其它

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1 最速下降法

求下述类型复积分对于大 k 值的渐近逼近方法：

$$\int g(z)\exp[kf(z)]\mathrm{d}z \tag{1}$$

式中 $g(z)$ 和 $f(z)$ 与 k 无关；

现把 k 看作正实数，但对于 k 是复值结果一般也是正确的，这需要引述几点关于一个复变量 (ζ) 函数的渐近展开：

渐近展开定义：若

$$F(\zeta)=\sum_{m=0}^{n}\frac{a_m}{\zeta^m}+R_n(\zeta), \tag{2}$$

对于一个给定区间内的 $\arg\zeta$，当 $\zeta\to\infty$ 时，对所有的 n，$\zeta^n R_n(\zeta)\to 0$，$a_0,a_1,\cdots,a_n$ 是常数，则可写成

$$F(\zeta)\sim a_0+\frac{a_1}{\zeta}+\frac{a_2}{\zeta^2}+\cdots \tag{3}$$

右方称为对于该 $\arg\zeta$ 给定范围的 $F(\zeta)$ 的渐近展开；

若 $F(\zeta)$ 是两个函数 $G(\zeta)$ 和 $H(\zeta)$ 的商，则可写成

$$G(\zeta)\sim H(\zeta)\left(a_0+\frac{a_1}{\zeta}+\frac{a_2}{\zeta^2}+\cdots\right) \tag{4}$$

渐近展开的主要性质：

若 $|\zeta|$ 充分大时，(3) 式的级数停止或收敛，则该级数是渐近的；但对于 $|\zeta|$ 的任意值，它常常未能收敛。

一般地说，对于给定的 $F(\zeta)$，一种特殊的展开只对 $\arg\zeta$ 的一个特定范围成立；若它对所有的 $\arg\zeta$ 都成立，则它是收敛的。

对于给定的 $F(\zeta)$，在 $\arg\zeta$ 的适当范围内，渐近展开是唯一的，即 (3) 式中各系数都是唯一的；另一方面，任一函数的渐近展开也属于无穷多个其他函数；

两个函数乘积的渐近展开由它们各自的渐近展开相乘得到；

(3) 式可逐项积分，从而无条件地得出 $F(\zeta)$ 积分的渐近展开；

它也可以逐项微分，只要它存在，就得出 $F(\zeta)$ 微分的渐近展开；

一般，对于一个给定的 (充分大的)$|\zeta|$ 值，(3) 式各项的模开头逐项减小到一极小值，随后又增大；若对展开式求和到最小项前任何项，则误差具有第一个省略项的数量级；$|\zeta|$ 值越大，可用的准确度越高；在物理应用这，通常只用第一项就足够了；

得出 (1) 式按 k 的逆幂渐近展开的方法的根据是将它与下述形式的积分联系起来：

$$\int_0^\infty h(\mu)\exp[-k\mu^2]\mathrm{d}\mu \tag{5}$$

为了能导出 (5) 式的渐近展开，把 $h(\mu)$ 展开为 μ 的升幂级数，并逐项积分；

Watson 引理：

设

$$h(\mu)=\frac{1}{\mu^\alpha}\sum_{s=0}^{\infty}c_s\mu^{\beta s}, \tag{6}$$

其收敛半径为 ρ，β 是正实数，且 α 的实部小于 1；设存在一个实数 d，使得对所有大于 ρ 的实数值 μ，$\mu^\alpha\exp[-d\mu^2]h(\mu)$ 有界，则

$$\begin{aligned}&\frac{1}{2k^{(1-\alpha)/2}}\left\{c_0\Gamma\left(\frac{-\alpha+1}{2}\right)+c_1\Gamma\left(\frac{\beta-\alpha+1}{2}\right)\frac{1}{k^{\beta/2}}\right.\\&\left.+c_2\Gamma\left(\frac{2\beta-\alpha+1}{2}\right)\frac{1}{k^{\beta}}+c_3\Gamma\left(\frac{3\beta-\alpha+1}{2}\right)\frac{1}{k^{3\beta/2}}+\cdots\right\}\end{aligned} \tag{7}$$

