

代码库

Quasar

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1 数论

1.1 快速求逆元

返回结果:

$$x^{-1}(\text{mod})$$

使用条件: $x \in [0, \text{mod})$ 并且 x 与 mod 互质

```

1 LL inv(LL a, LL p) {
2     LL d, x, y;
3     exgcd(a, p, d, x, y);
4     return d == 1 ? (x + p) % p : -1;
5 }
```

1.2 莫比乌斯反演

```

1 #include<cstdio>
2 #include<string>
3 #include<cstring>
4 #include<algorithm>
5 using namespace std;
6 int mu[100001], prime[100001];
7 bool check[100001];
8 int tot;
9 inline void findmu()
10 {
11     memset(check, false, sizeof(check));
12     mu[1]=1;
13     int i, j;
14     for(i=2; i<=100000; i++)
15     {
16         if(!check[i])
17         {
18             tot++;
19             prime[tot]=i;
20             mu[i]=-1;
21         }
22         for(j=1; j<=tot; j++)
23         {
24             if(i*prime[j]>100000)
25                 break;
26             check[i*prime[j]]=true;
27             if(i%prime[j]==0)
28             {
29                 mu[i*prime[j]]=0;
30                 break;
31             }
32             else
33                 mu[i*prime[j]]=-mu[i];

```

```

34     }
35 }
36 }
37 int sum[100001];
38 //找 [1,n],[1,m] 内互质的数的对数
39 inline long long solve(int n,int m)
40 {
41     long long ans=0;
42     if(n>m)
43         swap(n,m);
44     int i,la=0;
45     for(i=1;i<=n;i=la+1)
46     {
47         la=min(n/(n/i),m/(m/i));
48         ans+=(long long)(sum[la]-sum[i-1])*(n/i)*(m/i);
49     }
50     return ans;
51 }
52 int main()
53 {
54     //freopen("b.in","r",stdin);
55     // freopen("b.out","w",stdout);
56     findmu();
57     sum[0]=0;
58     int i;
59     for(i=1;i<=100000;i++)
60         sum[i]=sum[i-1]+mu[i];
61     int a,b,c,d,k;
62     int T;
63     scanf("%d",&T);
64     while(T--)
65     {
66         scanf("%d%d%d%d%d",&a,&b,&c,&d,&k);
67         long long ans=0;
68         ans=solve(b/k,d/k)-solve((a-1)/k,d/k)-solve(b/k,(c-1)/k)+solve((a-1)/k,(c-1)/k);
69         printf("%lld\n",ans);
70     }
71     return 0;
72 }

```

1.3 杜教筛

```

1 //51nod 约数个数和
2 const int Mo = 1e9 + 7;
3 const int N = 1e6 + 10;
4 const int inv2 = 500000004;
5
6 map<ll, ll> mp;
7 int cnt, notp[N], pri[N], mu[N], s[N];

```



```

8 void Prime(){
9     mu[1] = 1;
10    for(int i = 2; i < N; ++i){
11        if(!notp[i]) pri[++cnt] = i, mu[i] = -1;
12        for(int j = 1; i * pri[j] < N; ++j){
13            notp[i * pri[j]] = 1;
14            if(i % pri[j] == 0){
15                mu[i * pri[j]] = 0;
16                break;
17            }
18            else mu[i * pri[j]] = -mu[i];
19        }
20    }
21    for(int i = 1; i < N; ++i){
22        s[i] = (s[i - 1] + 1LL * i * mu[i]) % Mo;
23    }
24 }
25 ll calc(ll l, ll r){
26     return 1LL * (l + r) * (r - l + 1) % Mo * inv2 % Mo;
27 }
28 ll solve1(ll n){
29     if(n < N) return (ll)s[n];
30     if(mp.count(n)) return mp[n];
31     ll res = 1;
32     for(ll i = 2, j; i <= n; i = j + 1){
33         j = n / (n / i);
34         res = (res - calc(i, j) * solve1(n / i)) % Mo;
35     }
36     return mp[n] = res;
37 }
38 ll solve2(ll n){
39     ll p = 0, q = 0;
40     for(ll i = 1, j; i <= n; i = j + 1){
41         j = n / (n / i);
42         p = (p + calc(i, j) % Mo * (n / i)) % Mo;
43         q = (q + (j - i + 1) * (n / i) % Mo * (n / i + 1)) % Mo;
44     }
45     return p * q % Mo * inv2 % Mo;
46 }
47 int main(){
48     Prime();
49     ll n, ans = 0;
50     scanf("%lld", &n);
51     for(ll i = 1, j; i <= n; i = j + 1){
52         j = n / (n / i);
53         ans = (ans + (solve1(j) - solve1(i - 1)) * solve2(n / i)) % Mo;
54     }
55     ans = (ans % Mo + Mo) % Mo;
56     printf("%lld\n", ans);

