Parabolic Equations

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0.1 Three Dimensional Schemes

Consider the partial differential equation

$$v_t = \nu \nabla^2 v + F(x, y, z, t) \ , \tag{1}$$

and the obvious FTCS explicit scheme for approximating the solution of equation is

$$u_{jkl}^{n+1} = u_{jkl}^n + (r_x \delta_x^2 + r_y \delta_y^2 + r_z \delta_z^2) u_{jkl}^n + F_{jkl}^n \ . \tag{2}$$

The difference scheme is a $\mathcal{O}(\Delta t) + O(\Delta x^2) + O(\Delta y^2) + O(\Delta z^2)$ order approximation of partial differential equation. And the difference scheme is conditionally stable, with stability condition $r_x + r_y + r_z \leqslant 1/2$. The three dimensional BTCS scheme

$$u_{jkl}^{n+1} - (r_x \delta_x^2 + r_y \delta_y^2 + r_z \delta_z^2) u_{jkl}^{n+1} = u_{jkl}^n + F_{jkl}^n \ . \tag{3}$$

and the three dimensional Crank-Nicolson scheme

$$u_{jkl}^{n+1} - \frac{1}{2}(r_x \delta_x^2 + r_y \delta_y^2 + r_z \delta_z^2) u_{jkl}^{n+1} = u_{jkl}^n + \frac{1}{2}(r_x \delta_x^2 + r_y \delta_y^2 + r_z \delta_z^2) u_{jkl}^n + \frac{1}{2}\left(F_{jkl}^n + F_{jkl}^{n+1}\right) \tag{4}$$

are both unconditionally stable schemes which are $\mathcal{O}(\Delta t)+O(\Delta x^2)+O(\Delta y^2)+O(\Delta z^2)$ and $\mathcal{O}(\Delta t^2)+O(\Delta x^2)+O(\Delta y^2)+O(\Delta z^2)$, respectively. One approach is to use three dimensional Peaceman-Rachford scheme (along with the rationalization of using $\Delta t/3$ as time steps). But the three dimensional Peaceman-Rachford scheme is not unconditionally stable. And the three dimensional Peaceman-Rachford scheme is only first order accurate in time.

Factor the left hand side of equation of three dimensional Crank-Nicolson scheme

$$\left(1-\frac{r_x}{2}\delta_x^2\right)\left(1-\frac{r_y}{2}\delta_y^2\right)\left(1-\frac{r_z}{2}\delta_z^2\right)u_{jkl}^{n+1}\ , \tag{5}$$

If add

$$\left[\frac{r_x r_y}{4}\delta_x^2\delta_y^2+\frac{r_x r_z}{4}\delta_x^2\delta_z^2+\frac{r_y r_z}{4}\delta_y^2\delta_z^2\right]\left(u_{jkl}^{n+1}-u_{jkl}^n\right)-\frac{r_x r_y r_z}{8}\delta_x^2\delta_y^2\delta_z^2\left(u_{jkl}^{n+1}+u_{jkl}^n\right)\ , \tag{6}$$

to the left hand side of equation.

$$\begin{aligned}&\left(1-\frac{r_x}{2}\delta_x^2\right)\left(1-\frac{r_y}{2}\delta_y^2\right)\left(1-\frac{r_z}{2}\delta_z^2\right)u_{jkl}^{n+1}=\\&\left(1+\frac{r_x}{2}\delta_x^2\right)\left(1+\frac{r_y}{2}\delta_y^2\right)\left(1+\frac{r_z}{2}\delta_z^2\right)u_{jkl}^{n}+\frac{1}{2}\left(F_{jkl}^n+F_{jkl}^{n+1}\right)\ .\end{aligned} \tag{7}$$