是 (5) 式的渐近展开；Γ 表示伽马函数；

为了能改变积分变量，以便用一个或多个 (5) 式类型的积分来表示 (1) 式，须用一些线段组成积分路线，沿着其中每一线段，$f(z)$ 的虚部为常数，而 $f(z)$ 的实部单调下降到 $-\infty$；若不是这样，则第一步要适当改变路线；路线的改变由复平面内积分的基本规则所支配；这里只说明，怎样才能利用一些具有所需性质的线段使路线闭合，并已假定可尝试用标准方法对任何余下的围道积分求值；

2 稳相法

3 Dyson Series

3.1 Hamiltonian derivation

[1] We reproduce the position-space Feynman rules using time-dependent perturbation theory. Instead of assuming that the quantum field satisfies the Euler-Lagrange equations, we

instead assume its dynamics is determined by a Hamiltonian H by the Heisenberg equations of motion $i\partial_t\phi(x) = [\phi, H]$. The formal solution of this equation is

$$\phi(\boldsymbol{x}, t) = S(t, t_0)^\dagger \phi(\boldsymbol{x}) S(t, t_0) \ , \tag{8}$$

where $S(t, t_0)$ is the time-evolution operator (the S-matrix) that satisfies

$$i\partial_t S(t, t_0) = H(t) S(t, t_0) \ . \tag{9}$$

These are the dynamical equations in the Heisenberg picture where all the time dependence is in operators. States including the vacuum state $|\Omega\rangle$ in the Heisenberg picture are, by definition, time independent.

The Hamiltonian can either be defined at any given time as a functional of the fields $\phi(\boldsymbol{x})$ and $\pi(\boldsymbol{x})$ or equivalently as a functional of the creation and annihilation operators $a_p^\dagger$ and a_p. We will not need an explicit form of the Hamiltonian for this derivation so we just assume it is some time-dependent operator $H(t)$.

The first step in time-dependent perturbation theory is to write the Hamiltonian as

$$H(t) = H_0 + V(t) \ , \tag{10}$$

where the time evolution induced by H_0 can be solved exactly and V is small in some sense. For example, H_0 could be the free Hamiltonian, which is time independent, and V might be a ϕ^3 interaction:

$$V(t) = \int \mathrm{d}^3x \frac{g}{3!}\phi(\boldsymbol{x}, t)^3 \ . \tag{11}$$

The operators $\phi(\boldsymbol{x}, t), H, H_0$ and V are all in the Heisenberg picture.

Next, we need to change to the interaction picture. In the interaction picture the fields evolve only with H_0. The interaction picture fields are just what we had been calling (and will continue to call) the free fields:

$$\phi_0(\boldsymbol{x}, t) = e^{iH_0(t-t_0)}\phi(\boldsymbol{x})e^{-iH_0(t-t_0)} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p e^{-ipx} + a_p^\dagger e^{ipx}) \ . \tag{12}$$

To be precise, $\phi(\boldsymbol{x})$ is the Schrödinger picture field, which does not change with time. The free fields are equal to the Schrödinger picture fields and also to the Heisenberg picture fields, by definition, at a single reference time, which we call t_0.

The Heisenberg picture fields are related to the free fields by

$$\begin{aligned}\phi(\boldsymbol{x}, t) &= S^\dagger(t, t_0)e^{-iH_0(t-t_0)}\phi_0(\boldsymbol{x}, t)e^{iH_0(t-t_0)}S(t, t_0) \\ &= U^\dagger(t, t_0)\phi_0(\boldsymbol{x}, t)U(t, t_0) \ .\end{aligned} \tag{13}$$

The operator $U(t,t_0) \equiv e^{iH_0(t-t_0)}S(t,t_0)$ therefore relates the full Heisenberg picture fields to the free fields at the same time t. The evolution begins from the time t_0 where the fields in the two pictures (and the Schrödinger picture) are equal.

a differential equation for $U(t,t_0)$ is

$$\begin{aligned} i\partial_t U(t,t_0) &= i\left(\partial_t e^{iH_0(t-t_0)}\right)S(t,t_0) + e^{iH_0(t-t_0)} i\partial_t S(t,t_0) \\ &= -e^{iH_0(t-t_0)}H_0 S(t,t_0) + e^{iH_0(t-t_0)}H(t)S(t,t_0) \\ &= e^{iH_0(t-t_0)}[-H_0 + H(t)]e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}S(t,t_0) \\ &= V_I(t)U(t,t_0) \ , \end{aligned} \tag{14}$$

where $V_I(t) \equiv e^{iH_0(t-t_0)}V(t)e^{-iH_0(t-t_0)}$ is the original Heisenberg picture potential $V(t)$, now expressed in the interaction picture.