```

```

57     return 0;
58 }

```

1.4 拉格朗日乘数法

```

1  void Init(){
2     inv[0] = 1;
3     for(int i = 1; i < maxn; ++i) inv[i] = 1LL * i * inv[i - 1] % Mo;
4     inv[maxn - 1] = Pow(inv[maxn - 1], Mo - 2);
5     for(int i = maxn - 2; i >= 0; --i) inv[i] = 1LL * inv[i + 1] * (i + 1) % Mo;
6     for(int i = 1; i < maxn; ++i) f[0][i] = 1;
7     for(int k = 1; k < maxn; ++k){
8         for(int i = 1; i < maxn; ++i){
9             f[k][i] = 1LL * f[k - 1][i] * i % Mo;
10        }
11    }
12    for(int k = 0; k < maxn; ++k){
13        for(int i = 1; i < maxn; ++i){
14            f[k][i] = (f[k][i] + f[k][i - 1]) % Mo;
15        }
16    }
17 }
18 int calc(int nn, int k){
19     if(nn < maxn) return f[k][nn];
20     int tmp = 1, res = 0;
21     for(int i = 1; i < k + 3; ++i) tmp = 1LL * tmp * (nn - i) % Mo;
22     for(int i = 1; i < k + 3; ++i){
23         res = (res + 1LL * tmp * Pow(nn - i, Mo - 2) % Mo * inv[i - 1] % Mo * inv[k + 2 - i] %
24             ↪ Mo * ((k + 2 - i) & 1 ? -1 : 1) * f[k][i]) % Mo;
25     }
26     return (res + Mo) % Mo;
27 }

```

1.5 扩展欧几里德算法

返回结果:

$$ax + by = \gcd(a, b)$$

时间复杂度: $\mathcal{O}(n \log n)$

```

1  LL exgcd(LL a, LL b, LL &x, LL &y) {
2     if(!b) {
3         x = 1;
4         y = 0;
5         return a;
6     } else {
7         LL d = exgcd(b, a % b, x, y);
8         LL t = x;
9         x = y;
10        y = t - a / b * y;

```

```

11         return d;
12     }
13 }

```

1.6 中国剩余定理

返回结果:

$$x \equiv r_i \pmod{p_i} \quad (0 \leq i < n)$$

使用条件: p_i 需两两互质

```

1 LL china(int n, int *a, int *m) {
2     LL M = 1, d, x = 0, y;
3     for(int i = 0; i < n; i++)
4         M *= m[i];
5     for(int i = 0; i < n; i++) {
6         LL w = M / m[i];
7         d = exgcd(m[i], w, d, y);
8         y = (y % M + M) % M;
9         x = (x + y * w % M * a[i]) % M;
10    }
11    while(x < 0) x += M;
12    return x;
13 }

```

1.7 中国剩余定理 2

```

1 //merge Ax=B and ax=b to A'x=B'
2 void merge(LL &A, LL &B, LL a, LL b){
3     LL x, y;
4     sol(A, -a, b-B, x, y);
5     A=lcm(A, a);
6     B=(a*y+b)%A;
7     B=(B+A)%A;
8 }

```

1.8 组合数取模

```

1 LL prod = 1, P;
2 pair<LL, LL> comput(LL n, LL p, LL k) {
3     if(n <= 1) return make_pair(0, 1);
4     LL ans = 1, cnt = 0;
5     ans = pow(prod, n / P, P);
6     cnt = n / p;
7     pair<LL, LL> res = comput(n / p, p, k);
8     cnt += res.first;
9     ans = ans * res.second % P;
10    for(int i = n - n % P + 1; i <= n; i++)
11        if(i % p)

