Eigenvalues and Eigenvectors

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1 Eigenvectors and Eigenvalues

If A is a stochastic matrix, then the steady-state vector $\boldsymbol{q}$ for A satisfies the equation $A\boldsymbol{x} = \boldsymbol{x}$, i.e. $A\boldsymbol{q} = 1 \cdot \boldsymbol{q}$.

Definition

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\boldsymbol{x}$ of $A\boldsymbol{x} = \lambda\boldsymbol{x}$; such an $\boldsymbol{x}$ is called an eigenvector corresponding to λ. An eigenvector must be nonzero, but an eigenvalue may be zero.

Although row reduction can be used to find eigenvalues, but it can not be used to find eigenvectors. An echelon form of a matrix $\mathbf{A}$ usually does not display the eigenvalues of $\mathbf{A}$.

λ is an eigenvalue of $\mathbf{A}$ if and only if the equation

$$(A - \lambda\boldsymbol{I})\boldsymbol{x} = \mathbf{0} \tag{1}$$

has a nontrivial solution. The set of all solutions of (1) is just the null space of the matrix $A\lambda I$. This set is a subspace of $\mathbb{R}^n$ and is called the eigenspace of $\mathbf{A}$ corresponding to λ. The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

Theorem

The eigenvalues of a triangular matrix are the entries on its main diagonal.

0 is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A}$ is not invertible.

Theorem

If $\mathbf{v}_1, \cdots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_r$ of an $n \times n$ matrix $\mathbf{A}$, then the set $\{\mathbf{v}1, \cdots, \mathbf{v}_r\}$ is linearly independent.

2 Characteristic Equation

Let $\mathbf{A}$ be an $n \times n$ matrix, U be any echelon form obtained from $\mathbf{A}$ by row replacements and row interchanges (without scaling), and r be the number of such row interchanges. The determinant of $\mathbf{A}$, written as det A, is $(1)^r$ times the product of the diagonal entries $u_{11}, \cdots, u_{nn}$ in U. If $\mathbf{A}$ is invertible, then $u_{11}, \cdots, u_{nn}$ are all pivots (because $\mathbf{A} \sim I_n$ and the u_{ii} have not been scaled to 1's). Otherwise, at least unn is zero, and the product $u_{11} \cdots u_{nn}$ is zero.

2.1 Determinants

Let A be an $n \times n$ matrix, U be any echelon form obtained from A by row replacements and row interchanges (without scaling), and r be the number of such row interchanges. Then the determinant of A, written as det A, is $(-1)^r$ times the product of the diagonal entries $u_{11}, \cdots, u_{nn}$ in U. If A is invertible, then $u_{11}, \cdots, u_{nn}$ are all pivots (because

$A \sim I_n$ and the u_{ii} have not been scaled to 1's). Otherwise, at least u_{nn} is zero, and the product $u_{11} \cdots u_{nn}$ is zero.

$$\det A = \begin{cases} (-1)^r \cdot (\text{product of pvots in } U), & \text{when } A \text{ is invertible} \\ \\ 0, & \text{when } A \text{ is not invertible} \end{cases} \tag{2}$$

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if :

s. The number 0 is not an eigenvalue of A.

t. The determinant of A is not zero.

Properties of Determinants

Let A and B be an $n \times n$ matrix.

a. A is invertible if and only if det $A \neq 0$.

b. det $AB = (\det A)(\det B)$.

c. det $A^T =$ det A.

d. If A is triangular, then det A is the product of the entries on the main diagonal of A.

e. A row replacement operation on A does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

2.2 The Characteristic Equation

The scalar equation det $(A - \lambda \boldsymbol{I}) = 0$ is called the characteristic equation of A.

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation

$$\det(A - \lambda \boldsymbol{I}) = 0 \; .$$

3 Diagonalization

The eigenvalue-eigenvector information contained within a matrix A can be displayed in a useful factorization of the form $A = PDP^{-1}$.

A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. Such a basis is called an eigenvector basis.

3.1 Diagonalizing Matrices

Theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

3.2 Matrices Whose Eigenvalues Are Not Distinct

Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \cdots, \lambda_p$.

a. For $1 \leqslant k \leqslant p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n, and this happens if and only if the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

c. If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \cdots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.

4 Eigenvectors and Linear Transformations

Any linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be implemented via left-multiplication by a matrix A, called the standard matrix of T.

4.1 The Matrix of a Linear Transformation

Let V be an n-dimensional vector space, W an m-dimensional vector space, and T any linear transformation from V to W. To associate a matrix with T, choose (ordered) bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, respectively.

4.2 Linear Transformations from V into V

4.3 Linear Transformations on $\mathbb{R}^n$

Diagonal Matrix Representation

Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$.

4.4 Similarity of Matrix Representations

5 Complex Eigenvalues

6 Discrete Dynamical Systems

7 Applications to Differential Equations

8 Iterative Estimates for Eigenvalues