代码库

Blazar

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目录

1 数论

1.1 快速求逆元

返回结果：

$$x^{-1}(mod)$$

使用条件：$x \in [0, mod)$ 并且 x 与 mod 互质

```
LL inv(LL a,LL p){
        LL d,x,y;
        d=exgcd(a,p,x,y);
        return d==1?(x+p)%p:-1;
}
```

1.2 莫比乌斯反演

```
#include<cstdio>
#include<string>
#include<cstring>
#include<algorithm>
using namespace std;
int mu[100001],prime[100001];
bool check[100001];
int tot;
inline void findmu()
{
     memset(check,false,sizeof(check));
     mu[1]=1;
     int i,j;
     for(i=2;i<=100000;i++)
     {
          if(!check[i])
          {
               tot++;
               prime[tot]=i;
               mu[i]=-1;
          }
          for(j=1;j<=tot;j++)
          {
               if(i*prime[j]>100000)
                    break;
               check[i*prime[j]]=true;
```

```
                if(i%prime[j]==0)
                {
                    mu[i*prime[j]]=0;
                    break;
                }
                else
                    mu[i*prime[j]]=-mu[i];
            }
        }
}
int sum[100001];
//找 [1,n],[1,m] 内互质的数的对数
inline long long solve(int n,int m)
{
    long long ans=0;
    if(n>m)
        swap(n,m);
    int i,la=0;
    for(i=1;i<=n;i=la+1)
    {
        la=min(n/(n/i),m/(m/i));
        ans+=(long long)(sum[la]-sum[i-1])*(n/i)*(m/i);
    }
    return ans;
}
int main()
{
    //freopen("b.in","r",stdin);
   // freopen("b.out","w",stdout);
    findmu();
    sum[0]=0;
    int i;
    for(i=1;i<=100000;i++)
        sum[i]=sum[i-1]+mu[i];
    int a,b,c,d,k;
    int T;
    scanf("%d",&T);
    while(T--)
    {
        scanf("%d%d%d%d%d",&a,&b,&c,&d,&k);
        long long ans=0;
        ans=solve(b/k,d/k)-solve((a-1)/k,d/k)-solve(b/k,(c-1)/k)+solve((a-1)/k,(c-1)/k);
```

```
        printf("%lld\n",ans);
    }
    return 0;
}
```

1.3 扩展欧几里德算法

返回结果：

$$ax + by = gcd(a, b)$$

时间复杂度：$\mathcal{O}(nlogn)$

```
LL exgcd(LL a,LL b,LL &x,LL &y){
        if(!b){
                x=1;y=0;return a;
        }else{
                LL d=exgcd(b,a%b,x,y);
                LL t=x;x=y;y=t-a/b*y;
                return d;
        }
}
```

1.4 中国剩余定理

返回结果：

$$x \equiv r_i(mod\ \ p_i)\ \ (0 \leq i < n)$$

使用条件：p_i 需两两互质

```
LL china(int n,int *a,int *m){
        LL M=1,d,x=0,y;
        for(int i=0;i<n;i++)
                M*=m[i];
        for(int i=0;i<n;i++){
                LL w=M/m[i];
                d=exgcd(m[i],w,d,y);
                y=(y%M+M)%M;
                x=(x+y*w%M*a[i])%M;
        }
        while(x<0)x+=M;
        return x;
}
```

1.5 中国剩余定理 2

```
//merge Ax=B and ax=b to A'x=B'
void merge(LL &A,LL &B,LL a,LL b){
    LL x,y;
    sol(A,-a,b-B,x,y);
    A=lcm(A,a);
    B=(a*y+b)%A;
    B=(B+A)%A;
}
```

1.6 组合数取模

```
LL prod=1,P;
pair<LL,LL> comput(LL n,LL p,LL k){
    if(n<=1)return make_pair(0,1);
    LL ans=1,cnt=0;
    ans=pow(prod,n/P,P);
    cnt=n/p;
    pair<LL,LL>res=comput(n/p,p,k);
    cnt+=res.first;
    ans=ans*res.second%P;
    for(int i=n-n%P+1;i<=n;i++)if(i%p){

        ans=ans*i%P;
    }
    return make_pair(cnt,ans);
}
pair<LL,LL> calc(LL n,LL p,LL k){
    prod=1;P=pow(p,k,1e18);
    for(int i=1;i<P;i++)if(i%p)prod=prod*i%P;
    pair<LL,LL> res=comput(n,p,k);
//  res.second=res.second*pow(p,res.first%k,P)%P;
//  res.first-=res.first%k;
    return res;
}
LL calc(LL n,LL m,LL p,LL k){
    pair<LL,LL>A,B,C;
    LL P=pow(p,k,1e18);
    A=calc(n,p,k);
    B=calc(m,p,k);
    C=calc(n-m,p,k);
```

```
    LL ans=1;
    ans=pow(p,A.first-B.first-C.first,P);
    ans=ans*A.second%P*inv(B.second,P)%P*inv(C.second,P)%P;
    return ans;
}
```

1.7 扩展小步大步

```
LL solve2(LL a,LL b,LL p){
    //a^x=b (mod p)
    b%=p;
    LL e=1%p;
    for(int i=0;i<100;i++){
        if(e==b)return i;
        e=e*a%p;
    }
    int r=0;
    while(gcd(a,p)!=1){
        LL d=gcd(a,p);
        if(b%d)return -1;
        p/=d;b/=d;b=b*inv(a/d,p);
        r++;
    }LL res=BSGS(a,b,p);
    if(res==-1)return -1;
    return res+r;
}
```

1.8 卢卡斯定理

```
LL Lucas(LL n,LL m,LL p){
        LL ans=1;
        while(n&&m){
                LL a=n%p,b=m%p;
                if(a<b)return 0;
                ans=(ans*C(a,b,p))%p;
                n/=p;m/=p;
        }return ans%p;
}
```

1.9 小步大步

返回结果：

$$a^x = b\ (mod\ p)$$

使用条件：p 为质数

时间复杂度：$\mathcal{O}(\sqrt{n})$

```
LL BSGS(LL a,LL b,LL p){
	LL m=sqrt(p)+.5,v=inv(pw(a,m,p),p),e=1;
	map<LL,LL>hash;hash[1]=0;
	for(int i=1;i<m;i++)
		e=e*a%p,hash[e]=i;
	for(int i=0;i<=m;i++){
		if(hash.count(b))return i*m+hash[b];
		b=b*v%p;
	}return -1;
}
```

1.10 Miller Rabin 素数测试

```
const int BASE[12] = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37};
bool check(long long n,int base) {
    long long n2=n-1,res;
    int s=0;
    while(n2%2==0) n2>>=1,s++;
    res=pw(base,n2,n);
    if((res==1)||(res==n-1)) return 1;
    while(s--) {
        res=mul(res,res,n);
        if(res==n-1) return 1;
    }
    return 0; // n is not a strong pseudo prime
}
bool isprime(const long long &n) {
    if(n==2)
        return true;
    if(n<2 || n%2==0)
        return false;
    for(int i=0;i<12&&BASE[i]<n;i++){
        if(!check(n,BASE[i]))
            return false;
    }
```

```
    return true;
}
```

1.11 Pollard Rho 大数分解

时间复杂度：$\mathcal{O}(n^{1/4})$

```
LL prho(LL n,LL c){
        LL i=1,k=2,x=rand()%(n-1)+1,y=x;
        while(1){
                i++;x=(x*x%n+c)%n;
                LL d=__gcd((y-x+n)%n,n);
                if(d>1&&d<n)return d;
                if(y==x)return n;
                if(i==k)y=x,k<<=1;
        }
}
void factor(LL n,vector<LL>&fat){
        if(n==1)return;
        if(isprime(n)){
                fat.push_back(n);
                return;
        }LL p=n;
        while(p>=n)p=prho(p,rand()%(n-1)+1);
        factor(p,fat);
        factor(n/p,fat);
}
```

1.12 快速数论变换（zky）

返回结果：

$$c_i = \sum_{0 \le j \le i} a_j \cdot b_{i-j}(mod) \ \ (0 \le i < n)$$

使用说明：$magic$ 是 mod 的原根

时间复杂度：$\mathcal{O}(nlogn)$

```
/*
{(mod,G)}={(81788929,7),(101711873,3),(167772161,3)
                        ,(377487361,7),(998244353,3),(1224736769,3)
                        ,(1300234241,3),(1484783617,5)}
*/
int mo=998244353,G=3;
```

```
void NTT(int a[],int n,int f){
	for(register int i=0;i<n;i++)
		if(i<rev[i])
			swap(a[i],a[rev[i]]);
	for (register int i=2;i<=n;i<<=1){
		static int exp[maxn];
		exp[0]=1;exp[1]=pw(G,(mo-1)/i);
		if(f==-1)exp[1]=pw(exp[1],mo-2);
		for(register int k=2;k<(i>>1);k++)
			exp[k]=1LL*exp[k-1]*exp[1]%mo;
		for(register int j=0;j<n;j+=i){
			for(register int k=0;k<(i>>1);k++){
				register int &pA=a[j+k],&pB=a[j+k+(i>>1)];
				register int A=pA,B=1LL*pB*exp[k]%mo;
				pA=(A+B)%mo;
				pB=(A-B+mo)%mo;
			}
		}
	}
	if(f==-1){
		int rv=pw(n,mo-2)%mo;
		for(int i=0;i<n;i++)
			a[i]=1LL*a[i]*rv%mo;
	}
}
void mul(int m,int a[],int b[],int c[]){
	int n=1,len=0;
	while(n<m)n<<=1,len++;
	for (int i=1;i<n;i++)
		rev[i]=(rev[i>>1]>>1)|((i&1)<<(len-1));
	NTT(a,n,1);
	NTT(b,n,1);
	for(int i=0;i<n;i++)
		c[i]=1LL*a[i]*b[i]%mo;
	NTT(c,n,-1);
}
```

1.13 原根

```
vector<LL>fct;
bool check(LL x,LL g){
	for(int i=0;i<fct.size();i++)
```

```
                if(pw(g,(x-1)/fct[i],x)==1)
                        return 0;
        return 1;
}
LL findrt(LL x){
        LL tmp=x-1;
        for(int i=2;i*i<=tmp;i++){
                if(tmp%i==0){
                        fct.push_back(i);
                        while(tmp%i==0)tmp/=i;
                }
        }if(tmp>1)fct.push_back(tmp);
        // x is 1,2,4,p^n,2p^n
        // x has phi(phi(x)) primitive roots
        for(int i=2;i<int(1e9);i++)if(check(x,i))
                return i;
        return -1;
}
const int BASE[12] = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37};
bool check(long long n,int base) {
    long long n2=n-1,res;
    int s=0;
    while(n2%2==0) n2>>=1,s++;
    res=pw(base,n2,n);
    if((res==1)||(res==n-1)) return 1;
    while(s--) {
        res=mul(res,res,n);
        if(res==n-1) return 1;
    }
    return 0; // n is not a strong pseudo prime
}
bool isprime(const long long &n) {
    if(n==2)
        return true;
    if(n<2 || n%2==0)
        return false;
    for(int i=0;i<12&&BASE[i]<n;i++){
        if(!check(n,BASE[i]))
            return false;
    }
    return true;
}
```

1.14 线性递推

```
//已知 a_0, a_1, ..., a_{m-1}||
     a_n = c_0 * a_{n-m} + ... + c_{m-1} * a_{n-1}||
     求 a_n = v_0 * a_0 + v_1 * a_1 + ... + v_{m-1} * a_{m-1}||

void linear_recurrence(long long n, int m, int a[], int c[], int p) {
	long long v[M] = {1 % p}, u[M << 1], msk = !!n;
	for(long long i(n); i > 1; i >>= 1) {
		msk <<= 1;
	}
	for(long long x(0); msk; msk >>= 1, x <<= 1) {
		fill_n(u, m << 1, 0);
		int b(!!(n & msk));
		x |= b;
		if(x < m) {
			u[x] = 1 % p;
		}else {
			for(int i(0); i < m; i++) {
				for(int j(0), t(i + b); j < m; j++, t++) {
					u[t] = (u[t] + v[i] * v[j]) % p;
				}
			}
			for(int i((m << 1) - 1); i >= m; i--) {
				for(int j(0), t(i - m); j < m; j++, t++) {
					u[t] = (u[t] + c[j] * u[i]) % p;
				}
			}
		}
		copy(u, u + m, v);
	}
	//a[n] = v[0] * a[0] + v[1] * a[1] + ... + v[m - 1] * a[m - 1].
	for(int i(m); i < 2 * m; i++) {
		a[i] = 0;
		for(int j(0); j < m; j++) {
			a[i] = (a[i] + (long long)c[j] * a[i + j - m]) % p;
		}
	}
	for(int j(0); j < m; j++) {
		b[j] = 0;
		for(int i(0); i < m; i++) {
			b[j] = (b[j] + v[i] * a[i + j]) % p;
```

```
			}
		}
		for(int j(0); j < m; j++) {
			a[j] = b[j];
		}
}
```

1.15 线性筛

```
void sieve(){
	f[1]=mu[1]=phi[1]=1;
	for(int i=2;i<maxn;i++){
		if(!minp[i]){
			minp[i]=i;
			minpw[i]=i;
			mu[i]=-1;
			phi[i]=i-1;
			f[i]=i-1;
			p[++p[0]]=i;//Case 1 prime
		}
		for(int j=1;j<=p[0]&&(LL)i*p[j]<maxn;j++){
			minp[i*p[j]]=p[j];
			if(i%p[j]==0){
				//Case 2 not coprime
				minpw[i*p[j]]=minpw[i]*p[j];
				phi[i*p[j]]=phi[i]*p[j];
				mu[i*p[j]]=0;
				if(i==minpw[i]){
					f[i*p[j]]=i*p[j]-i;//Special Case for f(p^k)
				}else{
					f[i*p[j]]=f[i/minpw[i]]*f[minpw[i]*p[j]];
				}
				break;
			}else{
				//Case 3 coprime
				minpw[i*p[j]]=p[j];
				f[i*p[j]]=f[i]*f[p[j]];
				phi[i*p[j]]=phi[i]*(p[j]-1);
				mu[i*p[j]]=-mu[i];
			}
		}
	}
```

```
}
```

1.16 直线下整点个数

返回结果：

$$\sum_{0\le i<n} \lfloor \frac{a+b\cdot i}{m} \rfloor$$

使用条件：$n,m>0$，$a,b\ge 0$

时间复杂度：$\mathcal{O}(nlogn)$

```
//calc \sum_{i=0}^{n-1} [(a+bi)/m]
// n,a,b,m >0
LL solve(LL n,LL a,LL b,LL m){
    if(b==0)
        return n*(a/m);
    if(a>=m || b>=m)
        return n*(a/m)+(n-1)*n/2*(b/m)+solve(n,a%m,b%m,m);
    return solve((a+b*n)/m,(a+b*n)%m,m,b);
}
```

2 数值

2.1 高斯消元

```
void Gauss(){
        int r,k;
        for(int i=0;i<n;i++){
                r=i;
                for(int j=i+1;j<n;j++)
                        if(fabs(A[j][i])>fabs(A[r][i]))r=j;
                if(r!=i)for(int j=0;j<=n;j++)swap(A[i][j],A[r][j]);
                for(int k=i+1;k<n;k++){
                        double f=A[k][i]/A[i][i];
                        for(int j=i;j<=n;j++)A[k][j]-=f*A[i][j];
                }
        }
        for(int i=n-1;i>=0;i--){
                for(int j=i+1;j<n;j++)
                        A[i][n]-=A[j][n]*A[i][j];
                A[i][n]/=A[i][i];
        }
```

```
        for(int i=0;i<n-1;i++)
                cout<<fixed<<setprecision(3)<<A[i][n]<<" ";
        cout<<fixed<<setprecision(3)<<A[n-1][n];
}
bool Gauss(){
        for(int i=1;i<=n;i++){
                int r=0;
                for(int j=i;j<=m;j++)
                if(a[j][i]){r=j;break;}
                if(!r)return 0;
                ans=max(ans,r);
                swap(a[i],a[r]);
                for(int j=i+1;j<=m;j++)
                if(a[j][i])a[j]^=a[i];
        }for(int i=n;i>=1;i--){
                for(int j=i+1;j<=n;j++)if(a[i][j])
                a[i][n+1]=a[i][n+1]^a[j][n+1];
        }return 1;
}
LL Gauss(){
        for(int i=0;i<n;i++)for(int j=0;j<n;j++)A[i][j]%=m;
        for(int i=0;i<n;i++)for(int j=0;j<n;j++)A[i][j]=(A[i][j]+m)%m;
        LL ans=n%2?-1:1;
        for(int i=0;i<n;i++){
                for(int j=i+1;j<n;j++){
                        while(A[j][i]){
                                LL t=A[i][i]/A[j][i];
                                for(int k=0;k<n;k++)
                                A[i][k]=(A[i][k]-A[j][k]*t%m+m)%m;
                                swap(A[i],A[j]);
                                ans=-ans;
                        }
                }ans=ans*A[i][i]%m;
        }return (ans%m+m)%m;
}
int Gauss(){//求秩
        int r,now=-1;
        int ans=0;
        for(int i = 0; i <n; i++){
                r = now + 1;
                for(int j = now + 1; j < m; j++)
                        if(fabs(A[j][i]) > fabs(A[r][i]))
```

```
                        r = j;
            if (!sgn(A[r][i])) continue;
            ans++;
            now++;
            if(r != now)
                for(int j = 0; j < n; j++)
                    swap(A[r][j], A[now][j]);

            for(int k = now + 1; k < m; k++){
                double t = A[k][i] / A[now][i];
                for(int j = 0; j < n; j++){
                    A[k][j] -= t * A[now][j];
                }
            }
        }
    return ans;
}
```

2.2 快速傅立叶变换

返回结果：

$$c_i = \sum_{0 \le j \le i} a_j \cdot b_{i-j} \quad (0 \le i < n)$$

时间复杂度：$\mathcal{O}(nlogn)$

```
typedef complex<double> cp;
const double pi = acos(-1);
void FFT(vector<cp>&num,int len,int ty){
    for(int i=1,j=0;i<len-1;i++){
        for(int k=len;j^=k>>=1,~j&k;);
        if(i<j)
            swap(num[i],num[j]);
    }
    for(int h=0;(1<<h)<len;h++){
        int step=1<<h,step2=step<<1;
        cp w0(cos(2.0*pi/step2),ty*sin(2.0*pi/step2));
        for(int i=0;i<len;i+=step2){
            cp w(1,0);
            for(int j=0;j<step;j++){
                cp &x=num[i+j+step];
                cp &y=num[i+j];
                cp d=w*x;
                x=y-d;
```

```
                y=y+d;
                w=w*w0;
            }
        }
    }
    if(ty==-1)
        for(int i=0;i<len;i++)
            num[i]=cp(num[i].real()/(double)len,num[i].imag());
}
vector<cp> mul(vector<cp>a,vector<cp>b){
    int len=a.size()+b.size();
    while((len&-len)!=len)len++;
    while(a.size()<len)a.push_back(cp(0,0));
    while(b.size()<len)b.push_back(cp(0,0));
    FFT(a,len,1);
    FFT(b,len,1);
    vector<cp>ans(len);
    for(int i=0;i<len;i++)
        ans[i]=a[i]*b[i];
    FFT(ans,len,-1);
    return ans;
}
```

2.3 单纯形法求解线性规划

返回结果：

$$max\{c_{1\times m}\cdot x_{m\times 1}\ |\ x_{m\times 1}\geq 0_{m\times 1},a_{n\times m}\cdot x_{m\times 1}\leq b_{n\times 1}\}$$

```
namespace LP{
        const int maxn=233;
        double a[maxn][maxn];
        int Ans[maxn],pt[maxn];
        int n,m;
        void pivot(int l,int i){
                double t;
                swap(Ans[l+n],Ans[i]);
                t=-a[l][i];
                a[l][i]=-1;
                for(int j=0;j<=n;j++)a[l][j]/=t;
                for(int j=0;j<=m;j++){
                        if(a[j][i]&&j!=l){
                                t=a[j][i];
```

```
                a[j][i]=0;
                for(int k=0;k<=n;k++)a[j][k]+=t*a[l][k];
            }
        }
    }
    vector<double> solve(vector<vector<double> >A,vector<double>B,vector<double>C){
        n=C.size();
        m=B.size();
        for(int i=0;i<C.size();i++)
            a[0][i+1]=C[i];
        for(int i=0;i<B.size();i++)
            a[i+1][0]=B[i];

        for(int i=0;i<m;i++)
            for(int j=0;j<n;j++)
                a[i+1][j+1]=-A[i][j];

        for(int i=1;i<=n;i++)Ans[i]=i;

        double t;
        for(;;){
            int l=0;t=-eps;
            for(int j=1;j<=m;j++)if(a[j][0]<t)t=a[l=j][0];
            if(!l)break;
            int i=0;
            for(int j=1;j<=n;j++)if(a[l][j]>eps){i=j;break;}
            if(!i){
                puts("Infeasible");
                return vector<double>();
            }
            pivot(l,i);
        }
        for(;;){
            int i=0;t=eps;
            for(int j=1;j<=n;j++)if(a[0][j]>t)t=a[0][i=j];
            if(!i)break;
            int l=0;
            t=1e30;
            for(int j=1;j<=m;j++)if(a[j][i]<-eps){
                double tmp;
                tmp=-a[j][0]/a[j][i];
                if(t>tmp)t=tmp,l=j;
```

```
                        }
                        if(!l){
                                puts("Unbounded");
                                return vector<double>();
                        }
                        pivot(l,i);
                }
                vector<double>x;
                for(int i=n+1;i<=n+m;i++)pt[Ans[i]]=i-n;
                for(int i=1;i<=n;i++)x.push_back(pt[i]?a[pt[i]][0]:0);
                return x;
        }
}
```

2.4 自适应辛普森

```
double area(const double &left, const double &right) {
    double mid = (left + right) / 2;
    return (right - left) * (calc(left) + 4 * calc(mid) + calc(right)) / 6;
}

double simpson(const double &left, const double &right,
               const double &eps, const double &area_sum) {
    double mid = (left + right) / 2;
    double area_left = area(left, mid);
    double area_right = area(mid, right);
    double area_total = area_left + area_right;
    if (std::abs(area_total - area_sum) < 15 * eps) {
        return area_total + (area_total - area_sum) / 15;
    }
    return simpson(left, mid, eps / 2, area_left)
         + simpson(mid, right, eps / 2, area_right);
}

double simpson(const double &left, const double &right, const double &eps) {
    return simpson(left, right, eps, area(left, right));
}
```

2.5 多项式求根

```cpp
const double eps=1e-12;
double a[10][10];
typedef vector<double> vd;
int sgn(double x) { return x < -eps ? -1 : x > eps; }
double mypow(double x,int num){
        double ans=1.0;
        for(int i=1;i<=num;++i)ans*=x;
        return ans;
}
double f(int n,double x){
        double ans=0;
        for(int i=n;i>=0;--i)ans+=a[n][i]*mypow(x,i);
        return ans;
}
double getRoot(int n,double l,double r){
        if(sgn(f(n,l))==0)return l;
        if(sgn(f(n,r))==0)return r;
        double temp;
        if(sgn(f(n,l))>0)temp=-1;else temp=1;
        double m;
        for(int i=1;i<=10000;++i){
                m=(l+r)/2;
                double mid=f(n,m);
                if(sgn(mid)==0){
                        return m;
                }
                if(mid*temp<0)l=m;else r=m;
        }
        return (l+r)/2;
}
vd did(int n){
        vd ret;
        if(n==1){
                ret.push_back(-1e10);
                ret.push_back(-a[n][0]/a[n][1]);
                ret.push_back(1e10);
                return ret;
        }
        vd mid=did(n-1);
        ret.push_back(-1e10);
        for(int i=0;i+1<mid.size();++i){
                int t1=sgn(f(n,mid[i])),t2=sgn(f(n,mid[i+1]));
```

```
                if(t1*t2>0)continue;
                ret.push_back(getRoot(n,mid[i],mid[i+1]));
        }
        ret.push_back(1e10);
        return ret;
}
int main(){
        int n; scanf("%d",&n);
        for(int i=n;i>=0;--i){
                scanf("%lf",&a[n][i]);
        }
        for(int i=n-1;i>=0;--i)
                for(int j=0;j<=i;++j)a[i][j]=a[i+1][j+1]*(j+1);
        vd ans=did(n);
        sort(ans.begin(),ans.end());
        for(int i=1;i+1<ans.size();++i)printf("%.10f\n",ans[i]);
        return 0;
}
```

3 数据结构

3.1 平衡的二叉查找树

3.1.1 Treap

```
#include<bits/stdc++.h>
using namespace std;
const int maxn=1e5+5;
#define sz(x) (x?x->siz:0)
struct Treap{
        struct node{
                int key,val;
                int siz,s;
                node *c[2];
                node(int v=0){
                        val=v;
                        key=rand();
                        siz=1,s=1;
                        c[0]=c[1]=0;
                }
                void rz(){siz=s;if(c[0])siz+=c[0]->siz;if(c[1])siz+=c[1]->siz;}
        }pool[maxn],*cur,*root;
```

```
	Treap(){cur=pool;}
	node* newnode(int val){return *cur=node(val),cur++;}
	void rot(node *&t,int d){
		if(!t->c[d])t=t->c[!d];
		else{
			node *p=t->c[d];t->c[d]=p->c[!d];
			p->c[!d]=t;t->rz();p->rz();t=p;
		}
	}
	void insert(node *&t,int x){
		if(!t){t=newnode(x);return;}
		if(t->val==x){t->s++;t->siz++;return;}
		insert(t->c[x>t->val],x);
		if(t->key<t->c[x>t->val]->key)
			rot(t,x>t->val);
		else t->rz();
	}
	void del(node *&t,int x){
		if(!t)return;
		if(t->val==x){
			if(t->s>1){t->s--;t->siz--;return;}
			if(!t->c[0]||!t->c[1]){
				if(!t->c[0])t=t->c[1];
				else t=t->c[0];
				return;
			}
			int d=t->c[0]->key<t->c[1]->key;
			rot(t,d);
			del(t,x);
			return;
		}
		del(t->c[x>t->val],x);
		t->rz();
	}
	int pre(node *t,int x){
		if(!t)return INT_MIN;
		int ans=pre(t->c[x>t->val],x);
		if(t->val<x)ans=max(ans,t->val);
		return ans;
	}
	int nxt(node *t,int x){
		if(!t)return INT_MAX;
```

```
                int ans=nxt(t->c[x>=t->val],x);
                if(t->val>x)ans=min(ans,t->val);
                return ans;
        }
        int rank(node *t,int x){
                if(!t)return 0;
                if(t->val==x)return sz(t->c[0]);
                if(t->val<x)return sz(t->c[0])+t->s+rank(t->c[1],x);
                if(t->val>x)return rank(t->c[0],x);
        }
        int kth(node *t,int x){
                if(sz(t->c[0])>=x)return kth(t->c[0],x);
                if(sz(t->c[0])+t->s>=x)return t->val;
                return kth(t->c[1],x-t->s-sz(t->c[0]));
        }
        void deb(node *t){
                if(!t)return;
                deb(t->c[0]);
                printf("%d ",t->val);
                deb(t->c[1]);
        }
        void insert(int x){insert(root,x);}
        void del(int x){del(root,x);}
        int pre(int x){return pre(root,x);}
        int nxt(int x){return nxt(root,x);}
        int rank(int x){return rank(root,x);}
        int kth(int x){return kth(root,x);}
        void deb(){deb(root);puts("");}
}T;
int main(){
    srand(12121);
    int m;
    scanf("%d",&m);
    while(m--){
        int opt,x;
        scanf("%d",&opt);
        switch(opt){
            case 1:
                scanf("%d",&x);
                T.insert(x);
                break;
            case 2:
```

```
                scanf("%d",&x);
                T.del(x);
                break;
            case 3:
                scanf("%d",&x);
                printf("%d\n",T.rank(x)+1);
                break;
            case 4:
                scanf("%d",&x);
                printf("%d\n",T.kth(x));
                break;
            case 5:
                scanf("%d",&x);
                printf("%d\n",T.pre(x));
                break;
            case 6:
                scanf("%d",&x);
                printf("%d\n",T.nxt(x));
                break;
        }
    }
        return 0;
}
```

3.1.2 Splay

```
void Rotate(int x, int c){
        int y = T[x].c[c];
        int z = T[y].c[1 - c];

        if (T[x].fa){
                if (T[T[x].fa].c[0] == x) T[T[x].fa].c[0] = y;
                else T[T[x].fa].c[1] = y;
        }

        T[z].fa = x; T[x].c[c] = z;
        T[y].fa = T[x].fa; T[x].fa = y; T[y].c[1 - c] = x;

        Update(x);
        Update(y);
}

```

```
int stack[M], fx[M];

void Splay(int x, int fa){
        int top = 0;
        for (int  u = x; u != fa; u = T[u].fa)
                stack[++top] = u;
        for (int i = 2; i <= top; i++)
                if (T[stack[i]].c[0] == stack[i - 1]) fx[i] = 0;
                else fx[i] = 1;

        for (int i = 2; i <= top; i += 2){
                if (i == top) Rotate(stack[i], fx[i]);
                else {
                        if (fx[i] == fx[i + 1]){
                                Rotate(stack[i + 1], fx[i + 1]);
                                Rotate(stack[i], fx[i]);
                        } else {
                                Rotate(stack[i], fx[i]);
                                Rotate(stack[i + 1], fx[i + 1]);
                        }
                }
        }

        if (fa == 0) Root = x;
}
```

3.2 坚固的数据结构

3.2.1 坚固的平衡树

```
#define sz(x) (x?x->siz:0)
struct node{
    int siz,key;
    LL val,sum;
    LL mu,a,d;
    node *c[2],*f;
    void split(int ned,node *&p,node *&q);
    node* rz(){
        sum=val;siz=1;
        if(c[0])sum+=c[0]->sum,siz+=c[0]->siz;
        if(c[1])sum+=c[1]->sum,siz+=c[1]->siz;
        return this;
```

```
    }
    void make(LL _mu,LL _a,LL _d){
        sum=sum*_mu+_a*siz+_d*siz*(siz-1)/2;
        val=val*_mu+_a+_d*sz(c[0]);
        mu*=_mu;a=a*_mu+_a;d=d*_mu+_d;
    }
    void pd(){
        if(mu==1&&a==0&&d==0)return;
        if(c[0])c[0]->make(mu,a,d);
        if(c[1])c[1]->make(mu,a+d+d*sz(c[0]),d);
        mu=1;a=d=0;
    }
    node(){mu=1;}
}nd[maxn*2],*root;
node *merge(node *p,node *q){
    if(!p||!q)return p?p->rz():(q?q->rz():0);
    p->pd();q->pd();
    if(p->key<q->key){
        p->c[1]=merge(p->c[1],q);
        return p->rz();
    }else{
        q->c[0]=merge(p,q->c[0]);
        return q->rz();
    }
}
void node::split(int ned,node *&p,node *&q){
    if(!ned){p=0;q=this;return;}
    if(ned==siz){p=this;q=0;return;}
    pd();
    if(sz(c[0])>=ned){
        c[0]->split(ned,p,q);c[0]=0;rz();
        q=merge(q,this);
    }else{
        c[1]->split(ned-sz(c[0])-1,p,q);c[1]=0;rz();
        p=merge(this,p);
    }
}
int main(){
    for(int i=1;i<=n;i++){
        nd[i].val=in();
        nd[i].key=rand();
        nd[i].rz();
```

```
        root=merge(root,nd+i);
    }
}
```

3.2.2 坚固的字符串

1. ext 库中的 rope

```
#include <ext/rope>

using __gnu_cxx::crope;
using __gnu_cxx::rope;

crope a, b;

int main(void) {
    a = b.substr(pos, len);     // [pos, pos + len)
    a = b.substr(pos);          // [pos, pos]
    b.c_str();                  // might lead to memory leaks
    b.delete_c_str();           // delete the c_str that created before
    a.insert(pos, text);        // insert text before position pos
    a.erase(pos, len);          // erase [pos, pos + len)
}
```

2. 可持久化平衡树实现的 rope

```
class Rope {
private:
    class Node {
    public:
        Node *left, *right;
        int size;
        char key;

        Node(char key = 0, Node *left = NULL, Node *right = NULL)
                : key(key), left(left), right(right) {
            update();
        }

        void update() {
            size = (left ? left->size : 0) + 1 + (right ? right->size : 0);
        }