```

```

12         ans = ans * i % P;
13     return make_pair(cnt, ans);
14 }
15 pair<LL, LL> calc(LL n, LL p, LL k) {
16     prod = 1;
17     P = pow(p, k, 1e18);
18     for(int i = 1; i < P; i++)
19         if(i % p)
20             prod = prod * i % P;
21     pair<LL, LL> res = comput(n, p, k);
22     return res;
23 }
24 LL calc(LL n, LL m, LL p, LL k) {
25     pair<LL, LL> A, B, C;
26     LL P = pow(p, k, 1e18);
27     A = calc(n, p, k);
28     B = calc(m, p, k);
29     C = calc(n - m, p, k);
30     LL ans = 1;
31     ans = pow(p, A.first - B.first - C.first, P);
32     ans = ans * A.second % P * inv(B.second, P) % P * inv(C.second, P) % P;
33     return ans;
34 }

```

1.9 扩展小步大步

```

1 LL exBSGS(LL a, LL b, LL p) {
2     //  $a^x = b \pmod p$ 
3     b %= p;
4     LL e = 1 % p;
5     for(int i = 0; i < 100; i++) {
6         if(e == b) return i;
7         e = e * a % p;
8     }
9     int r = 0;
10    while(gcd(a, p) != 1) {
11        LL d = gcd(a, p);
12        if(b % d) return -1;
13        p /= d;
14        b /= d;
15        b = b * inv(a / d, p);
16        r++;
17    }
18    LL res = BSGS(a, b, p);
19    if(res == -1) return -1;
20    return res + r;
21 }

```

1.10 卢卡斯定理

```

1 LL Lucas(LL n, LL m, LL p) {
2     LL ans = 1;
3     while(n && m) {
4         LL a = n % p, b = m % p;
5         if(a < b) return 0;
6         ans = (ans * C(a, b, p)) % p;
7         n /= p;
8         m /= p;
9     }
10    return ans % p;
11 }

```

1.11 小步大步

返回结果:

$$a^x = b \pmod{p}$$

使用条件: p 为质数

时间复杂度: $\mathcal{O}(\sqrt{n})$

```

1 LL BSGS(LL a, LL b, LL p) {
2     LL m = sqrt(p) + .5, v = inv(pw(a, m, p), p), e = 1;
3     map<LL, LL> hash;
4     hash[1] = 0;
5     for(int i = 1; i < m; i++)
6         e = e * a % p, hash[e] = i;
7     for(int i = 0; i <= m; i++) {
8         if(hash.count(b))
9             return i * m + hash[b];
10        b = b * v % p;
11    }
12    return -1;
13 }

```

1.12 Miller Rabin 素数测试

```

1 const int BASE[12] = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37};
2 bool check(long long n, int base) {
3     long long n2 = n - 1, res;
4     int s = 0;
5     while(n2 % 2 == 0) n2 >>= 1, s++;
6     res = pw(base, n2, n);
7     if((res == 1) || (res == n - 1)) return 1;
8     while(s-- > 0) {
9         res = mul(res, res, n);
10        if(res == n - 1) return 1;
11    }
12    return 0; // n is not a strong pseudo prime

```

```

13 }
14 bool isprime(const long long &n) {
15     if(n == 2)
16         return true;
17     if(n < 2 || n % 2 == 0)
18         return false;
19     for(int i = 0; i < 12 && BASE[i] < n; i++) {
20         if(!check(n, BASE[i]))
21             return false;
22     }
23     return true;
24 }

```

1.13 Pollard Rho 大数分解

时间复杂度: $\mathcal{O}(n^{1/4})$

```

1 LL prho(LL n, LL c) {
2     LL i = 1, k = 2, x = rand() % (n - 1) + 1, y = x;
3     while(1) {
4         i++;
5         x = (x * x % n + c) % n;
6         LL d = __gcd((y - x + n) % n, n);
7         if(d > 1 && d < n) return d;
8         if(y == x) return n;
9         if(i == k) y = x, k <<= 1;
10    }
11 }
12 void factor(LL n, vector<LL>&fat) {
13     if(n == 1) return;
14     if(isprime(n)) {
15         fat.push_back(n);
16         return;
17     }
18     LL p = n;
19     while(p >= n) p = prho(p, rand() % (n - 1) + 1);
20     factor(p, fat);
21     factor(n / p, fat);
22 }

```

1.14 快速数论变换

返回结果:

$$c_i = \sum_{0 \leq j \leq i} a_j \cdot b_{i-j}(\text{mod}) \quad (0 \leq i < n)$$

