# The Einstein field equations

June 5, 2017

## 1 Purpose and justification of the field equations

[1] The Newtonian equation is

$$\nabla^2\phi = 4\pi G\rho \ , \tag{1}$$

where $\rho$ is the density of mass. Its solution for a point particle of mass $m$ is

$$\phi = -\frac{Gm}{r} \ , \tag{2}$$

which is dimensionless in units where $c = 1$.

$\rho$ is the energy density as measured by only one observer, the MCRF (momentarily comoving reference frame). Other observers measure the energy density to be the component $T^{00}$ in their own reference frames. If using $\rho$ as the source of the field, then it means one class of observers is preferred above all others, namely those for whom $\rho$ is the energy density. However, all coordinate systems on an equal footing. An invariant theory can avoid introducing preferred coordinate systems by using the whole of the stress-energy tensor $\mathbf{T}$ as the source of the gravitational field. The generalization of Eq. (1) to relativity would then have

$$\mathbf{O}(\mathbf{g}) = k\mathbf{T} \ , \tag{3}$$

where $k$ is a constant and $\mathbf{O}$ is a differential operator on the metric tensor $\mathbf{g}$.

$\{\mathbf{O}^{\alpha\beta}\}$ must be the components of a $\binom{2}{0}$ tensor and must be combinations of $g_{\mu\nu,\lambda\sigma}$, $g_{\mu\nu,\lambda}$, and $g_{\mu\nu}$. Ricci tensor $R^{\alpha\beta}$ satisfies these conditions. Any tensor of the form

$$O^{\alpha\beta} = R^{\alpha\beta} + \mu g^{\alpha\beta} R + \Lambda g^{\alpha\beta} \ , \tag{4}$$

satisfies these conditions, if $\mu$ and $\Lambda$ are constants. $\mu$ can be determined by the Einstein equivalence principle which demands local conservation of energy and momentum

$$T^{\alpha\beta}{}_{;\beta} = 0 \ . \tag{5}$$

This equation must be true for any metric tensor, then

$$O^{\alpha\beta}{}_{;\beta} = 0 \ . \tag{6}$$

Since $g^{\alpha\beta}{}_{;\mu} = 0$,

$$(R^{\alpha\beta} + \mu g^{\alpha\beta} R)_{;\beta} = 0 \ . \tag{7}$$

$\mu = -\dfrac{1}{2}$ if it is to be an identity for arbitrary $g_{\alpha\beta}$.

$$G^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta} \ , \tag{8}$$

$$\mathbf{G} + \Lambda\mathbf{g} = k\mathbf{T} \tag{9}$$

which are called the field equations of GR, or Einstein's field equations. Constant $k$ can be determined by demanding that Newton's gravitational field equation comes out right, but that $\Lambda$ remains arbitrary.

### 1.1 Geometrized units

$c = G = 1$ and $1 = G/c^2 = 7.425 \times 10^{-28}$ m kg$^{-1}$.

# 2 Einsteins equations

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta} \ , \tag{10}$$

where $k = 8\pi$ and constant $\Lambda$ is called the cosmological constant. The value of $k$ is obtained by demanding that Einstein's equations predict the correct behavior of planets in the solar system. Since $\{g_{\alpha\beta}\}$ are the components of a tensor in some coordinate system, a change in coordinates induces a change in them. There are four coordinates, so there are four arbitrary functional degrees of freedom among the ten $g_{\alpha\beta}$. It should be impossible, therefore, to determine all ten $g_{\alpha\beta}$ from any initial data, since the coordinates to the future of the initial moment can be changed arbitrarily. Einsteins equations have exactly this property: the Bianchi identities

$$G^{\alpha\beta}{}_{;\beta} = 0 \tag{11}$$

mean that there are four differential identities (one for each value of $\alpha$ above) among the ten $G^{\alpha\beta}$. These ten are not independent, and the ten Einstein equations are really only six independent differential equations for the six functions among the ten $g_{\alpha\beta}$ that characterize the geometry independently of the coordinates.

# 3 Einstein's equations for weak gravitational fields

## References

[1] B. Schutz. *A First Course in General Relativity.* May 2009.