```

```
    std::string to_string() {
        return (left ? left->to_string() : "") + key
             + (right ? right->to_string() : "");
    }
};

bool random(int a, int b) {
    return rand() % (a + b) < a;
}

Node* merge(Node *x, Node *y) {
    if (!x) {
        return y;
    }
    if (!y) {
        return x;
    }
    if (random(x->size, y->size)) {
        return new Node(x->key, x->left, merge(x->right, y));
    } else {
        return new Node(y->key, merge(x, y->left), y->right);
    }
}

std::pair<Node*, Node*> split(Node *x, int size) {
    if (!x) {
        return std::make_pair<Node*, Node*>(NULL, NULL);
    }
    if (size == 0) {
        return std::make_pair<Node*, Node*>(NULL, x);
    }
    if (size > x->size) {
        return std::make_pair<Node*, Node*>(x, NULL);
    }
    if (x->left && size <= x->left->size) {
        std::pair<Node*, Node*> part =
            split(x->left, size);
        return std::make_pair(part.first, new Node(x->key, part.second, x->right));
    } else {
        std::pair<Node*, Node*> part =
            split(x->right, size - (x->left ? x->left->size : 0) - 1);
        return std::make_pair(new Node(x->key, x->left, part.first), part.second);
```

```
        }
    }

    Node* build(const std::string &text, int left, int right) {
        if (left > right) {
            return NULL;
        }
        int mid = left + right >> 1;
        return new Node(text[mid],
                        build(text, left, mid - 1),
                        build(text, mid + 1, right));
    }

public:
    Node *root;

    Rope() {
        root = NULL;
    }

    Rope(const std::string &text) {
        root = build(text, 0, (int)text.length() - 1);
    }

    Rope(const Rope &other) {
        root = other.root;
    }

    Rope& operator = (const Rope &other) {
        if (this == &other) {
            return *this;
        }
        root = other.root;
        return *this;
    }

    int size() {
        return root ? root->size : 0;
    }

    void insert(int pos, const std::string &text) {
        if (pos < 0 || pos > size()) {
```

```
            throw "Out of range";
        }
        std::pair<Node*, Node*> part = split(root, pos);
        root = merge(merge(part.first, build(text, 0, (int)text.length() - 1)),
                     part.second);
    }

    void erase(int left, int right) {
        if (left < 0 || left >= size() ||
            right < 1 || right > size()) {
            throw "Out of range";
        }
        if (left >= right) {
            return;
        }
        std::pair<Node*, Node*> part = split(root, left);
        root = merge(part.first, split(part.second, right - left).second);
    }

    std::string substr(int left, int right) {
        if (left < 0 || left >= size() ||
            right < 1 || right > size()) {
            throw "Out of range";
        }
        if (left >= right) {
            return "";
        }
        return split(split(root, left).second, right - left).first->to_string();
    }

    void copy(int left, int right, int pos) {
        if (left < 0 || left >= size() ||
            right < 1 || right > size() ||
            pos < 0 || pos > size()) {
            throw "Out of range";
        }
        if (left >= right) {
            return;
        }
        std::pair<Node*, Node*> part = split(root, pos);
        root = merge(merge(part.first,
                           split(split(root, left).second, right - left).first),
```

```
                                  part.second);
        }
};
```

3.2.3 坚固的左偏树

```
int Merge(int x, int y){
  if (x == 0 || y == 0) return x + y;
  if (Heap[x].Key < Heap[y].Key) swap(x, y);
  Heap[x].Ri = Merge(Heap[x].Ri, y);
  if (Heap[Heap[x].Le].Dis < Heap[Heap[x].Ri].Dis) swap(Heap[x].Le, Heap[x].Ri);
  if (Heap[x].Ri == 0) Heap[x].Dis = 0;
  else Heap[x].Dis = Heap[Heap[x].Ri].Dis + 1;
  return x;
}

for (int i = 0; i <= n; i++){
        Heap[i].Le = Heap[i].Ri = 0;
        Heap[i].Dis = 0;
        Heap[i].Key = Cost[i];
}
Heap[0].Dis = -1;
```

3.2.4 不坚固的斜堆

```
struct node;
node *Null,*root[maxn];
struct node{
        node* c[2];
        int val,ind;
        node(int _val=0,int _ind=0){
                val=_val;c[0]=c[1]=Null;ind=_ind;
        }
};
node* merge(node *p,node *q){
        if(p==Null)return q;
        if(q==Null)return p;
        if(p->val>q->val)swap(p,q);
        p->c[1]=merge(p->c[1],q);
        swap(p->c[0],p->c[1]);
        return p;
```

```
}

Null=new node(0);
Null->c[0]=Null->c[1]=Null;
```

3.3 树上的魔术师

3.3.1 轻重树链剖分 (zky)

```
vector<int>G[maxn];
int fa[maxn],top[maxn],siz[maxn],son[maxn],mp[maxn],z,dep[maxn];
void dfs(int u){
        siz[u]=1;
        for(int i=0;i<G[u].size();i++){
                int v=G[u][i];
                if(v!=fa[u]){
                        fa[v]=u;dep[v]=dep[u]+1;
                        dfs(v);
                        siz[u]+=siz[v];
                        if(siz[son[u]]<siz[v])son[u]=v;
                }
        }
}
void build(int u,int tp){
        top[u]=tp;mp[u]=++z;
        if(son[u])build(son[u],tp);
        for(int v,i=0;i<G[u].size();i++)if((v=G[u][i])!=son[u]&&v!=fa[u])build(v,v);
}
```

3.3.2 Link Cut Tree(zky)

```
struct LCT{
    struct node{
        bool rev;
        int mx,val;
        node *f,*c[2];
        bool d(){return this==f->c[1];}
        bool rt(){return !f||(f->c[0]!=this&&f->c[1]!=this);}
        void sets(node *x,int d){pd();if(x)x->f=this;c[d]=x;rz();}
        void makerv(){rev^=1;swap(c[0],c[1]);}
        void pd(){
            if(rev){
```

```
                if(c[0])c[0]->makerv();
                if(c[1])c[1]->makerv();
                rev=0;
            }
        }
        void rz(){
            mx=val;
            if(c[0])mx=max(mx,c[0]->mx);
            if(c[1])mx=max(mx,c[1]->mx);
        }
}nd[int(1e4)+1];
void rot(node *x){
    node *y=x->f;if(!y->rt())y->f->pd();
    y->pd();x->pd();bool d=x->d();
    y->sets(x->c[!d],d);
    if(y->rt())x->f=y->f;
    else y->f->sets(x,y->d());
    x->sets(y,!d);
}
void splay(node *x){
    while(!x->rt())
        if(x->f->rt())rot(x);
        else if(x->d()==x->f->d())rot(x->f),rot(x);
        else rot(x),rot(x);
}
node* access(node *x){
    node *y=0;
    for(;x;x=x->f){
        splay(x);
        x->sets(y,1);y=x;
    }return y;
}
void makert(node *x){
    access(x)->makerv();
    splay(x);
}
void link(node *x,node *y){
    makert(x);
    x->f=y;
    access(x);
}
void cut(node *x,node *y){
```

```
        makert(x);access(y);splay(y);
        y->c[0]=x->f=0;
        y->rz();
    }
    void link(int x,int y){link(nd+x,nd+y);}
    void cut(int x,int y){cut(nd+x,nd+y);}
}T;
```

3.3.3 AAA Tree

```
#define rep(i,a,n) for(int i=a;i<n;i++)
int n,m;
struct info{
    int mx,mn,sum,sz;
    info(){}
    info(int mx,int mn,int sum,int sz):
        mx(mx),mn(mn),sum(sum),sz(sz){}
    void deb(){printf("sum:%d size:%d",(int)sum,sz);}
};
struct flag{
    int mul,add;
    flag(){mul=1;}
    flag(int mul,int add):
        mul(mul),add(add){}
    bool empty(){return mul==1&&add==0;}
};
info operator+(const info &a,const flag &b) {
    return a.sz?info(a.mx*b.mul+b.add,a.mn*b.mul+b.add,a.sum*b.mul+b.add*a.sz,a.sz):a;
}
info operator+(const info &a,const info &b) {
    return info(max(a.mx,b.mx),min(a.mn,b.mn),a.sum+b.sum,a.sz+b.sz);
}
flag operator+(const flag &a,const flag &b) {
    return flag(a.mul*b.mul,a.add*b.mul+b.add);
}
struct node{
    node *c[4],*f;
    flag Cha,All;
    info cha,sub,all;
    bool rev,inr;
    int val;
    void makerev(){rev^=1;swap(c[0],c[1]);}
```

```
    void makec(const flag &a){
        Cha=Cha+a;cha=cha+a;val=val*a.mul+a.add;
        all=cha+sub;
    }
    void makes(const flag &a,bool _=1){
        All=All+a;all=all+a;sub=sub+a;
        if(_)makec(a);
    }
    void rz(){
        cha=all=sub=info(-(1<<30),1<<30,0,0);
        if(!inr)all=cha=info(val,val,val,1);
        rep(i,0,2)if(c[i])cha=cha+c[i]->cha,sub=sub+c[i]->sub;
        rep(i,0,4)if(c[i])all=all+c[i]->all;
        rep(i,2,4)if(c[i])sub=sub+c[i]->all;
    }
    void pd(){
        if(rev){
            if(c[0])c[0]->makerev();
            if(c[1])c[1]->makerev();
            rev=0;
        }
        if(!All.empty()){
            rep(i,0,4)if(c[i])c[i]->makes(All,i>=2);
            All=flag(1,0);
        }
        if(!Cha.empty()){
            rep(i,0,2)if(c[i])c[i]->makec(Cha);
            Cha=flag(1,0);
        }

    }
    node *C(int i){if(c[i])c[i]->pd();return c[i];}
    bool d(int ty){return f->c[ty+1]==this;}
    int D(){rep(i,0,4)if(f->c[i]==this)return i;}
    void sets(node *x,int d){if(x)x->f=this;c[d]=x;}
    bool rt(int ty){
        if(ty==0)return !f||(f->c[0]!=this&&f->c[1]!=this);
        else return !f||!f->inr||!inr;
    }
}nd[maxn*2],*cur=nd+maxn,*pool[maxn],**Cur=pool;
int _cnt;
node *newnode(){
```

```
    _cnt++;
    node *x=(Cur==pool)?cur++:*(--Cur);
    rep(i,0,4)x->c[i]=0;x->f=0;
    x->All=x->Cha=flag(1,0);
    x->all=x->cha=info(-(1<<30),(1<<30),0,0);
    x->inr=1;x->rev=0;x->val=0;
    return x;
}
void dele(node *x){*(Cur++)=x;}
void rot(node *x,int ty){
    node *p=x->f;int d=x->d(ty);
    if(!p->f)x->f=0;else p->f->sets(x,p->D());
    p->sets(x->c[!d+ty],d+ty);x->sets(p,!d+ty);p->rz();
}
void splay(node *x,int ty=0){
    while(!x->rt(ty)){
        if(x->f->rt(ty))rot(x,ty);
        else if(x->d(ty)==x->f->d(ty))rot(x->f,ty),rot(x,ty);
        else rot(x,ty),rot(x,ty);
    }x->rz();
}
void add(node *u,node *w){
    w->pd();
    rep(i,2,4)if(!w->c[i]){w->sets(u,i);return;}
    node *x=newnode(),*v;
    for(v=w;v->c[2]->inr;v=v->C(2));
    x->sets(v->c[2],2);x->sets(u,3);
    v->sets(x,2);splay(x,2);
}
void del(node *w){
    if(w->f->inr){
        w->f->f->sets(w->f->c[5-w->D()],w->f->D());
        dele(w->f);splay(w->f->f,2);
    }else w->f->sets(0,w->D());
    w->f=0;
}
void access(node *w){
    static node *sta[maxn];
    static int top=0;
    node *v=w,*u;
    for(u=w;u;u=u->f)sta[top++]=u;
    while(top)sta[--top]->pd();
```

```
    splay(w);
    if(w->c[1])u=w->c[1],w->c[1]=0,add(u,w),w->rz();
    while(w->f){
        for(u=w->f;u->inr;u=u->f);
        splay(u);
        if(u->c[1])w->f->sets(u->c[1],w->D()),splay(w->f,2);
        else del(w);
        u->sets(w,1);
        (w=u)->rz();
    }splay(v);
}
void makert(node *x){
    access(x);x->makerev();
}
node *findp(node *u){
    access(u);u=u->C(0);
    while(u&&u->c[1])u=u->C(1);
    return u;
}
node *findr(node *u){for(;u->f;u=u->f);return u;}
node* cut(node *u){
    node *v=findp(u);
    if(v)access(v),del(u),v->rz();
    return v;
}
void link(node *u,node *v) {
    node* p=cut(u);
    if(findr(u)!=findr(v))p=v;
    if(p)access(p),add(u,p),p->rz();
}
int main(){
//  freopen("bzoj3153.in","r",stdin);
    n=getint();m=getint();
    static int _u[maxn],_v[maxn];
    rep(i,1,n)_u[i]=getint(),_v[i]=getint();
    rep(i,1,n+1){
        nd[i].val=getint();
        nd[i].rz();
    }
    rep(i,1,n)makert(nd+_u[i]),link(nd+_u[i],nd+_v[i]);
    int root=getint();
    makert(nd+root);
```

```
//  deb();
    int x,y,z;
    node *u,*v;
    while(m--){
        int k=getint();x=getint();
        u=nd+x;
        if(k==0||k==3||k==4||k==5||k==11){
            access(u);
            if(k==3||k==4||k==11){
                int ans=u->val;
                rep(i,2,4)if(u->c[i]){
                    info res=u->c[i]->all;
                    if(k==3) ans=min(ans,res.mn);
                    else if(k==4) ans=max(ans,res.mx);
                    else if(k==11) ans+=res.sum;
                }printf("%d\n",ans);
            }else{
                y=getint();
                flag fg(k==5,y);
                u->val=u->val*fg.mul+fg.add;
                rep(i,2,4)if(u->c[i])u->c[i]->makes(fg);
                u->rz();
            }
        }else if(k==2||k==6||k==7||k==8||k==10){
            y=getint();
            makert(u),access(nd+y),splay(u);
            if (k==7||k==8||k==10) {
                info ans=u->cha;
                if (k==7) printf("%d\n",ans.mn);
                else if (k==8) printf("%d\n",ans.mx);
                else printf("%d\n",ans.sum);
            }else u->makec(flag(k==6,getint()));
            makert(nd+root);
        }else if(k==9)link(u,nd+getint());
        else if(k==1)makert(u),root=x;
    }
    return 0;
}
```

3.4 ST

```
for (int i = 1; i <= n; i++)
        Log[i] = int(log2(i));

for (int i = 1; i <= n; i++)
        Rmq[i][0] = i;

for (int k = 1; (1 << k) <= n; k++)
        for (int i = 1; i + (1 << k) - 1 <= n; i++){
                int x = Rmq[i][k - 1], y = Rmq[i + (1 << (k - 1))][k - 1];
                if (a[x] < a[y])
                        Rmq[i][k] = x;
                else
                        Rmq[i][k] = y;
        }

int Smallest(int l, int r){
        int k = Log[r - l + 1];

        int x = Rmq[l][k];
        int y = Rmq[r - (1 << k) + 1][k];

        if (a[x] < a[y]) return x;
        else return y;
}
```

3.5 可持久化线段树

```
struct node1 {
        int L, R, Lson, Rson, Sum;
} tree[N * 40];
int root[N], a[N], b[N];
int tot, n, m;
int Real[N];
int Same(int x) {
        ++tot;
        tree[tot] = tree[x];
        return tot;
}
int build(int L, int R) {
        ++tot;
        tree[tot].L = L;
        tree[tot].R = R;
```

```
        tree[tot].Lson = tree[tot].Rson = tree[tot].Sum = 0;
        if (L == R) return tot;
        int s = tot;
        int mid = (L + R) >> 1;
        tree[s].Lson = build(L, mid);
        tree[s].Rson = build(mid + 1, R);
        return s;
}
int Ask(int Lst, int Cur, int L, int R, int k) {
        if (L == R) return L;
        int Mid = (L + R) >> 1;
        int Left = tree[tree[Cur].Lson].Sum - tree[tree[Lst].Lson].Sum;
        if (Left >= k) return Ask(tree[Lst].Lson, tree[Cur].Lson, L, Mid, k);
        k -= Left;
        return Ask(tree[Lst].Rson, tree[Cur].Rson, Mid + 1, R, k);
}
int Add(int Lst, int pos) {
        int root = Same(Lst);
        tree[root].Sum++;
        if (tree[root].L == tree[root].R) return root;
        int mid = (tree[root].L + tree[root].R) >> 1;
        if (pos <= mid) tree[root].Lson = Add(tree[root].Lson, pos);
        else tree[root].Rson = Add(tree[root].Rson, pos);
        return root;
}
int main() {
        scanf("%d%d", &n, &m);
        int up = 0;
        for (int i = 1; i <= n; i++){
                scanf("%d", &a[i]);
                b[i] = a[i];
        }
        sort(b + 1, b + n + 1);
        up = unique(b + 1, b + n + 1) - b - 1;
        for (int i = 1; i <= n; i++){
                int tmp = lower_bound(b + 1, b + up + 1, a[i]) - b;
                Real[tmp] = a[i];
                a[i] = tmp;
        }
        tot = 0;
        root[0] = build(1, up);
        for (int i = 1; i <= n; i++){
```

```
            root[i] = Add(root[i - 1], a[i]);
        }
        for (int i = 1; i <= m; i++){
            int u, v, w;
            scanf("%d%d%d", &u, &v, &w);
            printf("%d\n", Real[Ask(root[u - 1], root[v], 1, up, w)]);
        }
        return 0;
}
```

3.6 可持久化 Trie

```
int Pre[N];
int n, q, Len, cnt, Lstans;
char s[N];
int First[N], Last[N];
int Root[N];
int Trie_tot;
struct node{
    int To[30];
    int Lst;
}Trie[N];
int tot;
struct node1{
    int L, R, Lson, Rson, Sum;
}tree[N * 25];
int Build(int L, int R){
    ++tot;
    tree[tot].L = L;
    tree[tot].R = R;
    tree[tot].Lson = tree[tot].Rson = tree[tot].Sum = 0;
    if (L == R) return tot;
    int s = tot;
    int mid = (L + R) >> 1;
    tree[s].Lson = Build(L, mid);
    tree[s].Rson = Build(mid + 1, R);
    return s;
}
int Same(int x){
    ++tot;
    tree[tot] = tree[x];
    return tot;
```

```
}
int Add(int Lst, int pos){
    int s = Same(Lst);
    tree[s].Sum++;
    if (tree[s].L == tree[s].R) return s;
    int Mid = (tree[s].L + tree[s].R) >> 1;
    if (pos <= Mid) tree[s].Lson = Add(tree[Lst].Lson, pos);
    else tree[s].Rson = Add(tree[Lst].Rson, pos);
    return s;
}

int Ask(int Lst, int Cur, int L, int R, int pos){
    if (L >= pos) return 0;
    if (R < pos) return tree[Cur].Sum - tree[Lst].Sum;
    int Mid = (L + R) >> 1;
    int Ret = Ask(tree[Lst].Lson, tree[Cur].Lson, L, Mid, pos);
    Ret += Ask(tree[Lst].Rson, tree[Cur].Rson, Mid + 1, R, pos);
    return Ret;
}

int main(){
    while (scanf("%d", &n) == 1){
        for (int i = 1; i <= Trie_tot; i++){
            for (int j = 1; j <= 26; j++)
                Trie[i].To[j] = 0;
            Trie[i].Lst = 0;
        }
        Trie_tot = 1;
        cnt = 0;
        for (int ii = 1; ii <= n; ii++){
            scanf("%s", s + 1);
            Len = strlen(s + 1);
            int Cur = 1;
            First[ii] = cnt + 1;
            for (int i = 1; i <= Len; i++){
                int ch = s[i] - 'a' + 1;
                if (Trie[Cur].To[ch] == 0){
                    ++Trie_tot;
                    Trie[Cur].To[ch] = Trie_tot;
                }
                Cur = Trie[Cur].To[ch];
                Pre[++cnt] = Trie[Cur].Lst;
```

```
                Trie[Cur].Lst = ii;
            }
            Last[ii] = cnt;
        }
        tot = 0;
        Root[0] = Build(0, n);
        for (int i = 1; i <= cnt; i++){
            Root[i] = Add(Root[i - 1], Pre[i]);
        }
        Lstans = 0;
        scanf("%d", &q);
        for (int ii = 1; ii <= q; ii++){
            int L, R;
            scanf("%d%d", &L, &R);
            L = (L + Lstans) % n + 1;
            R = (R + Lstans) % n + 1;
            if (L > R) swap(L, R);
            int Ret = Ask(Root[First[L] - 1], Root[Last[R]], 0, n, L);
            printf("%d\n", Ret);
            Lstans = Ret;
        }
    }
    return 0;
}
```

3.7 k-d 树

```
long long norm(const long long &x) {
    //    For manhattan distance
    return std::abs(x);
    //    For euclid distance
    return x * x;
}

struct Point {
    int x, y, id;

    const int& operator [] (int index) const {
        if (index == 0) {
            return x;
        } else {
            return y;
```

```
        }
    }

    friend long long dist(const Point &a, const Point &b) {
        long long result = 0;
        for (int i = 0; i < 2; ++i) {
            result += norm(a[i] - b[i]);
        }
        return result;
    }
} point[N];

struct Rectangle {
    int min[2], max[2];

    Rectangle() {
        min[0] = min[1] = INT_MAX;
        max[0] = max[1] = INT_MIN;
    }

    void add(const Point &p) {
        for (int i = 0; i < 2; ++i) {
            min[i] = std::min(min[i], p[i]);
            max[i] = std::max(max[i], p[i]);
        }
    }

    long long dist(const Point &p) {
        long long result = 0;
        for (int i = 0; i < 2; ++i) {
            //    For minimum distance
            result += norm(std::min(std::max(p[i], min[i]), max[i]) - p[i]);
            //    For maximum distance
            result += std::max(norm(max[i] - p[i]), norm(min[i] - p[i]));
        }
        return result;
    }
};

struct Node {
    Point seperator;
    Rectangle rectangle;
```

```
    int child[2];

    void reset(const Point &p) {
        seperator = p;
        rectangle = Rectangle();
        rectangle.add(p);
        child[0] = child[1] = 0;
    }
} tree[N << 1];

int size, pivot;

bool compare(const Point &a, const Point &b) {
    if (a[pivot] != b[pivot]) {
        return a[pivot] < b[pivot];
    }
    return a.id < b.id;
}

int build(int l, int r, int type = 1) {
    pivot = type;
    if (l >= r) {
        return 0;
    }
    int x = ++size;
    int mid = l + r >> 1;
    std::nth_element(point + l, point + mid, point + r, compare);
    tree[x].reset(point[mid]);
    for (int i = l; i < r; ++i) {
        tree[x].rectangle.add(point[i]);
    }
    tree[x].child[0] = build(l, mid, type ^ 1);
    tree[x].child[1] = build(mid + 1, r, type ^ 1);
    return x;
}

int insert(int x, const Point &p, int type = 1) {
    pivot = type;
    if (x == 0) {
        tree[++size].reset(p);
        return size;
    }
```

```
    tree[x].rectangle.add(p);
    if (compare(p, tree[x].seperator)) {
        tree[x].child[0] = insert(tree[x].child[0], p, type ^ 1);
    } else {
        tree[x].child[1] = insert(tree[x].child[1], p, type ^ 1);
    }
    return x;
}

//    For minimum distance
void query(int x, const Point &p, std::pair<long long, int> &answer, int type = 1) {
    pivot = type;
    if (x == 0 || tree[x].rectangle.dist(p) > answer.first) {
        return;
    }
    answer = std::min(answer,
             std::make_pair(dist(tree[x].seperator, p), tree[x].seperator.id));
    if (compare(p, tree[x].seperator)) {
        query(tree[x].child[0], p, answer, type ^ 1);
        query(tree[x].child[1], p, answer, type ^ 1);
    } else {
        query(tree[x].child[1], p, answer, type ^ 1);
        query(tree[x].child[0], p, answer, type ^ 1);
    }
}

std::priority_queue<std::pair<long long, int> > answer;

void query(int x, const Point &p, int k, int type = 1) {
    pivot = type;
    if (x == 0 ||
        (int)answer.size() == k && tree[x].rectangle.dist(p) > answer.top().first) {
        return;
    }
    answer.push(std::make_pair(dist(tree[x].seperator, p), tree[x].seperator.id));
    if ((int)answer.size() > k) {
        answer.pop();
    }
    if (compare(p, tree[x].seperator)) {
        query(tree[x].child[0], p, k, type ^ 1);
        query(tree[x].child[1], p, k, type ^ 1);
    } else {
```

```
        query(tree[x].child[1], p, k, type ^ 1);
        query(tree[x].child[0], p, k, type ^ 1);
    }
}
```

3.8 莫队算法

```
struct node{
        int l, r, id;
        friend bool operator < (const node &a, const node &b){
                if (a.l / Block == b.l / Block) return a.r / Block < b.r / Block;
                return a.l / Block < b.l / Block;
        }
}q[N];
Block = int(sqrt(n));
for (int i = 1; i <= m; i++){
        scanf("%d%d", &q[i].l, &q[i].r);
        q[i].id = i;
}
sort(q + 1, q + 1 + m);
Cur = a[1]; /// Hints: adjust by yourself
Le = Ri = 1;
for (int i = 1; i <= m; i++){
        while (q[i].r > Ri) Ri++, ChangeRi(1, Le, Ri);
        while (q[i].l > Le) ChangeLe(-1, Le, Ri), Le++;
        while (q[i].l < Le) Le--, ChangeLe(1, Le, Ri);
        while (q[i].r < Ri) ChangeRi(-1, Le, Ri), Ri--;
        Ans[q[i].id] = Cur;
}
```

3.9 树上在线莫队

```
bool operator<(qes a,qes b){
    if(dfn[a.x]/B!=dfn[b.x]/B)return dfn[a.x]/B<dfn[b.x]/B;
    if(dfn[a.y]/B!=dfn[b.y]/B)return dfn[a.y]/B<dfn[b.y]/B;
    if(a.tm/B!=b.tm/B)return a.tm/B<b.tm/B;
    return a.tm<b.tm;
}
void vxor(int x){
    if(vis[x])ans-=(LL)W[cnt[col[x]]]*V[col[x]],cnt[col[x]]--;
    else cnt[col[x]]++,ans+=(LL)W[cnt[col[x]]]*V[col[x]];
```

```
    vis[x]^=1;
}
void change(int x,int y){
    if(vis[x]){
        vxor(x);col[x]=y;vxor(x);
    }else col[x]=y;
}
void TimeMachine(int tar){//XD
    for(int i=now+1;i<=tar;i++)change(C[i].x,C[i].y);
    for(int i=now;i>tar;i--)change(C[i].x,C[i].pre);
    now=tar;
}
void vxor(int x,int y){
    while(x!=y)if(dep[x]>dep[y])vxor(x),x=fa[x];
    else vxor(y),y=fa[y];
}
    for(int i=1;i<=q;i++){
        int ty=getint(),x=getint(),y=getint();
        if(ty&&dfn[x]>dfn[y])swap(x,y);
        if(ty==0) C[++Csize]=(oper){x,y,pre[x],i},pre[x]=y;
        else Q[Qsize+1]=(qes){x,y,Qsize+1,Csize},Qsize++;
    }sort(Q+1,Q+1+Qsize);
    int u=Q[1].x,v=Q[1].y;
    TimeMachine(Q[1].tm);
    vxor(Q[1].x,Q[1].y);
    int LCA=lca(Q[1].x,Q[1].y);
    vxor(LCA);anss[Q[1].id]=ans;vxor(LCA);
    for(int i=2;i<=Qsize;i++){
        TimeMachine(Q[i].tm);
        vxor(Q[i-1].x,Q[i].x);
        vxor(Q[i-1].y,Q[i].y);
        int LCA=lca(Q[i].x,Q[i].y);
        vxor(LCA);
        anss[Q[i].id]=ans;
        vxor(LCA);
    }
```

3.10 整体二分

```
struct BIT{
        LL d[maxn];
        inline int lowbit(int x){return x&-x;}
```

```
	LL get(int x){
		LL ans=0;
		while(x)ans+=d[x],x-=lowbit(x);
		return ans;
	}
	void updata(int x,LL f){
		while(x<=m)d[x]+=f,x+=lowbit(x);
	}
	void add(int l,int r,LL f){
		updata(l,f);
		updata(r+1,-f);
	}
}T,T2;
int anss[maxn],wana[maxn];
struct qes{
	LL x,y,z;
	qes(LL _x=0,LL _y=0,LL _z=0):
		x(_x),y(_y),z(_z){}
}q[maxn],p[maxn];
bool part(qes &q){
	if(q.y+q.z>=wana[q.x])return 1;
	q.z+=q.y;q.y=0;return 0;
}
void solve(int lef,int rig,int l,int r){
	if(l==r){
		for(int i=lef;i<=rig;i++)if(anss[p[i].x]!=-1)
		anss[p[i].x]=l;return;
	}int mid=(l+r)>>1;
	for(int i=l;i<=mid;i++){
		if(q[i].x<=q[i].y)T.add(q[i].x,q[i].y,q[i].z);
		else T.add(1,q[i].y,q[i].z),T.add(q[i].x,m,q[i].z);
	}for(int i=lef;i<=rig;i++){
		p[i].y=0;
		for(int j=0;j<O[p[i].x].size()&&p[i].y<=int(1e9)+1;j++)
		p[i].y+=T.get(O[p[i].x][j]);
	}for(int i=l;i<=mid;i++){
		if(q[i].x<=q[i].y)T.add(q[i].x,q[i].y,-q[i].z);
		else T.add(1,q[i].y,-q[i].z),T.add(q[i].x,m,-q[i].z);
	}int dv=stable_partition(p+lef,p+rig+1,part)-p-1;
	if(lef<=dv)
	solve(lef,dv,l,mid);
	if(dv+1<=rig)
```

```
        solve(dv+1,rig,mid+1,r);
}
```

3.11 树状数组 kth

```
int find(int k){
    int cnt=0,ans=0;
    for(int i=22;i>=0;i--){
        ans+=(1<<i);
        if(ans>n || cnt+d[ans]>=k)ans-=(1<<i);
        else cnt+=d[ans];
    }
    return ans+1;
}
```

3.12 虚树

```
int a[maxn*2],sta[maxn*2];
int top=0,k;
void build(){
    top=0;
    sort(a,a+k,bydfn);
    k=unique(a,a+k)-a;
    sta[top++]=1;_n=k;
    for(int i=0;i<k;i++){
        int LCA=lca(a[i],sta[top-1]);
        while(dep[LCA]<dep[sta[top-1]]){
            if(dep[LCA]>=dep[sta[top-2]]){
                add_edge(LCA,sta[--top]);
                if(sta[top-1]!=LCA)sta[top++]=LCA;
                break;
            }add_edge(sta[top-2],sta[top-1]);top--;
        }if(sta[top-1]!=a[i])sta[top++]=a[i];
    }
    while(top>1)
        add_edge(sta[top-2],sta[top-1]),top--;
        for(int i=0;i<k;i++)inr[a[i]]=1;
}
```

3.13 点分治 (zky)

```
int siz[maxn],f[maxn],dep[maxn],cant[maxn],root,All,d[maxn];
void makert(int u,int fa){
    siz[u]=1;f[u]=0;
    for(int i=0;i<G[u].size();i++){
        edge e=G[u][i];
        if(e.v!=fa&&!cant[e.v]){
            dep[e.v]=dep[u]+1;
            makert(e.v,u);
            siz[u]+=siz[e.v];
            f[u]=max(f[u],siz[e.v]);
        }
    }f[u]=max(f[u],All-f[u]);
    if(f[root]>f[u])root=u;
}
void dfs(int u,int fa){
        //Gain data
    for(int i=0;i<G[u].size();i++){
        edge e=G[u][i];
        if(e.v==fa||cant[e.v])continue;
        d[e.v]=d[u]+e.w;
        dfs(e.v,u);
    }
}
void calc(int u){
        d[u]=0;
    for(int i=0;i<G[u].size();i++){
        edge e=G[u][i];
        if(cant[e.v])continue;
        d[e.v]=e.w;
        dfs(e.v,u);

    }
}
void solve(int u){
    calc(u);cant[u]=1;
    for(int i=0;i<G[u].size();i++){
        edge e=G[u][i];
        if(cant[e.v])continue;
        All=siz[e.v];
        f[root=0]=n+1;
        makert(e.v,0);
        solve(root);
```

```
    }
}
All=n
f[root=0]=n+1;
makert(1,1);
solve(root);
```