使用说明: *magic* 是 *mod* 的原根

时间复杂度: $\mathcal{O}(n \log n)$

```

1 /*
2 {(mod,G)}={ (81788929,7), (101711873,3), (167772161,3)

```

```

3      , (377487361, 7), (998244353, 3), (1224736769, 3)
4      , (1300234241, 3), (1484783617, 5)}
5  */
6  int mo = 998244353, G = 3;
7  void NTT(int a[], int n, int f) {
8      for(register int i = 0; i < n; i++)
9          if(i < rev[i])
10             swap(a[i], a[rev[i]]);
11     for (register int i = 2; i <= n; i <= 1) {
12         static int exp[maxn];
13         exp[0] = 1;
14         exp[1] = pw(G, (mo - 1) / i);
15         if(f == -1) exp[1] = pw(exp[1], mo - 2);
16         for(register int k = 2; k < (i >> 1); k++)
17             exp[k] = 1LL * exp[k - 1] * exp[1] % mo;
18         for(register int j = 0; j < n; j += i) {
19             for(register int k = 0; k < (i >> 1); k++) {
20                 register int &pA = a[j + k], &pB = a[j + k + (i >> 1)];
21                 register int A = pA, B = 1LL * pB * exp[k] % mo;
22                 pA = (A + B) % mo;
23                 pB = (A - B + mo) % mo;
24             }
25         }
26     }
27     if(f == -1) {
28         int rv = pw(n, mo - 2) % mo;
29         for(int i = 0; i < n; i++)
30             a[i] = 1LL * a[i] * rv % mo;
31     }
32 }
33 void mul(int m, int a[], int b[], int c[]) {
34     int n = 1, len = 0;
35     while(n < m) n <= 1, len++;
36     for (int i = 1; i < n; i++)
37         rev[i] = (rev[i >> 1] >> 1) | ((i & 1) << (len - 1));
38     NTT(a, n, 1);
39     NTT(b, n, 1);
40     for(int i = 0; i < n; i++)
41         c[i] = 1LL * a[i] * b[i] % mo;
42     NTT(c, n, -1);
43 }

```

1.15 原根

```

1  vector<LL> fct;
2  bool check(LL x, LL g) {
3      for(int i = 0; i < fct.size(); i++)
4          if(pw(g, (x - 1) / fct[i], x) == 1)
5              return 0;

```

```

6     return 1;
7 }
8 LL findrt(LL x) {
9     LL tmp = x - 1;
10    for(int i = 2; i * i <= tmp; i++) {
11        if(tmp % i == 0) {
12            fct.push_back(i);
13            while(tmp % i == 0) tmp /= i;
14        }
15    }
16    if(tmp > 1) fct.push_back(tmp);
17    // x is 1,2,4,p^n,2p^n
18    // x has phi(phi(x)) primitive roots
19    for(int i = 2; i < int(1e9); i++)
20        if(check(x, i))
21            return i;
22    return -1;
23 }
24 const int BASE[12] = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37};
25 bool check(long long n, int base) {
26     long long n2 = n - 1, res;
27     int s = 0;
28     while(n2 % 2 == 0) n2 >>= 1, s++;
29     res = pw(base, n2, n);
30     if((res == 1) || (res == n - 1)) return 1;
31     while(s--) {
32         res = mul(res, res, n);
33         if(res == n - 1) return 1;
34     }
35     return 0; // n is not a strong pseudo prime
36 }
37 bool isprime(const long long &n) {
38     if(n == 2)
39         return true;
40     if(n < 2 || n % 2 == 0)
41         return false;
42     for(int i = 0; i < 12 && BASE[i] < n; i++) {
43         if(!check(n, BASE[i]))
44             return false;
45     }
46     return true;
47 }

```

1.16 线性递推

```

1 //已知  $a_0, a_1, \dots, a_{m-1} \setminus \setminus$ 
2  $a_n = c_0 * a_{n-m} + \dots + c_{m-1} * a_{n-1} \setminus \setminus$ 
3 求  $a_n = v_0 * a_0 + v_1 * a_1 + \dots + v_{m-1} * a_{m-1} \setminus \setminus$ 
4

