Vibrating Systems

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1 Vibrations of Coupled Mass Points

[1] Consider the free vibration of two mass points, fixed to two walls by springs of equal spring constant. The two mass points shall have equal masses. The displacements from the rest positions are denoted by x_1 and x_2, respectively. We consider only vibrations along the line connecting the mass points. When displacing the mass 1 from the rest position, there acts the force $-kx_1$ by the spring fixed to the wall, and the force $+k(x_2 - x_1)$ by the spring connecting the two mass points.

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) \ , \tag{1}$$

$$m\ddot{x}_2 = -kx_2 + k(x_2 - x_1) \ . \tag{2}$$

The frequencies that are equal for all particles are called eigenfrequencies. The related vibrational states are called eigen- or normal vibrations. These definitions are correspondingly generalized for a N-particle system. We use the ansatz

$$x_1 = A_1 \cos \omega t \ , \tag{3}$$

$$x_2 = A_2 \cos \omega t \ , \tag{4}$$

i.e., both particles shall vibrate with the same frequency ω.

$$A_1(-m\omega^2 + 2k) - A_2 k = 0 \ , \tag{5}$$

$$-A_1 k + A_2(-m\omega^2 + 2k) = 0 \ . \tag{6}$$

The system of equations has nontrivial solutions for the amplitudes only if the determinant of coefficients D vanishes:

$$D = \begin{vmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + 2k \end{vmatrix} = (-m\omega^2 + 2k)^2 - k^2 = 0 \ .$$

An equation for determining the frequencies is

$$\omega^4 - 4\frac{k}{m}\omega^2 + 3\frac{k^2}{m^2} = 0 \ .$$

The positive solutions of the equation are the frequencies

$$\omega_1 = \sqrt{\frac{3k}{m}} \ ,$$
$$\omega_2 = \sqrt{\frac{k}{m}} \ .$$

These frequencies are called eigenfrequencies of the system. The corresponding vibrations are called eigenvibrations or normal vibrations.

$$A_1 = -A_2 \text{ for } \omega_1 = \sqrt{\frac{3k}{m}} \ , \tag{7}$$

$$A_1 = A_2 \text{ for } \omega_2 = \sqrt{\frac{k}{m}} \ . \tag{8}$$

The two mass points vibrate in-phase with the lower frequency ω_2, and with the higher frequency ω_1 against each other. The number of normal vibrations equals the number of coordinates (degrees of freedom) which are necessary for a complete description of the system. This leads to a determinant of rank N for ω^2, and therefore in general to N normal frequencies.

The general motion of the mass points corresponds to a superposition of the normal modes with different phase and amplitude.

$$x_1(t) = C_1 \cos(\omega_1 t + \varphi_1) + C_2 \cos(\omega_2 t + \varphi_2) \ , \tag{9}$$

$$x_2(t) = -C_1 \cos(\omega_1 t + \varphi_1) + C_2 \cos(\omega_2 t + \varphi_2) \ . \tag{10}$$

Here, we already utilized the result that x_1 and x_2 have opposite-equal amplitudes for a pure ω_1-vibration, and equal amplitudes for pure ω_2-vibrations.

2 The Vibrating String

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3 Coupled Oscillations

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3.1 Two Coupled Harmonic Oscillators

3.2 Weak Coupling

3.3 General Problem of Coupled Oscillations

Consider a conservative system described in terms of a set of generalized coordinates q_k and the time t. If the system has n degrees of freedom, $k = 1, 2, \cdots, n$. Specify that a configuration of stable equilibrium exists for the system and that at equilibrium the generalized coordinates have values q_{k0}. In such a configuration, Lagrange's equations are satisfied by

$$q_k = q_{k0} \ , \ \dot{q}_k = 0 \ , \ \ddot{q}_k = 0 \ , \ k = 1, 2, \cdots, n$$

Every nonzero term of the form $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k}$ must contain at least either $\dot{q}_k$ or $\ddot{q}_k$, so all such terms vanish at equilibrium. From Lagrange's equation,

$$\left.\frac{\partial L}{\partial q_k}\right|_0 = \left.\frac{\partial T}{\partial q_k}\right|_0 - \left.\frac{\partial U}{\partial q_k}\right|_0 = 0 \tag{11}$$

where 0 designates the quantity is evaluated at equilibrium.