If everything commuted, the solution to Eq. (14) would be $U(t,t_0) = \exp\left(-i\int_{t_0}^t V_I(t')\mathrm{d}t'\right)$. But $V_I(t_1)$ does not necessarily commute with $V_I(t_2)$, so this is not the right answer. It turns out that the right answer is very similar:

$$U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^t \mathrm{d}t' V_I(t')\right]\right\} \ , \tag{15}$$

where $T\{\}$ is the time-ordering operator. This solution works because time ordering effectively makes everything inside commute:

$$T\{A\cdots B\cdots\} = T\{B\cdots A\cdots\} \ . \tag{16}$$

Since it has the right boundary conditions, namely $U(t,t) = 1$, this solution is unique.

Time ordering of an exponential is defined in the obvious way through its expansion:

$$U(t,t_0) = 1 - i\int_{t_0}^t \mathrm{d}t' V_I(t') - \frac{1}{2}\int_{t_0}^t \mathrm{d}t' \int_{t_0}^t \mathrm{d}t'' T\{V_I(t')V_I(t'')\} + \cdots \tag{17}$$

This is known as a Dyson series. Dyson defined the time-ordered product and this series in his classic paper. In that paper he showed the equivalence of old-fashioned perturbation theory or, more exactly, the interaction picture method developed by Schwinger and Tomonaga based on time-dependent perturbation theory, and Feynman's method, involving space-time diagrams, which we are about to get to.

3.1.1 Perturbative solution for the Dyson series

Removing the subscript on V for simplicity, the differential equation we want to solve is

$$i\partial_t U(t,t_0) = V(t)U(t,t_0) \ . \tag{18}$$

Integrating this equation lets us write it in an equivalent form:

$$U(t,t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' V(t') U(t', t_0) \ , \tag{19}$$

where 1 is the appropriate integration constant so that $U(t_0, t_0) = 1$.

Solve the integral equation order-by-order in V. At zeroth order in V,

$$U(t,t_0) = 1 \ . \tag{20}$$

To first order in V,

$$U(t,t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' V(t') + \cdots \ . \tag{21}$$

To second order,

$$\begin{aligned} U(t,t_0) &= 1 - i\int_{t_0}^{t} \mathrm{d}t' V(t') \left[1 - i\int_{t_0}^{t'} \mathrm{d}t'' V(t'') + \cdots\right] \\ &= 1 - i\int_{t_0}^{t} \mathrm{d}t' V(t') + (-i)^2 \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' V(t') V(t'') + \cdots \ . \end{aligned} \tag{22}$$

The second integral has $t_0 < t'' < t' < t$, which is the same as $t_0 < t'' < t$ and $t'' < t' < t$. So it can also be written as

$$\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' V(t')V(t'') = \int_{t_0}^{t} \mathrm{d}t'' \int_{t''}^{t} \mathrm{d}t' V(t')V(t'') = \int_{t'}^{t} \mathrm{d}t'' \int_{t_0}^{t} \mathrm{d}t' V(t'')V(t') \tag{23}$$

where we have relabeled $t'' \leftrightarrow t'$ and swapped the order of the integrals to form. Averaging the first and third form gives

$$\begin{aligned} \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' V(t')V(t'') &= \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t' \left[\int_{t_0}^{t'} \mathrm{d}t'' V(t')V(t'') + \int_{t'}^{t} \mathrm{d}t'' V(t'')V(t')\right] \\ &= \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t} \mathrm{d}t'' T\left\{V(t')V(t'')\right\} \ . \end{aligned} \tag{24}$$

Thus

$$U(t,t_0) = 1 - i\int_{t_0}^{t} \mathrm{d}t' V(t') + \frac{(-i)^2}{2}\int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t} \mathrm{d}t'' T\left\{V(t')V(t'')\right\} + \cdots \ . \tag{25}$$

Continuing this way, we find, restoring the subscript on V, that

$$U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} \mathrm{d}t' V_I(t')\right]\right\} \ . \tag{26}$$

3.2 Perturbation Theory

[2] The technique that has historically been most useful in calculating the S-matrix is perturbation theory, an expansion in powers of the interaction term V in the Hamiltonian $H = H_0 + V$.

