Coulomb Scattering

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From Coulomb's law

$$\boldsymbol{F} = m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = m\ddot{\boldsymbol{r}} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r^2}\hat{\boldsymbol{e}}_r\ , \tag{1}$$

where

$$\begin{aligned}
\boldsymbol{r} &= r\hat{\boldsymbol{e}}_r = r(\cos\theta\ \hat{i} + \sin\theta\ \hat{j}) \\
\hat{\boldsymbol{e}}_r &= (\cos\theta\ \hat{i} + \sin\theta\ \hat{j}) \\
\hat{\boldsymbol{e}}_\theta &= (-\sin\theta\ \hat{i} + \cos\theta\ \hat{j}) \\
\dot{\hat{\boldsymbol{e}}}_r &= \dot{\theta}(-\sin\theta\ \hat{i} + \cos\theta\ \hat{j}) = \dot{\theta}\hat{\boldsymbol{e}}_\theta \\
\dot{\hat{\boldsymbol{e}}}_\theta &= \dot{\theta}(-\cos\theta\ \hat{i} - \sin\theta\ \hat{j}) = -\dot{\theta}\hat{\boldsymbol{e}}_r
\end{aligned}$$

thus

$$\begin{aligned}
\dot{\boldsymbol{r}} &= \dot{r}\hat{\boldsymbol{e}}_r + r\dot{\hat{\boldsymbol{e}}}_r \\
&= \dot{r}(\cos\theta\ \hat{i} + \sin\theta\ \hat{j}) + r\dot{\theta}(-\sin\theta\ \hat{i} + \cos\theta\ \hat{j}) \\
&= \dot{r}\hat{\boldsymbol{e}}_r + r\dot{\theta}\hat{\boldsymbol{e}}_\theta
\end{aligned}$$

and

$$\begin{aligned}
\ddot{\boldsymbol{r}} &= (\dot{r}\hat{\boldsymbol{e}}_r + r\dot{\theta}\hat{\boldsymbol{e}}_\theta)' \\
&= \ddot{r}\hat{\boldsymbol{e}}_r + \dot{r}\dot{\hat{\boldsymbol{e}}}_r + \dot{r}\dot{\theta}\hat{\boldsymbol{e}}_\theta + r\ddot{\theta}\hat{\boldsymbol{e}}_\theta + r\dot{\theta}\dot{\hat{\boldsymbol{e}}}_\theta \\
&= \ddot{r}\hat{\boldsymbol{e}}_r + \dot{r}\dot{\hat{\boldsymbol{e}}}_r + \dot{r}\dot{\theta}\hat{\boldsymbol{e}}_\theta + r\ddot{\theta}\hat{\boldsymbol{e}}_\theta + r\dot{\theta}\dot{\hat{\boldsymbol{e}}}_\theta \\
&= \ddot{r}\hat{\boldsymbol{e}}_r + \dot{r}\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{r}\dot{\theta}\hat{\boldsymbol{e}}_\theta + r\ddot{\theta}\hat{\boldsymbol{e}}_\theta - r\dot{\theta}^2\hat{\boldsymbol{e}}_r \\
&= (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{e}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{e}}_\theta
\end{aligned}$$

Since Coulomb's force points to $\hat{\boldsymbol{e}}_r$,

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{2}$$

$$\mathrm{d}\ln r^2 + \mathrm{d}\ln\dot{\theta} = 0$$

$$r^2\dot{\theta} = \text{const.} \tag{3}$$

It is the conservation of angular momentum.