We assume that the equations connecting the generalized coordinates and the regular coordinates do not explicitly contain the time, that is,

$$x_{\alpha,i} = x_{\alpha,i}(q_j) \text{ or } q_j = q_j(x_{\alpha,i}) \tag{12}$$

The kinetic energy is a homogeneous quadratic function of the generalized velocities

$$T = \frac{1}{2}\sum_{j,k} m_{jk}\dot{q}_j\dot{q}_k \tag{13}$$

$$\left.\frac{\partial T}{\partial q_k}\right|_0 = 0\ , \quad k = 1, 2, \cdots, n \tag{14}$$

$$\left.\frac{\partial U}{\partial q_k}\right|_0 = 0\ , \quad k = 1, 2, \cdots, n \tag{15}$$

The equations of motion are

$$\sum_j (A_{jk}q_j + m_{jk}\ddot{q}_j) = 0\ . \tag{16}$$

This is a set of n second-order linear homogeneous differential equations with constant coefficients.

The equations of motion become

$$\sum_j (A_{jk} - \omega^2 m_{jk})a_j = 0\ . \tag{17}$$

where the common factor $\exp[i(\omega t - \delta)]$ has been canceled. This is a set of n linear, homogeneous, algebraic equations that the a_j must satisfy. For a nontrivial solution to exist, the determinant of the coefficients must vanish:

$$|A_{jk} - \omega^2 m_{jk}| = 0 \ . \tag{18}$$

$$\begin{vmatrix} A_{11} - \omega^2 m_{11} & A_{12} - \omega^2 m_{12} & A_{13} - \omega^2 m_{13} \\ A_{12} - \omega^2 m_{12} & A_{22} - \omega^2 m_{22} & A_{23} - \omega^2 m_{23} \\ A_{13} - \omega^2 m_{13} & A_{23} - \omega^2 m_{23} & A_{33} - \omega^2 m_{33} \\ \vdots & \vdots & \vdots \end{vmatrix} = 0 \tag{19}$$

where the symmetry of the A_{jk} and m_{jk} has been explicitly included.

The equation represented by this determinant is called the characteristic equation or secular equation of the system and is an equation of degree n in ω^2. There are in general n roots we may label ω_r^2. The ω_r are called the characteristic frequencies or eigenfrequencies of the system. (In some situations, two or more of the ω_r can be equal. This is the phenomenon of degeneracy.) Each of the roots of the characteristic equation may be substituted into Equ. (17) to determine the ratios $a_1 : a_2 : a_3 \cdots : a_n$ for each value of ω_r. Because there are n values of ω_r, we can construct n sets of ratios of the a_j. Each of the sets defines the components of n-dimensional vector $\boldsymbol{a}_r$, called an eigenvector of the system. $\boldsymbol{a}_r$ is the eigenvector associated with the eigenfrequency ω_r. Designate by a_{jr} the jth component of the r-th eigenvector.

The principle of superposition applies for the differential equation, write the general solution for q_j as a linear combination of the solutions for each of the n values of r

$$q_j(t) = \sum_r a_{jr} e^{i(\omega_r t - \delta_r)} \tag{20}$$

Because it is only the real part of $q_j(t)$ that is physically meaningful,

$$q_j(t) = \mathbf{Re} \sum_r a_{jr} e^{i(\omega_r t - \delta_r)} = \sum_r a_{jr} \cos(\omega_r t - \delta_r) \tag{21}$$

The motion of the coordinate q_j is compound of motions with each of the n values of the frequencies ω_r. The q_j are not the normal coordinates.

3.4 Orthogonality of the Eigenvectors

3.5 Normal Coordinates

3.6 Molecular Vibrations

3.7 Three Linearly Coupled Plane Pendula

3.8 The Loaded String

Consider an elastic string(or a spring) on which a number of identical particles are placed at regular intervals. The ends of the string are constrained to remain stationary. Let the mass of each of the n particles be m, and let the spacing between particle at equilibrium be d. The length of the string is $L = (n+1)d$.

References

[1] W. Greiner. *Classical Mechanics: Systems of Particles and Hamiltonian Dynamics*. Classical theoretical physics. Springer Berlin Heidelberg, 2009.

[2] Stephen Thornton and Jerry Marion. *Classical Dynamics of Particles and Systems*. 5th edition edition, 2004.