# CHEBYSHEV POLYNOMIALS

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[1] The Chebyshev equation

$$(1-x^2)y'' - xy' + n^2 y = 0 \ , \tag{1}$$

can be converted to an equation of Sturm-Liouville form by multiplying by the integrating factor $(1-x^2)^{-1/2}$.

$$\left[(1-x^2)^{1/2} y'\right]' + n^2 (1-x^2)^{-1/2} y = 0 \ . \tag{2}$$

The solutions, the Chebyshev polynomials $T_n(x)$, are given by a Rodrigues' formula:

$$T_n(x) = \frac{(-2)^n n! (1-x^2)^{1/2}}{(2n)!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (1-x^2)^{n-1/2} \ . \tag{3}$$

Their orthogonality over the range $-1 \leqslant x \leqslant 1$ and their normalisation are given by

$$\int_{-1}^{1} (1-x^2)^{-1/2} T_m(x) T_n(x) \mathrm{d}x = \begin{cases} 0 & \text{for} \quad m \neq n \ , \\ \pi/2 & \text{for} \quad n = m \neq 0 \ , \\ \pi & \text{for} \quad n = m = 0 \ , \end{cases} \tag{4}$$

and their generating function is

$$G(x,h) = \frac{1-xh}{1-2xh+h^2} = \sum_{n=0}^{\infty} T_n(x)h^2 \ . \tag{5}$$

[2] The generating function for the Legendre polynomials can be generalized to

$$\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x)t^n \ , \tag{6}$$

where the coefficients $C_n^{(\alpha)}(x)$ are known as the ultraspherical polynomials (also called Gegenbauer polynomials). For $\alpha = 1/2$, we recover the Legendre polynomials. The special cases $\alpha = 0$ and $\alpha = 1$ yield two types of Chebyshev polynomials. The primary importance of the Chebyshev polynomials is in numerical analysis.

# 1 Type I Polynomials

With $\alpha = 1$, $C_n^{(1)}(x)$ is written as $U_n(x)$,

# 2 Type II Polynomials

# References

[1] K.F. Riley, M.P. Hobson, and S.J. Bence. *Mathematical Methods for Physics and Engineering: A Comprehensive Guide*. Cambridge University Press, 2006.

[2] George B. Arfken and Hans J. Weber. *Mathematical Methods for Physicists, Seventh Edition: A Comprehensive Guide*. Academic Press, 7 edition, January 2012.