The S-matrix is

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2i\pi\delta(E_\beta - E_\alpha)T^\dagger_{\beta\alpha}$$
$$T^\dagger_{\beta\alpha} = (\Phi_\beta, V\Psi^\dagger_\alpha) \ ,$$

where $\Psi^\dagger_\alpha$ satisfies the Lippmann-Schwinger equation

$$\Psi^\dagger_\alpha = \Phi_\alpha + \int \mathrm{d}\gamma \frac{T^\dagger_{\gamma\alpha}\Phi_\gamma}{E_\alpha - E_\gamma + i\epsilon} \ . \tag{27}$$

Operating on this equation with V and taking the scalar product with Φ_β yields an integral equation for $T^\dagger$

$$T^\dagger_{\beta\alpha} = V_{\beta\alpha} + \int \mathrm{d}\gamma \frac{V_{\beta\gamma}T^\dagger_{\gamma\alpha}}{E_\alpha - E_\gamma + i\epsilon} \ , \tag{28}$$

where

$$V_{\beta\alpha} \equiv (\Phi_\beta, V\Phi_\alpha) \ . \tag{29}$$

The perturbation series for $T^\dagger_{\gamma\alpha}$ is obtained by iteration from Eq. (28)

$$T^\dagger_{\beta\alpha} = V_{\beta\alpha} + \int \mathrm{d}\gamma \frac{V_{\beta\gamma}V_{\gamma\alpha}}{E_\alpha - E_\gamma + i\epsilon} + \int \mathrm{d}\gamma \mathrm{d}\gamma' \frac{V_{\beta\gamma}V_{\gamma\gamma'}V_{\gamma'\alpha}}{(E_\alpha - E_{\gamma'} + i\epsilon)(E_\alpha - E_\gamma + i\epsilon)} + \cdots \ . \tag{30}$$

The method of calculation based on Eq. (30), which dominated calculations of the S-matrix in the 1930s, is today known as old-fashioned perturbation theory.

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau, \tau_0) = V(\tau)U(\tau, \tau_0) \ , \tag{31}$$

where

$$V(t) \equiv \exp(iH_0t)V\exp(-iH_0t) \tag{32}$$

(Operators with this sort of time-dependence are said to be defined in the interaction picture, to distinguish their time-dependence from the time-dependence $O_H(t) = \exp(iHt)O_H\exp(-iHt)$ required in the Heisenberg picture of quantum mechanics.)

The time-ordered product of n Vs is a sum over all $n!$ permutations of the Vs, each of which gives the same integral over all $t_1, t_2, \cdots t_n$,

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}t_1\mathrm{d}t_2 \cdots \mathrm{d}t_n T\{V(t_1)\cdots V(t_n)\} \ . \tag{33}$$

This is sometimes known as the Dyson series. This series can be summed if the $V(t)$ at different times all commute. The sum is then

$$S = \exp\left(-i\int_{-\infty}^{\infty} \mathrm{d}tV(t)\right) \ . \tag{34}$$

4 Sokhotski - Plemelj theorem

Let C be a smooth closed simple curve in the plane, and φ an analytic function on C. Then the Cauchy-type integral

$$\frac{1}{2\pi i}\int_C \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} \ , \tag{35}$$

defines two analytic functions of z, ϕ_i inside C and ϕ_e outside. The Sokhotski-Plemelj formulas relate the limiting boundary values of these two analytic functions at a point z on C and the Cauchy principal value $\mathcal{P}$ of the integral:

$$\phi_i(z) = \frac{1}{2\pi i}\mathcal{P}\int_C \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2}\varphi(z) \ , \tag{36}$$

$$\phi_e(z) = \frac{1}{2\pi i}\mathcal{P}\int_C \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2}\varphi(z) \ . \tag{37}$$

Subsequent generalizations relaxed the smoothness requirements on curve C and the function ϕ.