3.14 元芳树

```
#include<bits/stdc++.h>
using namespace std;
const int maxn=1e4+1e4+233;
const int BIT=18;
int n,m,q;
struct edge{
        int u,v,w;
        bool operator==(edge oth)const{
                return u==oth.u && v==oth.v && w==oth.w;
        }
        bool operator!=(edge oth)const{
                return !(*this==oth);
        }
};
vector<edge>G[maxn],T[maxn];

int dfn[maxn],low[maxn],tot,rlen[maxn];
bool ins[maxn];
stack<edge>S;
int Rcnt=0;
vector<edge>ring[maxn];
vector<int>bel[maxn],sum[maxn],dis[maxn];
int fa[maxn][BIT];
int dep[maxn],dep2[maxn],fw[maxn];
vector<pair<int,int> >ind[maxn];
map<pair<int,int>,int>Mw;
pair<int,int>pack(int a,int b){
        if(a>b)swap(a,b);
        return make_pair(a,b);
}
void tarjan(int u){
        dfn[u]=low[u]=++tot;
        for(int i=0;i<G[u].size();i++){
```

```
        edge e=G[u][i];
        if(dfn[e.v])
            low[u]=min(low[u],dfn[e.v]);
        else{
            S.push(e);
            tarjan(e.v);
            if(low[e.v]==dfn[u]){

                if(S.top()==e){
                    fa[e.v][0]=u;
                    fw[e.v]=e.w;
                    S.pop();
                    continue;
                }

                Rcnt++;
                edge ed;
                do{
          ed=S.top();S.pop();
          ring[Rcnt].push_back(ed);
        }while(ed!=e);
          reverse(ring[Rcnt].begin(),ring[Rcnt].end());
        int last=ring[Rcnt].back().v;
          ring[Rcnt].push_back((edge){last,u,Mw[pack(last,u)]});
            }
            low[u]=min(low[u],low[e.v]);
        }
    }
}
void up(int u){
    if(dep[u]||u==1)return ;
    if(fa[u][0])up(fa[u][0]);
    dep[u]=dep[fa[u][0]]+1;
    fw[u]+=fw[fa[u][0]];
}
void build(){
    S.push((edge){0,1,0});
    tarjan(1);

    for(int i=1;i<=Rcnt;i++){
        rlen[i]=0;
        sum[i].resize(ring[i].size());
```

```
			dis[i].resize(ring[i].size());
			for(int j=0;j<ring[i].size();j++){
				rlen[i]+=ring[i][j].w;
				ind[i].push_back(make_pair(ring[i][j].u,j));
			}
			sum[i][0]=0;
			fw[i+n]=0;
			fa[i+n][0]=ring[i][0].u;
			for(int j=1;j<ring[i].size();j++){
				sum[i][j]=sum[i][j-1]+ring[i][j-1].w;
				dis[i][j]=min(sum[i][j],rlen[i]-sum[i][j]);
				fw[ring[i][j].u]=dis[i][j];
				fa[ring[i][j].u][0]=i+n;
			}
			sort(ind[i].begin(),ind[i].end());
		}

		for(int i=1;i<=n+Rcnt;i++)
			up(i);

		for(int j=1;j<BIT;j++)
		for(int i=1;i<=n+Rcnt;i++)if(fa[i][j-1])
			fa[i][j]=fa[fa[i][j-1]][j-1];

}
pair<int,int>second_lca;
int lca(int u,int v){
	if(dep[u]<dep[v])swap(u,v);
	int d=dep[u]-dep[v];
	for(int i=0;i<BIT;i++)if(d>>i&1)
		u=fa[u][i];
	if(u==v)return u;
	for(int i=BIT-1;i>=0;i--)if(fa[u][i]!=fa[v][i]){
		u=fa[u][i];
		v=fa[v][i];
	}
	second_lca=make_pair(u,v);
	return fa[u][0];
}
int main(){

	freopen("bzoj2125.in","r",stdin);
```

```cpp

	scanf("%d%d%d",&n,&m,&q);
	for(int i=1;i<=m;i++){
		int u,v,w;scanf("%d%d%d",&u,&v,&w);
		G[u].push_back((edge){u,v,w});
		G[v].push_back((edge){v,u,w});
		Mw[pack(u,v)]=w;
	}

	build();
	while(q--){
		int u,v;
		scanf("%d%d",&u,&v);
		int LCA=lca(u,v);
		if(LCA<=n)printf("%d\n",fw[u]+fw[v]-2*fw[LCA]);
		else{
			if(dep[u]<dep[v])swap(u,v);
			int R=LCA-n;
			int uu=second_lca.first;
			int vv=second_lca.second;
			int ans=fw[u]-fw[uu]+fw[v]-fw[vv];
			int uid,vid;
			uid=lower_bound(ind[R].begin(),ind[R].end(),make_pair(uu,-1))->second;
			vid=lower_bound(ind[R].begin(),ind[R].end(),make_pair(vv,-1))->second;
			ans+=min(abs(sum[R][uid]-sum[R][vid]),rlen[R]-abs(sum[R][uid]-sum[R][vid]));
			printf("%d\n",ans);
		}
	}
	return 0;
}
```

4 图论

4.1 强连通分量

```cpp
int stamp, comps, top;
int dfn[N], low[N], comp[N], stack[N];

void tarjan(int x) {
    dfn[x] = low[x] = ++stamp;
    stack[top++] = x;
    for (int i = 0; i < (int)edge[x].size(); ++i) {
```

```cpp
            int y = edge[x][i];
            if (!dfn[y]) {
                tarjan(y);
                low[x] = std::min(low[x], low[y]);
            } else if (!comp[y]) {
                low[x] = std::min(low[x], dfn[y]);
            }
        }
        if (low[x] == dfn[x]) {
            comps++;
            do {
                int y = stack[--top];
                comp[y] = comps;
            } while (stack[top] != x);
        }
}

void solve() {
    stamp = comps = top = 0;
    std::fill(dfn, dfn + n, 0);
    std::fill(comp, comp + n, 0);
    for (int i = 0; i < n; ++i) {
        if (!dfn[i]) {
            tarjan(i);
        }
    }
}
```

4.2 2-SAT 问题

```cpp
int stamp, comps, top;
int dfn[N], low[N], comp[N], stack[N];

void add(int x, int a, int y, int b) {
    edge[x << 1 | a].push_back(y << 1 | b);
}

void tarjan(int x) {
    dfn[x] = low[x] = ++stamp;
    stack[top++] = x;
    for (int i = 0; i < (int)edge[x].size(); ++i) {
        int y = edge[x][i];
```

```cpp
            if (!dfn[y]) {
                tarjan(y);
                low[x] = std::min(low[x], low[y]);
            } else if (!comp[y]) {
                low[x] = std::min(low[x], dfn[y]);
            }
        }
        if (low[x] == dfn[x]) {
            comps++;
            do {
                int y = stack[--top];
                comp[y] = comps;
            } while (stack[top] != x);
        }
}

bool solve() {
    int counter = n + n + 1;
    stamp = top = comps = 0;
    std::fill(dfn, dfn + counter, 0);
    std::fill(comp, comp + counter, 0);
    for (int i = 0; i < counter; ++i) {
        if (!dfn[i]) {
            tarjan(i);
        }
    }
    for (int i = 0; i < n; ++i) {
        if (comp[i << 1] == comp[i << 1 | 1]) {
            return false;
        }
        answer[i] = (comp[i << 1 | 1] < comp[i << 1]);
    }
    return true;
}
```

4.3 二分图最大匹配

4.3.1 Hungary 算法

时间复杂度：$\mathcal{O}(V \cdot E)$

```cpp
vector<int>G[maxn];
int Link[maxn],vis[maxn],T;
```

```cpp
bool find(int x){
	for(int i=0;i<G[x].size();i++){
		int v=G[x][i];
		if(vis[v]==T)continue;
		vis[v]=T;
		if(!Link[v]||find(Link[v])){
			Link[v]=x;
			return 1;
		}
	}return 0;
}
int Hungarian(int n){
	int ans=0;
	memset(Link,0,sizeof Link);
	for(int i=1;i<=n;i++){
		T++;
		ans+=find(i);
	}return ans;
}
```

4.3.2 Hopcroft Karp 算法

时间复杂度：$\mathcal{O}(\sqrt{V} \cdot E)$

```cpp
int matchx[N], matchy[N], level[N];

bool dfs(int x) {
    for (int i = 0; i < (int)edge[x].size(); ++i) {
        int y = edge[x][i];
        int w = matchy[y];
        if (w == -1 || level[x] + 1 == level[w] && dfs(w)) {
            matchx[x] = y;
            matchy[y] = x;
            return true;
        }
    }
    level[x] = -1;
    return false;
}

int solve() {
    std::fill(matchx, matchx + n, -1);
    std::fill(matchy, matchy + m, -1);
```

```
    for (int answer = 0; ; ) {
        std::vector<int> queue;
        for (int i = 0; i < n; ++i) {
            if (matchx[i] == -1) {
                level[i] = 0;
                queue.push_back(i);
            } else {
                level[i] = -1;
            }
        }
        for (int head = 0; head < (int)queue.size(); ++head) {
            int x = queue[head];
            for (int i = 0; i < (int)edge[x].size(); ++i) {
                int y = edge[x][i];
                int w = matchy[y];
                if (w != -1 && level[w] < 0) {
                    level[w] = level[x] + 1;
                    queue.push_back(w);
                }
            }
        }
        int delta = 0;
        for (int i = 0; i < n; ++i) {
            if (matchx[i] == -1 && dfs(i)) {
                delta++;
            }
        }
        if (delta == 0) {
            return answer;
        } else {
            answer += delta;
        }
    }
}
```

4.4 二分图最大权匹配

时间复杂度：$\mathcal{O}(V^4)$

```
int labelx[N], labely[N], match[N], slack[N];
bool visitx[N], visity[N];

```

```cpp
bool dfs(int x) {
    visitx[x] = true;
    for (int y = 0; y < n; ++y) {
        if (visity[y]) {
            continue;
        }
        int delta = labelx[x] + labely[y] - graph[x][y];
        if (delta == 0) {
            visity[y] = true;
            if (match[y] == -1 || dfs(match[y])) {
                match[y] = x;
                return true;
            }
        } else {
            slack[y] = std::min(slack[y], delta);
        }
    }
    return false;
}

int solve() {
    for (int i = 0; i < n; ++i) {
        match[i] = -1;
        labelx[i] = INT_MIN;
        labely[i] = 0;
        for (int j = 0; j < n; ++j) {
            labelx[i] = std::max(labelx[i], graph[i][j]);
        }
    }
    for (int i = 0; i < n; ++i) {
        while (true) {
            std::fill(visitx, visitx + n, 0);
            std::fill(visity, visity + n, 0);
            for (int j = 0; j < n; ++j) {
                slack[j] = INT_MAX;
            }
            if (dfs(i)) {
                break;
            }
            int delta = INT_MAX;
            for (int j = 0; j < n; ++j) {
                if (!visity[j]) {
```

```
                    delta = std::min(delta, slack[j]);
                }
            }
            for (int j = 0; j < n; ++j) {
                if (visitx[j]) {
                    labelx[j] -= delta;
                }
                if (visity[j]) {
                    labely[j] += delta;
                } else {
                    slack[j] -= delta;
                }
            }
        }
    }
    int answer = 0;
    for (int i = 0; i < n; ++i) {
        answer += graph[match[i]][i];
    }
    return answer;
}
```

4.5 最大流 (dinic)

时间复杂度：$\mathcal{O}(V^2 \cdot E)$

```
struct edge{int u,v,cap,flow;};
vector<edge>edges;
vector<int>G[maxn];
int s,t;
int cur[maxn],d[maxn];
void add(int u,int v,int cap){
        edges.push_back((edge){u,v,cap,0});
        G[u].push_back(edges.size()-1);
        edges.push_back((edge){v,u,0,0});
        G[v].push_back(edges.size()-1);
}
bool bfs(){
        static int vis[maxn];
        memset(vis,0,sizeof vis);vis[s]=1;
        queue<int>q;q.push(s);d[s]=0;
        while(!q.empty()){
                int u=q.front();q.pop();
```

```
			for(int i=0;i<G[u].size();i++){
				edge e=edges[G[u][i]];if(vis[e.v]||e.cap==e.flow)continue;
				d[e.v]=d[u]+1;vis[e.v]=1;q.push(e.v);
			}
	}return vis[t];
}
int dfs(int u,int a){
	if(u==t||!a)return a;
	int flow=0,f;
	for(int &i=cur[u];i<G[u].size();i++){
		edge e=edges[G[u][i]];
		if(d[e.v]==d[u]+1&&(f=dfs(e.v,min(a,e.cap-e.flow)))>0){
			edges[G[u][i]].flow+=f;
			edges[G[u][i]^1].flow-=f;
			flow+=f;a-=f;if(!a)break;
		}
	}return flow;
}
int dinic(){
	int flow=0,x;
	while(bfs()){
		memset(cur,0,sizeof cur);
		while(x=dfs(s,INT_MAX)){
			flow+=x;
			memset(cur,0,sizeof cur);
		}
	}return flow;
}
```

4.6 最大流（sap）

时间复杂度：$\mathcal{O}(V^2 \cdot E)$

```
int g[T], adj[M], nxt[M], f[M];
int cnt[T], dist[T], cur[T], fa[T], dat[T];
void Ins(int x, int y, int ff, int rf){
	adj[++tot] = y; nxt[tot] = g[x]; g[x] = tot; f[tot] = ff;
	adj[++tot] = x; nxt[tot] = g[y]; g[y] = tot; f[tot] = rf;
}
int sap(int s, int t){
	int x, sum;
	for (int i = 1; i <= t; i++){
		dist[i] = 1;
```

```
            cur[i] = g[i];
            fa[i] = 0;
            dat[i] = 0;
            cnt[i] = 0;
    }
    cnt[0] = 1; cnt[1] = t - 1;
    dist[t] = 0;
    dat[s] = INF;
    x = s;
    sum = 0;
    while (1){
            int p;
            for (p = cur[x]; p; p = nxt[p]){
                    if (f[p] > 0 && dist[adj[p]] == dist[x] - 1) break;
            }
            if (p > 0){
                    cur[x] = p;
                    fa[adj[p]] = p;
                    dat[adj[p]] = min(dat[x], f[p]);
                    x = adj[p];
                    if (x == t){
                            sum += dat[x];
                            while (x != s){
                                    f[fa[x]] -= dat[t];
                                    f[fa[x] ^ 1] += dat[t];
                                    x = adj[fa[x] ^ 1];
                             }
                    }
            } else {
                    cnt[dist[x]] --;
                    if (cnt[dist[x]] == 0) return sum;
                    dist[x] = t + 1;
                    for (int p = g[x]; p; p = nxt[p]){
                            if (f[p] > 0 && dist[adj[p]] + 1 < dist[x]){
                                    dist[x] = dist[adj[p]] + 1;
                                    cur[x] = p;
                            }
                    }
                    cnt[dist[x]]++;
                    if (dist[s] > t) return sum;
                    if (x != s) x = adj[fa[x] ^ 1];
             }
```

```
        }
}
/*
tot = 1
edges' id start from 2
remember to clean g
t is the number of points
*/
```

4.7 最小费用最大流

4.7.1 稀疏图

时间复杂度：$\mathcal{O}(V \cdot E^2)$

```
struct EdgeList {
    int size;
    int last[N];
    int succ[M], other[M], flow[M], cost[M];
    void clear(int n) {
        size = 0;
        std::fill(last, last + n, -1);
    }
    void add(int x, int y, int c, int w) {
        succ[size] = last[x];
        last[x] = size;
        other[size] = y;
        flow[size] = c;
        cost[size++] = w;
    }
} e;

int n, source, target;
int prev[N];

void add(int x, int y, int c, int w) {
    e.add(x, y, c, w);
    e.add(y, x, 0, -w);
}

bool augment() {
    static int dist[N], occur[N];
    std::vector<int> queue;
```

```cpp
    std::fill(dist, dist + n, INT_MAX);
    std::fill(occur, occur + n, 0);
    dist[source] = 0;
    occur[source] = true;
    queue.push_back(source);
    for (int head = 0; head < (int)queue.size(); ++head) {
        int x = queue[head];
        for (int i = e.last[x]; ~i; i = e.succ[i]) {
            int y = e.other[i];
            if (e.flow[i] && dist[y] > dist[x] + e.cost[i]) {
                dist[y] = dist[x] + e.cost[i];
                prev[y] = i;
                if (!occur[y]) {
                    occur[y] = true;
                    queue.push_back(y);
                }
            }
        }
        occur[x] = false;
    }
    return dist[target] < INT_MAX;
}

std::pair<int, int> solve() {
    std::pair<int, int> answer = std::make_pair(0, 0);
    while (augment()) {
        int number = INT_MAX;
        for (int i = target; i != source; i = e.other[prev[i] ^ 1]) {
            number = std::min(number, e.flow[prev[i]]);
        }
        answer.first += number;
        for (int i = target; i != source; i = e.other[prev[i] ^ 1]) {
            e.flow[prev[i]] -= number;
            e.flow[prev[i] ^ 1] += number;
            answer.second += number * e.cost[prev[i]];
        }
    }
    return answer;
}
```

4.7.2 稠密图

使用条件：费用非负

时间复杂度：$\mathcal{O}(V \cdot E^2)$

```
struct EdgeList {
    int size;
    int last[N];
    int succ[M], other[M], flow[M], cost[M];
    void clear(int n) {
        size = 0;
        std::fill(last, last + n, -1);
    }
    void add(int x, int y, int c, int w) {
        succ[size] = last[x];
        last[x] = size;
        other[size] = y;
        flow[size] = c;
        cost[size++] = w;
    }
} e;

int n, source, target, flow, cost;
int slack[N], dist[N];
bool visit[N];

void add(int x, int y, int c, int w) {
    e.add(x, y, c, w);
    e.add(y, x, 0, -w);
}

bool relabel() {
    int delta = INT_MAX;
    for (int i = 0; i < n; ++i) {
        if (!visit[i]) {
            delta = std::min(delta, slack[i]);
        }
        slack[i] = INT_MAX;
    }
    if (delta == INT_MAX) {
        return true;
    }
    for (int i = 0; i < n; ++i) {
```

```
        if (visit[i]) {
            dist[i] += delta;
        }
    }
    return false;
}

int dfs(int x, int answer) {
    if (x == target) {
        flow += answer;
        cost += answer * (dist[source] - dist[target]);
        return answer;
    }
    visit[x] = true;
    int delta = answer;
    for (int i = e.last[x]; ~i; i = e.succ[i]) {
        int y = e.other[i];
        if (e.flow[i] > 0 && !visit[y]) {
            if (dist[y] + e.cost[i] == dist[x]) {
                int number = dfs(y, std::min(e.flow[i], delta));
                e.flow[i] -= number;
                e.flow[i ^ 1] += number;
                delta -= number;
                if (delta == 0) {
                    dist[x] = INT_MIN;
                    return answer;
                }
            } else {
                slack[y] = std::min(slack[y], dist[y] + e.cost[i] - dist[x]);
            }
        }
    }
    return answer - delta;
}

std::pair<int, int> solve() {
    flow = cost = 0;
    std::fill(dist, dist + n, 0);
    do {
        do {
            fill(visit, visit + n, 0);
        } while (dfs(source, INT_MAX));
```

```
    } while (!relabel());
    return std::make_pair(flow, cost);
}
```

4.8 一般图最大匹配

时间复杂度：$\mathcal{O}(V^3)$

```
int match[N], belong[N], next[N], mark[N], visit[N];
std::vector<int> queue;

int find(int x) {
    if (belong[x] != x) {
        belong[x] = find(belong[x]);
    }
    return belong[x];
}

void merge(int x, int y) {
    x = find(x);
    y = find(y);
    if (x != y) {
        belong[x] = y;
    }
}

int lca(int x, int y) {
    static int stamp = 0;
    stamp++;
    while (true) {
        if (x != -1) {
            x = find(x);
            if (visit[x] == stamp) {
                return x;
            }
            visit[x] = stamp;
            if (match[x] != -1) {
                x = next[match[x]];
            } else {
                x = -1;
            }
        }
```

```
            std::swap(x, y);
        }
}

void group(int a, int p) {
    while (a != p) {
        int b = match[a], c = next[b];
        if (find(c) != p) {
            next[c] = b;
        }
        if (mark[b] == 2) {
            mark[b] = 1;
            queue.push_back(b);
        }
        if (mark[c] == 2) {
            mark[c] = 1;
            queue.push_back(c);
        }
        merge(a, b);
        merge(b, c);
        a = c;
    }
}

void augment(int source) {
    queue.clear();
    for (int i = 0; i < n; ++i) {
        next[i] = visit[i] = -1;
        belong[i] = i;
        mark[i] = 0;
    }
    mark[source] = 1;
    queue.push_back(source);
    for (int head = 0; head < (int)queue.size() && match[source] == -1; ++head) {
        int x = queue[head];
        for (int i = 0; i < (int)edge[x].size(); ++i) {
            int y = edge[x][i];
            if (match[x] == y || find(x) == find(y) || mark[y] == 2) {
                continue;
            }
            if (mark[y] == 1) {
                int r = lca(x, y);
```

```
                if (find(x) != r) {
                    next[x] = y;
                }
                if (find(y) != r) {
                    next[y] = x;
                }
                group(x, r);
                group(y, r);
            } else if (match[y] == -1) {
                next[y] = x;
                for (int u = y; u != -1; ) {
                    int v = next[u];
                    int mv = match[v];
                    match[v] = u;
                    match[u] = v;
                    u = mv;
                }
                break;
            } else {
                next[y] = x;
                mark[y] = 2;
                mark[match[y]] = 1;
                queue.push_back(match[y]);
            }
        }
    }
}

int solve() {
    std::fill(match, match + n, -1);
    for (int i = 0; i < n; ++i) {
        if (match[i] == -1) {
            augment(i);
        }
    }
    int answer = 0;
    for (int i = 0; i < n; ++i) {
        answer += (match[i] != -1);
    }
    return answer;
}
```

4.9 无向图全局最小割

时间复杂度：$\mathcal{O}(V^3)$

注意事项：处理重边时，应该对边权累加

```cpp
int node[N], dist[N];
bool visit[N];

int solve(int n) {
    int answer = INT_MAX;
    for (int i = 0; i < n; ++i) {
        node[i] = i;
    }
    while (n > 1) {
        int max = 1;
        for (int i = 0; i < n; ++i) {
            dist[node[i]] = graph[node[0]][node[i]];
            if (dist[node[i]] > dist[node[max]]) {
                max = i;
            }
        }
        int prev = 0;
        memset(visit, 0, sizeof(visit));
        visit[node[0]] = true;
        for (int i = 1; i < n; ++i) {
            if (i == n - 1) {
                answer = std::min(answer, dist[node[max]]);
                for (int k = 0; k < n; ++k) {
                    graph[node[k]][node[prev]] =
                        (graph[node[prev]][node[k]] += graph[node[k]][node[max]]);
                }
                node[max] = node[--n];
            }
            visit[node[max]] = true;
            prev = max;
            max = -1;
            for (int j = 1; j < n; ++j) {
                if (!visit[node[j]]) {
                    dist[node[j]] += graph[node[prev]][node[j]];
                    if (max == -1 || dist[node[max]] < dist[node[j]]) {
                        max = j;
                    }
                }
```

```
            }
        }
    }
    return answer;
}
```

4.10 有根树的同构

时间复杂度：$\mathcal{O}(VlogV)$

```
const unsigned long long MAGIC = 4423;

unsigned long long magic[N];
std::pair<unsigned long long, int> hash[N];

void solve(int root) {
    magic[0] = 1;
    for (int i = 1; i <= n; ++i) {
        magic[i] = magic[i - 1] * MAGIC;
    }
    std::vector<int> queue;
    queue.push_back(root);
    for (int head = 0; head < (int)queue.size(); ++head) {
        int x = queue[head];
        for (int i = 0; i < (int)son[x].size(); ++i) {
            int y = son[x][i];
            queue.push_back(y);
        }
    }
    for (int index = n - 1; index >= 0; --index) {
        int x = queue[index];
        hash[x] = std::make_pair(0, 0);

        std::vector<std::pair<unsigned long long, int> > value;
        for (int i = 0; i < (int)son[x].size(); ++i) {
            int y = son[x][i];
            value.push_back(hash[y]);
        }
        std::sort(value.begin(), value.end());

        hash[x].first = hash[x].first * magic[1] + 37;
        hash[x].second++;
```

```
        for (int i = 0; i < (int)value.size(); ++i) {
            hash[x].first = hash[x].first * magic[value[i].second] + value[i].first;
            hash[x].second += value[i].second;
        }
        hash[x].first = hash[x].first * magic[1] + 41;
        hash[x].second++;
    }
}
```

4.11 弦图判定

```
int n, m, first[1001], l, next[2000001], where[2000001],f[1001], a[1001], c[1001], L[1001], R[1001],
v[1001], idx[1001], pos[1001];
bool b[1001][1001];

inline void makelist(int x, int y){
    where[++l] = y;
    next[l] = first[x];
    first[x] = l;
}

bool cmp(const int &x, const int &y){
    return(idx[x] < idx[y]);
}

int main(){
    for (;;)
    {
        n = read(); m = read();
        if (!n && !m) return 0;
        memset(first, 0, sizeof(first)); l = 0;
        memset(b, false, sizeof(b));
        for (int i = 1; i <= m; i++)
        {
            int x = read(), y = read();
            if (x != y && !b[x][y])
            {
               b[x][y] = true; b[y][x] = true;
               makelist(x, y); makelist(y, x);
            }
        }
        memset(f, 0, sizeof(f));
```

```
        memset(L, 0, sizeof(L));
        memset(R, 255, sizeof(R));
        L[0] = 1; R[0] = n;
        for (int i = 1; i <= n; i++) c[i] = i, pos[i] = i;
        memset(idx, 0, sizeof(idx));
        memset(v, 0, sizeof(v));
        for (int i = n; i; --i)
        {
            int now = c[i];
            R[f[now]]--;
            if (R[f[now]] < L[f[now]]) R[f[now]] = -1;
            idx[now] = i; v[i] = now;
            for (int x = first[now]; x; x = next[x])
                if (!idx[where[x]])
                {
                   swap(c[pos[where[x]]], c[R[f[where[x]]]]);
                   pos[c[pos[where[x]]]] = pos[where[x]];
                   pos[where[x]] = R[f[where[x]]];
                   L[f[where[x]] + 1] = R[f[where[x]]]--;
                   if (R[f[where[x]]] < L[f[where[x]]]) R[f[where[x]]] = -1;
                   if (R[f[where[x]] + 1] == -1)
                       R[f[where[x]] + 1] = L[f[where[x]] + 1];
                   ++f[where[x]];
                }
        }
        bool ok = true;
        //v 是完美消除序列.
        for (int i = 1; i <= n && ok; i++)
        {
            int cnt = 0;
            for (int x = first[v[i]]; x; x = next[x])
                if (idx[where[x]] > i) c[++cnt] = where[x];
            sort(c + 1, c + cnt + 1, cmp);
            bool can = true;
            for (int j = 2; j <= cnt; j++)
                if (!b[c[1]][c[j]])
                {
                    ok = false;
                    break;
                }
        }
        if (ok) printf("Perfect\n");
```

```
        else printf("Imperfect\n");
        printf("\n");
    }
}
```

4.12 弦图求最大团

```
int n, m, first[100001], next[2000001], where[2000001], l, L[100001], R[100001], c[100001], f[100001
pos[100001], idx[100001], v[100001], ans;

inline void makelist(int x, int y){
    where[++l] = y;
    next[l] = first[x];
    first[x] = l;
}

int read(){
    char ch;
    for (ch = getchar(); ch < '0' || ch > '9'; ch = getchar());
    int cnt = 0;
    for (; ch >= '0' && ch <= '9'; ch = getchar()) cnt = cnt * 10 + ch - '0';
    return(cnt);
}

int main(){
    //freopen("1006.in", "r", stdin);
    //freopen("1006.out", "w", stdout);
    memset(first, 0, sizeof(first)); l = 0;
    n = read(); m = read();
    for (int i = 1; i <= m; i++)
    {
        int x, y;
        x = read(); y = read();
        makelist(x, y); makelist(y, x);
    }
    memset(L, 0, sizeof(L));
    memset(R, 255, sizeof(R));
    memset(f, 0, sizeof(f));
    memset(idx, 0, sizeof(idx));
    for (int i = 1; i <= n; i++) c[i] = i, pos[i] = i;
    L[0] = 1; R[0] = n; ans = 0;
    for (int i = n; i; --i)
```

```
    {
        int now = c[i], cnt = 1;
        idx[now] = i; v[i] = now;
        if (--R[f[now]] < L[f[now]]) R[f[now]] = -1;
        for (int x = first[now]; x; x = next[x])
            if (!idx[where[x]])
            {
                swap(c[pos[where[x]]], c[R[f[where[x]]]]);
                pos[c[pos[where[x]]]] = pos[where[x]];
                pos[where[x]] = R[f[where[x]]];
                L[f[where[x]] + 1] = R[f[where[x]]]--;
                if (R[f[where[x]]] < L[f[where[x]]]) R[f[where[x]]] = -1;
                if (R[f[where[x]] + 1] == -1) R[f[where[x]] + 1] = L[f[where[x]] + 1];
                ++f[where[x]];
            }
            else ++cnt;
        ans = max(ans, cnt);
    }
    printf("%d\n", ans);
}
```

4.13 最大团搜索

```
// mc[i] 代表只用 i-n 号点的答案
// g 代表连通性
void dfs(int size) {
        int i, j, k;
        if (len[size] == 0) {
                if (size > ans) {
                        ans = size;
                        found = true;
                }
                return;
        }
        for (k = 0; k < len[size] && !found; k ++) {
                if (size + len[size] - k <= ans) break;
                i = list[size][k];
                if (size + mc[i] <= ans) break;
                for (j = k + 1, len[size + 1] = 0; j < len[size]; j ++)
                        if (g[i][list[size][j]]) list[size + 1][len[size + 1] ++] = list[size][j];
                dfs(size + 1);
                if (found) {
```

```
                        prin[size + 1] = i;
                }
        }
}
void work() {
        int i, j;
        mc[n] = ans = 1;
        ansi = 1;
        for (i = n - 1; i; i --) {
                found = false;
                len[1] = 0;
                for (j = i + 1; j <= n; j ++) if (g[i][j]) list[1][len[1]++] = j;
                dfs(1);
                mc[i] = ans;
                if (found) prin[1] = i;
        }
}
void print() {
        printf("%d\n", ans);
        for (int i = 1; i < ans; i ++) printf("%d ", prin[i]);
        printf("%d\n", prin[ans]);
}
```

4.14 极大团计数

```
bool g[N][N];
int ne[N], ce[N], list[N][N], ans;
void dfs(int size) {
        if (ans > 1000) return;
        int i, j, k, t, cnt, best = 0;
        bool bb;
        if (ne[size] == ce[size]) {
                if (ce[size] == 0) ++ans;
                return;
        }
        for (t = 0, i = 1; i <= ne[size]; ++i) {
                for (cnt = 0, j = ne[size] + 1; j <= ce[size]; ++j)
                        if (!g[list[size][i]][list[size][j]]) ++cnt;
                if (t == 0 || cnt < best) t = i, best = cnt;
        }
        if (t && best <= 0) return;
        for (k = ne[size] + 1; k <= ce[size]; ++k) {
```

```
				if (t > 0) {
					for (i = k; i <= ce[size]; ++i)
						if (!g[list[size][t]][list[size][i]]) break;
					swap(list[size][k], list[size][i]);
				}
				i = list[size][k];
				ne[size + 1] = ce[size + 1] = 0;
				for (j = 1; j < k; ++j)
					if (g[i][list[size][j]])
						list[size + 1][++ne[size + 1]] = list[size][j];
				for (ce[size + 1] = ne[size + 1], j = k + 1; j <= ce[size]; ++j)
					if (g[i][list[size][j]]) list[size + 1][++ce[size + 1]] = list[size][j];
				dfs(size + 1);
				++ne[size];
				--best;
				for (j = k + 1, cnt = 0; j <= ce[size]; ++j)
					if (!g[i][list[size][j]]) ++cnt;
				if (t == 0 || cnt < best) t = k, best = cnt;
				if (t && best <= 0) break;
		}
}
int main(){
		int n, m;
		while (scanf("%d%d", &n, &m) == 2) {
				for (int i = 1; i <= n; ++i)
					for (int j = 1; j <= n; ++j)
						g[i][j] = false;
				while (m--) {
					int x, y;
					scanf("%d%d", &x, &y);
					g[x][y] = g[y][x] = true;
				}
				ne[0] = 0;
				ce[0] = 0;
				for (int i = 1; i <= n; ++i)
					list[0][++ce[0]] = i;
				ans = 0;
				dfs(0);
				if (ans > 1000) puts("Too many maximal sets of friends.");
				else printf("%d\n", ans);
		}
		return 0;
```

```
}
```

