Matrix Algebra

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1 Matrix Operations

The diagonal entries in an $m \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \cdots$, and they form the main diagonal of A. A diagonal matrix is a square matrix whose nondiagonal entries are zero. An example is the $n \times n$ identity matrix, I_n. An $m \times n$ matrix whose entries are all zero is a zero matrix and is written as 0. The size of a zero matrix is usually clear from the context.

1.1 Sums and Scalar Multiples

Theorem

Let $A, B, andC$ be matrices of the same size, and let r and s be scalars.

$$a. A + B = B + A \;, \qquad d. r(A + B) = rA + rB \tag{1}$$

$$b. (A + B) + C = A + (B + C) \;, \qquad e. (r + s)A = rA + sA \tag{2}$$

$$c. A + 0 = A \;, \qquad f. r(sA) = (rs)A \tag{3}$$

1.2 Matrix Multiplication

Theorem

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\boldsymbol{b}_1, \cdots, A\boldsymbol{b}_p$. That is,

$$AB = A[\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_p] = [A\boldsymbol{b}_1 A\boldsymbol{b}_2 \cdots A\boldsymbol{b}_p]$$

Multiplication of matrices corresponds to composition of linear transformations.

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

ROW-COLUMN RULE FOR COMPUTING AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \ .$$

1.3 Properties of Matrix Multiplication

Theorem

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$ for any scalar r

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

1.4 Powers of a Matrix

1.5 The Transpose of a Matrix

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Theorem

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

$a.\ (A^T)^T = A$

$b.\ (A+B)^T = A^T + B^T$

$c.$ For any scalar$r$, $(rA)^T = rA^T$

$d.\ (AB)^T = B^T A^T$

2 The Inverse of a Matrix

An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \text{ and } AC = I$$

where $I = I_n$, the $n \times n$ identity matrix. C is called an inverse of A, denoted by A^{-1}, then

$$A^{-1}A = AA^{-1} = I\ .$$

A matrix that is not invertible is sometimes called a singular matrix, and an invertible matrix is called a nonsingular matrix.

Theorem

Let $A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $ad - bc = 0$, then A is not invertible.

The quantity $ad - bc$ is called the determinant of A, i.e.

$$\det A = ad - bc \ .$$

Theorem

If A is an invertible $n \times n$ matrix, then for each $\boldsymbol{b}$ in $\mathbb{R}^n$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Theorem

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

2.1 Elementary Matrices

An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

2.2 An Algorithm for Finding A^{-1}

ALGORITHM FOR FINDING A^{-1}

Row reduce the augmented matrix $[A\ I]$. If A is row equivalent to I, then $[A\ I]$ is row equivalent to $[I A^{-1}]$. Otherwise, A does not have an inverse.

2.3 Another View of Matrix Inversion

3 Characterizations of Invertible Matrices

4 Partitioned Matrices

5 Matrix Factorizations

6 The Leontief Input–Output Model

7 Applications to Computer Graphics

8 Subspaces of $\mathbb{R}^n$

9 Dimension and Rank