunov's Direct Method

2.5 Simple Critical Points of Nonlinear Systems

2.6 Nonlinear Mechanics. Conservative Systems

2.7 Periodic Solutions. The Poincare-Bendixson Theorem