4.15 最小树形图

```
int n, m, used[N], pass[N], eg[N], more, queue[N];
double g[N][N];

void combine(int id, double &sum) {
        int tot = 0, from, i, j, k;
        for (; id != 0 && !pass[id]; id = eg[id]) {
                queue[tot++] = id;
                pass[id] = 1;
        }

        for (from = 0; from < tot && queue[from] != id; from++);
        if  (from == tot) return;
        more = 1;
        for (i = from; i < tot; i++) {
                sum += g[eg[queue[i]]][queue[i]];
                if (i != from) {
                        used[queue[i]] = 1;
                        for (j = 1; j <= n; j++) if (!used[j]) {
                                if (g[queue[i]][j] < g[id][j]) g[id][j] = g[queue[i]][j];
                        }
                }
        }

        for (i = 1; i <= n; i++) if (!used[i] && i != id) {
                for (j = from; j < tot; j++) {
                        k = queue[j];
                        if (g[i][id] > g[i][k] - g[eg[k]][k]) g[i][id] = g[i][k] - g[eg[k]][k];
                }
        }
}

double mdst(int root) {
        int i, j, k;
        double sum = 0;
        memset(used, 0, sizeof(used));
        for (more = 1; more; ) {
                more = 0;
                memset(eg, 0, sizeof(eg));
```

```
                for (i = 1; i <= n; i++) if (!used[i] && i != root) {
                        for (j = 1, k = 0; j <= n; j++) if (!used[j] && i != j)
                                if (k == 0 || g[j][i] < g[k][i]) k = j;
                        eg[i] = k;
                }

                memset(pass, 0, sizeof(pass));
                for (i = 1; i <= n; i++) if (!used[i] && !pass[i] && i != root) combine(i, sum);
        }

        for (i = 1; i <= n; i++) if (!used[i] && i != root) sum += g[eg[i]][i];
        return sum;
}
```

4.16 度限制生成树

```
int n, m, S, K, ans , cnt , Best[N], fa[N], FE[N];
int f[N], p[M], t[M], c[M], o, Cost[N];
bool u[M], d[M];
pair<int, int> MinCost[N];
struct Edge {
        int a, b, c;
        bool operator < (const Edge & E) const { return c < E.c; }
}E[M];
vector<int> SE;
inline int F(int x) {
        return fa[x] == x ? x : fa[x] = F(fa[x]);
}
inline void AddEdge(int a, int b, int C) {
        p[++o] = b; c[o] = C;
        t[o] = f[a]; f[a] = o;
}
void dfs(int i, int father) {
        fa[i] = father;
        if (father == S) Best[i] = -1;
        else {
                Best[i] = i;
                if (~Best[father] && Cost[Best[father]] > Cost[i]) Best[i] = Best[father];
        }
        for (int j = f[i]; j; j = t[j])
        if (!d[j] && p[j] != father) {
                Cost[p[j]] = c[j];
```

```
            FE[p[j]] = j;
            dfs(p[j], i);
        }
}
inline bool Kruskal() {
    cnt = n - 1,ans = 0; o = 1;
    for (int i = 1; i <= n; i++) fa[i] = i, f[i] = 0;
    sort(E + 1, E + m + 1);
    for (int i = 1; i <= m; i++) {
        if (E[i].b == S) swap(E[i].a, E[i].b);
        if (E[i].a != S && F(E[i].a) != F(E[i].b)) {
            fa[F(E[i].a)] = F(E[i].b);
            ans += E[i].c;
            cnt --;
            u[i] = true;
            AddEdge(E[i].a, E[i].b, E[i].c);
            AddEdge(E[i].b, E[i].a, E[i].c);
        }
    }
    for (int i = 1; i <= n; i++) MinCost[i] = make_pair(INF, INF);
    for (int i = 1; i <= m; i++)
    if (E[i].a == S) {
        SE.push_back(i);
        MinCost[F(E[i].b)] = min(MinCost[F(E[i].b)], make_pair(E[i].c, i));
    }
    int dif = 0;
    for (int i = 1; i <= n; i++)
    if (i != S && fa[i] == i) {
        if (MinCost[i].second == INF) return false;
        if (++ dif > K) return false;
        dfs(E[MinCost[i].second].b, S);
        u[MinCost[i].second] = true;
        ans += MinCost[i].first;
    }
    return true;
}
bool Solve() {
    memset(d,false,sizeof d);
    memset(u,false,sizeof u);
    if (!Kruskal()) return false;
    for (int i = cnt + 1; i <= K && i <= n; i++) {
        int MinD = INF, MinID = -1;
```

```
                for (int j = (int) SE.size() - 1; j >= 0; j--)
                if (u[SE[j]])
                        SE.erase(SE.begin() + j);
                for (int j = 0; j < (int) SE.size(); j++) {
                        int tmp = E[SE[j]].c - Cost[Best[E[SE[j]].b]];
                        if (tmp < MinD) {
                                MinD = tmp;
                                MinID= SE[j];
                        }
                }
                if (MinID == -1) return true;
                if (MinD >= 0) break;
                ans += MinD;
                u[MinID] = true;
                d[FE[Best[E[MinID].b]]] = d[FE[Best[E[MinID].b]] ^ 1] = true;
                dfs(E[MinID].b, S);
        }
        return true;
}
int main(){
        Solve();
        return 0;
}
```

4.17 哈密尔顿回路（ORE 性质的图）

ORE 性质：

$$\forall x, y \in V \wedge (x, y) \notin E \quad s.t. \quad deg_x + deg_y \geq n$$

返回结果：从顶点 1 出发的一个哈密尔顿回路

使用条件：$n \geq 3$

```
int left[N], right[N], next[N], last[N];

void cover(int x) {
    left[right[x]] = left[x];
    right[left[x]] = right[x];
}

int adjacent(int x) {
    for (int i = right[0]; i <= n; i = right[i]) {
        if (graph[x][i]) {
            return i;
```

```
        }
    }
    return 0;
}

std::vector<int> solve() {
    for (int i = 1; i <= n; ++i) {
        left[i] = i - 1;
        right[i] = i + 1;
    }
    int head, tail;
    for (int i = 2; i <= n; ++i) {
        if (graph[1][i]) {
            head = 1;
            tail = i;
            cover(head);
            cover(tail);
            next[head] = tail;
            break;
        }
    }
    while (true) {
        int x;
        while (x = adjacent(head)) {
            next[x] = head;
            head = x;
            cover(head);
        }
        while (x = adjacent(tail)) {
            next[tail] = x;
            tail = x;
            cover(tail);
        }
        if (!graph[head][tail]) {
            for (int i = head, j; i != tail; i = next[i]) {
                if (graph[head][next[i]] && graph[tail][i]) {
                    for (j = head; j != i; j = next[j]) {
                        last[next[j]] = j;
                    }
                    j = next[head];
                    next[head] = next[i];
                    next[tail] = i;
```

```
                    tail = j;
                    for (j = i; j != head; j = last[j]) {
                        next[j] = last[j];
                    }
                    break;
                }
            }
        }
        next[tail] = head;
        if (right[0] > n) {
            break;
        }
        for (int i = head; i != tail; i = next[i]) {
            if (adjacent(i)) {
                head = next[i];
                tail = i;
                next[tail] = 0;
                break;
            }
        }
    }
    std::vector<int> answer;
    for (int i = head; ; i = next[i]) {
        if (i == 1) {
            answer.push_back(i);
            for (int j = next[i]; j != i; j = next[j]) {
                answer.push_back(j);
            }
            answer.push_back(i);
            break;
        }
        if (i == tail) {
            break;
        }
    }
    return answer;
}
```

4.18 必经点树

```
vector<int>G[maxn],rG[maxn],dom[maxn];
int n,m;
```

```
int dfn[maxn],rdfn[maxn],dfs_c,semi[maxn],idom[maxn],fa[maxn];
struct ufsets{
    int fa[maxn],best[maxn];
    int find(int x){
        if(fa[x]==x)
            return x;
        int f=find(fa[x]);
        if(dfn[semi[best[x]]]>dfn[semi[best[fa[x]]]])
            best[x]=best[fa[x]];
        fa[x]=f;
        return f;
    }
    int getbest(int x){
        find(x);
        return best[x];
    }
    void init(){
        for(int i=1;i<=n;i++)
            fa[i]=best[i]=i;
    }
}uf;
void init(){
    uf.init();
    for(int i=1;i<=n;i++){
        semi[i]=i;
        idom[i]=0;
        fa[i]=0;
        dfn[i]=rdfn[i]=0;
    }
    dfs_c=0;
}
void dfs(int u){
    dfn[u]=++dfs_c;
    rdfn[dfn[u]]=u;
    for(int i=0;i<G[u].size();i++){
        int v=G[u][i];
        if(!dfn[v]){
            fa[v]=u;
            dfs(v);
        }
    }
}
```

```cpp

void tarjan(){
    for(int i=n;i>1;i--){
        int tmp=1e9;
        int y=rdfn[i];
        for(int i=0;i<rG[y].size();i++){
            int x=rG[y][i];
            tmp=min(tmp,dfn[semi[uf.getbest(x)]]);
        }
        semi[y]=rdfn[tmp];
        int x=fa[y];
        dom[semi[y]].push_back(y);
        uf.fa[y]=x;
        for(int i=0;i<dom[x].size();i++){
            int z=dom[x][i];
            if(dfn[semi[uf.getbest(z)]]<dfn[x])
                idom[z]=uf.getbest(z);
            else
                idom[z]=semi[z];
        }
        dom[x].clear();
    }
    semi[rdfn[1]]=1;
    for(int i=2;i<=n;i++){
        int x=rdfn[i];
        if(idom[x]!=semi[x])
            idom[x]=idom[idom[x]];

    }
    idom[rdfn[1]]=0;
}
init();
dfs(1);
tarjan();
```

5 字符串

5.1 模式匹配

5.1.1 KMP 算法

```cpp
void build(char *pattern) {
    int length = (int)strlen(pattern + 1);
    fail[0] = -1;
    for (int i = 1, j; i <= length; ++i) {
        for (j = fail[i - 1]; j != -1 && pattern[i] != pattern[j + 1]; j = fail[j]);
        fail[i] = j + 1;
    }
}

void solve(char *text, char *pattern) {
    int length = (int)strlen(text + 1);
    for (int i = 1, j; i <= length; ++i) {
        for (j = match[i - 1]; j != -1 && text[i] != pattern[j + 1]; j = fail[j]);
        match[i] = j + 1;
    }
}
///Hint: 1 - Base
```

5.1.2 扩展 KMP 算法

返回结果：

$$next_i = lcp(text, text_{i \ldots n-1})$$

```cpp
void solve(char *text, int length, int *next) {
    int j = 0, k = 1;
    for (; j + 1 < length && text[j] == text[j + 1]; j++);
    next[0] = length - 1;
    next[1] = j;
    for (int i = 2; i < length; ++i) {
        int far = k + next[k] - 1;
        if (next[i - k] < far - i + 1) {
            next[i] = next[i - k];
        } else {
            j = std::max(far - i + 1, 0);
            for (; i + j < length && text[j] == text[i + j]; j++);
            next[i] = j;
            k = i;
        }
    }
}
/// 0 - Base
```

5.1.3 AC 自动机

```
struct Node{
	int Next[30], fail, mark;
}Tree[N];

void Init(){
	memset(Tree, 0, sizeof Tree);
	cnt = 1;

	for (int i = 1; i <= n; i++){
		char c;
		int now = 1;
		scanf("%s", s + 1);
		int Length = strlen(s + 1);
		for (int j = 1; j <= Length; j++){
			c = s[j];
			if (Tree[now].Next[c - 'a']) now = Tree[now].Next[c - 'a']; else
				Tree[now].Next[c - 'a'] = ++ cnt, now = cnt;
		}
	}
}

void Build_Ac(){
	int en = 0;
	Q[0] = 1;
	for (int fi = 0; fi <= en; fi++){
		int now = Q[fi];
		for (int next = 0; next < 26; next++)
			if (Tree[now].Next[next])
			{
				int k = Tree[now].Next[next];
				if (now == 1) Tree[k].fail = 1; else
				{
					int h = Tree[now].fail;
					while (h && !Tree[h].Next[next]) h = Tree[h].fail;
					if (!h) Tree[k].fail = 1;
					else Tree[k].fail = Tree[h].Next[next];
				}
				Q[++ en] = k;
			}
	}
```

```
}

/// Hints : when not match , fail = 1
```

5.2 后缀三姐妹

5.2.1 后缀数组

```
struct Sa{
	int heap[N],s[N],sa[N],r[N],tr[N],sec[N],m,cnt;
	int h[19][N];

	void Prep(){
		for (int i=1; i<=m; i++) heap[i]=0;
		for (int i=1; i<=n; i++) heap[s[i]]++;
		for (int i=2; i<=m; i++) heap[i]+=heap[i-1];
		for (int i=n; i>=1; i--) sa[heap[s[i]]--]=i;
		r[sa[1]]=1; cnt=1;
		for (int i=2; i<=n; i++){
			if (s[sa[i]]!=s[sa[i-1]]) cnt++;
			r[sa[i]]=cnt;
		}
		m=cnt;
	}

	void Suffix(){
		int j=1;
		while (cnt<n){
			cnt=0;
			for (int i=n-j+1; i<=n; i++) sec[++cnt]=i;
			for (int i=1; i<=n; i++) if (sa[i]>j)
				sec[++cnt]=sa[i]-j;
			for (int i=1; i<=n; i++) tr[i]=r[sec[i]];
			for (int i=1; i<=m; i++) heap[i]=0;
			for (int i=1; i<=n; i++) heap[tr[i]]++;
			for (int i=2; i<=m; i++) heap[i]+=heap[i-1];
			for (int i=n; i>=1; i--)
					sa[heap[tr[i]]--]=sec[i];
			tr[sa[1]]=1; cnt=1;
			for (int i=2; i<=n; i++){
				if ((r[sa[i]]!=r[sa[i-1]]) || (r[sa[i]+j]!=r[sa[i-1]+j]))
					cnt++;
```

```
                tr[sa[i]]=cnt;
            }
            for (int i=1; i<=n; i++) r[i]=tr[i];
            m=cnt; j=j+j;
        }
    }

    void Calc(){
        int k=0;
        for (int i=1; i<=n; i++){
            if (r[i]==1) continue;
            int j=sa[r[i]-1];
            while ((i+k<=n) && (j+k<=n) && (s[i+k]==s[j+k])) k++;
            h[0][r[i]]=k;
            if (k) k--;
        }
        for (int i=1; i<19; i++)
            for (int j=1; j+(1 << i)-1<=n; j++)
                h[i][j]=min(h[i-1][j],h[i-1][j + (1 << (i - 1)) + 1]);
    }

    int Query(int L,int R){
        L=r[L], R=r[R];
        if (L>R) swap(L,R);
        L++;
        int l0 = Lg[R-L+1];
        return min(h[l0][L],h[l0][R-(1 << l0)+1]);
    }

    void Work(){
        Prep(); Suffix(); Calc();
    }
}P,S;

/// Hints : 1 - Base
```

5.2.2 后缀数组 (dc3)

```
//`DC3 待排序的字符串放在 r 数组中，从 r[0] 到 r[n-1]，长度为 n，且最大值小于 m.`
//`约定除 r[n-1] 外所有的 r[i] 都大于 0, r[n-1]=0。`
//`函数结束后，结果放在 sa 数组中，从 sa[0] 到 sa[n-1]。`
//`r 必须开长度乘 3`
```

```
#define maxn 10000
#define F(x) ((x)/3+((x)%3==1?0:tb))
#define G(x) ((x)<tb?(x)*3+1:((x)-tb)*3+2)

int wa[maxn],wb[maxn],wv[maxn],wss[maxn];
int s[maxn*3],sa[maxn*3];
int c0(int *r,int a,int b)
{
        return r[a]==r[b]&&r[a+1]==r[b+1]&&r[a+2]==r[b+2];
}
int c12(int k,int *r,int a,int b)
{
        if(k==2) return r[a]<r[b]||r[a]==r[b]&&c12(1,r,a+1,b+1);
        else return r[a]<r[b]||r[a]==r[b]&&wv[a+1]<wv[b+1];
}
void sort(int *r,int *a,int *b,int n,int m)
{
        int i;
        for(i=0;i<n;i++) wv[i]=r[a[i]];
        for(i=0;i<m;i++) wss[i]=0;
        for(i=0;i<n;i++) wss[wv[i]]++;
        for(i=1;i<m;i++) wss[i]+=wss[i-1];
        for(i=n-1;i>=0;i--) b[--wss[wv[i]]]=a[i];
}
void dc3(int *r,int *sa,int n,int m)
{
        int i,j,*rn=r+n,*san=sa+n,ta=0,tb=(n+1)/3,tbc=0,p;
        r[n]=r[n+1]=0;
        for(i=0;i<n;i++)
                if(i%3!=0) wa[tbc++]=i;
        sort(r+2,wa,wb,tbc,m);
        sort(r+1,wb,wa,tbc,m);
        sort(r,wa,wb,tbc,m);
        for(p=1,rn[F(wb[0])]=0,i=1;i<tbc;i++)
                rn[F(wb[i])]=c0(r,wb[i-1],wb[i])?p-1:p++;
        if (p<tbc) dc3(rn,san,tbc,p);
        else for (i=0;i<tbc;i++) san[rn[i]]=i;
        for (i=0;i<tbc;i++)
                if(san[i]<tb) wb[ta++]=san[i]*3;
        if(n%3==1) wb[ta++]=n-1;
        sort(r,wb,wa,ta,m);
        for(i=0;i<tbc;i++)
```

```
                wv[wb[i]=G(san[i])]=i;
        for(i=0,j=0,p=0;i<ta && j<tbc;p++)
                sa[p]=c12(wb[j]%3,r,wa[i],wb[j])?wa[i++]:wb[j++];
        for(;i<ta;p++) sa[p]=wa[i++];
        for(;j<tbc;p++) sa[p]=wb[j++];
}

int main(){
        int n,m=0;
        scanf("%d",&n);
        for (int i=0;i<n;i++) scanf("%d",&s[i]),s[i]++,m=max(s[i]+1,m);
        printf("%d\n",m);
        s[n++]=0;
        dc3(s,sa,n,m);
        for (int i=0;i<n;i++) printf("%d ",sa[i]);printf("\n");
}
```

5.2.3 后缀自动机-多串 LCS

对一个串建后缀自动机，其他串在上面匹配，因为是求所有串的公共子串，所以每个点记录每个串最长匹配长度的最小值，最后找到所有点中最长的一个即可。一个注意事项就是，当走到一个点时，还要更新它的 parent 树上的祖先的匹配长度，数组开两倍啦啦啦！

```
struct Node{
        int len, fail;
        int To[30];
}T[N];
int Lst, Root, tot, ans;
char s[N];
int Len[N], Ans[N], Ord[N];
void Add(int x, int l){
        int Nt = ++tot, p = Lst;
        T[Nt].len = l;
        for (;p && !T[p].To[x]; p = T[p].fail) T[p].To[x] = Nt;
        if (!p) T[Nt].fail = Root; else
        if (T[T[p].To[x]].len == T[p].len + 1) T[Nt].fail = T[p].To[x];
        else{
                int q = ++tot, qt = T[p].To[x];
                T[q] = T[qt];
                T[q].len = T[p].len + 1;
                T[qt].fail = T[Nt].fail = q;
                for (;p && T[p].To[x] == qt; p = T[p].fail) T[p].To[x] = q;
        }
```

```
        Lst = Nt;
}
bool cmp(int a, int b){
        return T[a].len < T[b].len;
}
int main(){
        scanf("%s", s + 1);
        int n = strlen(s + 1);
        ans = n;
        Root = tot = Lst = 1;
        for (int i = 1; i <= n; i++)
                Add(s[i] - 'a' + 1, i);
        for (int i = 1; i <= tot; i++)
                Ord[i] = i;
        sort(Ord + 1, Ord + tot + 1, cmp);
        for (int i = 1; i <= tot; i++)
                Ans[i] = T[i].len;
        bool flag = 0;
        while (scanf("%s", s + 1) != EOF){
                flag = 1;
                int n = strlen(s + 1);
                int p = Root, len = 0;
                for (int i = 1; i <= tot; i++) Len[i] = 0;
                for (int i = 1; i <= n; i++){
                        int x = s[i] - 'a' + 1;
                        if (T[p].To[x]) len++, p = T[p].To[x];
                        else {
                                while (p && !T[p].To[x]) p = T[p].fail;
                                if (!p) p = Root, len = 0;
                                else len = T[p].len + 1, p = T[p].To[x];
                        }
                        Len[p] = max(Len[p], len);
                }
                for (int i = tot; i >= 1; i--){
                        int Cur = Ord[i];
                        Ans[Cur] = min(Ans[Cur], Len[Cur]);
                        if (Len[Cur] && T[Cur].fail)
                                Len[T[Cur].fail] = T[T[Cur].fail].len;
                }
        }
        if (flag){
                ans = 0;
```

```
                for (int i = 1; i <= tot; i++){
                        ans = max(ans, Ans[i]);
                }
        }
        printf("%d\n", ans);
        return 0;
}
```

5.2.4 后缀自动机-各长度字串出现次数最大值

给一个字符串 S，令 F(x) 表示 S 的所有长度为 x 的子串中，出现次数的最大值。 构建字符串的自动机，对于每个节点，right 集合大小就是出现次数，maxs 就是它代表的最长长度，那么我们用 |right(x)| 去更新 f[maxs[x]] 的值，最后从大到小用 f[i] 去更新 f[i-1] 的值即可

```
struct Node{
        int len, fail;
        int To[30];
}T[N];
int Lst, Root, tot, n;
char s[N];
int Ord[N], Ans[N], Ways[N], heap[N];
void Add(int x, int l){
        int Nt = ++tot, p = Lst;
        T[Nt].len = l;
        for (;p && !T[p].To[x]; p = T[p].fail) T[p].To[x] = Nt;
        if (!p) T[Nt].fail = Root; else
        if (T[T[p].To[x]].len == T[p].len + 1) T[Nt].fail = T[p].To[x];
        else{
                int q = ++tot, qt = T[p].To[x];
                T[q] = T[qt];
                T[q].len = T[p].len + 1;
                T[qt].fail = T[Nt].fail = q;
                for (;p && T[p].To[x] == qt; p = T[p].fail) T[p].To[x] = q;
        }
        Lst = Nt;
}
bool cmp(int a, int b){
        return T[a].len < T[b].len;
}
void sort(){
        for (int i = 1; i <= tot; i++) heap[T[i].len]++;
        for (int i = 1; i <= n; i++) heap[i] += heap[i-1];
        for (int i = 1; i <= tot; i++) Ord[heap[T[i].len]--]=i;
```

```
}
int main(){
	scanf("%s", s + 1);
	n = strlen(s + 1);
	Root = tot = Lst = 1;
	for (int i = 1; i <= n; i++)
		Add(s[i] - 'a' + 1, i);
	sort();
	memset(Ways , 0, sizeof(Ways));
	for (int i = 1, p = Root; i <= n; i++)
		p = T[p].To[s[i] - 'a' + 1], Ways[p] = 1;
	for (int i = tot; i >= 1; i--){
		int Cur = Ord[i];
		if (T[Cur].fail == 0) continue;
		Ways[T[Cur].fail] += Ways[Cur];
	}
	for (int i = 1; i <= tot; i++)
		Ans[T[i].len] = max(Ans[T[i].len], Ways[i]);
	for (int i = n; i >= 1; i--)
		Ans[i] = max(Ans[i + 1], Ans[i]);
	for (int i = 1; i <= n; i++)
		printf("%d\n", Ans[i]);
	return 0;
}
```

5.2.5 后缀自动机-两串 LCS

```
struct node{
	int len, fail;
	int To[27];
}T[N];
char a[N], b[N];
int Lst, Root, tot;
void add(int x, int l){
	int Nt = ++tot, p = Lst;
	T[Nt].len = l;
	for (;p && !T[p].To[x]; p = T[p].fail) T[p].To[x] = Nt;
	if (!p) T[Nt].fail = Root;
	else
	if (T[T[p].To[x]].len == T[p].len + 1) T[Nt].fail = T[p].To[x];
	else{
		int q = ++tot, qt = T[p].To[x];
```

```
                T[q] = T[qt];
                T[q].len  = T[p].len + 1;
                T[qt].fail = T[Nt].fail = q;
                for (;p && T[p].To[x] == qt; p = T[p].fail) T[p].To[x] = q;
        }
        Lst = Nt;
}
int main(){
        while (scanf("%s%s", a + 1, b + 1) == 2){
                int n = strlen(a + 1);
                Lst = Root = tot = 1;
                for (int i = 1; i <= n; i++)
                        add(a[i] - 'a' + 1, i);
                int m = strlen(b + 1);
                int p = Root, len = 0;
                int Ans = 0;
                for (int i = 1; i <= m; i++){
                        int x = b[i] - 'a' + 1;
                        if (T[p].To[x]) len++, p = T[p].To[x];
                        else {
                                while (p && !T[p].To[x]) p = T[p].fail;
                                if (!p) p = Root, len = 0;
                                else len = T[p].len + 1, p = T[p].To[x];
                        }
                        if (len > Ans) Ans = len;
                }
                printf("%d\n", Ans);
                for (int i = 1; i <= tot; i++){
                        T[i].len = T[i].fail = 0;
                        for (int j = 1; j <= 26; j++)
                                T[i].To[j] = 0;
                }
        }
        return 0;
}
//Hints£ºSAM + Longest common subsequence
```

5.3 回文三兄弟

5.3.1 马拉车

```cpp
void Manacher(){
	R[1] = 1;
	for (int i = 2, j = 1; i <= length; i++){
		if (j + R[j] <= i){
			R[i] = 0;
		} else {
			R[i] = min(R[j * 2 - i], j + R[j] - i);
		}
		while (i - R[i] >= 1 && i + R[i] <= length
			&& text[i - R[i]] == text[i + R[i]]){
			R[i]++;
		}
		if (i + R[i] > j + R[j]){
			j = i;
		}
	}
}
	length = 0;
	int n = strlen(s + 1);
	for (int i = 1; i <= n; i++){
		text[++length] = '*';
		text[++length] = s[i];
	}
	text[++length] = '*';
/// Hints: 1 - Base
```

5.3.2 回文自动机 (zky)

```cpp
struct PAM{
	int tot,last,str[maxn],nxt[maxn][26],n;
	int len[maxn],suf[maxn],cnt[maxn];
	int newnode(int l){
		len[tot]=l;
		return tot++;
	}
	void init(){
		tot=0;
		newnode(0);// tree0 is node 0
		newnode(-1);// tree-1 is node 1
		str[0]=-1;
		suf[0]=1;
	}
```

```
        int find(int x){
                while(str[n-len[x]-1]!=str[n])x=suf[x];
                return x;
        }
        void add(int c){
                str[++n]=c;
                int u=find(last);
                if(!nxt[u][c]){
                        int v=newnode(len[u]+2);
                        suf[v]=nxt[find(suf[u])][c];
                        nxt[u][c]=v;
                }last=nxt[u][c];
                cnt[last]++;
        }
        void count(){
                for(int i=tot-1;i>=0;i--)cnt[suf[i]]+=cnt[i];
        }
}P;
int main(){
        P.init();
        for(int i=0;i<n;i++)
                P.add(s[i]-'a');
        P.count();
```

5.4 循环串最小表示

```
string sol(char *s){
    int n=strlen(s);
    int i=0,j=1,k=0,p;
    while(i<n&&j<n&&k<n){
        int t=s[(i+k)%n]-s[(j+k)%n];
        if(t==0)k++;
        else if(t<0)j+=k+1,k=0;
        else i+=k+1,k=0;
        if(i==j)j++;
    }p=min(i,j);
    string S;
    for(int i=p;i<p+n;i++)S.push_back(s[i%n]);
    return S;
}
```

6 计算几何

6.1 二维几何基础

```cpp
inline int sign(double x) { return x < -EPS ? -1 : x > EPS; }
inline double sqr(double x) { return x * x; }

struct point {
	double x, y;
	point(double x = 0, double y = 0) : x(x), y(y) {}
	inline double length() const { return sqrt(x * x + y * y); }
	inline double norm() const { return length(); }
	inline double norm2() const { return x * x + y * y; }
	inline point unit() const {
		double len = length();
		return point(x / len, y / len);
	}
	inline point negate() const { return point(-x, -y); }
	inline point rot90() const {		// counter - clockwise
		return point(-y, x);
	}
	inline point _rot90() const {		// clockwise
		return point(y, -x);
	}
	inline point rotate(double theta) const {		// counter - clockwise
		double c = cos(theta), s = sin(theta);
		return point(x * c - y * s, x * s + y * c);
	}
	int get() { return scanf("%lf %lf", &x, &y); }
	void out() { printf("(%.5f, %.5f)\n", x, y); }
};

inline bool operator==(const point &a, const point &b) {
	return fabs(a.x - b.x) < EPS && fabs(a.y - b.y) < EPS;
}
inline bool operator!=(const point &a, const point &b) {
	return fabs(a.x - b.x) > EPS || fabs(a.y - b.y) > EPS;
}
inline bool operator<(const point &a, const point &b) {
	if (fabs(a.x - b.x) > EPS) return a.x < b.x;
	return a.y + EPS < b.y;
}
```

```
inline point operator+(const point &a, const point &b) {
        return point(a.x + b.x, a.y + b.y);
}
inline point operator-(const point &a, const point &b) {
        return point(a.x - b.x, a.y - b.y);
}
inline point operator*(const point &a, const double &b) {
        return point(a.x * b, a.y * b);
}
inline point operator/(const point &a, const double &b) {
        return point(a.x / b, a.y / b);
}
inline double det(const point &a, const point &b) {
        return a.x * b.y - b.x * a.y;
}
inline double dot(const point &a, const point &b) {
        return a.x * b.x + a.y * b.y;
}
inline double dis(const point &a, const point &b) {
        return sqrt(sqr(a.x - b.x) + sqr(a.y - b.y));
}
struct line {
        point s, t;
        line(point s = point(), point t = point()) : s(s), t(t) {}
        inline double length() const { return dis(s, t); }
};
//线段交点
//注意如果两条线段是共线的且有交点，那么 intersect_judgement 确实会返回 true,
//但是 line_intersect 会求错，所以这种情况需要特判.
inline bool point_on_line(const point &a, const line &b) {
        return sign(det(a - b.s, b.t - b.s)) == 0 && dot(b.s - a, b.t - a) < EPS;
}
inline bool two_side(const point &a, const point &b, const line &c) {
        return sign(det(a - c.s, c.t - c.s)) * sign(det(b - c.s, c.t - c.s)) < 0;
}
inline bool intersect_judgement(const line &a, const line &b) {
        if (point_on_line(b.s, a) || point_on_line(b.t, a)) return true;
        if (point_on_line(a.s, b) || point_on_line(a.t, b)) return true;
        return two_side(a.s, a.t, b) && two_side(b.s, b.t, a);
}
inline point line_intersect(const line &a, const line &b) {
        double s1 = det(a.t - a.s, b.s - a.s);
```

```cpp
        double s2 = det(a.t - a.s, b.t - a.s);
        return (b.s * s2 - b.t * s1) / (s2 - s1);
}
//点到直线的距离
double point_to_line(const point &p, const line &l) {
        return fabs(det(l.t - l.s, p - l.s)) / dis(l.s, l.t);
}
inline double min_point_to_line(const point &a, const line &b) {
        if (dot(b.s - a, b.t - a) < EPS)
                return fabs(det(b.s - a, b.t - a) / b.length());
        return min(dis(a, b.s), dis(a, b.t));
}
//点在多边形内
bool in_polygon(const point &p, const vector<point> &poly) {
        int n = (int)poly.size();
        int counter = 0;
        for (int i = 0; i < n; ++i) {
                point a = poly[i], b = poly[(i + 1) % n];
                if (point_on_line(p, line(a, b))) return false;        // bounded excluded
                int x = sign(det(p - a, b - a));
                int y = sign(a.y - p.y);
                int z = sign(b.y - p.y);
                if (x > 0 && y <= 0 && z > 0) counter++;
                if (x < 0 && z <= 0 && y > 0) counter--;
        }
        return counter != 0;
}
//点到直线的投影
point project_to_line(const point &p, const line &l) {
        return l.s + (l.t - l.s) * (dot(p - l.s, l.t - l.s) / (l.t - l.s).norm2());
}
//圆类
struct circle {
        point center;
        double radius;
        circle(point center = point(), double radius = 0)
                        : center(center), radius(radius) {}
};
inline bool operator==(const circle &a, const circle &b) {
        return a.center == b.center && fabs(a.radius - b.radius) < EPS;
}
inline bool operator!=(const circle &a, const circle &b) {
```

```cpp
	return a.center != b.center || fabs(a.radius - b.radius) > EPS;
}
inline bool in_circle(const point &p, const circle &c) {
	return dis(p, c.center) < c.radius + EPS;
}
//圆的生成函数
circle make_circle(const point &a, const point &b) {
	return circle((a + b) / 2, dis(a, b) / 2);
}
circle make_circle(const point &a, const point &b, const point &c) {
	point center = circumcenter(a, b, c);
	return circle(center, dis(center, a));
}
//点到圆的切线
pair<line, line> tangent(const point &p, const circle &c) {
	circle a = make_circle(p, c.center);
	return make_pair(circle_intersect(a, c), circle_intersect(c, a));
}
//直线与圆的交点
//返回 AB 方向的第一个交点
point line_circle_intersect(const line &l, const circle &c) {
	double x = sqrt(sqr(c.radius) - sqr(point_to_line(c.center, l)));
	return project_to_line(c.center, l) + (l.s - l.t).unit() * x;
}
//圆与圆的交点
point circle_intersect(const circle &a, const circle &b) {		// get another point using circle_intersect(b, a)
	point r = (b.center - a.center).unit();
	double d = dis(a.center, b.center);
	double x = .5 * ((sqr(a.radius) - sqr(b.radius)) / d + d);
	double h = sqrt(sqr(a.radius) - sqr(x));
	return a.center + r * x + r.rot90() * h;
}
```