```



```

5 void linear_recurrence(long long n, int m, int a[], int c[], int p) {
6     long long v[M] = {1 % p}, u[M << 1], msk = !!n;
7     for(long long i(n); i > 1; i >>= 1) {
8         msk <<= 1;
9     }
10    for(long long x(0); msk; msk >>= 1, x <<= 1) {
11        fill_n(u, m << 1, 0);
12        int b(!!(n & msk));
13        x |= b;
14        if(x < m) {
15            u[x] = 1 % p;
16        } else {
17            for(int i(0); i < m; i++) {
18                for(int j(0), t(i + b); j < m; j++, t++) {
19                    u[t] = (u[t] + v[i] * v[j]) % p;
20                }
21            }
22            for(int i((m << 1) - 1); i >= m; i--) {
23                for(int j(0), t(i - m); j < m; j++, t++) {
24                    u[t] = (u[t] + c[j] * u[i]) % p;
25                }
26            }
27        }
28        copy(u, u + m, v);
29    }
30    //a[n] = v[0] * a[0] + v[1] * a[1] + ... + v[m - 1] * a[m - 1].
31    for(int i(m); i < 2 * m; i++) {
32        a[i] = 0;
33        for(int j(0); j < m; j++) {
34            a[i] = (a[i] + (long long)c[j] * a[i + j - m]) % p;
35        }
36    }
37    for(int j(0); j < m; j++) {
38        b[j] = 0;
39        for(int i(0); i < m; i++) {
40            b[j] = (b[j] + v[i] * a[i + j]) % p;
41        }
42    }
43    for(int j(0); j < m; j++) {
44        a[j] = b[j];
45    }
46 }

```

1.17 线性筛

```

1 void sieve() {
2     f[1] = mu[1] = phi[1] = 1;
3     for(int i = 2; i < maxn; i++) {
4         if(!minp[i]) {

```

```

5         minp[i] = i;
6         minpw[i] = i;
7         mu[i] = -1;
8         phi[i] = i - 1;
9         f[i] = i - 1;
10        p[++p[0]] = i; // Case 1 prime
11    }
12    for(int j = 1; j <= p[0] && (LL)i * p[j] < maxn; j++) {
13        minp[i * p[j]] = p[j];
14        if(i % p[j] == 0) {
15            // Case 2 not coprime
16            minpw[i * p[j]] = minpw[i] * p[j];
17            phi[i * p[j]] = phi[i] * p[j];
18            mu[i * p[j]] = 0;
19            if(i == minpw[i]) {
20                f[i * p[j]] = i * p[j] - i; // Special Case for f(p^k)
21            } else {
22                f[i * p[j]] = f[i / minpw[i]] * f[minpw[i] * p[j]];
23            }
24            break;
25        } else {
26            // Case 3 coprime
27            minpw[i * p[j]] = p[j];
28            f[i * p[j]] = f[i] * f[p[j]];
29            phi[i * p[j]] = phi[i] * (p[j] - 1);
30            mu[i * p[j]] = -mu[i];
31        }
32    }
33 }
34 }
```

1.18 直线下整点个数

返回结果:

$$\sum_{0 \leq i < n} \lfloor \frac{a + b \cdot i}{m} \rfloor$$

使用条件: $n, m > 0, a, b \geq 0$

时间复杂度: $\mathcal{O}(n \log n)$

```

1 //calc \sum_{i=0}^{n-1} [(a+bi)/m]
2 // n,a,b,m > 0
3 LL solve(LL n, LL a, LL b, LL m) {
4     if(b == 0)
5         return n * (a / m);
6     if(a >= m || b >= m)
7         return n * (a / m) + (n - 1) * n / 2 * (b / m) + solve(n, a % m, b % m, m);
8     return solve((a + b * n) / m, (a + b * n) % m, m, b);
9 }
```



```

47         swap(A[i],A[j]);
48         ans=-ans;
49     }
50     }ans=ans*A[i][i]%m;
51 }return (ans%m+m)%m;
52 }
53 int Gauss(){//求秩
54     int r,now=-1;
55     int ans=0;
56     for(int i = 0; i < n; i++){
57         r = now + 1;
58         for(int j = now + 1; j < m; j++)
59             if(fabs(A[j][i]) > fabs(A[r][i]))
60                 r = j;
61         if (!sgn(A[r][i])) continue;
62         ans++;
63         now++;
64         if(r != now)
65             for(int j = 0; j < n; j++)
66                 swap(A[r][j], A[now][j]);
67
68         for(int k = now + 1; k < m; k++){
69             double t = A[k][i] / A[now][i];
70             for(int j = 0; j < n; j++){
71                 A[k][j] -= t * A[now][j];
72             }
73         }
74     }
75     return ans;
76 }