$$m\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{r^2}\hat{\boldsymbol{e}}_r \tag{4}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}\theta} = \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{mr^2\dot{\theta}}\hat{\boldsymbol{e}}_r \tag{5}$$

performing integration on both sides,

$$\int_{\boldsymbol{v}_1}^{\boldsymbol{v}_2}\mathrm{d}\boldsymbol{v} = \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{mr^2\dot{\theta}}\int_\pi^\theta \hat{\boldsymbol{e}}_r\mathrm{d}\theta\ , \tag{6}$$

$$(\boldsymbol{v}_2 - \boldsymbol{v}_1) = \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{mr^2\dot{\theta}}\int_\pi^\theta (\cos\theta\ \hat{\imath} + \sin\theta\ \hat{\jmath})\mathrm{d}\theta\ ,$$

$$(\boldsymbol{v}_2 - \boldsymbol{v}_1) = \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{mr^2\dot{\theta}}(\sin\theta\ \hat{\imath} - (1+\cos\theta)\ \hat{\jmath}) \tag{7}$$

dot product $\boldsymbol{v}_1$ on both sides,

$$(\boldsymbol{v}_2 - \boldsymbol{v}_1)\cdot\boldsymbol{v}_1 = \frac{1}{4\pi\epsilon_0}\frac{Z_1Z_2e^2}{mr^2\dot{\theta}}(\sin\theta\ \hat{\imath} - (1+\cos\theta)\ \hat{\jmath})\cdot\boldsymbol{v}_1 \tag{8}$$

the initial velocity direction is $\boldsymbol{v}_1 = v\,\hat{i}$,

$$v(\cos\theta - 1) \;=\; \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{m r^2 \dot{\theta}}\sin\theta \tag{9}$$

due to $mr^2\dot{\theta} = -mvb$, where b is compact parameter,

$$\begin{aligned} b &= \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{mv^2}\frac{\sin\theta}{(1-\cos\theta)} \\ &= \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{mv^2}\cot\frac{\theta}{2} \\ &= \frac{1}{2}\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 E}\cot\frac{\theta}{2} \end{aligned} \tag{10}$$

$a \equiv \dfrac{Z_1 Z_2 e^2}{4\pi\epsilon_0 E}$

$$b = \frac{a}{2}\cot\frac{\theta}{2} \tag{11}$$

Consider the scattering of two structureless charged particles by Coulomb interaction, which is called Coulomb scattering or Rutherford scattering. The two charged particles represent the alpha particle and the nucleus of the target atom. Since the mass of electrons is extremely small, one can ignore the scattering by electrons. One to one correspondence holds between the impact parameter b and the scattering angle θ. The alpha particles which pass the area $2\pi b\mathrm{d}b$ of the impact parameter between b and $b + \mathrm{d}b$ are scattered to the region of solid angle $\mathrm{d}\Omega = 2\pi\sin\theta\mathrm{d}\theta$ around the scattering angle θ. The differential cross section is defined as the number of particles scattered to the region of solid angle $\mathrm{d}\Omega$ when there exists one incident particle per unit time and unit area.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = \frac{2\pi b}{|\mathrm{d}\theta/\mathrm{d}b|}\ . \tag{12}$$

Using

$$b = a\cot\left(\frac{\theta}{2}\right)\ , \tag{13}$$

$$a = \frac{Z_1 Z_2 e^2}{\mu\nu^2}\ , \tag{14}$$

which holds between the impact parameter b and the scattering angle θ in the case of Coulomb scattering,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\sigma_R}{\mathrm{d}\Omega} \equiv \frac{1}{2\pi \sin\theta} \frac{\mathrm{d}\sigma}{\mathrm{d}\theta} = \frac{a^2}{4} \frac{1}{\sin^4 \theta/2} \ . \tag{15}$$

μ is the reduced mass, and v is the speed of the relative motion in the asymptotic region, i.e., at the beginning of scattering. The characteristics of the Coulomb scattering are that the forward scattering is strong, but also that backward scattering takes place with a certain probability as well.

The distance of closest approach d and the scattering angle θ or the impact parameter b is related by

$$d = a \left[1 + \csc\left(\frac{\theta}{2}\right) \right] = a + \sqrt{a^2 + b^2} \ . \tag{16}$$

for the Rutherford scattering.