6.2 快速凸包

```cpp
//水平序凸包
inline bool turn_left(const point &a, const point &b, const point &c) {
	return det(b - a, c - a) > EPS;
}
inline bool turn_right(const point &a, const point &b, const point &c) {
	return det(b - a, c - a) < -EPS;
}
```

```cpp
inline vector<point> convex_hull(vector<point> a) {
	int n = (int)a.size(), cnt = 0;
	sort(a.begin(), a.end());
	vector<point> ret;
	for (int i = 0; i < n; ++i) {
		while (cnt > 1 && turn_left(ret[cnt - 2], a[i], ret[cnt - 1])) {
			--cnt;
			ret.pop_back();
		}
		ret.push_back(a[i]);
		++cnt;
	}
	int fixed = cnt;
	for (int i = n - 1; i >= 0; --i) {
		while (cnt > fixed && turn_left(ret[cnt - 2], a[i], ret[cnt - 1])) {
			--cnt;
			ret.pop_back();
		}
		ret.push_back(a[i]);
		++cnt;
	}
	// this algorithm will preserve the points which are collineation
	// the lowest point will occur twice , i.e. ret.front () == ret.back ()
	return ret;
}
```

6.3 半平面交

```cpp
//半平面交
inline bool two_side(const point &a, const point &b, const line &c) {
	return sign(det(a - c.s, c.t - c.s)) * sign(det(b - c.s, c.t - c.s)) < 0;
}
vector<point> cut(const vector<point> &c, line p) {
	vector<point> ret;
	if (c.empty()) return ret;
	for (int i = 0; i < (int)c.size(); ++i) {
		int j = (i + 1) % (int)c.size();
		if (!turn_right(p.s, p.t, c[i])) ret.push_back(c[i]);
		if (two_side(c[i], c[j], p))
			ret.push_back(line_intersubsection(p, line(c[i], c[j])));
	}
	return ret;
```

```cpp
}
static const double BOUND = 1e5;
/*
convex .clear ();
convex . push_back ( point (-BOUND , -BOUND ));
convex . push_back ( point (BOUND , -BOUND ));
convex . push_back ( point (BOUND , -BOUND ));
convex . push_back ( point (BOUND , -BOUND ));
convex = cut(convex , line(point , point));
Judgement : convex . empty ();
*/
//高效半平面交
// plane[] 按照法向量 (逆时针 90 度) 极角排序，去除平行半平面
inline bool turn_left(const line &l, const point &p) {
        return turn_left(l.s, l.t, p);
}
vector<line> half_plane_intersect(const vector<line> &h) {
        int fore = 0, rear = -1;
        vector<line> ret;
        for (int i = 0; i < (int)h.size(); ++i) {
                while (fore < rear &&
                                        !turn_left(h[i], line_intersect(ret[rear - 1], ret[rear]))
                        --rear;
                while (fore < rear &&
                                        !turn_left(h[i], line_intersect(ret[fore], ret[fore + 1]))
                        ++fore;
                ++rear;
                ret.push_back(h[i]);
        }
        while (rear - fore > 1 &&
                                !turn_left(ret[fore], line_intersect(ret[rear - 1], ret[rear])))
                --rear;
        while (rear - fore > 1 &&
                                !turn_left(ret[rear], line_intersect(ret[fore], ret[fore + 1])))
                ++fore;
        if (rear - fore < 2) return vector<line>();
        return ret;
}
```

6.4 三角形的心

```
//三角形的内心
point incenter(const point &a, const point &b, const point &c) {
        double p = (a - b).length() + (b - c).length() + (c - a).length();
        return (a * (b - c).length() + b * (c - a).length() + c * (a - b).length()) /
                                  p;
}
//三角形的外心
point circumcenter(const point &a, const point &b, const point &c) {
        point p = b - a, q = c - a, s(dot(p, p) / 2, dot(q, q) / 2);
        double d = det(p, q);
        return a + point(det(s, point(p.y, q.y)), det(point(p.x, q.x), s)) / d;
}
//三角形的垂心
point orthocenter(const point &a, const point &b, const point &c) {
        return a + b + c - circumcenter(a, b, c) * 2.0;
}
```

6.5 圆与多边形面积交

```
// 求扇形面积
double getSectorArea(const Point &a, const Point &b, const double &r) {
        double c = (2.0 * r * r - sqrdist(a, b)) / (2.0 * r * r);
        double alpha = acos(c);
        return r * r * alpha / 2.0;
}
// 求二次方程 ax^2 + bx + c = 0 的解
std::pair<double, double> getSolution(const double &a, const double &b, const double &c) {
        double delta = b * b - 4.0 * a * c;
        if (dcmp(delta) < 0) return std::make_pair(0, 0);
        else return std::make_pair((-b - sqrt(delta)) / (2.0 * a), (-b + sqrt(delta)) / (2.0 * a));
}
// 直线与圆的交点
std::pair<Point, Point> getIntersection(const Point &a, const Point &b, const double &r) {
        Point d = b - a;
        double A = dot(d, d);
        double B = 2.0 * dot(d, a);
        double C = dot(a, a) - r * r;
        std::pair<double, double> s = getSolution(A, B, C);
        return std::make_pair(a + d * s.first, a + d * s.second);
}
// 原点到线段 AB 的距离
double getPointDist(const Point &a, const Point &b) {
```

```cpp
	Point d = b - a;
	int sA = dcmp(dot(a, d)), sB = dcmp(dot(b, d));
	if (sA * sB <= 0) return det(a, b) / dist(a, b);
	else return std::min(dist(a), dist(b));
}
// a 和 b 和原点组成的三角形与半径为 r 的圆的交的面积
double getArea(const Point &a, const Point &b, const double &r) {
	double dA = dot(a, a), dB = dot(b, b), dC = getPointDist(a, b), ans = 0.0;
	if (dcmp(dA - r * r) <= 0 && dcmp(dB - r * r) <= 0) return det(a, b) / 2.0;
	Point tA = a / dist(a) * r;
	Point tB = b / dist(b) * r;
	if (dcmp(dC - r) > 0) return getSectorArea(tA, tB, r);
	std::pair<Point, Point> ret = getIntersection(a, b, r);
	if (dcmp(dA - r * r) > 0 && dcmp(dB - r * r) > 0) {
		ans += getSectorArea(tA, ret.first, r);
		ans += det(ret.first, ret.second) / 2.0;
		ans += getSectorArea(ret.second, tB, r);
		return ans;
	}
	if (dcmp(dA - r * r) > 0) return det(ret.first, b) / 2.0 + getSectorArea(tA, ret.first, r);
	else return det(a, ret.second) / 2.0 + getSectorArea(ret.second, tB, r);
}
// 求圆与多边形的交的主过程
double getArea(int n, Point *p, const Point &c, const double r)  {
	double ret = 0.0;
	for (int i = 0; i < n; i++) {
		int sgn = dcmp(det(p[i] - c, p[(i + 1) % n] - c));
		if (sgn > 0) ret += getArea(p[i] - c, p[(i + 1) % n] - c, r);
		else ret -= getArea(p[(i + 1) % n] - c, p[i] - c, r);
	}
	return fabs(ret);
}
```

6.6 圆并求面积

注意事项：复杂度 $\mathcal{O}(n^2 \log n)$

```cpp
struct arc {
	double theta;
	int delta;
	point p;
	arc(){};
	arc(const double &theta, const point &p, int d)
```

```
                            : theta(theta), p(p), delta(d) {}
};

vector<arc> vec;
vector<double> ans;
vector<point> center;
int cnt = 0;

inline bool operator<(const arc &a, const arc &b) {
        return a.theta + EPS < b.theta;
}

inline void psh(const double t1, const point p1, const double t2,
                                                                const point p2) {
        if (t2 + EPS < t1) cnt++;
        vec.push_back(arc(t1, p1, 1));
        vec.push_back(arc(t2, p2, -1));
}

inline double cub(const double &x) { return x * x * x; }
inline void combine(int d, const double &area, const point &o) {
        if (sign(area) == 0) return;
        center[d] = (center[d] * ans[d] + o * area) * (1 / (ans[d] + area));
        ans[d] += area;
}

void area(vector<circle> &cir) {
        int n = cir.size();
        vector<bool> f;
        f.resize(n);
        vec.clear();
        cnt = 0;
        for (int i = 0; i < n; i++) {
                f[i] = true;
                for (int j = 0; j < n; j++)
                        if (i != j) {
                                if ((cir[i] == cir[j] && i < j) ||
                                                (cir[i] != cir[j] && cir[i].radius < cir[j].radius +
                                                 (cir[i].center - cir[j].center).length() <
                                                                fabs(cir[i].radius - cir[j].radius)
                                        f[i] = false;
                                        break;
```

```
                        }
                }
        }
        int n1 = 0;
        for (int i = 0; i < n; i++)
                if (f[i]) cir[n1++] = cir[i];
        n = n1;
        ans.clear();
        center.clear();
        ans.resize(n + 1);
        center.resize(n + 1);
        point dvd;
        for (int i = 0; i < n; i++) {
                dvd = cir[i].center - point(cir[i].radius, 0);
                vec.clear();
                vec.push_back(arc(-PI, dvd, 1));
                cnt = 0;
                for (int j = 0; j < n; j++)
                        if (j != i) {
                                double d = (cir[j].center - cir[i].center).norm2();
                                if (d < sqr(cir[j].radius - cir[i].radius) + EPS) {
                                        if (cir[i].radius + i * EPS < cir[j].radius + j * EPS)
                                                psh(-PI, dvd, PI, dvd);
                                } else if (d + EPS < sqr(cir[j].radius + cir[i].radius)) {
                                        double lambda =
                                                        0.5 * (1 + (sqr(cir[i].radius) - sqr(cir[j].
                                        point cp(cir[i].center + (cir[j].center - cir[i].center) * l
                                        point nor((cir[j].center - cir[i].center)._rot90().unit() *
                                                                        (sqrt(sqr(cir[i].rac
                                        point frm(cp + nor);
                                        point to(cp - nor);
                                        psh(atan2((frm - cir[i].center).y, (frm - cir[i].center).x),
                                                        atan2((to - cir[i].center).y, (to - cir[i].c
                                }
                        }
                sort(vec.begin() + 1, vec.end());
                vec.push_back(arc(PI, dvd, -1));
                for (int j = 0; j + 1 < vec.size(); j++) {
                        cnt += vec[j].delta;
                        double theta(vec[j + 1].theta - vec[j].theta);
                        double area(sqr(cir[i].radius) * theta * 0.5);
                        combine(cnt, area, cir[i].center +
```

```
                        combine(cnt, -sqr(cir[i].radius) * sin(theta) * 0.5,
                                                                (cir[i].center + vec[j].p + vec[j + 1].p) /
                        combine(cnt, det(vec[j].p, vec[j + 1].p) * 0.5,
                                                                (vec[j].p + vec[j + 1].p) / 3.);
                }
        }
}
```

6.7 最小覆盖圆

```
circle minimum_circle(vector<point> p) {
        circle ret;
        random_shuffle(p.begin(), p.end());
        for (int i = 0; i < (int)p.size(); ++i)
                if (!in_circle(p[i], ret)) {
                        ret = circle(p[i], 0);
                        for (int j = 0; j < i; ++j)
                                if (!in_circle(p[j], ret)) {
                                        ret = make_circle(p[j], p[i]);
                                        for (int k = 0; k < j; ++k)
                                                if (!in_circle(p[k], ret)) ret = make_circle(p[i], p
                                }
                }
        return ret;
}
```

6.8 最小覆盖球

```
double eps(1e-8);
int sign(const double & x) {
        return (x > eps) - (x + eps < 0);
}
bool equal(const double & x, const double & y) {
        return x + eps > y and y + eps > x;
}
struct Point {
        double x, y, z;
        Point() {
```

```
        }
        Point(const double & x, const double & y, const double & z) : x(x), y(y), z(z){
        }
        void scan() {
                scanf("%lf%lf%lf", &x, &y, &z);
        }
        double sqrlen() const {
                return x * x + y * y + z * z;
        }
        double len() const {
                return sqrt(sqrlen());
        }
        void print() const {
                printf("(%lf %lf %lf)\n", x, y, z);
        }
} a[33];
Point operator + (const Point & a, const Point & b) {
        return Point(a.x + b.x, a.y + b.y, a.z + b.z);
}
Point operator - (const Point & a, const Point & b) {
        return Point(a.x - b.x, a.y - b.y, a.z - b.z);
}
Point operator * (const double & x, const Point & a) {
        return Point(x * a.x, x * a.y, x * a.z);
}
double operator % (const Point & a, const Point & b) {
        return a.x * b.x + a.y * b.y + a.z * b.z;
}
Point operator * (const Point & a, const Point & b) {
        return Point(a.y * b.z - a.z * b.y, a.z * b.x - a.x * b.z, a.x * b.y - a.y * b.x);
}
struct Circle {
        double r;
        Point o;
        Circle() {
                o.x = o.y = o.z = r = 0;
        }
        Circle(const Point & o, const double & r) : o(o), r(r) {
        }
        void scan() {
                o.scan();
                scanf("%lf", &r);
```

```
        }
        void print() const {
                o.print();
                printf("%lf\n", r);
        }
};
struct Plane {
        Point nor;
        double m;
        Plane(const Point & nor, const Point & a) : nor(nor){
                m = nor % a;
        }
};
Point intersect(const Plane & a, const Plane & b, const Plane & c) {
        Point c1(a.nor.x, b.nor.x, c.nor.x), c2(a.nor.y, b.nor.y, c.nor.y), c3(a.nor.z, b.nor.z, c.n
        return 1 / ((c1 * c2) % c3) * Point((c4 * c2) % c3, (c1 * c4) % c3, (c1 * c2) % c4);
}
bool in(const Point & a, const Circle & b) {
        return sign((a - b.o).len() - b.r) <= 0;
}
bool operator < (const Point & a, const Point & b) {
        if(!equal(a.x, b.x)) {
                return a.x < b.x;
        }
        if(!equal(a.y, b.y)) {
                return a.y < b.y;
        }
        if(!equal(a.z, b.z)) {
                return a.z < b.z;
        }
        return false;
}
bool operator == (const Point & a, const Point & b) {
        return equal(a.x, b.x) and equal(a.y, b.y) and equal(a.z, b.z);
}
vector<Point> vec;
Circle calc() {
        if(vec.empty()) {
                return Circle(Point(0, 0, 0), 0);
        }else if(1 == (int)vec.size()) {
                return Circle(vec[0], 0);
        }else if(2 == (int)vec.size()) {
```

```
                return Circle(0.5 * (vec[0] + vec[1]), 0.5 * (vec[0] - vec[1]).len());
        }else if(3 == (int)vec.size()) {
                double r((vec[0] - vec[1]).len() * (vec[1] - vec[2]).len() * (vec[2] - vec[0]).len()
                return Circle(intersect(Plane(vec[1] - vec[0], 0.5 * (vec[1] + vec[0])),
                                               Plane(vec[2] - vec[1], 0.5 * (vec[2] + vec[1])),
                                        Plane((vec[1] - vec[0]) * (vec[2] - vec[0]), vec[0])), r);
        }else {
                Point o(intersect(Plane(vec[1] - vec[0], 0.5 * (vec[1] + vec[0])),
                                  Plane(vec[2] - vec[0], 0.5 * (vec[2] + vec[0])),
                                  Plane(vec[3] - vec[0], 0.5 * (vec[3] + vec[0]))));
                return Circle(o, (o - vec[0]).len());
        }
}
Circle miniBall(int n) {
        Circle res(calc());
        for(int i(0); i < n; i++) {
                if(!in(a[i], res)) {
                        vec.push_back(a[i]);
                        res = miniBall(i);
                        vec.pop_back();
                        if(i) {
                                Point tmp(a[i]);
                                memmove(a + 1, a, sizeof(Point) * i);
                                a[0] = tmp;
                        }
                }
        }
        return res;
}
int main() {
        int n;
        for(;;) {
                scanf("%d", &n);
                if(!n) {
                        break;
                }
                for(int i(0); i < n; i++) {
                        a[i].scan();
                }
                sort(a, a + n);
                n = unique(a, a + n) - a;
                vec.clear();
```

```
			printf("%.10f\n", miniBall(n).r);
		}
}
```

6.9 三维几何基础

```
int dcmp(const double &x) {
	return fabs(x) < eps ? 0 : (x > 0 ? 1 : -1);
}

struct TPoint{
	double x, y, z;
	TPoint() {}
	TPoint(double x, double y, double z) : x(x), y(y), z(z) {}
	TPoint operator +(const TPoint &p)const {
		return TPoint(x + p.x, y + p.y, z + p.z);
	}
	TPoint operator -(const TPoint &p)const {
		return TPoint(x - p.x, y - p.y, z - p.z);
	}
	TPoint operator *(const double &p)const {
		return TPoint(x * p, y * p, z * p);
	}
	TPoint operator /(const double &p)const {
		return TPoint(x / p, y / p, z / p);
	}
	bool operator <(const TPoint &p)const {
		int dX = dcmp(x - p.x), dY = dcmp(y - p.y), dZ = dcmp(z - p.z);
		return dX < 0 || (dX == 0 && (dY < 0 || (dY == 0 && dZ < 0)));
	}
	bool read() {
		return scanf("%lf%lf%lf", &x, &y, &z) == 3;
	}
};

double sqrdist(const TPoint &a) {
	double ret = 0;
	ret += a.x * a.x;
	ret += a.y * a.y;
	ret += a.z * a.z;
	return ret;
}
```

```cpp
double sqrdist(const TPoint &a, const TPoint &b) {
        double ret = 0;
        ret += (a.x - b.x) * (a.x - b.x);
        ret += (a.y - b.y) * (a.y - b.y);
        ret += (a.z - b.z) * (a.z - b.z);
        return ret;
}
double dist(const TPoint &a) {
        return sqrt(sqrdist(a));
}
double dist(const TPoint &a, const TPoint &b) {
        return sqrt(sqrdist(a, b));
}
TPoint det(const TPoint &a, const TPoint &b) {
        TPoint ret;
        ret.x = a.y * b.z - b.y * a.z;
        ret.y = a.z * b.x - b.z * a.x;
        ret.z = a.x * b.y - b.x * a.y;
        return ret;
}
double dot(const TPoint &a, const TPoint &b) {
        double ret = 0;
        ret += a.x * b.x;
        ret += a.y * b.y;
        ret += a.z * b.z;
        return ret;
}
double detdot(const TPoint &a, const TPoint &b, const TPoint &c, const TPoint &d) {
        return dot(det(b - a, c - a), d - a);
}
```

6.10 三维凸包

```cpp
struct Triangle{
        TPoint a, b, c;
        Triangle() {}
        Triangle(TPoint a, TPoint b, TPoint c) : a(a), b(b), c(c) {}
        double getArea() {
                TPoint ret = det(b - a, c - a);
                return dist(ret) / 2.0;
        }
};
```

```cpp
namespace Convex_Hull {
        struct Face{
                int a, b, c;
                bool isOnConvex;
                Face() {}
                Face(int a, int b, int c) : a(a), b(b), c(c) {}
        };

        int nFace, left, right, whe[MAXN][MAXN];
        Face queue[MAXF], tmp[MAXF];

        bool isVisible(const std::vector<TPoint> &p, const Face &f, const TPoint &a) {
                return dcmp(detdot(p[f.a], p[f.b], p[f.c], a)) > 0;
        }

        bool init(std::vector<TPoint> &p) {
                bool check = false;
                for (int i = 1; i < (int)p.size(); i++) {
                        if (dcmp(sqrdist(p[0], p[i]))) {
                                std::swap(p[1], p[i]);
                                check = true;
                                break;
                        }
                }
                if (!check) return false;
                check = false;
                for (int i = 2; i < (int)p.size(); i++) {
                        if (dcmp(sqrdist(det(p[i] - p[0], p[1] - p[0])))) {
                                std::swap(p[2], p[i]);
                                check = true;
                                break;
                        }
                }
                if (!check) return false;
                check = false;
                for (int i = 3; i < (int)p.size(); i++) {
                        if (dcmp(detdot(p[0], p[1], p[2], p[i]))) {
                                std::swap(p[3], p[i]);
                                check = true;
                                break;
                        }
                }
```

```
        if (!check) return false;
        for (int i = 0; i < (int)p.size(); i++)
            for (int j = 0; j < (int)p.size(); j++) {
                whe[i][j] = -1;
            }
        return true;
    }

    void pushface(const int &a, const int &b, const int &c) {
        nFace++;
        tmp[nFace] = Face(a, b, c);
        tmp[nFace].isOnConvex = true;
        whe[a][b] = nFace;
        whe[b][c] = nFace;
        whe[c][a] = nFace;
    }

    bool deal(const std::vector<TPoint> &p, const std::pair<int, int> &now, const TPoint &base)
        int id = whe[now.second][now.first];
        if (!tmp[id].isOnConvex) return true;
        if (isVisible(p, tmp[id], base)) {
            queue[++right] = tmp[id];
            tmp[id].isOnConvex = false;
            return true;
        }
        return false;
    }

    std::vector<Triangle> getConvex(std::vector<TPoint> &p) {
        static std::vector<Triangle> ret;
        ret.clear();
        if (!init(p)) return ret;
        if (!isVisible(p, Face(0, 1, 2), p[3])) pushface(0, 1, 2); else pushface(0, 2, 1);
        if (!isVisible(p, Face(0, 1, 3), p[2])) pushface(0, 1, 3); else pushface(0, 3, 1);
        if (!isVisible(p, Face(0, 2, 3), p[1])) pushface(0, 2, 3); else pushface(0, 3, 2);
        if (!isVisible(p, Face(1, 2, 3), p[0])) pushface(1, 2, 3); else pushface(1, 3, 2);
        for (int a = 4; a < (int)p.size(); a++) {
            TPoint base = p[a];
            for (int i = 1; i <= nFace; i++) {
                if (tmp[i].isOnConvex && isVisible(p, tmp[i], base)) {
                    left = 0, right = 0;
                    queue[++right] = tmp[i];
```

```
                    tmp[i].isOnConvex = false;
                    while (left < right) {
                        Face now = queue[++left];
                        if (!deal(p, std::make_pair(now.a, now.b), base)) pu
                        if (!deal(p, std::make_pair(now.b, now.c), base)) pu
                        if (!deal(p, std::make_pair(now.c, now.a), base)) pu
                    }
                    break;
                }
            }
        }
        for (int i = 1; i <= nFace; i++) {
            Face now = tmp[i];
            if (now.isOnConvex) {
                ret.push_back(Triangle(p[now.a], p[now.b], p[now.c]));
            }
        }
        return ret;
    }
};

int n;
std::vector<TPoint> p;
std::vector<Triangle> answer;

int main() {
    scanf("%d", &n);
    for (int i = 1; i <= n; i++) {
        TPoint a;
        a.read();
        p.push_back(a);
    }
    answer = Convex_Hull::getConvex(p);
    double areaCounter = 0.0;
    for (int i = 0; i < (int)answer.size(); i++) {
        areaCounter += answer[i].getArea();
    }
    printf("%.3f\n", areaCounter);
    return 0;
}
```

6.11 三维绕轴旋转

注意事项： 逆时针绕轴 AB 旋转 θ 角。

```
Matrix getTrans(const double &a, const double &b, const double &c) {
    Matrix ret;
    ret.a[0][0] = 1; ret.a[0][1] = 0; ret.a[0][2] = 0; ret.a[0][3] = 0;
    ret.a[1][0] = 0; ret.a[1][1] = 1; ret.a[1][2] = 0; ret.a[1][3] = 0;
    ret.a[2][0] = 0; ret.a[2][1] = 0; ret.a[2][2] = 1; ret.a[2][3] = 0;
    ret.a[3][0] = a; ret.a[3][1] = b; ret.a[3][2] = c; ret.a[3][3] = 1;
    return ret;
}
Matrix getRotate(const double &a, const double &b, const double &c, const double &theta) {
    Matrix ret;
    ret.a[0][0] = a * a * (1 - cos(theta)) + cos(theta);
    ret.a[0][1] = a * b * (1 - cos(theta)) + c * sin(theta);
    ret.a[0][2] = a * c * (1 - cos(theta)) - b * sin(theta);
    ret.a[0][3] = 0;

    ret.a[1][0] = b * a * (1 - cos(theta)) - c * sin(theta);
    ret.a[1][1] = b * b * (1 - cos(theta)) + cos(theta);
    ret.a[1][2] = b * c * (1 - cos(theta)) + a * sin(theta);
    ret.a[1][3] = 0;

    ret.a[2][0] = c * a * (1 - cos(theta)) + b * sin(theta);
    ret.a[2][1] = c * b * (1 - cos(theta)) - a * sin(theta);
    ret.a[2][2] = c * c * (1 - cos(theta)) + cos(theta);
    ret.a[2][3] = 0;

    ret.a[3][0] = 0;
    ret.a[3][1] = 0;
    ret.a[3][2] = 0;
    ret.a[3][3] = 1;
    return ret;
}
Matrix getRotate(const double &ax, const double &ay, const double &az, const double &bx, const doubl
    double l = dist(Point(0, 0, 0), Point(bx, by, bz));
    Matrix ret = getTrans(-ax, -ay, -az);
    ret = ret * getRotate(bx / l, by / l, bz / l, theta);
    ret = ret * getTrans(ax, ay, az);
    return ret;
}
```

6.12 Delaunay 三角剖分

```
/*
Delaunay Triangulation 随机增量算法 :
节点数至少为点数的 6 倍, 空间消耗较大注意计算内存使用
建图的过程在 build 中, 注意初始化内存池和初始三角形的坐标范围 (Triangulation::LOTS)
Triangulation::find 返回包含某点的三角形
Triangulation::add_point 将某点加入三角剖分
某个 Triangle 在三角剖分中当且仅当它的 has_children 为 0
如果要找到三角形 u 的邻域, 则枚举它的所有 u.edge[i].tri, 该条边的两个点为 u.p[(i+1)%3], u.p[(i+2)%3]
*/
const int N = 100000 + 5, MAX_TRIS = N * 6;
const double EPSILON = 1e-6, PI = acos(-1.0);
struct Point {
	double x,y; Point():x(0),y(0){} Point(double x, double y):x(x),y(y){}
	bool operator ==(Point const& that)const {return x==that.x&&y==that.y;}
};
inline double sqr(double x) { return x*x; }
double dist_sqr(Point const& a, Point const& b){return sqr(a.x-b.x)+sqr(a.y-b.y);}
bool in_circumcircle(Point const& p1, Point const& p2, Point const& p3, Point const& p4) {
	double u11 = p1.x - p4.x, u21 = p2.x - p4.x, u31 = p3.x - p4.x;
	double u12 = p1.y - p4.y, u22 = p2.y - p4.y, u32 = p3.y - p4.y;
	double u13 = sqr(p1.x) - sqr(p4.x) + sqr(p1.y) - sqr(p4.y);
	double u23 = sqr(p2.x) - sqr(p4.x) + sqr(p2.y) - sqr(p4.y);
	double u33 = sqr(p3.x) - sqr(p4.x) + sqr(p3.y) - sqr(p4.y);
	double det = -u13*u22*u31 + u12*u23*u31 + u13*u21*u32 - u11*u23*u32 - u12*u21*u33 + u11*u22*
	return det > EPSILON;
}
double side(Point const& a, Point const& b, Point const& p) { return (b.x-a.x)*(p.y-a.y) - (b.y-a.y
typedef int SideRef; struct Triangle; typedef Triangle* TriangleRef;
struct Edge {
	TriangleRef tri; SideRef side; Edge() : tri(0), side(0) {}
	Edge(TriangleRef tri, SideRef side) : tri(tri), side(side) {}
};
struct Triangle {
	Point p[3]; Edge edge[3]; TriangleRef children[3]; Triangle() {}
	Triangle(Point const& p0, Point const& p1, Point const& p2) {
		p[0]=p0;p[1]=p1;p[2]=p2;children[0]=children[1]=children[2]=0;
	}
	bool has_children() const { return children[0] != 0; }
	int num_children() const {
		return children[0] == 0 ? 0
```

```
            : children[1] == 0 ? 1
            : children[2] == 0 ? 2 : 3;
    }
    bool contains(Point const& q) const {
        double a=side(p[0],p[1],q), b=side(p[1],p[2],q), c=side(p[2],p[0],q);
        return a >= -EPSILON && b >= -EPSILON && c >= -EPSILON;
    }
} triange_pool[MAX_TRIS], *tot_triangles;
void set_edge(Edge a, Edge b) {
    if (a.tri) a.tri->edge[a.side] = b;
    if (b.tri) b.tri->edge[b.side] = a;
}
class Triangulation {
    public:
        Triangulation() {
            const double LOTS = 1e6;
            the_root = new(tot_triangles++) Triangle(Point(-LOTS,-LOTS),Point(+LOTS,-LOT
        }
        TriangleRef find(Point p) const { return find(the_root,p); }
        void add_point(Point const& p) { add_point(find(the_root,p),p); }
    private:
        TriangleRef the_root;
        static TriangleRef find(TriangleRef root, Point const& p) {
            for( ; ; ) {
                if (!root->has_children()) return root;
                else for (int i = 0; i < 3 && root->children[i] ; ++i)
                        if (root->children[i]->contains(p))
                            {root = root->children[i]; break;}
            }
        }
        void add_point(TriangleRef root, Point const& p) {
            TriangleRef tab,tbc,tca;
            tab = new(tot_triangles++) Triangle(root->p[0], root->p[1], p);
            tbc = new(tot_triangles++) Triangle(root->p[1], root->p[2], p);
            tca = new(tot_triangles++) Triangle(root->p[2], root->p[0], p);
            set_edge(Edge(tab,0),Edge(tbc,1));set_edge(Edge(tbc,0),Edge(tca,1));
            set_edge(Edge(tca,0),Edge(tab,1));set_edge(Edge(tab,2),root->edge[2]);
            set_edge(Edge(tbc,2),root->edge[0]);set_edge(Edge(tca,2),root->edge[1]);
            root->children[0]=tab;root->children[1]=tbc;root->children[2]=tca;
            flip(tab,2); flip(tbc,2); flip(tca,2);
        }
        void flip(TriangleRef tri, SideRef pi) {
```

```
                        TriangleRef trj = tri->edge[pi].tri; int pj = tri->edge[pi].side;
                        if(!trj||!in_circumcircle(tri->p[0],tri->p[1],tri->p[2],trj->p[pj])) return;
                        TriangleRef trk = new(tot_triangles++) Triangle(tri->p[(pi+1)%3], trj->p[pj]
                        TriangleRef trl = new(tot_triangles++) Triangle(trj->p[(pj+1)%3], tri->p[pi]
                        set_edge(Edge(trk,0), Edge(trl,0));
                        set_edge(Edge(trk,1), tri->edge[(pi+2)%3]); set_edge(Edge(trk,2), trj->edge[
                        set_edge(Edge(trl,1), trj->edge[(pj+2)%3]); set_edge(Edge(trl,2), tri->edge[
                        tri->children[0]=trk;tri->children[1]=trl;tri->children[2]=0;
                        trj->children[0]=trk;trj->children[1]=trl;trj->children[2]=0;
                        flip(trk,1); flip(trk,2); flip(trl,1); flip(trl,2);
                }
};
int n; Point ps[N];
void build(){
        tot_triangles = triange_pool; cin >> n;
        for(int i = 0; i < n; ++ i) scanf("%lf%lf",&ps[i].x,&ps[i].y);
        random_shuffle(ps, ps + n); Triangulation tri;
        for(int i = 0; i < n; ++ i) tri.add_point(ps[i]);
}
```