```

2.2 线性基

```

1  const int N = 65;
2
3  LL bin[N], bas[N];
4  int pos[N], num;
5
6  void add(long long x, int m)
7  {
8      for(int j = m; j >= 0; j--)
9          if((x & bin[j]) && pos[j])
10             x ^= bas[pos[j]];
11     if(x == 0)
12         return;
13     for(int j = m; j >= 0; j--)
14         if(x & bin[j])
15             {
16                 pos[j] = ++num;

```

```

17     bas[num] = x;
18     break;
19 }
20 }
21
22 int work(long long *a, int n, int m)
23 {
24     num = 0;
25     memset(pos, 0, sizeof(pos));
26     for(int i = 1; i <= n; i++)
27         add(a[i], m);
28     return num;
29 }

```

2.3 快速傅立叶变换

返回结果:

$$c_i = \sum_{0 \leq j \leq i} a_j \cdot b_{i-j} \quad (0 \leq i < n)$$

时间复杂度: $\mathcal{O}(n \log n)$

```

1  typedef complex<double> cp;
2  const double pi = acos(-1);
3  void FFT(vector<cp>&num, int len, int ty){
4      for(int i=1, j=0; i<len-1; i++){
5          for(int k=len; j^=k>>=1, ~j&k;);
6          if(i<j)
7              swap(num[i], num[j]);
8      }
9      for(int h=0; (1<<h)<len; h++){
10         int step=1<<h, step2=step<<1;
11         cp w0(cos(2.0*pi/step2), ty*sin(2.0*pi/step2));
12         for(int i=0; i<len; i+=step2){
13             cp w(1, 0);
14             for(int j=0; j<step; j++){
15                 cp &x=num[i+j+step];
16                 cp &y=num[i+j];
17                 cp d=w*x;
18                 x=y-d;
19                 y=y+d;
20                 w=w*w0;
21             }
22         }
23     }
24     if(ty== -1)
25         for(int i=0; i<len; i++)
26             num[i]=cp(num[i].real()/((double)len), num[i].imag());
27 }
28 vector<cp> mul(vector<cp>a, vector<cp>b){

```

```

29     int len=a.size()+b.size();
30     while((len&-len)!=len)len++;
31     while(a.size()<len)a.push_back(cp(0,0));
32     while(b.size()<len)b.push_back(cp(0,0));
33     FFT(a,len,1);
34     FFT(b,len,1);
35     vector<cp>ans(len);
36     for(int i=0;i<len;i++)
37         ans[i]=a[i]*b[i];
38     FFT(ans,len,-1);
39     return ans;
40 }

```

2.4 1e9+7 FFT

```

1  // double 精度对  $10^9 + 7$  取模最多可以做到  $2^{20}$ 
2  const int MOD = 1000003;
3  const double PI = acos(-1);
4  typedef complex<double> Complex;
5  const int N = 65536, L = 15, MASK = (1 << L) - 1;
6  Complex w[N];
7  void FFTInit() {
8      for (int i = 0; i < N; ++i)
9          w[i] = Complex(cos(2 * i * PI / N), sin(2 * i * PI / N));
10 }
11 void FFT(Complex p[], int n) {
12     for (int i = 1, j = 0; i < n - 1; ++i) {
13         for (int s = n; j ^= s >>= 1, ~j & s;);
14         if (i < j) swap(p[i], p[j]);
15     }
16     for (int d = 0; (1 << d) < n; ++d) {
17         int m = 1 << d, m2 = m * 2, rm = n >> (d + 1);
18         for (int i = 0; i < n; i += m2) {
19             for (int j = 0; j < m; ++j) {
20                 Complex &p1 = p[i + j + m], &p2 = p[i + j];
21                 Complex t = w[rm * j] * p1;
22                 p1 = p2 - t, p2 = p2 + t;
23             } } }
24 }
25 Complex A[N], B[N], C[N], D[N];
26 void mul(int a[N], int b[N]) {
27     for (int i = 0; i < N; ++i) {
28         A[i] = Complex(a[i] >> L, a[i] & MASK);
29         B[i] = Complex(b[i] >> L, b[i] & MASK);
30     }
31     FFT(A, N), FFT(B, N);
32     for (int i = 0; i < N; ++i) {
33         int j = (N - i) % N;
34         Complex da = (A[i] - conj(A[j])) * Complex(0, -0.5),