It is equivalent to

$$\begin{aligned}&\left(1-\frac{r_x}{2}\delta_x^2\right)\left(1-\frac{r_y}{2}\delta_y^2\right)\left(1-\frac{r_z}{2}\delta_z^2\right)\left(u_{jkl}^{n+1}-u_{jkl}^n\right)=\\&\left(r_x\delta_x^2+r_y\delta_y^2+r_z\delta_z^2\right)u_{jkl}^n+\frac{r_x r_y r_z}{4}\delta_x^2\delta_y^2\delta_z^2u_{jkl}^n+\frac{1}{2}\left(F_{jkl}^n+F_{jkl}^{n+1}\right)\ .\end{aligned} \tag{8}$$

Drop the $\delta_x^2\delta_y^2\delta_z^2$ term and get the following form of the Douglas-Gunn scheme

$$\left(1-\frac{r_x}{2}\delta_x^2\right)\Delta u^*=\left(r_x\delta_x^2+r_y\delta_y^2+r_z\delta_z^2\right)u_{jkl}^n+\frac{1}{2}\left(F_{jkl}^n+F_{jkl}^{n+1}\right)\ , \tag{9}$$

$$\left(1-\frac{r_y}{2}\delta_y^2\right)\Delta u^{**}=\Delta u^*\ , \tag{10}$$

$$\left(1-\frac{r_z}{2}\delta_z^2\right)\Delta u=\Delta u^{**}\ , \tag{11}$$

$$\Delta u=u_{jkl}^{n+1}-u_{jkl}^n \tag{12}$$

The Douglas-Gunn scheme is accurate of order of $\mathcal{O}(\Delta t^2) + O(\Delta x^2) + O(\Delta y^2) + O(\Delta z^2)$.

The discrete Fourier transforms of the nonhomogeneous version of equations are

$$\left(1 + 2r_x \sin^2\frac{\xi}{2}\right)\widehat{\Delta u^*} = \left(-4r_x \sin^2\frac{\xi}{2} - 4r_y \sin^2\frac{\eta}{2} - 4r_z \sin^2\frac{\zeta}{2}\right)\hat{u}^n \ , \tag{13}$$

$$\left(1 + 2r_y \sin^2\frac{\eta}{2}\right)\widehat{\Delta u^{**}} = \widehat{\Delta u^*} \ , \tag{14}$$

$$\left(1 + 2r_z \sin^2\frac{\zeta}{2}\right)\widehat{\Delta u} = \widehat{\Delta u^{**}} \ , \tag{15}$$

$$\widehat{\Delta u} = \widehat{u}^{n+1} - \widehat{u}^n \ , \tag{16}$$

then

$$\begin{aligned}\widehat{u}^{n+1} = & \left[1 - 2r_x \sin^2\frac{\xi}{2} - 2r_y \sin^2\frac{\eta}{2} - 2r_z \sin^2\frac{\zeta}{2} + 4r_x r_y \sin^2\frac{\xi}{2}\sin^2\frac{\eta}{2}\right. \\ & \left. +4r_x r_z \sin^2\frac{\xi}{2}\sin^2\frac{\eta}{2} + 4r_y r_z \sin^2\frac{\eta}{2}\sin^2\frac{\zeta}{2} + 8r_x r_y r_z \sin^2\frac{\xi}{2}\sin^2\frac{\eta}{2}\sin^2\frac{\zeta}{2}\right] / \\ & \left[\left(1 + 2r_x \sin^2\frac{\xi}{2}\right)\left(1 + 2r_y \sin^2\frac{\eta}{2}\right)\left(1 + 2r_z \sin^2\frac{\zeta}{2}\right)\right]\widehat{u}^n = \rho(\xi, \eta, \zeta)\widehat{u}^n \ .\end{aligned} \tag{17}$$

The above expression is in the general form

$$\frac{1 - a - b - c + d + e + f + g}{1 + a + b + c + d + e + f + g} \ ,$$

where $a, \cdots, g$ are all positive and it is easy to see that

$$-1 \leqslant \frac{1 - a - b - c + d + e + f + g}{1 + a + b + c + d + e + f + g} \leqslant 1 \ .$$

Likewise, $|\rho(\xi, \eta, \zeta)| \leqslant 1$. Hence, the Douglas-Gunn scheme is unconditionally stable. Since it is both consistent and unconditionally stable, the Douglas-Gunn scheme is convergent.