7 其他

7.1 斯坦纳树

```
priority_queue<pair<int, int> > Q;

// m is key point
// n is all point

for (int s = 0; s < (1 << m); s++){
        for (int i = 1; i <= n; i++){
                for (int s0 = (s&(s-1)); s0 ; s0=(s&(s0-1))){
                                f[s][i] = min(f[s][i], f[s0][i] + f[s - s0][i]);
                        }
        }
        for (int i = 1; i <= n; i++) vis[i] = 0;
    while (!Q.empty()) Q.pop();
        for (int i = 1; i <= n; i++){
                Q.push(mp(-f[s][i], i));
        }
        while (!Q.empty()){
                while (!Q.empty() && Q.top().first != -f[s][Q.top().second]) Q.pop();
```

```
                if (Q.empty()) break;
                int Cur = Q.top().second; Q.pop();
                for (int p = g[Cur]; p; p = nxt[p]){
                        int y = adj[p];
                        if ( f[s][y] > f[s][Cur] + 1){
                                f[s][y] = f[s][Cur] + 1;
                                Q.push(mp(-f[s][y], y));
                        }
                }
        }
}
```

7.2 无敌的读入优化

```
namespace Reader {
        const int L = (1 << 20) + 5;
        char buffer[L], *S, *T;
        __inline bool getchar(char &ch) {
                if (S == T) {
                        T = (S = buffer) + fread(buffer, 1, L, stdin);
                        if (S == T) {
                                ch = EOF;
                                return false;
                        }
                }
                ch = *S ++;
                return true;
        }
        __inline bool getint(int &x) {
                char ch;
                for (; getchar(ch) && (ch < '0' || ch > '9'); );
                if (ch == EOF) return false;
                x = ch - '0';
                for (; getchar(ch), ch >= '0' && ch <= '9'; )
                        x = x * 10 + ch - '0';
                return true;
        }
}
Reader::getint(x);
Reader::getint(y);
```

7.3 最小树形图

```
const int maxn=1100;

int n,m , g[maxn][maxn] , used[maxn] , pass[maxn] , eg[maxn] , more , queue[maxn];

void combine (int id , int &sum ) {
	int tot = 0 , from , i , j , k ;
	for ( ; id!=0 && !pass[ id ] ; id=eg[id] ) {
		queue[tot++]=id ; pass[id]=1;
	}
	for ( from=0; from<tot && queue[from]!=id ; from++);
	if  ( from==tot ) return ;
	more = 1 ;
	for ( i=from ; i<tot ; i++) {
		sum+=g[eg[queue[i]]][queue[i]] ;
		if ( i!=from ) {
			used[queue[i]]=1;
			for ( j = 1 ; j <= n ; j++) if ( !used[j] )
				if ( g[queue[i]][j]<g[id][j] ) g[id][j]=g[queue[i]][j] ;
		}
	}
	for ( i=1; i<=n ; i++) if ( !used[i] && i!=id ) {
		for ( j=from ; j<tot ; j++){
			k=queue[j];
			if ( g[i][id]>g[i][k]-g[eg[k]][k] ) g[i][id]=g[i][k]-g[eg[k]][k];
		}
	}
}

int mdst( int root ) { // return the total length of MDST
	int i , j , k , sum = 0 ;
	memset ( used , 0 , sizeof ( used ) ) ;
	for ( more =1; more ; ) {
		more = 0 ;
		memset (eg,0,sizeof(eg)) ;
		for ( i=1 ; i <= n ; i ++) if ( !used[i] && i!=root ) {
			for ( j=1 , k=0 ; j <= n ; j ++) if ( !used[j] && i!=j )
				if ( k==0 || g[j][i] < g[k][i] ) k=j ;
			eg[i] = k ;
		}
		memset(pass,0,sizeof(pass));
```

```
                for ( i=1; i<=n ; i++) if ( !used[i] && !pass[i] && i!= root ) combine ( i , sum ) ;
        }
        for ( i =1; i<=n ; i ++) if ( !used[i] && i!= root ) sum+=g[eg[i]][i];
        return sum ;
}
```

7.4 DLX

```
int n,m,K;
struct DLX{
        int L[maxn],R[maxn],U[maxn],D[maxn];
        int sz,col[maxn],row[maxn],s[maxn],H[maxn];
        bool vis[233];
        int ans[maxn],cnt;
        void init(int m){
                for(int i=0;i<=m;i++){
                        L[i]=i-1;R[i]=i+1;
                        U[i]=D[i]=i;s[i]=0;
                }
                memset(H,-1,sizeof H);
                L[0]=m;R[m]=0;sz=m+1;
        }
        void Link(int r,int c){
                U[sz]=c;D[sz]=D[c];U[D[c]]=sz;D[c]=sz;
                if(H[r]<0)H[r]=L[sz]=R[sz]=sz;
                else{
                        L[sz]=H[r];R[sz]=R[H[r]];
                        L[R[H[r]]]=sz;R[H[r]]=sz;
                }
                s[c]++;col[sz]=c;row[sz]=r;sz++;
        }
        void remove(int c){
                for(int i=D[c];i!=c;i=D[i])
                        L[R[i]]=L[i],R[L[i]]=R[i];
        }
        void resume(int c){
                for(int i=U[c];i!=c;i=U[i])
                        L[R[i]]=R[L[i]]=i;
        }
        int A(){
                int res=0;
                memset(vis,0,sizeof vis);
```

```
			for(int i=R[0];i;i=R[i])if(!vis[i]){
				vis[i]=1;res++;
				for(int j=D[i];j!=i;j=D[j])
					for(int k=R[j];k!=j;k=R[k])
						vis[col[k]]=1;
			}
			return res;
		}
		void dfs(int d,int &ans){
			if(R[0]==0){ans=min(ans,d);return;}
			if(d+A()>=ans)return;
			int tmp=23333,c;
			for(int i=R[0];i;i=R[i])
				if(tmp>s[i])tmp=s[i],c=i;
			for(int i=D[c];i!=c;i=D[i]){
				remove(i);
				for(int j=R[i];j!=i;j=R[j])remove(j);
				dfs(d+1,ans);
				for(int j=L[i];j!=i;j=L[j])resume(j);
				resume(i);
			}
		}
		void del(int c){//exactly cover
		L[R[c]]=L[c];R[L[c]]=R[c];
			for(int i=D[c];i!=c;i=D[i])
				for(int j=R[i];j!=i;j=R[j])
					U[D[j]]=U[j],D[U[j]]=D[j],--s[col[j]];
	}
	void add(int c){  //exactly cover
		R[L[c]]=L[R[c]]=c;
			for(int i=U[c];i!=c;i=U[i])
				for(int j=L[i];j!=i;j=L[j])
					++s[col[U[D[j]]=D[U[j]]=j]];
	}
		bool dfs2(int k){//exactly cover
		if(!R[0]){
			cnt=k;return 1;
		}
		int c=R[0];
			for(int i=R[0];i;i=R[i])
				if(s[c]>s[i])c=i;
		del(c);
```

```
                for(int i=D[c];i!=c;i=D[i]){
                        for(int j=R[i];j!=i;j=R[j])
                                del(col[j]);
            ans[k]=row[i];if(dfs2(k+1))return true;
                        for(int j=L[i];j!=i;j=L[j])
                                add(col[j]);
        }
        add(c);
                return 0;
        }
}dlx;
int main(){
        dlx.init(n);
        for(int i=1;i<=m;i++)
                for(int j=1;j<=n;j++)
                        if(dis(station[i],city[j])<mid-eps)
                                dlx.Link(i,j);
                        dlx.dfs(0,ans);
}
```

7.5 插头 DP

```
int n,m,l;
struct L{
    int d[11];
    int& operator[](int x){return d[x];}
    int mc(int x){
        int an=1;
        if(d[x]==1){
            for(x++;x<l;x++)if(d[x]){
                an=an+(d[x]==1?1:-1);
                if(!an)return x;
            }
        }else{
            for(x--;x>=0;x--)if(d[x]){
                an=an+(d[x]==2?1:-1);
                if(!an)return x;
            }
        }
    }
    int h(){int an=0;for(int i=l-1;i>=0;i--)an=an*3+d[i];return an;}
    L s(int x,int y){
```

```
        L S=*this;
        S[x]=y;return S;
    }
    L operator>>(int _){
        L S=*this;
        for(int i=l-1;i>=1;i--)S[i]=S[i-1];
        S[0]=0;return S;
    }
};
struct Int{
    int len;
    int a[40];
    Int(){len=1;memset(a,0,sizeof a);}
    Int operator+=(const Int &o){
        int l=max(len,o.len);
        for(int i=0;i<l;i++)
            a[i]=a[i]+o.a[i];
        for(int i=0;i<l;i++)
            a[i+1]+=a[i]/10,a[i]%=10;
        if(a[l])l++;len=l;
        return *this;
    }
    void print(){
        for(int i=len-1;i>=0;i--)
            printf("%d",a[i]);
        puts("");
    }
};
struct hashtab{
    int sz;
    int tab[177147];
    Int w[177147];
    L s[177147];
    hashtab(){memset(tab,-1,sizeof tab);}
    void cl(){
        for(int i=0;i<sz;i++)tab[s[i].h()]=-1;
        sz=0;
    }
    Int& operator[](L S){
        int h=S.h();
        if(tab[h]==-1)tab[h]=sz,s[sz]=S,w[sz]=Int(),sz++;
        return w[tab[h]];
```

```
    }
}f[2];
bool check(L S){
    int cn1=0,cn2=0;
    for(int i=0;i<l;i++){
        cn1+=S[i]==1;
        cn2+=S[i]==2;
    }return cn1==1&&cn2==1;
}
int main(){
    Int One;One.a[0]=1;
    scanf("%d%d",&n,&m);if(n<m)swap(n,m);l=m+1;
    if(n==1||m==1){puts("1");return 0;}
    int cur=0;f[cur].cl();
    for(int i=1;i<=n;i++){
        for(int j=1;j<=m;j++){
            if(i==1&&j==1){
                f[cur][L().s(0,1).s(1,2)]+=One;
                continue;
            }
            cur^=1;f[cur].cl();
            for(int k=0;k<f[!cur].sz;k++){
                L S=f[!cur].s[k];Int w=f[!cur][S];
                int d1=S[j-1],d2=S[j];
                if(d1==0&&d2==0){
                    if(i!=n&&j!=m)f[cur][S.s(j-1,1).s(j,2)]+=w;
                }else
                if(d1==0||d2==0){
                    if(i!=n)f[cur][S.s(j-1,d1|d2).s(j,0)]+=w;
                    if(j!=m)f[cur][S.s(j-1,0).s(j,d1|d2)]+=w;
                }else
                if(d1==1&&d2==2){
                    if(i==n&&j==m&&check(S))
                        (w+=w).print();
                }else
                if(d1==2&&d2==1){
                    f[cur][S.s(j-1,0).s(j,0)]+=w;
                }else
                if((d1==1&&d2==1)||(d1==2&&d2==2)){
                    int m1=S.mc(j),m2=S.mc(j-1);
                    f[cur][S.s(j-1,0).s(j,0).s(m1,1).s(m2,2)]+=w;
                }
```

```
            }
        }
        cur^=1;f[cur].cl();
        for(int k=0;k<f[!cur].sz;k++){
            L S=f[!cur].s[k];Int w=f[!cur][S];
            f[cur][S>>1]=w;
        }
    }
    return 0;
}
```

7.6 某年某月某日是星期几

```
int solve(int year, int month, int day) {
    int answer;
    if (month == 1 || month == 2) {
        month += 12;
        year--;
    }
    if ((year < 1752) || (year == 1752 && month < 9) ||
        (year == 1752 && month == 9 && day < 3)) {
        answer = (day + 2 * month + 3 * (month + 1) / 5 + year + year / 4 + 5) % 7;
    } else {
        answer = (day + 2 * month + 3 * (month + 1) / 5 + year + year / 4
               - year / 100 + year / 400) % 7;
    }
    return answer;
}
```

7.7 枚举大小为 k 的子集

使用条件：$k > 0$

```
void solve(int n, int k) {
    for (int comb = (1 << k) - 1; comb < (1 << n); ) {
        // ...
        int x = comb & -comb, y = comb + x;
        comb = (((comb & ~y) / x) >> 1) | y;
    }
}
```

7.8 环状最长公共子串

```
int n, a[N << 1], b[N << 1];

bool has(int i, int j) {
    return a[(i - 1) % n] == b[(j - 1) % n];
}

const int DELTA[3][2] = {{0, -1}, {-1, -1}, {-1, 0}};

int from[N][N];

int solve() {
    memset(from, 0, sizeof(from));
    int ret = 0;
    for (int i = 1; i <= 2 * n; ++i) {
        from[i][0] = 2;
        int left = 0, up = 0;
        for (int j = 1; j <= n; ++j) {
            int upleft = up + 1 + !!from[i - 1][j];
            if (!has(i, j)) {
                upleft = INT_MIN;
            }
            int max = std::max(left, std::max(upleft, up));
            if (left == max) {
                from[i][j] = 0;
            } else if (upleft == max) {
                from[i][j] = 1;
            } else {
                from[i][j] = 2;
            }
            left = max;
        }
        if (i >= n) {
            int count = 0;
            for (int x = i, y = n; y; ) {
                int t = from[x][y];
                count += t == 1;
                x += DELTA[t][0];
                y += DELTA[t][1];
            }
            ret = std::max(ret, count);
```

```cpp
                int x = i - n + 1;
                from[x][0] = 0;
                int y = 0;
                while (y <= n && from[x][y] == 0) {
                    y++;
                }
                for (; x <= i; ++x) {
                    from[x][y] = 0;
                    if (x == i) {
                        break;
                    }
                    for (; y <= n; ++y) {
                        if (from[x + 1][y] == 2) {
                            break;
                        }
                        if (y + 1 <= n && from[x + 1][y + 1] == 1) {
                            y++;
                            break;
                        }
                    }
                }
            }
        }
    return ret;
}
```

7.9 LLMOD

```cpp
LL multiplyMod(LL a, LL b, LL P) { // `需要保证 a 和 b 非负`
        LL t = (a * b - LL((long double)a / P * b + 1e-3) * P) % P;
        return t < 0 : t + P : t;
}
```

8 vimrc

```vim
autocmd BufNewFile *.cpp exec ":call Setfilehead()"
func! Setfilehead()
    call append(0, '// Create: '.strftime("%Y-%m-%d %H:%M:%S"))
endfunc

colo morning
set fdm=syntax
```

```
set foldlevel=100

set ruler
set number
set smartindent
set autoindent
set tabstop=4
set softtabstop=4
set shiftwidth=4
set hlsearch
set incsearch
set autoread
set backspace=2
set mouse=a
set autochdir
set makeprg=g++\ %:r.cpp\ -o\ %:r\ -g\ -std=c++11\ -Wall\ -Wextra\ -Wconversion

syntax on

nmap <C-A> ggVG
vmap <C-C> "+y
noremap <C-V> "+P

filetype plugin indent on

autocmd FileType cpp set cindent
autocmd FileType cpp map <F9> :make<CR>
autocmd FileType cpp map <C-F9> :!g++ %:r.cpp -o %:r -g -O2 -std=c++11 -Wall -Wextra<CR>
autocmd FileType cpp map <F8> :!time ./%:r < %:r.in <CR>
autocmd FileType cpp map <F5> :!time ./%:r <CR>
autocmd FileType cpp map <F10> :!gdb ./%:r <CR>

autocmd FileType python set smartindent autoindent
autocmd FileType python map <F5> :!python ./%<CR>

map <F3> :vnew %:r.in <CR>
map <F4> :!gedit % <CR>
```

9 常用结论

9.1 上下界网络流

$B(u,v)$ 表示边 (u,v) 流量的下界，$C(u,v)$ 表示边 (u,v) 流量的上界，$F(u,v)$ 表示边 (u,v) 的流量。设 $G(u,v)=F(u,v)-B(u,v)$，显然有

$$0 \le G(u,v) \le C(u,v)-B(u,v)$$

无源汇的上下界可行流

建立超级源点 S^* 和超级汇点 T^*，对于原图每条边 (u,v) 在新网络中连如下三条边：$S^* \to v$，容量为 $B(u,v)$；$u \to T^*$，容量为 $B(u,v)$；$u \to v$，容量为 $C(u,v) - B(u,v)$。最后求新网络的最大流，判断从超级源点 S^* 出发的边是否都满流即可，边 (u,v) 的最终解中的实际流量为 $G(u,v) + B(u,v)$。

有源汇的上下界可行流

从汇点 T 到源点 S 连一条上界为 ∞，下界为 0 的边。按照**无源汇的上下界可行流**一样做即可，流量即为 $T \to S$ 边上的流量。

有源汇的上下界最大流

1. 在**有源汇的上下界可行流**中，从汇点 T 到源点 S 的边改为连一条上界为 ∞，下届为 x 的边。x 满足二分性质，找到最大的 x 使得新网络存在**无源汇的上下界可行流**即为原图的最大流。

2. 从汇点 T 到源点 S 连一条上界为 ∞，下界为 0 的边，变成无源汇的网络。按照**无源汇的上下界可行流**的方法，建立超级源点 S^* 和超级汇点 T^*，求一遍 $S^* \to T^*$ 的最大流，再将从汇点 T 到源点 S 的这条边拆掉，求一次 $S \to T$ 的最大流即可。

有源汇的上下界最小流

1. 在**有源汇的上下界可行流**中，从汇点 T 到源点 S 的边改为连一条上界为 x，下界为 0 的边。x 满足二分性质，找到最小的 x 使得新网络存在**无源汇的上下界可行流**即为原图的最小流。

2. 按照**无源汇的上下界可行流**的方法，建立超级源点 S^* 与超级汇点 T^*，求一遍 $S^* \to T^*$ 的最大流，但是注意这一次不加上汇点 T 到源点 S 的这条边，即不使之改为无源汇的网络去求解。求完后，再加上那条汇点 T 到源点 S 上界 ∞ 的边。因为这条边下界为 0，所以 S^*，T^* 无影响，再直接求一次 $S^* \to T^*$ 的最大流。若超级源点 S^* 出发的边全部满流，则 $T \to S$ 边上的流量即为原图的最小流，否则无解。

9.2 上下界费用流

来源：BZOJ 3876 设汇 t，源 s，超级源 S，超级汇 T，本质是每条边的下界为 `1`，上界为 `MAX`，跑一遍有源汇的上下界最小费用最小流。（因为上界无穷大，所以只要满足所有下界的最小费用最小流）

1. 对每个点 x：从 x 到 t 连一条费用为 `0`，流量为 `MAX` 的边，表示可以任意停止当前的剧情（接下来的剧情从更优的路径去走，画个样例就知道了）

2. 对于每一条边权为 `z` 的边 `x->y`：

   - 从 `S` 到 `y` 连一条流量为 `1`，费用为 `z` 的边，代表这条边至少要被走一次。
   - 从 `x` 到 `y` 连一条流量为 `MAX`，费用为 `z` 的边，代表这条边除了至少走的一次之外还可以随便走。

- 从 x 到 T 连一条流量为 1，费用为 0 的边。(注意是每一条 x->y 的边都连，或者你可以记下 x 的出边数 Kx，连一次流量为 Kx，费用为 0 的边)。

建完图后从 S 到 T 跑一遍费用流，即可。(当前跑出来的就是满足上下界的最小费用最小流了)

9.3 弦图相关

1. 团数 $\leq$ 色数，弦图团数 = 色数
2. 设 $next(v)$ 表示 $N(v)$ 中最前的点. 令 w* 表示所有满足 $A \in B$ 的 w 中最后的一个点，判断 $v \cup N(v)$ 是否为极大团，只需判断是否存在一个 w，满足 $Next(w) = v$ 且 $|N(v)| + 1 \leq |N(w)|$ 即可.
3. 最小染色：完美消除序列从后往前依次给每个点染色，给每个点染上可以染的最小的颜色
4. 最大独立集：完美消除序列从前往后能选就选
5. 弦图最大独立集数 = 最小团覆盖数，最小团覆盖：设最大独立集为 $\{p_1, p_2, \ldots, p_t\}$，则 $\{p_1 \cup N(p_1), \ldots, p_t \cup N(p_t)\}$ 为最小团覆盖

9.4 Bernoulli 数

1. 初始化：$B_0(n) = 1$
2. 递推公式：
$$B_m(n) = n^m - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k(n)}{m-k+1}$$
3. 应用：
$$\sum_{k=1}^{n} k^m = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} n^{m+1-k}$$

10 常见错误

1. 数组或者变量类型开错，例如将 double 开成 int;
2. 函数忘记返回返回值;
3. 初始化数组没有初始化完全;
4. 对空间限制判断不足导致 MLE;

11 测试列表

1. 检测评测机是否开 O2;
2. 检测 __int128 以及 __float128 是否能够使用;
3. 检测是否能够使用 C++11;
4. 检测是否能够使用 Ext Lib;

5. 检测程序运行所能使用的内存大小；
6. 检测程序运行所能使用的栈大小；
7. 检测是否有代码长度限制；
8. 检测是否能够正常返 Runtime Error（assertion、return 1、空指针）；
9. 查清楚厕所方位和打印机方位；

12 Java

12.1 基础模板

```java
import java.io.*;
import java.util.*;
import java.math.*;
public class Main {
    public static void main(String[] args) {
        InputStream inputStream = System.in;
        OutputStream outputStream = System.out;
        InputReader in = new InputReader(inputStream);
        PrintWriter out = new PrintWriter(outputStream);
    }
}
public static class edge implements Comparable<edge>{
        public int u,v,w;
        public int compareTo(edge e){
                return w-e.w;
        }
}
public static class cmp implements Comparator<edge>{
        public int compare(edge a,edge b){
                if(a.w<b.w)return 1;
                if(a.w>b.w)return -1;
                return 0;
        }
}
class InputReader {
    public BufferedReader reader;
    public StringTokenizer tokenizer;

    public InputReader(InputStream stream) {
        reader = new BufferedReader(new InputStreamReader(stream), 32768);
```

```java
            tokenizer = null;
        }

        public String next() {
            while (tokenizer == null || !tokenizer.hasMoreTokens()) {
                try {
                    tokenizer = new StringTokenizer(reader.readLine());
                } catch (IOException e) {
                    throw new RuntimeException(e);
                }
            }
            return tokenizer.nextToken();
        }

        public int nextInt() {
            return Integer.parseInt(next());
        }

        public long nextLong() {
            return Long.parseLong(next());
        }
    }
```

12.2 样例代码

```java
import java.io.*;
import java.math.*;
import java.util.*;

public class Main{
        public static long max(long a,long b){
                if(a>b)return a;
                return b;
        }
        public static long Calc(long A, int x){
                long Ret = (A / (1L << x)) * (1L << (x - 1));

                Ret += max(A % (1L << x) - (1L << (x - 1)) + 1, 0L);

                return Ret;
        }
        private static class InputReader {
```

```
        public BufferedReader rea;
        public StringTokenizer tok;

        public InputReader(InputStream stream) {
            rea = new BufferedReader(new InputStreamReader(stream), 32768);
            tok = null;
        }

        public String next() {
            while (tok == null || !tok.hasMoreTokens()) {
                try {
                    tok = new StringTokenizer(rea.readLine());
                } catch (IOException e) {
                    throw new RuntimeException(e);
                }
            }
            return tok.nextToken();
        }

        public int nextInt() {
            return Integer.parseInt(next());
        }

        public long nextLong() {
            return Long.parseLong(next());
        }
    }

      public static void main(String arg[]){
              InputReader cin = new InputReader(System.in);
              //Scanner cin = new Scanner(System.in);
              int N = 70;

              long k[] = new long[N];
              int n;

              while(true){
                      n=cin.nextInt();
                      if (n == 0) break;

                      for (int i = 1; i <= n; i ++){
```

```
            k[i]=cin.nextLong();
            //System.out.println(k[i]);
        }

        long Len;
        BigInteger Sum;
        long A, B;
        long AnsA = -1, AnsB = -1;
        int Ans = 0;

        for (int i = -1; i <= 1; i++){
            Len = k[1] * 2 + i;

            if(Len<=0)continue;

            Sum = BigInteger.ZERO;
            for (int j = 1; j <= n; j++){
                Sum = Sum.add(BigInteger.valueOf(1L << (j - 1)) .multiply(Bi
            }
            //System.out.println(Sum);

            long x = Len;

            if ((Sum.multiply(BigInteger.valueOf(2))) .mod (BigInteger.valueOf(L

            long y = Sum.multiply(BigInteger.valueOf(2)).divide(BigInteger.value
            if ((y - x + 1) % 2 > 0) continue;
            A = (y - x + 1) / 2;
            if ((x + y - 1) % 2 > 0) continue;
            B = (x + y - 1) / 2;

            if (A < 1 || A > (long)1e18) continue;
            if (B < 1 || B > (long)1e18) continue;

            int flag = 1;

            long Cnt;
            long Cnt_B;
            long Cnt_A;

            for (int j = 1; j <= n; j++){
                Cnt_B = Calc(B, j);
```

```
                    Cnt_A = Calc(A - 1, j);

                    if (Cnt_B - Cnt_A != k[j]){
                        flag = 0;
                        break;
                    }
                }

                if (flag==1){
                    Ans++;
                    //printf("%lld %lld\n", A, B);
                    //System.out.println(A+" "+B);
                    AnsA = A;
                    AnsB = B;
                }
            }

            if (Ans == 0) System.out.println("None");//puts("None");
            else
            if (Ans == 1)
                System.out.println(AnsA+" "+AnsB);//cout << AnsA << ' ' << AnsB << e
            else
                System.out.println("Many");//puts("Many");
        }

    }
}
```

compact1, compact2, compact3

java.math

Class BigInteger

java.lang.Object
    java.lang.Number
        java.math.BigInteger

All Implemented Interfaces:

`Serializable, Comparable<BigInteger>`

```
public class BigInteger
extends Number
implements Comparable<BigInteger>
```

Immutable arbitrary-precision integers. All operations behave as if BigIntegers were represented in two's-complement notation (like Java's primitive integer types). BigInteger provides analogues to all of Java's primitive integer operators, and all relevant methods from java.lang.Math. Additionally, BigInteger provides operations for modular arithmetic, GCD calculation, primality testing, prime generation, bit manipulation, and a few other miscellaneous operations.

Semantics of arithmetic operations exactly mimic those of Java's integer arithmetic operators, as defined in *The Java Language Specification*. For example, division by zero throws an `ArithmeticException`, and division of a negative by a positive yields a negative (or zero) remainder. All of the details in the Spec concerning overflow are ignored, as BigIntegers are made as large as necessary to accommodate the results of an operation.

Semantics of shift operations extend those of Java's shift operators to allow for negative shift distances. A right-shift with a negative shift distance results in a left shift, and vice-versa. The unsigned right shift operator (>>>) is omitted, as this operation makes little sense in combination with the "infinite word size" abstraction provided by this class.

Semantics of bitwise logical operations exactly mimic those of Java's bitwise integer operators. The binary operators (`and`, `or`, `xor`) implicitly perform sign extension on the shorter of the two operands prior to performing the operation.

Comparison operations perform signed integer comparisons, analogous to those performed by Java's relational and equality operators.

Modular arithmetic operations are provided to compute residues, perform exponentiation, and compute multiplicative inverses. These methods always return a non-negative result, between `0` and `(modulus - 1)`, inclusive.

Bit operations operate on a single bit of the two's-complement representation of their operand. If necessary, the operand is sign- extended so that it contains the designated bit. None of the single-bit operations can produce a BigInteger with a different sign from the BigInteger being operated on, as they affect only a single bit, and the "infinite word size" abstraction provided by this class ensures that there are infinitely many "virtual sign bits" preceding each BigInteger.

For the sake of brevity and clarity, pseudo-code is used throughout the descriptions of BigInteger methods. The pseudo-code expression `(i + j)` is shorthand for "a BigInteger whose value is that of the BigInteger `i` plus that of the BigInteger `j`." The pseudo-code expression `(i == j)` is shorthand for "`true` if and only if the BigInteger `i` represents the same value as the BigInteger `j`." Other pseudo-code expressions are interpreted similarly.

All methods and constructors in this class throw `NullPointerException` when passed a null object reference for any input parameter. BigInteger must support values in the range $-2^{\texttt{Integer.MAX\_VALUE}}$ (exclusive) to $+2^{\texttt{Integer.MAX\_VALUE}}$ (exclusive) and may support values outside of that range. The range of probable prime values is limited and may be less than the full supported positive range of `BigInteger`. The range must be at least 1 to $2^{500000000}$.

Implementation Note:

BigInteger constructors and operations throw ArithmeticException when the result is out of the supported range of $-2^{\texttt{Integer.MAX\_VALUE}}$ (exclusive) to $+2^{\texttt{Integer.MAX\_VALUE}}$ (exclusive).

Since:

JDK1.1

See Also:

BigDecimal, Serialized Form

Field Summary

Fields

Modifier and Type	Field and Description
static **BigInteger**	**ONE** The BigInteger constant one.
static **BigInteger**	**TEN** The BigInteger constant ten.
static **BigInteger**	**ZERO** The BigInteger constant zero.

Constructor Summary

Constructors

Constructor and Description
BigInteger(byte[] val) Translates a byte array containing the two's-complement binary representation of a BigInteger into a BigInteger.
BigInteger(int signum, byte[] magnitude) Translates the sign-magnitude representation of a BigInteger into a BigInteger.

`BigInteger(int bitLength, int certainty, Random rnd)`
Constructs a randomly generated positive BigInteger that is probably prime, with the specified bitLength.

`BigInteger(int numBits, Random rnd)`
Constructs a randomly generated BigInteger, uniformly distributed over the range 0 to $(2^{numBits} - 1)$, inclusive.

`BigInteger(String val)`
Translates the decimal String representation of a BigInteger into a BigInteger.

`BigInteger(String val, int radix)`
Translates the String representation of a BigInteger in the specified radix into a BigInteger.

Method Summary

All Methods Static Methods Instance Methods Concrete Methods

Modifier and Type	Method and Description
`BigInteger`	`abs()` Returns a BigInteger whose value is the absolute value of this BigInteger.
`BigInteger`	`add(BigInteger val)` Returns a BigInteger whose value is `(this + val)`.
`BigInteger`	`and(BigInteger val)` Returns a BigInteger whose value is `(this & val)`.
`BigInteger`	`andNot(BigInteger val)` Returns a BigInteger whose value is `(this & ~val)`.
`int`	`bitCount()` Returns the number of bits in the two's complement representation of this BigInteger that differ from its sign bit.
`int`	`bitLength()` Returns the number of bits in the minimal two's-complement representation of this BigInteger, *excluding* a sign bit.
`byte`	`byteValueExact()` Converts this `BigInteger` to a `byte`, checking for lost information.
`BigInteger`	`clearBit(int n)` Returns a BigInteger whose value is equivalent to this BigInteger with the designated bit cleared.
`int`	`compareTo(BigInteger val)` Compares this BigInteger with the specified BigInteger.
`BigInteger`	`divide(BigInteger val)`

	Returns a BigInteger whose value is `(this / val)`.
BigInteger[]	**divideAndRemainder**(**BigInteger** val) Returns an array of two BigIntegers containing `(this / val)` followed by `(this % val)`.
double	**doubleValue**() Converts this BigInteger to a `double`.
boolean	**equals**(**Object** x) Compares this BigInteger with the specified Object for equality.
BigInteger	**flipBit**(int n) Returns a BigInteger whose value is equivalent to this BigInteger with the designated bit flipped.
float	**floatValue**() Converts this BigInteger to a `float`.
BigInteger	**gcd**(**BigInteger** val) Returns a BigInteger whose value is the greatest common divisor of `abs(this)` and `abs(val)`.
int	**getLowestSetBit**() Returns the index of the rightmost (lowest-order) one bit in this BigInteger (the number of zero bits to the right of the rightmost one bit).
int	**hashCode**() Returns the hash code for this BigInteger.
int	**intValue**() Converts this BigInteger to an `int`.
int	**intValueExact**() Converts this `BigInteger` to an `int`, checking for lost information.
boolean	**isProbablePrime**(int certainty) Returns `true` if this BigInteger is probably prime, `false` if it's definitely composite.
long	**longValue**() Converts this BigInteger to a `long`.
long	**longValueExact**() Converts this `BigInteger` to a `long`, checking for lost information.
BigInteger	**max**(**BigInteger** val) Returns the maximum of this BigInteger and `val`.
BigInteger	**min**(**BigInteger** val) Returns the minimum of this BigInteger and `val`.
BigInteger	**mod**(**BigInteger** m) Returns a BigInteger whose value is `(this mod m)`.