```

```

35         db = (A[i] + conj(A[j])) * Complex(0.5, 0),
36         dc = (B[i] - conj(B[j])) * Complex(0, -0.5),
37         dd = (B[i] + conj(B[j])) * Complex(0.5, 0);
38         C[j] = da * dd + da * dc * Complex(0, 1);
39         D[j] = db * dd + db * dc * Complex(0, 1);
40     }
41     FFT(C, N), FFT(D, N);
42     for (int i = 0; i < N; ++i) {
43         long long da = (long long)(C[i].imag() / N + 0.5) % MOD,
44         db = (long long)(C[i].real() / N + 0.5) % MOD,
45         dc = (long long)(D[i].imag() / N + 0.5) % MOD,
46         dd = (long long)(D[i].real() / N + 0.5) % MOD;
47         a[i] = ((dd << (L * 2)) + ((db + dc) << L) + da) % MOD;
48     }
49 }

```

2.5 单纯形法求解线性规划

返回结果:

$$\max\{c_{1 \times m} \cdot x_{m \times 1} \mid x_{m \times 1} \geq 0_{m \times 1}, a_{n \times m} \cdot x_{m \times 1} \leq b_{n \times 1}\}$$

```

1  namespace LP{
2      const int maxn=233;
3      double a[maxn][maxn];
4      int Ans[maxn],pt[maxn];
5      int n,m;
6      void pivot(int l,int i){
7          double t;
8          swap(Ans[l+n],Ans[i]);
9          t=-a[l][i];
10         a[l][i]=-1;
11         for(int j=0;j<=n;j++)a[l][j]/=t;
12         for(int j=0;j<=m;j++){
13             if(a[j][i]&&j!=1){
14                 t=a[j][i];
15                 a[j][i]=0;
16                 for(int k=0;k<=n;k++)a[j][k]+=t*a[l][k];
17             }
18         }
19     }
20     vector<double> solve(vector<vector<double> >A,vector<double>B,vector<double>C){
21         n=C.size();
22         m=B.size();
23         for(int i=0;i<C.size();i++)
24             a[0][i+1]=C[i];
25         for(int i=0;i<B.size();i++)
26             a[i+1][0]=B[i];
27
28         for(int i=0;i<m;i++)
29             for(int j=0;j<n;j++)

```

```

30         a[i+1][j+1]=-A[i][j];
31
32     for(int i=1;i<=n;i++)Ans[i]=i;
33
34     double t;
35     for(;;){
36         int l=0;t=-eps;
37         for(int j=1;j<=m;j++)if(a[j][0]<t)t=a[l=j][0];
38         if(!l)break;
39         int i=0;
40         for(int j=1;j<=n;j++)if(a[l][j]>eps){i=j;break;}
41         if(!i){
42             puts("Infeasible");
43             return vector<double>();
44         }
45         pivot(l,i);
46     }
47     for(;;){
48         int i=0;t=eps;
49         for(int j=1;j<=n;j++)if(a[0][j]>t)t=a[0][i=j];
50         if(!i)break;
51         int l=0;
52         t=1e30;
53         for(int j=1;j<=m;j++)if(a[j][i]<-eps){
54             double tmp;
55             tmp=-a[j][0]/a[j][i];
56             if(t>tmp)t=tmp,l=j;
57         }
58         if(!l){
59             puts("Unbounded");
60             return vector<double>();
61         }
62         pivot(l,i);
63     }
64     vector<double>x;
65     for(int i=n+1;i<=n+m;i++)pt[Ans[i]]=i-n;
66     for(int i=1;i<=n;i++)x.push_back(pt[i]?a[pt[i]][0]:0);
67     return x;
68 }
69 }

```

2.6 自适应辛普森

```

1 double area(const double &left, const double &right) {
2     double mid = (left + right) / 2;
3     return (right - left) * (calc(left) + 4 * calc(mid) + calc(right)) / 6;
4 }
5
6 double simpson(const double &left, const double &right,