Let f be a complex-valued function which is defined and continuous on the real line, and let a and b be real constants with $a < 0 < b$. Then

$$\lim_{\varepsilon\to 0^+}\int_a^b \frac{f(x)}{x \pm i\varepsilon}\,dx = \mp i\pi f(0) + \mathcal{P}\int_a^b \frac{f(x)}{x}\,dx, \tag{38}$$

where $\mathcal{P}$ denotes the Cauchy principal value. (Note that this version makes no use of analyticity.)

$$\lim_{\varepsilon\to 0^+}\int_a^b \frac{f(x)}{x \pm i\varepsilon}\,dx = \mp i\pi \lim_{\varepsilon\to 0^+}\int_a^b \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}f(x)\,dx + \lim_{\varepsilon\to 0^+}\int_a^b \frac{x^2}{x^2+\varepsilon^2}\,\frac{f(x)}{x}\,dx \ . \tag{39}$$

For the first term, $\varepsilon/\pi(x^2+\varepsilon^2)$ is a nascent delta function, and therefore approaches a Dirac delta function in the limit. Therefore, the first term equals $\mp i\pi f(0)$.

For the second term, the factor $x^2/(x^2+\varepsilon^2)$ approaches 1 for $|x| \gg \varepsilon$, approaches 0 for $|x| \ll \varepsilon$, and is exactly symmetric about 0. Therefore, in the limit, it turns the integral into a Cauchy principal value integral.

4.1 The Sokhotski-Plemelj formula

The Sokhotski-Plemelj formula is a relation between the following generalized functions (also called distributions),

$$\lim_{\varepsilon\to 0}\frac{1}{x \pm i\varepsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x) \ , \tag{40}$$

where $\varepsilon > 0$ is an infinitesimal real quantity. This identity formally makes sense only when first multiplied by a function $f(x)$ that is smooth and non-singular in a neighborhood of the origin, and then integrated over a range of x containing the origin. We shall also assume that $f(x) \to 0$ sufficiently fast as $x \to \pm\infty$ in order that integrals evaluated over the entire real line are convergent. Moreover, all surface terms at $\pm\infty$ that arise when integrating by parts are assumed to vanish.

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x\pm i\varepsilon} = \mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x} \mp i\pi f(0)\ , \tag{41}$$

where the Cauchy principal value integral is defined as:

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x} \equiv \lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{f(x)\mathrm{d}x}{x} + \int_{\delta}^{\infty}\frac{f(x)\mathrm{d}x}{x}\right\}\ , \tag{42}$$

assuming $f(x)$ is regular in a neighborhood of the real axis and vanishes as $|x| \to 0$.

A generalization is

$$\lim_{\varepsilon\to 0}\frac{1}{x-x_0\pm i\varepsilon} = \mathcal{P}\frac{1}{x-x_0} \mp i\pi\delta(x-x_0)\ , \tag{43}$$

where

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x-x_0} \equiv \lim_{\delta\to 0}\left\{\int_{-\infty}^{x_0-\delta}\frac{f(x)\mathrm{d}x}{x-x_0} + \int_{x_0+\delta}^{\infty}\frac{f(x)\mathrm{d}x}{x-x_0}\right\} \tag{44}$$

4.2 Derivation 1

$$\frac{1}{x\pm i\varepsilon} = \frac{x\mp i\varepsilon}{x^2+\varepsilon^2}\ , \tag{45}$$

where ε is is a positive infinitesimal quantity. Thus, for any smooth function that is non-singular in a neighborhood of the origin,

$$\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x\pm i\varepsilon} = \int_{-\infty}^{\infty}\frac{xf(x)\mathrm{d}x}{x^2+\varepsilon^2} \mp i\varepsilon\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x^2+\varepsilon^2} \tag{46}$$

$$\int_{-\infty}^{\infty}\frac{xf(x)\mathrm{d}x}{x^2+\varepsilon^2} = \int_{-\infty}^{-\delta}\frac{xf(x)\mathrm{d}x}{x^2+\varepsilon^2} + \int_{\delta}^{\infty}\frac{xf(x)\mathrm{d}x}{x^2+\varepsilon^2} + \int_{-\delta}^{\delta}\frac{xf(x)\mathrm{d}x}{x^2+\varepsilon^2} \tag{47}$$

In the first two integrals, it is safe to take the limit $\varepsilon \to 0$. In the third integral, if δ is small enough, then we can approximate $f(x) \simeq f(0)$ for values of $|x| < \delta$.

$$\int_{-\infty}^{\infty}\frac{xf(x)\mathrm{d}x}{x^2+\varepsilon^2} = \lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{f(x)\mathrm{d}x}{x} + \int_{\delta}^{\infty}\frac{f(x)\mathrm{d}x}{x}\right\} + f(0)\int_{-\delta}^{\delta}\frac{x\mathrm{d}x}{x^2+\varepsilon^2}\ . \tag{48}$$

