Second Order Ordinary Differential Equation

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1 Second Order Linear Equations

The general second order linear differential equation is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = R(x)\ , \tag{1}$$

or

$$y'' + P(x)y' + Q(x)y = R(x) \tag{2}$$

Theorem A

Let $P(x)$, $Q(x)$, and $R(x)$ be continuous functions on a closed interval $[a, b]$. If x_0 is any point in $[a, b]$, and if y_0 and y'_0 are any numbers whatever, then equation (1) has one and only one solution $y(x)$ on the entire interval such that $y(x_0) = y_0$ and $y'(x_0) = y_0$.

If $R(x)$ is identically zero, then (1) reduces to the homogeneous equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)y = 0\ , \tag{3}$$

If $R(x)$ is not identically zero, then (1) is said to be nonhomogeneous.

Theorem B

If y_g is the general solution of the reduced equation (3) and y_p is any particular solution of the complete equation (1), then $y_g + y_p$ is the general solution of (1).

Theorem C

If $y_1(x)$ and $y_2(x)$ are any two solutions of (3), then

$$c_1 y_1(x) + c_2 y_2(x) \tag{4}$$

is also a solution for any constants c_1 and c_2 .

Any linear combination of two solutions of the homogeneous equation (3) also a solution. The solution (4) is commonly called a linear combination of the solutions $y_1(x)$ and $y_2(x)$.

1.1 The General Solution of the Homogeneous Equation

If two functions $f(x)$ and $g(x)$ are defined on an interval $[a, b]$ and have the property that one is a constant multiple of the other, then they are said to be linearly dependent on $[a, b]$. Otherwise, if neither is a constant multiple of the other, they are called linearly independent. It is worth noting that if $f(x)$ is identically zero, then $f(x)$ and $g(x)$ are linearly dependent for every function $g(x)$, since $f(x) = 0 \cdot g(x)$.

Theorem A

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{5}$$

on the interval $[a, b]$. Then

$$c_1 y_1(x) + c_2 y_2(x) \tag{6}$$

is the general solution of equation (5) on $[a, b]$, in the sense that every solution of (5) on this interval can be obtained from (6) by a suitable choice of the arbitrary constants c_1 and c_2.

The Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x) \tag{7}$$

Lemma 1

If $y_1(x)$ and $y_2(x)$ are any two solutions of equation (5) on $[a, b]$, then their Wronskian $W = W(y_1, y_2)$ is either identically zero or never zero on $[a, b]$.

Lemma 2

If $y_1(x)$ and $y_2(x)$ are two solutions of equation (5) on $[a, b]$, then they are linearly dependent on this interval if and only if their Wronskian $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ is identically zero.

1.2 The Use of A Known Solution to Find Another

$$y'' + P(x)y' + Q(x)y = 0 \tag{8}$$

Assume that $y_1(x)$ is a known nonzero solution of (8), so that $cy_1(x)$ is also a solution for any constant c.

1.3 The Homogeneous Equation with Constant Coefficients

If $P(x)$ and $Q(x)$ are constants p and q:

$$y'' + py' + qy = 0 \tag{9}$$