The Quantum Theory of Radiation

October 22, 2017

1 The Euler-Lagrange Equations

[1] The canonical variables $q_N(t)$ in general field theories are fields $\psi_n(\boldsymbol{x}, t)$, for which N is a compound index, including a discrete label n indicating the type of field and a spatial coordinate $\boldsymbol{x}$. Correspondingly, the Lagrangian $L(t)$ is a functional of $\psi_n(\boldsymbol{x}, t)$ and $\dot{\psi}_n(\boldsymbol{x}, t)$, depending on the form of all of the functions $\psi_n(\boldsymbol{x}, t)$ and $\dot{\psi}_n(\boldsymbol{x}, t)$ for all $\boldsymbol{x}$, but at a fixed time t. In consequence, the partial derivatives with respect to q_N and $\dot{q}_N$ in the equations of motion must be interpreted as functional derivatives with respect to $\psi_n(\boldsymbol{x}, t)$ and $\dot{\psi}_n(\boldsymbol{x}, t)$, so that these equations read

$$\frac{\partial}{\partial t}\left(\frac{\delta L(t)}{\delta \dot{\psi}_n(\boldsymbol{x}, t)}\right) = \frac{\delta L(t)}{\delta \psi_n(\boldsymbol{x}, t)} \tag{1}$$

where the functional derivatives $\delta L/\delta\dot{\psi}_n$ and $\delta L/\delta\psi_n$ are defined so that the change in the Lagrangian produced by independent infinitesimal changes $\delta\psi_n(\boldsymbol{x}, t)$ and $\delta\dot{\psi}_n(\boldsymbol{x}, t)$ in $\psi_n(\boldsymbol{x}, t)$ and $\dot{\psi}_n(\boldsymbol{x}, t)$ at a fixed time t is

$$\delta L(t) = \sum_n \int \mathrm{d}^3x \left[\frac{\delta L(t)}{\delta\psi_n(\boldsymbol{x}, t)}\delta\psi_n(\boldsymbol{x}, t) + \frac{\delta L(t)}{\delta\dot{\psi}_n(\boldsymbol{x}, t)}\delta\dot{\psi}_n(\boldsymbol{x}, t)\right] . \tag{2}$$

The canonical conjugate to $\psi_n(\boldsymbol{x}, t)$ is

$$\pi_n(\boldsymbol{x}, t) = \frac{\delta L(t)}{\delta\dot{\psi}_n(\boldsymbol{x}, t)} , \tag{3}$$

and in a theory with no constraints, the canonical commutation relations are

$$[\psi_n(\boldsymbol{x},t),\pi_m(\boldsymbol{y},t)] = i\hbar\delta_{nm}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \ , \tag{4}$$

$$[\psi_n(\boldsymbol{x},t),\psi_m(\boldsymbol{y},t)] = [\pi_n(\boldsymbol{x},t),\pi_m(\boldsymbol{y},t)] = 0 \ . \tag{5}$$

Typically (though not always), the Lagrangian in a field theory will be an integral of a local Lagrangian density $\mathcal{L}$:

$$L(t) = \int \mathrm{d}^3x \mathcal{L}\left(\psi(\boldsymbol{x},t), \nabla\psi(\boldsymbol{x},t), \dot{\psi}(\boldsymbol{x},t)\right) \ . \tag{6}$$

The variation of the Lagrangian action due to infinitesimal changes in the ψ_n and their space and time derivatives that vanish for $|\boldsymbol{x}| \to \infty$ is

$$\delta L(t) = \int \mathrm{d}^3x \sum_n \left[\frac{\partial\mathcal{L}}{\partial\psi_n}\delta\psi_n + \sum_i \frac{\delta\mathcal{L}}{\partial(\partial_i\psi_n)}\frac{\partial}{\partial x_i}\delta\psi_n + \frac{\partial\mathcal{L}}{\delta\dot{\psi}_n}\frac{\partial}{\partial t}\delta\psi_n\right] \ .$$

Integrating by parts,

$$\delta L(t) = \int \mathrm{d}^3x \sum_n \left[\left(\frac{\partial\mathcal{L}}{\partial\psi_n} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial\mathcal{L}}{\partial(\partial_i\psi_n)}\right)\delta\psi_n + \frac{\partial\mathcal{L}}{\delta\dot{\psi}_n}\frac{\partial}{\partial t}\delta\psi_n\right] \ . \tag{7}$$

This may be expressed as formulas for the variational derivatives of the Lagrangian

$$\frac{\delta L}{\delta\psi_n} = \frac{\partial\mathcal{L}}{\partial\psi_n} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial\mathcal{L}}{\partial(\partial_i\psi_n)} \ , \tag{8}$$

$$\frac{\delta L}{\delta\dot{\psi}_n} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_n} \ . \tag{9}$$

The Euler-Lagrange field equations are

$$\frac{\partial\mathcal{L}}{\partial\psi_n} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial\mathcal{L}}{\partial(\partial_i\psi_n)} = \frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\psi}_n} \ . \tag{10}$$

In relativistically invariant theories,

$$\frac{\partial\mathcal{L}}{\partial\psi_n} = \sum_\mu \frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_n)} \ . \tag{11}$$

In theories with a local Lagrangian density, the field variable that is canonically conjugate to $\psi_n(\boldsymbol{x},t)$ is

$$\pi_n = \frac{\delta L}{\delta\dot{\psi}_n} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_n} \ . \tag{12}$$

References

[1] S. Weinberg. *Lectures on Quantum Mechanics*. September 2015.