However,

$$\int_{-\delta}^{\delta}\frac{x\mathrm{d}x}{x^2+\varepsilon^2} = 0\ , \tag{49}$$

since the integrand is an odd function of x that is being integrated symmetrically about the origin, and

$$\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x}\equiv\lim_{\delta\to 0}\left\{\int_{-\infty}^{-\delta}\frac{f(x)\mathrm{d}x}{x}+\int_{\delta}^{\infty}\frac{f(x)\mathrm{d}x}{x}\right\} \tag{50}$$

defines the principal value integral. Hence,

$$\int_{-\infty}^{\infty}\frac{xf(x)\mathrm{d}x}{x^2+\varepsilon^2}=\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x}\ . \tag{51}$$

Since ε is an infinitesimal quantity, the only significant contribution from

$$\varepsilon\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x^2+\varepsilon^2} \tag{52}$$

can come from the integration region where $x\simeq 0$, where the integrand behaves like ε^{-2}. Approximate $f(x)\simeq f(0)$,

$$\varepsilon\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x^2+\varepsilon^2}\simeq\varepsilon f(0)\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2+\varepsilon^2}=\pi f(0)\ , \tag{53}$$

where

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{x^2+\varepsilon^2}=\frac{1}{\varepsilon}\tan^{-1}\left(\frac{x}{\varepsilon}\right)\Big|_{-\infty}^{\infty}=\frac{\pi}{\varepsilon}\ . \tag{54}$$

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x\pm i\varepsilon}=\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x}\mp i\pi f(0)\ . \tag{55}$$

4.3 Derivation 2

Consider the following path of integration in the complex plane, denoted by C. C is the contour along the real axis from $-\infty$ to $-\delta$, followed by a semicircular path C_δ (of radius δ), followed by the contour along the real axis from δ to ∞. The infinitesimal quantity δ is assumed to be positive. Then

$$\int_C\frac{f(x)\mathrm{d}x}{x}=\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x}+\int_{C_\delta}\frac{f(x)\mathrm{d}x}{x} \tag{56}$$

In the limit of $\delta\to 0$, we can approximate $f(x)\simeq f(0)$ in the last integral. Noting that the contour C_δ can be parameterized as $x=\delta e^{i\theta}$ for $0\leqslant\theta\leqslant\pi$, we end up with

$$\lim_{\delta\to 0}\int_{C_\delta}\frac{f(x)\mathrm{d}x}{x}=f(0)\lim_{\delta\to 0}\int_{\pi}^{0}\frac{i\delta e^{i\delta}}{\delta e^{i\delta}}\mathrm{d}\theta=-i\pi f(0)\ . \tag{57}$$

Hence

$$\int_C\frac{f(x)\mathrm{d}x}{x}=\mathcal{P}\int_{-\infty}^{\infty}\frac{f(x)\mathrm{d}x}{x}-i\pi f(0)\ . \tag{58}$$

By deforming the contour C to a contour C' that consists of a straight line that runs from $-\infty+i\varepsilon$ to $\infty+i\varepsilon$, where ε is a positive infinitesimal (of the same order of magnitude as δ).

Assuming that $f(x)$ has no singularities in an infinitesimal neighborhood around the real axis, we are free to deform the contour C into C' without changing the value of the integral. It follows that

$$\int_C \frac{f(x)\mathrm{d}x}{x} = \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{f(x)\mathrm{d}x}{x} = \int_{-\infty}^{\infty} \frac{f(y+i\varepsilon)}{y+i\varepsilon}\mathrm{d}y \ , \tag{59}$$

Since ε is infinitesimal, we can approximate $f(y+i\varepsilon) \simeq f(y)$[1].

$$\int_C \frac{f(x)\mathrm{d}x}{x} = \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\mathrm{d}x \ . \tag{60}$$

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\mathrm{d}x = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{x} - i\pi f(0) \ . \tag{61}$$

