Curved manifolds

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1 On the relation of gravitation to curvature

The existence of inertial frames that fill all of spacetime: all of spacetime can be described by a single frame, all of whose coordinate points are always at rest relative to the origin, and all of whose clocks run at the same rate relative to the origins clock.

The metric of SR is defined physically by lengths of rods and readings of clocks.

In a nonuniform gravitational field, it is impossible to construct a frame in which the clocks all run at the same rate. The gravitational fields are incompatible with global SR: the ability to construct a global inertial frame. However in small regions of spacetime - regions small enough that nonuniformities of the gravitational forces are too small to measure - we can always construct a 'local' SR frame.

The clocks don't all run at the same rate in a gravitational field.

1.1 The gravitational redshift experiment

1.2 Nonexistence of a Lorentz frame at rest on Earth

1.3 The principle of equivalence

1.4 The redshift experiment again

1.5 Tidal forces

Nonuniformities in gravitational fields are called tidal forces, since they are the ones that raise tides. These tidal forces prevent the construction of global inertial frames.

1.6 The role of curvature

2 Tensor algebra in polar coordinates

3 Tensor calculus in polar coordinates

3.1 The Christoffel symbols

3.2 The covariant derivative

3.3 Divergence and Laplacian

3.4 Derivatives of one-forms and tensors of higher types

4 Christoffel symbols and the metric

5 Noncoordinate bases

6 Differentiable manifolds and tensors

A manifold is essentially a continuous space which looks locally like Euclidean space.

7 Riemannian manifolds

7.1 Lengths and volumes

7.2 Proof of the local-flatness theorem

8 Covariant differentiation

8.1 Divergence formula

9 Parallel-transport, geodesics, and curvature

10 The curvature tensor

11 Bianchi identities: Ricci and Einstein tensors

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda} = 0 \ . \tag{1}$$

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0 \ . \tag{2}$$

Since in our coordinates ${\Gamma^\mu}_{\alpha\beta} = 0$ at this point, this is equivalent to

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0 \ . \tag{3}$$

11.1 The Ricci tensor

The Ricci tensor is defined as

$$R_{\alpha\beta} := R^{\mu}{}_{\alpha\mu\beta} = R_{\beta\alpha} \ . \tag{4}$$

It is the contraction of $R^{\mu}{}_{\alpha\nu\beta}$ on the first and third indices. Other contractions would in principle also be possible: on the first and second, the first and fourth, etc. Because $R_{\alpha\beta\mu\nu}$ is antisymmetric on α and β and on μ and ν, all these contractions either vanish identically or reduce to $\pm R_{\alpha\beta}$. The Ricci tensor is essentially the only contraction of the Riemann tensor. It is a symmetric tensor. The Ricci scalar is defined as

$$R := g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} g^{\alpha\beta} R_{\alpha\mu\beta\nu} \ . \tag{5}$$