BigInteger	**modInverse**(BigInteger m) Returns a BigInteger whose value is (this^{-1} mod m).
BigInteger	**modPow**(BigInteger exponent, BigInteger m) Returns a BigInteger whose value is (this^{exponent} mod m).
BigInteger	**multiply**(BigInteger val) Returns a BigInteger whose value is (this * val).
BigInteger	**negate**() Returns a BigInteger whose value is (-this).
BigInteger	**nextProbablePrime**() Returns the first integer greater than this BigInteger that is probably prime.
BigInteger	**not**() Returns a BigInteger whose value is (~this).
BigInteger	**or**(BigInteger val) Returns a BigInteger whose value is (this | val).
BigInteger	**pow**(int exponent) Returns a BigInteger whose value is (this^{exponent}).
static BigInteger	**probablePrime**(int bitLength, Random rnd) Returns a positive BigInteger that is probably prime, with the specified bitLength.
BigInteger	**remainder**(BigInteger val) Returns a BigInteger whose value is (this % val).
BigInteger	**setBit**(int n) Returns a BigInteger whose value is equivalent to this BigInteger with the designated bit set.
BigInteger	**shiftLeft**(int n) Returns a BigInteger whose value is (this << n).
BigInteger	**shiftRight**(int n) Returns a BigInteger whose value is (this >> n).
short	**shortValueExact**() Converts this BigInteger to a short, checking for lost information.
int	**signum**() Returns the signum function of this BigInteger.
BigInteger	**subtract**(BigInteger val) Returns a BigInteger whose value is (this - val).
boolean	**testBit**(int n) Returns true if and only if the designated bit is set.
byte[]	**toByteArray**()

	Returns a byte array containing the two's-complement representation of this BigInteger.
String	toString() Returns the decimal String representation of this BigInteger.
String	toString(int radix) Returns the String representation of this BigInteger in the given radix.
static BigInteger	valueOf(long val) Returns a BigInteger whose value is equal to that of the specified long.
BigInteger	xor(BigInteger val) Returns a BigInteger whose value is (this ^ val).

Methods inherited from class java.lang.Number

byteValue, shortValue

Methods inherited from class java.lang.Object

clone, finalize, getClass, notify, notifyAll, wait, wait, wait

Field Detail

ZERO

`public static final BigInteger ZERO`

The BigInteger constant zero.

Since:

1.2

ONE

`public static final BigInteger ONE`

The BigInteger constant one.

Since:

1.2

TEN

`public static final BigInteger TEN`

The BigInteger constant ten.

compact1, compact2, compact3

java.util

Class TreeMap<K,V>

java.lang.Object
    java.util.AbstractMap<K,V>
        java.util.TreeMap<K,V>

Type Parameters:

`K` - the type of keys maintained by this map

`V` - the type of mapped values

All Implemented Interfaces:

`Serializable`, `Cloneable`, `Map<K,V>`, `NavigableMap<K,V>`, `SortedMap<K,V>`

```
public class TreeMap<K,V>
extends AbstractMap<K,V>
implements NavigableMap<K,V>, Cloneable, Serializable
```

A Red-Black tree based `NavigableMap` implementation. The map is sorted according to the natural ordering of its keys, or by a `Comparator` provided at map creation time, depending on which constructor is used.

This implementation provides guaranteed log(n) time cost for the `containsKey`, `get`, `put` and `remove` operations. Algorithms are adaptations of those in Cormen, Leiserson, and Rivest's *Introduction to Algorithms*.

Note that the ordering maintained by a tree map, like any sorted map, and whether or not an explicit comparator is provided, must be *consistent with `equals`* if this sorted map is to correctly implement the `Map` interface. (See `Comparable` or `Comparator` for a precise definition of *consistent with equals*.) This is so because the `Map` interface is defined in terms of the `equals` operation, but a sorted map performs all key comparisons using its `compareTo` (or `compare`) method, so two keys that are deemed equal by this method are, from the standpoint of the sorted map, equal. The behavior of a sorted map *is* well-defined even if its ordering is inconsistent with `equals`; it just fails to obey the general contract of the `Map` interface.

Note that this implementation is not synchronized. If multiple threads access a map concurrently, and at least one of the threads modifies the map structurally, it *must* be synchronized externally. (A structural modification is any operation that adds or deletes one or more mappings; merely changing the value associated with an existing key is not a structural modification.) This is typically accomplished by synchronizing on some object that naturally encapsulates the map. If no such object exists, the map should be "wrapped" using the `Collections.synchronizedSortedMap` method. This is best done at creation time, to prevent accidental unsynchronized access to the map:

```
SortedMap m = Collections.synchronizedSortedMap(new TreeMap(...));
```

The iterators returned by the `iterator` method of the collections returned by all of this class's "collection view methods" are *fail-fast*: if the map is structurally modified at any time after the iterator is created, in any way except through the iterator's own `remove` method, the iterator will throw a `ConcurrentModificationException`. Thus, in the face of concurrent modification, the iterator fails quickly and cleanly, rather than risking arbitrary, non-deterministic behavior at an undetermined time in the future.

Note that the fail-fast behavior of an iterator cannot be guaranteed as it is, generally speaking, impossible to make any hard guarantees in the presence of unsynchronized concurrent modification. Fail-fast iterators throw `ConcurrentModificationException` on a best-effort basis. Therefore, it would be wrong to write a program that depended on this exception for its correctness: *the fail-fast behavior of iterators should be used only to detect bugs.*

All `Map.Entry` pairs returned by methods in this class and its views represent snapshots of mappings at the time they were produced. They do **not** support the `Entry.setValue` method. (Note however that it is possible to change mappings in the associated map using `put`.)

This class is a member of the Java Collections Framework.

Since:

`1.2`

See Also:

`Map`, `HashMap`, `Hashtable`, `Comparable`, `Comparator`, `Collection`, `Serialized Form`

Nested Class Summary

Nested classes/interfaces inherited from class java.util.AbstractMap

`AbstractMap.SimpleEntry<K,V>`, `AbstractMap.SimpleImmutableEntry<K,V>`

Constructor Summary

Constructors

Constructor and Description
`TreeMap()` Constructs a new, empty tree map, using the natural ordering of its keys.
`TreeMap(Comparator<? super K> comparator)` Constructs a new, empty tree map, ordered according to the given comparator.
`TreeMap(Map<? extends K,? extends V> m)` Constructs a new tree map containing the same mappings as the given map, ordered according to the *natural ordering* of its keys.
`TreeMap(SortedMap<K,? extends V> m)` Constructs a new tree map containing the same mappings and using the same ordering as the specified sorted map.

Method Summary

All Methods Instance Methods Concrete Methods

Modifier and Type	Method and Description
Map.Entry<K,V>	**ceilingEntry**(K key) Returns a key-value mapping associated with the least key greater than or equal to the given key, or null if there is no such key.
K	**ceilingKey**(K key) Returns the least key greater than or equal to the given key, or null if there is no such key.
void	**clear**() Removes all of the mappings from this map.
Object	**clone**() Returns a shallow copy of this TreeMap instance.
Comparator<? super K>	**comparator**() Returns the comparator used to order the keys in this map, or null if this map uses the **natural ordering** of its keys.
boolean	**containsKey**(Object key) Returns true if this map contains a mapping for the specified key.
boolean	**containsValue**(Object value) Returns true if this map maps one or more keys to the specified value.
NavigableSet<K>	**descendingKeySet**() Returns a reverse order **NavigableSet** view of the keys contained in this map.
NavigableMap<K,V>	**descendingMap**() Returns a reverse order view of the mappings contained in this map.
Set<Map.Entry<K,V>>	**entrySet**() Returns a **Set** view of the mappings contained in this map.
Map.Entry<K,V>	**firstEntry**() Returns a key-value mapping associated with the least key in this map, or null if the map is empty.
K	**firstKey**() Returns the first (lowest) key currently in this map.
Map.Entry<K,V>	**floorEntry**(K key) Returns a key-value mapping associated with the greatest key less than or equal to the given key, or null if there is no such key.
K	**floorKey**(K key) Returns the greatest key less than or equal to the given key, or null if there is no such key.

	or null if there is no such key.
void	forEach(BiConsumer<? super K,? super V> action) Performs the given action for each entry in this map until all entries have been processed or the action throws an exception.
V	get(Object key) Returns the value to which the specified key is mapped, or null if this map contains no mapping for the key.
SortedMap<K,V>	headMap(K toKey) Returns a view of the portion of this map whose keys are strictly less than toKey.
NavigableMap<K,V>	headMap(K toKey, boolean inclusive) Returns a view of the portion of this map whose keys are less than (or equal to, if inclusive is true) toKey.
Map.Entry<K,V>	higherEntry(K key) Returns a key-value mapping associated with the least key strictly greater than the given key, or null if there is no such key.
K	higherKey(K key) Returns the least key strictly greater than the given key, or null if there is no such key.
Set<K>	keySet() Returns a Set view of the keys contained in this map.
Map.Entry<K,V>	lastEntry() Returns a key-value mapping associated with the greatest key in this map, or null if the map is empty.
K	lastKey() Returns the last (highest) key currently in this map.
Map.Entry<K,V>	lowerEntry(K key) Returns a key-value mapping associated with the greatest key strictly less than the given key, or null if there is no such key.
K	lowerKey(K key) Returns the greatest key strictly less than the given key, or null if there is no such key.
NavigableSet<K>	navigableKeySet() Returns a NavigableSet view of the keys contained in this map.
Map.Entry<K,V>	pollFirstEntry() Removes and returns a key-value mapping associated with the least key in this map, or null if the map is empty.
Map.Entry<K,V>	pollLastEntry() Removes and returns a key-value mapping associated with the greatest key in this map, or null if the map is empty.

the greatest key in this map, or null if the map is empty.

V	**put**(K key, V value) Associates the specified value with the specified key in this map.
void	**putAll**(Map<? extends K,? extends V> map) Copies all of the mappings from the specified map to this map.
V	**remove**(Object key) Removes the mapping for this key from this TreeMap if present.
V	**replace**(K key, V value) Replaces the entry for the specified key only if it is currently mapped to some value.
boolean	**replace**(K key, V oldValue, V newValue) Replaces the entry for the specified key only if currently mapped to the specified value.
void	**replaceAll**(BiFunction<? super K,? super V,? extends V> function) Replaces each entry's value with the result of invoking the given function on that entry until all entries have been processed or the function throws an exception.
int	**size**() Returns the number of key-value mappings in this map.
NavigableMap<K,V>	**subMap**(K fromKey, boolean fromInclusive, K toKey, boolean toInclusive) Returns a view of the portion of this map whose keys range from `fromKey` to `toKey`.
SortedMap<K,V>	**subMap**(K fromKey, K toKey) Returns a view of the portion of this map whose keys range from `fromKey`, inclusive, to `toKey`, exclusive.
SortedMap<K,V>	**tailMap**(K fromKey) Returns a view of the portion of this map whose keys are greater than or equal to `fromKey`.
NavigableMap<K,V>	**tailMap**(K fromKey, boolean inclusive) Returns a view of the portion of this map whose keys are greater than (or equal to, if `inclusive` is true) `fromKey`.
Collection<V>	**values**() Returns a **Collection** view of the values contained in this map.

Methods inherited from class java.util.AbstractMap

`equals, hashCode, isEmpty, toString`

13 gedit

```
Compile:
#!/bin/sh
full=$GEDIT_CURRENT_DOCUMENT_NAME
name=`echo $full | cut -d. -f1`
g++ $full -o $name -g -Wall

Debug:
#!/bin/bash
name=`echo $GEDIT_CURRENT_DOCUMENT_NAME | cut -d. -f1`
gnome-terminal -x bash -c "gdb ./$name"

Run:
#!/bin/bash
name=`echo $GEDIT_CURRENT_DOCUMENT_NAME | cut -d. -f1`
gnome-terminal -x bash -c "time ./$name;echo 'Press any key to continue'; read"
```

14 数学

14.1 常用数学公式

14.1.1 求和公式

1. $\sum_{k=1}^{n}(2k-1)^2=\frac{n(4n^2-1)}{3}$
2. $\sum_{k=1}^{n}k^3=[\frac{n(n+1)}{2}]^2$
3. $\sum_{k=1}^{n}(2k-1)^3=n^2(2n^2-1)$
4. $\sum_{k=1}^{n}k^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
5. $\sum_{k=1}^{n}k^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{12}$
6. $\sum_{k=1}^{n}k(k+1)=\frac{n(n+1)(n+2)}{3}$
7. $\sum_{k=1}^{n}k(k+1)(k+2)=\frac{n(n+1)(n+2)(n+3)}{4}$
8. $\sum_{k=1}^{n}k(k+1)(k+2)(k+3)=\frac{n(n+1)(n+2)(n+3)(n+4)}{5}$

14.1.2 斐波那契数列

1. $fib_0=0, fib_1=1, fib_n=fib_{n-1}+fib_{n-2}$
2. $fib_{n+2}\cdot fib_n-fib_{n+1}^2=(-1)^{n+1}$
3. $fib_{-n}=(-1)^{n-1}fib_n$

4. $fib_{n+k} = fib_k \cdot fib_{n+1} + fib_{k-1} \cdot fib_n$

5. $gcd(fib_m, fib_n) = fib_{gcd(m,n)}$

6. $fib_m | fib_n^2 \Leftrightarrow nfib_n | m$

14.1.3 错排公式

1. $D_n = (n-1)(D_{n-2} - D_{n-1})$

2. $D_n = n! \cdot (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!})$

14.1.4 莫比乌斯函数

$$\mu(n) = \begin{cases} 1 & 若n = 1 \\ (-1)^k & 若n无平方数因子，且n = p_1 p_2 \ldots p_k \\ 0 & 若n有大于1的平方数因数 \end{cases}$$

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & 若n = 1 \\ 0 & 其他情况 \end{cases}$$

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d})$$

$$g(x) = \sum_{n=1}^{[x]} f(\frac{x}{n}) \Leftrightarrow f(x) = \sum_{n=1}^{[x]} \mu(n) g(\frac{x}{n})$$

14.1.5 伯恩赛德引理

设 G 是一个有限群，作用在集合 X 上。对每个 g 属于 G，令 X^g 表示 X 中在 g 作用下的不动元素，轨道数（记作 $|X/G|$）由如下公式给出：

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

14.1.6 五边形数定理

设 $p(n)$ 是 n 的拆分数，有

$$p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k-1} p\left(n - \frac{k(3k-1)}{2}\right)$$

14.1.7 树的计数

1. 有根树计数：$n+1$ 个结点的有根树的个数为

$$a_{n+1} = \frac{\sum_{j=1}^{n} j \cdot a_j \cdot S_{n,j}}{n}$$

其中，

$$S_{n,j}=\sum_{i=1}^{n/j}a_{n+1-ij}=S_{n-j,j}+a_{n+1-j}$$

2. 无根树计数：当 n 为奇数时，n 个结点的无根树的个数为

$$a_n-\sum_{i=1}^{n/2}a_ia_{n-i}$$

当 n 为偶数时，n 个结点的无根树的个数为

$$a_n-\sum_{i=1}^{n/2}a_ia_{n-i}+\frac{1}{2}a_{\frac{n}{2}}(a_{\frac{n}{2}}+1)$$

3. n 个结点的完全图的生成树个数为

$$n^{n-2}$$

4. 矩阵 - 树定理：图 G 由 n 个结点构成，设 $\boldsymbol{A}[G]$ 为图 G 的邻接矩阵、$\boldsymbol{D}[G]$ 为图 G 的度数矩阵，则图 G 的不同生成树的个数为 $\boldsymbol{C}[G]=\boldsymbol{D}[G]-\boldsymbol{A}[G]$ 的任意一个 $n-1$ 阶主子式的行列式值。

14.1.8 欧拉公式

平面图的顶点个数、边数和面的个数有如下关系：

$$V-E+F=C+1$$

其中，V 是顶点的数目，E 是边的数目，F 是面的数目，C 是组成图形的连通部分的数目。当图是单连通图的时候，公式简化为：

$$V-E+F=2$$

14.1.9 皮克定理

给定顶点坐标均是整点（或正方形格点）的简单多边形，其面积 A 和内部格点数目 i、边上格点数目 b 的关系：

$$A=i+\frac{b}{2}-1$$

14.1.10 牛顿恒等式

设

$$\prod_{i=1}^{n}(x-x_i)=a_n+a_{n-1}x+\cdots+a_1x^{n-1}+a_0x^n$$

$$p_k=\sum_{i=1}^{n}x_i^k$$

则

$$a_0p_k + a_1p_{k-1} + \cdots + a_{k-1}p_1 + ka_k = 0$$

特别地，对于

$$|\boldsymbol{A} - \lambda\boldsymbol{E}| = (-1)^n(a_n + a_{n-1}\lambda + \cdots + a_1\lambda^{n-1} + a_0\lambda^n)$$

有

$$p_k = Tr(\boldsymbol{A}^k)$$

14.2 平面几何公式

14.2.1 三角形

1. 半周长

$$p = \frac{a+b+c}{2}$$

2. 面积

$$S = \frac{a \cdot H_a}{2} = \frac{ab \cdot sinC}{2} = \sqrt{p(p-a)(p-b)(p-c)}$$

3. 中线

$$M_a = \frac{\sqrt{2(b^2+c^2)-a^2}}{2} = \frac{\sqrt{b^2+c^2+2bc \cdot cosA}}{2}$$

4. 角平分线

$$T_a = \frac{\sqrt{bc \cdot [(b+c)^2 - a^2]}}{b+c} = \frac{2bc}{b+c}cos\frac{A}{2}$$

5. 高线

$$H_a = bsinC = csinB = \sqrt{b^2 - (\frac{a^2+b^2-c^2}{2a})^2}$$

6. 内切圆半径

$$\begin{aligned} r = \frac{S}{p} = \frac{arcsin\frac{B}{2} \cdot sin\frac{C}{2}}{sin\frac{B+C}{2}} = 4R \cdot sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2} \\ = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}} = p \cdot tan\frac{A}{2}tan\frac{B}{2}tan\frac{C}{2} \end{aligned}$$

7. 外接圆半径

$$R = \frac{abc}{4S} = \frac{a}{2sinA} = \frac{b}{2sinB} = \frac{c}{2sinC}$$

14.2.2 四边形

D_1, D_2 为对角线，M 对角线中点连线，A 为对角线夹角，p 为半周长

1. $a^2 + b^2 + c^2 + d^2 = D_1^2 + D_2^2 + 4M^2$

2. $S = \frac{1}{2}D_1D_2sinA$

3．对于圆内接四边形

$$ac + bd = D_1 D_2$$

4．对于圆内接四边形

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}$$

14.2.3 正 n 边形

R 为外接圆半径，r 为内切圆半径

1．中心角

$$A = \frac{2\pi}{n}$$

2．内角

$$C = \frac{n-2}{n}\pi$$

3．边长

$$a = 2\sqrt{R^2 - r^2} = 2R \cdot sin\frac{A}{2} = 2r \cdot tan\frac{A}{2}$$

4．面积

$$S = \frac{nar}{2} = nr^2 \cdot tan\frac{A}{2} = \frac{nR^2}{2} \cdot sinA = \frac{na^2}{4 \cdot tan\frac{A}{2}}$$

14.2.4 圆

1．弧长

$$l = rA$$

2．弦长

$$a = 2\sqrt{2hr - h^2} = 2r \cdot sin\frac{A}{2}$$

3．弓形高

$$h = r - \sqrt{r^2 - \frac{a^2}{4}} = r(1 - cos\frac{A}{2}) = \frac{1}{2} \cdot arctan\frac{A}{4}$$

4．扇形面积

$$S_1 = \frac{rl}{2} = \frac{r^2 A}{2}$$

5．弓形面积

$$S_2 = \frac{rl - a(r-h)}{2} = \frac{r^2}{2}(A - sinA)$$

14.2.5 棱柱

1．体积

$$V = Ah$$

A 为底面积，h 为高

2. 侧面积

$$S = lp$$

l 为棱长，p 为直截面周长

3. 全面积

$$T = S + 2A$$

14.2.6 棱锥

1. 体积

$$V = Ah$$

A 为底面积，h 为高

2. 正棱锥侧面积

$$S = lp$$

l 为棱长，p 为直截面周长

3. 正棱锥全面积

$$T = S + 2A$$

14.2.7 棱台

1. 体积

$$V = (A_1 + A_2 + \sqrt{A_1 A_2}) \cdot \frac{h}{3}$$

A_1, A_2 为上下底面积，h 为高

2. 正棱台侧面积

$$S = \frac{p_1 + p_2}{2} l$$

p_1, p_2 为上下底面周长，l 为斜高

3. 正棱台全面积

$$T = S + A_1 + A_2$$

14.2.8 圆柱

1. 侧面积

$$S = 2\pi rh$$

2. 全面积

$$T = 2\pi r(h + r)$$

3. 体积

$$V = \pi r^2 h$$

14.2.9　圆锥

1. 母线

$$l = \sqrt{h^2 + r^2}$$

2. 侧面积

$$S = \pi r l$$

3. 全面积

$$T = \pi r(l + r)$$

4. 体积

$$V = \frac{\pi}{3} r^2 h$$

14.2.10　圆台

1. 母线

$$l = \sqrt{h^2 + (r_1 - r_2)^2}$$

2. 侧面积

$$S = \pi(r_1 + r_2)l$$

3. 全面积

$$T = \pi r_1(l + r_1) + \pi r_2(l + r_2)$$

4. 体积

$$V = \frac{\pi}{3}(r_1^2 + r_2^2 + r_1 r_2)h$$

14.2.11　球

1. 全面积

$$T = 4\pi r^2$$

2. 体积

$$V = \frac{4}{3}\pi r^3$$

14.2.12　球台

1. 侧面积

$$S = 2\pi r h$$

2. 全面积

$$T = \pi(2rh + r_1^2 + r_2^2)$$

3. 体积

$$V = \frac{\pi h[3(r_1^2 + r_2^2) + h^2]}{6}$$

14.2.13 球扇形

1. 全面积

$$T = \pi r(2h + r_0)$$

h 为球冠高，r_0 为球冠底面半径

2. 体积

$$V = \frac{2}{3}\pi r^2 h$$

14.3 积分表

$\int \frac{1}{1+x^2} dx = \tan^{-1} x$

$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

$\int \frac{x}{a^2+x^2} dx = \frac{1}{2} \ln |a^2 + x^2|$

$\int \frac{x^2}{a^2+x^2} dx = x - a \tan^{-1} \frac{x}{a}$

$\int \sqrt{x^2 \pm a^2} dx = \frac{1}{2} x \sqrt{x^2 \pm a^2} \pm \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right|$

$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \tan^{-1} \frac{x}{\sqrt{a^2-x^2}}$

$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2} x \sqrt{x^2 \pm a^2} \mp \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right|$

$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right|$

$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}$

$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2}$

$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2}$

$\int \sqrt{ax^2 + bx + c} dx = \frac{b+2ax}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac-b^2}{8a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx^+ c)} \right|$

$\int x^n e^{ax} \mathrm{d}x = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a} \int x^{n-1} e^{ax} \mathrm{d}x$

$\int \sin^2 ax dx = \frac{x}{2} - \frac{1}{4a} \sin 2ax$

$\int \sin^3 ax dx = -\frac{3 \cos ax}{4a} + \frac{\cos 3ax}{12a}$

$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$

$\int \cos^3 ax dx = \frac{3 \sin ax}{4a} + \frac{\sin 3ax}{12a}$

$\int \tan ax dx = -\frac{1}{a} \ln \cos ax$

$\int \tan^2 ax dx = -x + \frac{1}{a} \tan ax$

$\int x \cos ax dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$

$\int x^2 \cos ax dx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax$

$\int x \sin ax dx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2}$

$\int x^2 \sin ax dx = \frac{2 - a^2 x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2}$

14.4 立体几何公式

14.4.1 球面三角公式

设 a,b,c 是边长，A,B,C 是所对的二面角，有余弦定理

$$cosa = cosb \cdot cosc + sinb \cdot sinc \cdot cosA$$

正弦定理

$$\frac{sinA}{sina} = \frac{sinB}{sinb} = \frac{sinC}{sinc}$$

三角形面积是 $A+B+C-\pi$

14.4.2 四面体体积公式

U,V,W,u,v,w 是四面体的 6 条棱，U,V,W 构成三角形，$(U,u),(V,v),(W,w)$ 互为对棱，则

$$V = \frac{\sqrt{(s-2a)(s-2b)(s-2c)(s-2d)}}{192uvw}$$

其中

$$\begin{cases} a & = & \sqrt{xYZ}, \\ b & = & \sqrt{yZX}, \\ c & = & \sqrt{zXY}, \\ d & = & \sqrt{xyz}, \\ s & = & a+b+c+d, \\ X & = & (w-U+v)(U+v+w), \\ x & = & (U-v+w)(v-w+U), \\ Y & = & (u-V+w)(V+w+u), \\ y & = & (V-w+u)(w-u+V), \\ Z & = & (v-W+u)(W+u+v), \\ z & = & (W-u+v)(u-v+W) \end{cases}$$

14.5 博弈游戏

14.6 巴什博奕

1. 只有一堆 n 个物品，两个人轮流从这堆物品中取物，规定每次至少取一个，最多取 m 个。最后取光者得胜。

2. 显然，如果 $n=m+1$，那么由于一次最多只能取 m 个，所以，无论先取者拿走多少个，后取者都能够一次拿走剩余的物品，后者取胜。因此我们发现了如何取胜的法则：如果 $n=(m+1)r+s$，（r 为任意自然数，$s \leq m$)，那么先取者要拿走 s 个物品，如果后取者拿走 $k(k \leq m)$ 个，那么先取者再拿走 $m+1-k$ 个，结果剩下 $(m+1)(r-1)$ 个，以后保持这样的取法，那么先取者肯定获胜。总之，要保持给对手留下 $(m+1)$ 的倍数，就能最后获胜。

14.7 威佐夫博弈

1. 有两堆各若干个物品，两个人轮流从某一堆或同时从两堆中取同样多的物品，规定每次至少取一个，多者不限，最后取光者得胜。

2. 判断一个局势 (a,b) 为奇异局势（必败态）的方法：

$$a_k = [k(1+\sqrt{5})/2]，b_k = a_k + k$$

14.8 阶梯博奕

1. 博弈在一列阶梯上进行，每个阶梯上放着自然数个点，两个人进行阶梯博弈，每一步则是将一个阶梯上的若干个点（至少一个）移到前面去，最后没有点可以移动的人输。

2. 解决方法：把所有奇数阶梯看成 N 堆石子，做 NIM。（把石子从奇数堆移动到偶数堆可以理解为拿走石子，就相当于几个奇数堆的石子在做 Nim）

14.9 图上删边游戏

14.9.1 链的删边游戏

1. 游戏规则：对于一条链，其中一个端点是根，两人轮流删边，脱离根的部分也算被删去，最后没边可删的人输。

2. 做法：$sg[i] = n - dist(i) - 1$（其中 n 表示总点数，$dist(i)$ 表示离根的距离）

14.9.2 树的删边游戏

1. 游戏规则：对于一棵有根树，两人轮流删边，脱离根的部分也算被删去，没边可删的人输。

2. 做法：叶子结点的 $sg = 0$，其他节点的 sg 等于儿子结点的 $sg + 1$ 的异或和。

14.9.3 局部连通图的删边游戏

1. 游戏规则：在一个局部连通图上，两人轮流删边，脱离根的部分也算被删去，没边可删的人输。局部连通图的构图规则是，在一棵基础树上加边得到，所有形成的环保证不共用边，且只与基础树有一个公共点。

2. 做法：去掉所有的偶环，将所有的奇环变为长度为 1 的链，然后做树的删边游戏。

14.10 常用数学公式

14.11 求和公式

1. $\sum_{k=1}^{n}(2k-1)^2 = \frac{n(4n^2-1)}{3}$

2. $\sum_{k=1}^{n} k^3 = [\frac{n(n+1)}{2}]^2$

3. $\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$

4. $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

5. $\sum_{k=1}^{n} k^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

6. $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

7. $\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

8. $\sum_{k=1}^{n} k(k+1)(k+2)(k+3) = \frac{n(n+1)(n+2)(n+3)(n+4)}{5}$

14.12 斐波那契数列

1. $fib_0 = 0, fib_1 = 1, fib_n = fib_{n-1} + fib_{n-2}$

2. $fib_{n+2} \cdot fib_n - fib_{n+1}^2 = (-1)^{n+1}$

3. $fib_{-n} = (-1)^{n-1} fib_n$

4. $fib_{n+k} = fib_k \cdot fib_{n+1} + fib_{k-1} \cdot fib_n$

5. $gcd(fib_m, fib_n) = fib_{gcd(m,n)}$

6. $fib_m | fib_n^2 \Leftrightarrow nfib_n | m$

14.13 错排公式

1. $D_n = (n-1)(D_{n-2} - D_{n-1})$

2. $D_n = n! \cdot (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!})$

14.14 莫比乌斯函数

$$\mu(n) = \begin{cases} 1 & \text{若}n = 1 \\ (-1)^k & \text{若}n\text{无平方数因子，且}n = p_1 p_2 \ldots p_k \\ 0 & \text{若}n\text{有大于1的平方数因数} \end{cases}$$

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{若}n = 1 \\ 0 & \text{其他情况} \end{cases}$$

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d) g(\frac{n}{d})$$

$$g(x) = \sum_{n=1}^{[x]} f(\frac{x}{n}) \Leftrightarrow f(x) = \sum_{n=1}^{[x]} \mu(n) g(\frac{x}{n})$$

14.15 Burnside 引理

设 G 是一个有限群，作用在集合 X 上。对每个 g 属于 G，令 X^g 表示 X 中在 g 作用下的不动元素，轨道数（记作 $|X/G|$）由如下公式给出：

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|.$$

14.16 五边形数定理

设 $p(n)$ 是 n 的拆分数，有

$$p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}}(-1)^{k-1}p\left(n-\frac{k(3k-1)}{2}\right)$$

14.17 树的计数

1. 有根树计数：$n+1$ 个结点的有根树的个数为

   $$a_{n+1} = \frac{\sum_{j=1}^{n} j\cdot a_j\cdot S_{n,j}}{n}$$

   其中，

   $$S_{n,j} = \sum_{i=1}^{n/j} a_{n+1-ij} = S_{n-j,j} + a_{n+1-j}$$

2. 无根树计数：当 n 为奇数时，n 个结点的无根树的个数为

   $$a_n - \sum_{i=1}^{n/2} a_i a_{n-i}$$

   当 n 为偶数时，n 个结点的无根树的个数为

   $$a_n - \sum_{i=1}^{n/2} a_i a_{n-i} + \frac{1}{2}a_{\frac{n}{2}}(a_{\frac{n}{2}}+1)$$

3. n 个结点的完全图的生成树个数为

   $$n^{n-2}$$

4. 矩阵 - 树定理：图 G 由 n 个结点构成，设 $\boldsymbol{A}[G]$ 为图 G 的邻接矩阵、$\boldsymbol{D}[G]$ 为图 G 的度数矩阵，则图 G 的不同生成树的个数为 $\boldsymbol{C}[G]=\boldsymbol{D}[G]-\boldsymbol{A}[G]$ 的任意一个 $n-1$ 阶主子式的行列式值。

14.18 欧拉公式

平面图的顶点个数、边数和面的个数有如下关系：

$$V - E + F = C + 1$$

其中，V 是顶点的数目，E 是边的数目，F 是面的数目，C 是组成图形的连通部分的数目。当图是单连通图的时候，公式简化为：

$$V - E + F = 2$$

14.19 皮克定理

给定顶点坐标均是整点（或正方形格点）的简单多边形，其面积 A 和内部格点数目 i、边上格点数目 b 的关系：

$$A = i + \frac{b}{2} - 1$$

14.20 牛顿恒等式

设

$$\prod_{i=1}^{n}(x - x_i) = a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n$$

$$p_k = \sum_{i=1}^{n} x_i^k$$

则

$$a_0 p_k + a_1 p_{k-1} + \cdots + a_{k-1} p_1 + k a_k = 0$$

特别地，对于

$$|\boldsymbol{A} - \lambda \boldsymbol{E}| = (-1)^n (a_n + a_{n-1}\lambda + \cdots + a_1 \lambda^{n-1} + a_0 \lambda^n)$$