4.4 Derivation 3

Starting from the definition of the Cauchy principal value, integrate by parts to obtain

$$\begin{aligned}
\int_{-\infty}^{-\delta} \frac{f(x)\mathrm{d}x}{x} &= f(x)\ln|x|\Big|_{-\infty}^{-\delta} - \int_{-\infty}^{-\delta} f'(x)\ln|x|\mathrm{d}x = f(-\varepsilon)\ln\varepsilon - \int_{-\infty}^{-\delta} f'(x)\ln|x|\mathrm{d}x \ , \\
\int_{\delta}^{\infty} \frac{f(x)\mathrm{d}x}{x} &= f(x)\ln|x|\Big|_{\delta}^{\infty} - \int_{\delta}^{\infty} f'(x)\ln|x|\mathrm{d}x = -f(\varepsilon)\ln\varepsilon - \int_{\delta}^{\infty} f'(x)\ln|x|\mathrm{d}x \ ,
\end{aligned}$$

where we have assumed that $f(x) \to 0$ sufficiently fast as $x \to \pm\infty$ so that the surface terms at $\pm\infty$ vanish.

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{x} = \lim_{\delta\to 0}\left\{[f(-\varepsilon) - f(\varepsilon)]\ln\delta - \int_{-\infty}^{-\delta} f'(x)\ln|x|\mathrm{d}x - \int_{\delta}^{\infty} f'(x)\ln|x|\mathrm{d}x\right\} \ . \tag{62}$$

Since $f(x)$ is differentiable and well behaved, we can define

$$g(x) \equiv \int_0^1 f'(xt)\mathrm{d}t = \frac{f(x)-f(0)}{x} \ , \tag{63}$$

which implies that $g(x)$ is smooth and non-singular and

$$f(x) = f(0) + xg(x) \ . \tag{64}$$

$$\begin{aligned}
\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{x} &= \lim_{\delta\to 0}\left\{-2g(x)\delta\ln\delta - \int_{-\infty}^{-\delta} f'(x)\ln|x|\mathrm{d}x - \int_{\delta}^{\infty} f'(x)\ln|x|\mathrm{d}x\right\} \\
&= -\int_{-\infty}^{\infty} f'(x)\ln|x|\mathrm{d}x \ .
\end{aligned}$$

[1] More precisely, we can expand $f(y+i\varepsilon)$ in a Taylor series about $\varepsilon = 0$ to obtain $f(y+i\varepsilon) = f(y) + \mathcal{O}(\varepsilon)$. At the end of the calculation, we may take $\varepsilon \to 0$, in which case the $\mathcal{O}(\varepsilon)$ terms vanish.

$\ln|x|$ is integrable at $x = 0$, so that the last integral is well-defined. Finally, we integrate by parts and drop the surface terms at $\pm\infty$ (under the usual assumption that $f'(x) \to 0$ sufficiently fast as $x \to \infty$). The end result is

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{x} = \int_{-\infty}^{\infty} f(x)\frac{\mathrm{d}\ln|x|}{\mathrm{d}x}\mathrm{d}x \ . \tag{65}$$

That is, we have derived the generalized function identity,

$$\frac{\mathrm{d}\ln|x|}{\mathrm{d}x} = \mathcal{P}\frac{1}{x} \ . \tag{66}$$

Begin with the definition of the principal value of the complex logarithm,

$$\mathrm{Ln}z = \ln|z| + i\mathrm{arg}\ z \ , \tag{67}$$

where arg z is the principal value of the argument (or phase) of the complex number z, with the convention that $-\pi < \mathrm{arg}z \leqslant \pi$. In particular, for real x and a positive infinitesimal ε,

$$\lim_{\varepsilon\to 0} \mathrm{Ln}(x \pm i\varepsilon) = \ln|x| \pm i\pi\Theta(-x) \ , \tag{68}$$

where $\Theta(x)$ is the Heaviside step function. Differentiating with respect to x immediately yields the Sokhotski-Plemelj formula,

$$\lim_{\varepsilon\to 0} \frac{1}{x \pm i\varepsilon} = \mathcal{P}\frac{1}{x} \mp i\pi\delta(x) \ , \tag{69}$$

where we have used

$$\frac{\mathrm{d}}{\mathrm{d}x}\Theta(-x) = -\frac{\mathrm{d}}{\mathrm{d}x}\Theta(x) = -\delta(x) \ . \tag{70}$$

References

[1] M. D. Schwartz. *Quantum Field Theory and the Standard Model*. March 2014.

[2] S. Weinberg. *The Quantum Theory of Fields*. June 1995.