有

$$p_k = \mathsf{Tr}(\boldsymbol{A}^k)$$

15 平面几何公式

15.1 三角形

1. 半周长

$$p = \frac{a+b+c}{2}$$

2. 面积

$$S = \frac{a \cdot H_a}{2} = \frac{ab \cdot sinC}{2} = \sqrt{p(p-a)(p-b)(p-c)}$$

3. 中线

$$M_a = \frac{\sqrt{2(b^2+c^2) - a^2}}{2} = \frac{\sqrt{b^2 + c^2 + 2bc \cdot cosA}}{2}$$

4. 角平分线

$$T_a = \frac{\sqrt{bc \cdot [(b+c)^2 - a^2]}}{b+c} = \frac{2bc}{b+c} cos\frac{A}{2}$$

5. 高线

$$H_a = bsinC = csinB = \sqrt{b^2 - (\frac{a^2+b^2-c^2}{2a})^2}$$

6. 内切圆半径

$$\begin{aligned} r &= \frac{S}{p} = \frac{arcsin\frac{B}{2} \cdot sin\frac{C}{2}}{sin\frac{B+C}{2}} = 4R \cdot sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2} \\ &= \sqrt{\frac{(p-a)(p-b)(p-c)}{p}} = p \cdot tan\frac{A}{2}tan\frac{B}{2}tan\frac{C}{2} \end{aligned}$$

7. 外接圆半径

$$R = \frac{abc}{4S} = \frac{a}{2sinA} = \frac{b}{2sinB} = \frac{c}{2sinC}$$

15.2 四边形

D_1, D_2 为对角线，M 对角线中点连线，A 为对角线夹角，p 为半周长

1. $a^2 + b^2 + c^2 + d^2 = D_1^2 + D_2^2 + 4M^2$

2. $S = \frac{1}{2}D_1D_2sinA$

3. 对于圆内接四边形

$$ac + bd = D_1D_2$$

4. 对于圆内接四边形

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}$$

15.3 正 n 边形

R 为外接圆半径，r 为内切圆半径

1. 中心角

$$A = \frac{2\pi}{n}$$

2. 内角

$$C = \frac{n-2}{n}\pi$$

3. 边长

$$a = 2\sqrt{R^2 - r^2} = 2R \cdot sin\frac{A}{2} = 2r \cdot tan\frac{A}{2}$$

4. 面积

$$S = \frac{nar}{2} = nr^2 \cdot tan\frac{A}{2} = \frac{nR^2}{2} \cdot sinA = \frac{na^2}{4 \cdot tan\frac{A}{2}}$$

15.4 圆

1. 弧长

$$l = rA$$

2. 弦长

$$a = 2\sqrt{2hr - h^2} = 2r \cdot sin\frac{A}{2}$$

3. 弓形高

$$h = r - \sqrt{r^2 - \frac{a^2}{4}} = r(1 - cos\frac{A}{2}) = \frac{1}{2} \cdot arctan\frac{A}{4}$$

4. 扇形面积

$$S_1 = \frac{rl}{2} = \frac{r^2A}{2}$$

5. 弓形面积

$$S_2 = \frac{rl - a(r - h)}{2} = \frac{r^2}{2}(A - sinA)$$

15.5 棱柱

1. 体积

$$V = Ah$$

A 为底面积，h 为高

2. 侧面积

$$S = lp$$

l 为棱长，p 为直截面周长

3. 全面积

$$T = S + 2A$$

15.6 棱锥

1. 体积

$$V = Ah$$

A 为底面积，h 为高

2. 正棱锥侧面积

$$S = lp$$

l 为棱长，p 为直截面周长

3. 正棱锥全面积

$$T = S + 2A$$

15.7 棱台

1. 体积

$$V = (A_1 + A_2 + \sqrt{A_1A_2}) \cdot \frac{h}{3}$$

A_1, A_2 为上下底面积，h 为高

2. 正棱台侧面积

$$S = \frac{p_1 + p_2}{2} l$$

p_1, p_2 为上下底面周长，l 为斜高

3. 正棱台全面积

$$T = S + A_1 + A_2$$

15.8 圆柱

1. 侧面积

$$S = 2\pi rh$$

2. 全面积

$$T = 2\pi r(h + r)$$

3. 体积

$$V = \pi r^2 h$$

15.9 圆锥

1. 母线

$$l = \sqrt{h^2 + r^2}$$

2. 侧面积

$$S = \pi rl$$

3. 全面积

$$T = \pi r(l + r)$$

4. 体积

$$V = \frac{\pi}{3} r^2 h$$

15.10 圆台

1. 母线

$$l = \sqrt{h^2 + (r_1 - r_2)^2}$$

2. 侧面积

$$S = \pi(r_1 + r_2)l$$

3. 全面积

$$T = \pi r_1(l + r_1) + \pi r_2(l + r_2)$$

4. 体积

$$V = \frac{\pi}{3}(r_1^2 + r_2^2 + r_1 r_2)h$$

15.11 球

1. 全面积

$$T = 4\pi r^2$$

2. 体积

$$V = \frac{4}{3}\pi r^3$$

15.12 球台

1. 侧面积

$$S = 2\pi rh$$

2. 全面积

$$T = \pi(2rh + r_1^2 + r_2^2)$$

3. 体积

$$V = \frac{\pi h[3(r_1^2 + r_2^2) + h^2]}{6}$$

15.13 球扇形

1. 全面积

$$T = \pi r(2h + r_0)$$

h 为球冠高，r_0 为球冠底面半径

2. 体积

$$V = \frac{2}{3}\pi r^2 h$$

16 立体几何公式

16.1 球面三角公式

设 a, b, c 是边长，A, B, C 是所对的二面角，有余弦定理

$$cosa = cosb \cdot cosc + sinb \cdot sinc \cdot cosA$$

正弦定理

$$\frac{sinA}{sina} = \frac{sinB}{sinb} = \frac{sinC}{sinc}$$

三角形面积是 $A + B + C - \pi$

16.2 四面体体积公式

U, V, W, u, v, w 是四面体的 6 条棱，U, V, W 构成三角形，$(U, u), (V, v), (W, w)$ 互为对棱，则

$$V = \frac{\sqrt{(s-2a)(s-2b)(s-2c)(s-2d)}}{192uvw}$$

其中

$$\begin{cases} a &= \sqrt{xYZ}, \\ b &= \sqrt{yZX}, \\ c &= \sqrt{zXY}, \\ d &= \sqrt{xyz}, \\ s &= a+b+c+d, \\ X &= (w-U+v)(U+v+w), \\ x &= (U-v+w)(v-w+U), \\ Y &= (u-V+w)(V+w+u), \\ y &= (V-w+u)(w-u+V), \\ Z &= (v-W+u)(W+u+v), \\ z &= (W-u+v)(u-v+W) \end{cases}$$

17 附录

17.1 NTT 素数及原根列表

Id	Primes	Primitive Root	Id	Primes	Primitive Root	Id	Primes	Primitive Root
1	7340033	3	38	311427073	7	75	786432001	7
2	13631489	15	39	330301441	22	76	799014913	13
3	23068673	3	40	347078657	3	77	800063489	3
4	26214401	3	41	359661569	3	78	802160641	11
5	28311553	5	42	361758721	29	79	818937857	5
6	69206017	5	43	377487361	7	80	824180737	5
7	70254593	3	44	383778817	5	81	833617921	13
8	81788929	7	45	387973121	6	82	850395137	3
9	101711873	3	46	399507457	5	83	862978049	3
10	104857601	3	47	409993217	3	84	880803841	26
11	111149057	3	48	415236097	5	85	883949569	7
12	113246209	7	49	447741953	3	86	897581057	3
13	120586241	6	50	459276289	11	87	899678209	7
14	132120577	5	51	463470593	3	88	907018241	3
15	136314881	3	52	468713473	5	89	913309697	3
16	138412033	5	53	469762049	3	90	918552577	5
17	141557761	26	54	493879297	10	91	919601153	3
18	147849217	5	55	531628033	5	92	924844033	5
19	155189249	6	56	576716801	6	93	925892609	3
20	158334977	3	57	581959681	11	94	935329793	3
21	163577857	23	58	595591169	3	95	938475521	3
22	167772161	3	59	597688321	11	96	940572673	7
23	169869313	5	60	605028353	3	97	943718401	7
24	185597953	5	61	635437057	11	98	950009857	7
25	186646529	3	62	639631361	6	99	957349889	6
26	199229441	3	63	645922817	3	100	962592769	7
27	204472321	19	64	648019969	17	101	972029953	10
28	211812353	3	65	655360001	3	102	975175681	17
29	221249537	3	66	666894337	5	103	976224257	3
30	230686721	6	67	683671553	3	104	985661441	3
31	246415361	3	68	710934529	17	105	998244353	3
32	249561089	3	69	715128833	3	106	1004535809	3
33	257949697	5	70	718274561	3	107	1007681537	3
34	270532609	22	71	740294657	3	108	1012924417	5
35	274726913	3	72	745537537	5	109	1045430273	3
36	290455553	3	73	754974721	11	110	1051721729	6
37	305135617	5	74	770703361	11	111	1053818881	7

17.2 cheat.pdf

Theoretical Computer Science Cheat Sheet

Definitions	
$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)$ $\forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0$ $\forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim\limits_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[n \atop k \right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{ n \atop k \right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle n \atop k \right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle n \atop k \right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n}\frac{1}{i}, \quad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \quad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\displaystyle\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\displaystyle\binom{n}{k} = \binom{n}{n-k}$,

4. $\displaystyle\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\displaystyle\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\displaystyle\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\displaystyle\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\displaystyle\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\displaystyle\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\displaystyle\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\displaystyle\left\{ n \atop 1 \right\} = \left\{ n \atop n \right\} = 1$,

12. $\displaystyle\left\{ n \atop 2 \right\} = 2^{n-1} - 1$, **13.** $\displaystyle\left\{ n \atop k \right\} = k\left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\}$,

14. $\displaystyle\left[n \atop 1 \right] = (n-1)!$, **15.** $\displaystyle\left[n \atop 2 \right] = (n-1)!H_{n-1}$, **16.** $\displaystyle\left[n \atop n \right] = 1$, **17.** $\displaystyle\left[n \atop k \right] \ge \left\{ n \atop k \right\}$,

18. $\displaystyle\left[n \atop k \right] = (n-1)\left[n-1 \atop k \right] + \left[n-1 \atop k-1 \right]$, **19.** $\displaystyle\left\{ n \atop n-1 \right\} = \left[n \atop n-1 \right] = \binom{n}{2}$, **20.** $\displaystyle\sum_{k=0}^{n}\left[n \atop k \right] = n!$, **21.** $\displaystyle C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\displaystyle\left\langle n \atop 0 \right\rangle = \left\langle n \atop n-1 \right\rangle = 1$, **23.** $\displaystyle\left\langle n \atop k \right\rangle = \left\langle n \atop n-1-k \right\rangle$, **24.** $\displaystyle\left\langle n \atop k \right\rangle = (k+1)\left\langle n-1 \atop k \right\rangle + (n-k)\left\langle n-1 \atop k-1 \right\rangle$,

25. $\displaystyle\left\langle 0 \atop k \right\rangle = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$ **26.** $\displaystyle\left\langle n \atop 1 \right\rangle = 2^n - n - 1$, **27.** $\displaystyle\left\langle n \atop 2 \right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $\displaystyle x^n = \sum_{k=0}^{n}\left\langle n \atop k \right\rangle\binom{x+k}{n}$, **29.** $\displaystyle\left\langle n \atop m \right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $\displaystyle m!\left\{ n \atop m \right\} = \sum_{k=0}^{n}\left\langle n \atop k \right\rangle\binom{k}{n-m}$,

31. $\displaystyle\left\langle n \atop m \right\rangle = \sum_{k=0}^{n}\left\{ n \atop k \right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\displaystyle\left\langle\!\left\langle n \atop 0 \right\rangle\!\right\rangle = 1$, **33.** $\displaystyle\left\langle\!\left\langle n \atop n \right\rangle\!\right\rangle = 0 \quad \text{for } n \ne 0$,

34. $\displaystyle\left\langle\!\left\langle n \atop k \right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle n-1 \atop k \right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle n-1 \atop k-1 \right\rangle\!\right\rangle$, **35.** $\displaystyle\sum_{k=0}^{n}\left\langle\!\left\langle n \atop k \right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\displaystyle\left\{ x \atop x-n \right\} = \sum_{k=0}^{n}\left\langle\!\left\langle n \atop k \right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\displaystyle\left\{ n+1 \atop m+1 \right\} = \sum_{k}\binom{n}{k}\left\{ k \atop m \right\} = \sum_{k=0}^{n}\left\{ k \atop m \right\}(m+1)^{n-k}$,

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Identities Cont.

38. $\left[{n+1 \atop m+1}\right] = \sum_k \left[{n \atop k}\right]\binom{k}{m} = \sum_{k=0}^{n} \left[{k \atop m}\right] n^{\underline{n-k}} = n! \sum_{k=0}^{n} \frac{1}{k!}\left[{k \atop m}\right]$,

39. $\left[{x \atop x-n}\right] = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n}$,

40. $\left\{{n \atop m}\right\} = \sum_k \binom{n}{k}\left\{{k+1 \atop m+1}\right\}(-1)^{n-k}$,

41. $\left[{n \atop m}\right] = \sum_k \left[{n+1 \atop k+1}\right]\binom{k}{m}(-1)^{m-k}$,

42. $\left\{{m+n+1 \atop m}\right\} = \sum_{k=0}^{m} k \left\{{n+k \atop k}\right\}$,

43. $\left[{m+n+1 \atop m}\right] = \sum_{k=0}^{m} k(n+k)\left[{n+k \atop k}\right]$,

44. $\binom{n}{m} = \sum_k \left\{{n+1 \atop k+1}\right\}\left[{k \atop m}\right](-1)^{m-k}$,

45. $(n-m)!\binom{n}{m} = \sum_k \left[{n+1 \atop k+1}\right]\left\{{k \atop m}\right\}(-1)^{m-k}$, for $n \geq m$,

46. $\left\{{n \atop n-m}\right\} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left[{m+k \atop k}\right]$,

47. $\left[{n \atop n-m}\right] = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left\{{m+k \atop k}\right\}$,

48. $\left\{{n \atop \ell+m}\right\}\binom{\ell+m}{\ell} = \sum_k \left\{{k \atop \ell}\right\}\left\{{n-k \atop m}\right\}\binom{n}{k}$,

49. $\left[{n \atop \ell+m}\right]\binom{\ell+m}{\ell} = \sum_k \left[{k \atop \ell}\right]\left[{n-k \atop m}\right]\binom{n}{k}$.

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \ldots, d_n$:
$$\sum_{i=1}^{n} 2^{-d_i} \leq 1,$$
and equality holds only if every internal node has 2 sons.

Recurrences

Master method:
$$T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then
$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then
$$T(n) = \Theta(n^{\log_b a}\log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \leq cf(n)$ for large n, then
$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence
$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have
$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get
$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find
$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$.
Summing factors (example): Consider the following recurrence
$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side
$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$\begin{aligned} 1\big(T(n) - 3T(n/2) &= n\big) \\ 3\big(T(n/2) - 3T(n/4) &= n/2\big) \\ \vdots \quad \vdots \quad & \vdots \\ 3^{\log_2 n - 1}\big(T(2) - 3T(1) &= 2\big) \end{aligned}$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get
$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have
$$\begin{aligned} n\sum_{i=0}^{m-1} c^i &= n\left(\frac{c^m - 1}{c-1}\right) \\ &= 2n(c^{\log_2 n} - 1) \\ &= 2n(c^{(k-1)\log_c n} - 1) \\ &= 2n^k - 2n, \end{aligned}$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider
$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that
$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find
$$\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:
1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:
$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:
$$\sum_{i\geq 0} g_{i+1}x^i = \sum_{i\geq 0} 2g_i x^i + \sum_{i\geq 0} x^i.$$

We choose $G(x) = \sum_{i\geq 0} x^i g_i$. Rewrite in terms of $G(x)$:
$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i\geq 0} x^i.$$

Simplify:
$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:
$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:
$$\begin{aligned} G(x) &= x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right) \\ &= x\left(2\sum_{i\geq 0} 2^i x^i - \sum_{i\geq 0} x^i\right) \\ &= \sum_{i\geq 0}(2^{i+1} - 1)x^{i+1}. \end{aligned}$$

So $g_i = 2^i - 1$.

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$\pi \approx 3.14159$, $e \approx 2.71828$, $\gamma \approx 0.57721$, $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803$, $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},\ B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$

$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$

$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520},\ \ldots$$

$$\ln n < H_n < \ln n + 1,$$

$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

1, 2, 6, 24, 120, 720, 5040, 40320, 362880, ...

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$

$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{n} k\binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x)\Pr[X = x].$$

If X continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$

$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$

$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if X and Y are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$

$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n}\Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n}\Pr[X_i] + \sum_{k=2}^{n}(-1)^{k+1}\sum_{i_i < \cdots < i_k}\Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$

$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

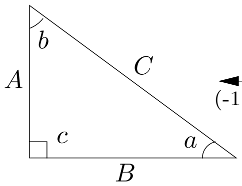
Geometric distribution:

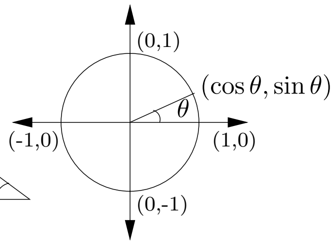
$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$

$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

Theoretical Computer Science Cheat Sheet

Trigonometry





Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:

$$\tfrac{1}{2}AB, \qquad \frac{AB}{A+B+C}.$$

Identities:

$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot\tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2\tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$

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sseiden@acm.org
http://www.csc.lsu.edu/~seiden

Matrices

Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{matrix} aei + bfg + cdh \\ - ceg - fha - ibd. \end{matrix}$$

Permanents:

$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

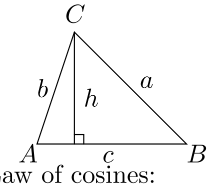
Identities:

$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

... in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:

$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab\sin C,$$
$$= \frac{c^2 \sin A \sin B}{2\sin C}.$$

Heron's formula:

$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a + b + c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:

$$\sin\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan\tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot\tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$

Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \bmod m_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$C \equiv r_n \bmod m_n$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i - 1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^{n} p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^{n} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\Big(\frac{a}{d}\Big).$$

Prime numbers:

$$p_n = n \ln n + n \ln\ln n - n + n \frac{\ln\ln n}{\ln n} + O\Big(\frac{n}{\ln n}\Big),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\Big(\frac{n}{(\ln n)^4}\Big).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \dots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embeded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k, \ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

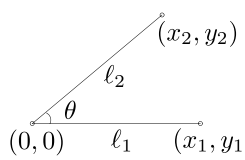
Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$
$$\big[|x_1 - x_0|^p + |y_1 - y_0|^p\big]^{1/p},$$
$$\lim_{p\to\infty} \big[|x_1 - x_0|^p + |y_1 - y_0|^p\big]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\tfrac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

Theoretical Computer Science Cheat Sheet

π

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:

$$\tfrac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:

$$\tfrac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \cdots$$

Newton's series:

$$\tfrac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:

$$\tfrac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:

$$\tfrac{\pi^2}{6} = \tfrac{1}{1^2} + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \cdots$$

$$\tfrac{\pi^2}{8} = \tfrac{1}{1^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \tfrac{1}{9^2} + \cdots$$

$$\tfrac{\pi^2}{12} = \tfrac{1}{1^2} - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

1. $\dfrac{d(cu)}{dx} = c\dfrac{du}{dx}$, **2.** $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$, **3.** $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$,

4. $\dfrac{d(u^n)}{dx} = nu^{n-1}\dfrac{du}{dx}$, **5.** $\dfrac{d(u/v)}{dx} = \dfrac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, **6.** $\dfrac{d(e^{cu})}{dx} = ce^{cu}\dfrac{du}{dx}$,

7. $\dfrac{d(c^u)}{dx} = (\ln c)c^u\dfrac{du}{dx}$, **8.** $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$,

9. $\dfrac{d(\sin u)}{dx} = \cos u\dfrac{du}{dx}$, **10.** $\dfrac{d(\cos u)}{dx} = -\sin u\dfrac{du}{dx}$,

11. $\dfrac{d(\tan u)}{dx} = \sec^2 u\dfrac{du}{dx}$, **12.** $\dfrac{d(\cot u)}{dx} = \csc^2 u\dfrac{du}{dx}$,

13. $\dfrac{d(\sec u)}{dx} = \tan u \, \sec u\dfrac{du}{dx}$, **14.** $\dfrac{d(\csc u)}{dx} = -\cot u \, \csc u\dfrac{du}{dx}$,

15. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, **16.** $\dfrac{d(\arccos u)}{dx} = \dfrac{-1}{\sqrt{1-u^2}}\dfrac{du}{dx}$,

17. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$, **18.** $\dfrac{d(\operatorname{arccot} u)}{dx} = \dfrac{-1}{1+u^2}\dfrac{du}{dx}$,

19. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **20.** $\dfrac{d(\operatorname{arccsc} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$,

21. $\dfrac{d(\sinh u)}{dx} = \cosh u\dfrac{du}{dx}$, **22.** $\dfrac{d(\cosh u)}{dx} = \sinh u\dfrac{du}{dx}$,

23. $\dfrac{d(\tanh u)}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$, **24.** $\dfrac{d(\coth u)}{dx} = -\operatorname{csch}^2 u\dfrac{du}{dx}$,

25. $\dfrac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \, \tanh u\dfrac{du}{dx}$, **26.** $\dfrac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \, \coth u\dfrac{du}{dx}$,

27. $\dfrac{d(\operatorname{arcsinh} u)}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$, **28.** $\dfrac{d(\operatorname{arccosh} u)}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$,

29. $\dfrac{d(\operatorname{arctanh} u)}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$, **30.** $\dfrac{d(\operatorname{arccoth} u)}{dx} = \dfrac{1}{u^2-1}\dfrac{du}{dx}$,

31. $\dfrac{d(\operatorname{arcsech} u)}{dx} = \dfrac{-1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$, **32.** $\dfrac{d(\operatorname{arccsch} u)}{dx} = \dfrac{-1}{|u|\sqrt{1+u^2}}\dfrac{du}{dx}$.

Integrals:

1. $\displaystyle\int cu\,dx = c\int u\,dx$, **2.** $\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx$,

3. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$, **4.** $\displaystyle\int \frac{1}{x}dx = \ln x$, **5.** $\displaystyle\int e^x\,dx = e^x$,

6. $\displaystyle\int \frac{dx}{1+x^2} = \arctan x$, **7.** $\displaystyle\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$,

8. $\displaystyle\int \sin x\,dx = -\cos x$, **9.** $\displaystyle\int \cos x\,dx = \sin x$,

10. $\displaystyle\int \tan x\,dx = -\ln|\cos x|$, **11.** $\displaystyle\int \cot x\,dx = \ln|\cos x|$,

12. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x|$, **13.** $\displaystyle\int \csc x\,dx = \ln|\csc x + \cot x|$,

14. $\displaystyle\int \arcsin\tfrac{x}{a}dx = \arcsin\tfrac{x}{a} + \sqrt{a^2-x^2}, \quad a > 0$, Calculus Cont.

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$

16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$

17. $\int \sin^2(ax) dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$

18. $\int \cos^2(ax) dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

19. $\int \sec^2 x \, dx = \tan x,$

20. $\int \csc^2 x \, dx = -\cot x,$

21. $\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx,$

22. $\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx,$

23. $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx, \quad n \neq 1,$

24. $\int \cot^n x \, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx, \quad n \neq 1,$

25. $\int \sec^n x \, dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx, \quad n \neq 1,$

26. $\int \csc^n x \, dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x \, dx, \quad n \neq 1,$

27. $\int \sinh x \, dx = \cosh x,$

28. $\int \cosh x \, dx = \sinh x,$

29. $\int \tanh x \, dx = \ln|\cosh x|,$

30. $\int \coth x \, dx = \ln|\sinh x|,$

31. $\int \operatorname{sech} x \, dx = \arctan \sinh x,$

32. $\int \operatorname{csch} x \, dx = \ln\left|\tanh \frac{x}{2}\right|,$

33. $\int \sinh^2 x \, dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$

34. $\int \cosh^2 x \, dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$

35. $\int \operatorname{sech}^2 x \, dx = \tanh x,$

36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$

37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln|a^2 - x^2|,$

38. $$\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \dfrac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \dfrac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$$

39. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln\left(x + \sqrt{a^2 + x^2}\right), \quad a > 0,$

40. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$

41. $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$

42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

43. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$

44. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a + x}{a - x}\right|,$

45. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2 - x^2}},$

46. $\int \sqrt{a^2 \pm x^2} \, dx = \frac{x}{2}\sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln\left|x + \sqrt{a^2 \pm x^2}\right|,$

47. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right|, \quad a > 0,$

48. $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln\left|\frac{x}{a + bx}\right|,$

49. $\int x\sqrt{a + bx} \, dx = \frac{2(3bx - 2a)(a + bx)^{3/2}}{15b^2},$

50. $\int \frac{\sqrt{a + bx}}{x} dx = 2\sqrt{a + bx} + a \int \frac{1}{x\sqrt{a + bx}} dx,$

51. $\int \frac{x}{\sqrt{a + bx}} dx = \frac{1}{\sqrt{2}} \ln\left|\frac{\sqrt{a + bx} - \sqrt{a}}{\sqrt{a + bx} + \sqrt{a}}\right|, \quad a > 0,$

52. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

53. $\int x\sqrt{a^2 - x^2} \, dx = -\frac{1}{3}(a^2 - x^2)^{3/2},$

54. $\int x^2\sqrt{a^2 - x^2} \, dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$

55. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|,$

56. $\int \frac{x \, dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$

57. $\int \frac{x^2 \, dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$

58. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln\left|\frac{a + \sqrt{a^2 + x^2}}{x}\right|,$

59. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$

60. $\int x\sqrt{x^2 \pm a^2} \, dx = \frac{1}{3}(x^2 \pm a^2)^{3/2},$

61. $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} \ln\left|\frac{x}{a + \sqrt{a^2 + x^2}}\right|,$

Calculus Cont.

62. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \tfrac{1}{a}\arccos\tfrac{a}{|x|}, \quad a>0,$

63. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x},$

64. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$

65. $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3},$

66. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac,\\[2ex] \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac, \end{cases}$$

67. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0,\\[2ex] \dfrac{1}{\sqrt{-a}}\arcsin\dfrac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0, \end{cases}$$

68. $$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

69. $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

70. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{-1}{\sqrt{c}}\ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0,\\[2ex] \dfrac{1}{\sqrt{-c}}\arcsin\dfrac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0, \end{cases}$$

71. $$\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{3}x^2-\tfrac{2}{15}a^2)(x^2+a^2)^{3/2},$$

72. $$\int x^n\sin(ax)\,dx = -\tfrac{1}{a}x^n\cos(ax) + \tfrac{n}{a}\int x^{n-1}\cos(ax)\,dx,$$

73. $$\int x^n\cos(ax)\,dx = \tfrac{1}{a}x^n\sin(ax) - \tfrac{n}{a}\int x^{n-1}\sin(ax)\,dx,$$

74. $$\int x^n e^{ax}\,dx = \frac{x^ne^{ax}}{a} - \tfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$$

75. $$\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2}\right),$$

76. $$\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$$

$$\begin{array}{llll}
x^1 = & x^{\underline{1}} & = & x^{\overline{1}}\\
x^2 = & x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{2}}-x^{\overline{1}}\\
x^3 = & x^{\underline{3}}+3x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{3}}-3x^{\overline{2}}+x^{\overline{1}}\\
x^4 = & x^{\underline{4}}+6x^{\underline{3}}+7x^{\underline{2}}+x^{\underline{1}} & = & x^{\overline{4}}-6x^{\overline{3}}+7x^{\overline{2}}-x^{\overline{1}}\\
x^5 = & x^{\underline{5}}+15x^{\underline{4}}+25x^{\underline{3}}+10x^{\underline{2}}+x^{\underline{1}} & = & x^{\underline{5}}-15x^{\overline{4}}+25x^{\overline{3}}-10x^{\overline{2}}+x^{\overline{1}}
\end{array}$$

$$\begin{array}{llll}
x^{\overline{1}} = & x^1 & x^{\underline{1}} = & x^1\\
x^{\overline{2}} = & x^2+x^1 & x^{\underline{2}} = & x^2-x^1\\
x^{\overline{3}} = & x^3+3x^2+2x^1 & x^{\underline{3}} = & x^3-3x^2+2x^1\\
x^{\overline{4}} = & x^4+6x^3+11x^2+6x^1 & x^{\underline{4}} = & x^4-6x^3+11x^2-6x^1\\
x^{\overline{5}} = & x^5+10x^4+35x^3+50x^2+24x^1 & x^{\underline{5}} = & x^5-10x^4+35x^3-50x^2+24x^1
\end{array}$$

Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1)-f(x),$$
$$\mathrm{E}\,f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x)+C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$\Delta(cu) = c\Delta u, \qquad \Delta(u+v) = \Delta u + \Delta v,$

$\Delta(uv) = u\Delta v + \mathrm{E}\,v\Delta u,$

$\Delta(x^{\underline{n}}) = nx^{\underline{n-1}},$

$\Delta(H_x) = x^{\underline{-1}}, \qquad \Delta(2^x) = 2^x,$

$\Delta(c^x) = (c-1)c^x, \qquad \Delta\binom{x}{m} = \binom{x}{m-1}.$

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x,$

$\sum(u+v)\,\delta x = \sum u\,\delta x + \sum v\,\delta x,$

$\sum u\Delta v\,\delta x = uv - \sum \mathrm{E}\,v\Delta u\,\delta x,$

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}, \qquad \sum x^{\underline{-1}}\,\delta x = H_x,$

$\sum c^x\,\delta x = \frac{c^x}{c-1}, \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}.$

Falling Factorial Powers:

$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$

$x^{\underline{0}} = 1,$

$x^{\underline{n}} = \dfrac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$

$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$

Rising Factorial Powers:

$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$

$x^{\overline{0}} = 1,$

$x^{\overline{n}} = \dfrac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$

$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$

Conversion:

$$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$$
$$= 1/(x+1)^{\overline{-n}},$$
$$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$$
$$= 1/(x-1)^{\underline{-n}},$$
$$x^n = \sum_{k=1}^n \left\{ n \atop k \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ n \atop k \right\}(-1)^{n-k}x^{\overline{k}},$$
$$x^{\underline{n}} = \sum_{k=1}^n \left[n \atop k \right](-1)^{n-k}x^k,$$
$$x^{\overline{n}} = \sum_{k=1}^n \left[n \atop k \right]x^k.$$

Theoretical Computer Science Cheat Sheet

Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2} f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} ix^i,$$

$$x^k \frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i},$$

$$\ln \frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1}\binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + x + 2x^2 + 6x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}}\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x} \ln \frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2}\left(\ln \frac{1}{1-x}\right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} = F_n x + F_{2n} x^2 + F_{3n} x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i,$$

$$\left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\left[{n \atop i} \right]x^i,$$

$$(e^x-1)^n = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}\frac{n!x^i}{i!},$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\left[{i \atop n} \right]\frac{n!x^i}{i!},$$

$$x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^i B_{2i}x^{2i}}{(2i)!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!},$$

$$\zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x},$$

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$

$$\zeta(x) = \prod_p \frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\textstyle\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\textstyle\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

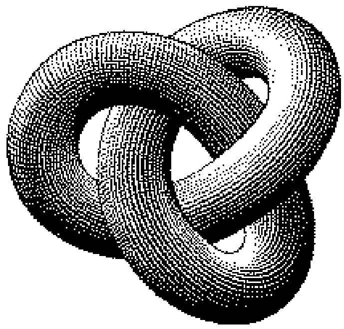
$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^i i!^2}{(i+1)(2i+1)!}x^{2i}.$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a\le b\le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad & \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n \end{aligned}$$

Let $A=(a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A \ne 0$. Let A_i be A with column i replaced by B. Then

$$x_i = \frac{\det A_i}{\det A}.$$

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

Fibonacci Numbers

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Definitions:

$$F_i = F_{i-1}+F_{i-2}, \quad F_0 = F_1 = 1,$$

$$F_{-i} = (-1)^{i-1}F_i,$$

$$F_i = \frac{1}{\sqrt{5}}\left(\phi^i - \hat{\phi}^i\right),$$

Cassini's identity: for $i>0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_kF_{n+1} + F_{k-1}F_n,$$

$$F_{2n} = F_nF_{n+1} + F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$

The Fibonacci number system: Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \ge k_{i+1}+2$ for all i, $1\le i<m$ and $k_m\ge 2$.

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)