```



```

7         const double &eps, const double &area_sum) {
8     double mid = (left + right) / 2;
9     double area_left = area(left, mid);
10    double area_right = area(mid, right);
11    double area_total = area_left + area_right;
12    if (std::abs(area_total - area_sum) < 15 * eps) {
13        return area_total + (area_total - area_sum) / 15;
14    }
15    return simpson(left, mid, eps / 2, area_left)
16        + simpson(mid, right, eps / 2, area_right);
17 }
18
19 double simpson(const double &left, const double &right, const double &eps) {
20     return simpson(left, right, eps, area(left, right));
21 }

```

2.7 NTT

```

1  const int maxn = (1 << 18) + 10;
2  const int G = 3, mod = 998244353, phi = mod - 1;
3  int rev[MAXN];
4  int Pow(int a, int b, int mo){
5      int res = 1;
6      while(b){
7          if(b & 1) res = 1LL * res * a % mo;
8          a = 1LL * a * a % mo; b >>= 1;
9      }
10     return res;
11 }
12 void NTT(int *a, int n, int type){
13     int i, j, k, w, wn, pa, pb;
14     for(i = 1; i < n; ++i) {
15         if(i > rev[i]) swap(a[i], a[rev[i]]);
16     }
17     for(k = 2; k <= n; k <<= 1){
18         wn = Pow(G, (type * phi / k % phi + phi) % phi, mod);
19         for(j = 0; j < n; j += k){
20             w = 1;
21             for(i = 0; i < (k >> 1); ++i, w = 1LL * w * wn % mod){
22                 pa = a[i + j];
23                 pb = 1LL * w * a[i + j + (k >> 1)] % mod;
24                 a[i + j] = (pa + pb) % mod;
25                 a[i + j + (k >> 1)] = (pa - pb) % mod;
26             }
27         }
28     }
29     if(type == -1){
30         int inv = Pow(n, phi - 1, mod);
31         for(int i = 0; i < n; ++i) a[i] = 1LL * a[i] * inv % mod;

```

```

32     }
33 }
34 void mul(int *a, int n, int *b, int m, int *c){
35     int K, N;
36     for(N = 1, K = 0; N < n + m - 1; N <= 1, K++); K--;
37     for(int i = 1; i < N; ++i){
38         rev[i] = (rev[i >> 1] >> 1) | ((i & 1) << K);
39     }
40     FFT(a, N, 1);
41     FFT(b, N, 1);
42     for(int i = 0; i < N; ++i) c[i] = 1LL * a[i] * b[i] % mod;
43     FFT(c, N, -1);
44 }

```

2.8 FFT

```

1  const int MAXN = (1 << 18) + 10;
2  const double pi = acos(-1);
3  int rev[MAXN];
4  struct Complex{
5      double x, y;
6      Complex(double _x = 0, double _y = 0){x = _x; y = _y;}
7  };
8  Complex operator + (const Complex &a, const Complex &b){
9      return Complex(a.x + b.x, a.y + b.y);
10 }
11 Complex operator - (const Complex &a, const Complex &b){
12     return Complex(a.x - b.x, a.y - b.y);
13 }
14 Complex operator * (const Complex &a, const Complex &b){
15     return Complex(a.x * b.x - a.y * b.y, a.x * b.y + b.x * a.y);
16 }
17 Complex operator * (const Complex &a, const double &b){
18     return Complex(a.x * b, a.y * b);
19 }
20 void FFT(Complex *a, int n, int type){
21     int i, j, k;
22     for(i = 1; i < n; ++i){
23         if(i > rev[i]) swap(a[i], a[rev[i]]);
24     }
25     Complex w, wn, pa, pb;
26     for(k = 2; k <= n; k <= 1){
27         wn = Complex(cos(2.0 * pi * type / k), sin(2.0 * pi * type / k));
28         for(j = 0; j < n; j += k){
29             for(i = 0, w = Complex(1); i < (k >> 1); ++i, w = w * wn){
30                 pa = a[i + j], pb = w * a[i + j + (k >> 1)];
31                 a[i + j] = pa + pb;
32                 a[i + j + (k >> 1)] = pa - pb;
33             }

```

```

34     }
35 }
36 if(type == -1){
37     double inv = 1.0 / n;
38     for(i = 0; i < n; ++i) a[i] = a[i] * inv;
39 }
40 }
41 void mul(Complex *a, int n, Complex *b, int m, Complex *c){
42     int K, N;
43     for(N = 1, K = 0; N < n + m - 1; N <= 1, K++); K--;
44     for(int i = 1; i < N; ++i){
45         rev[i] = (rev[i >> 1] >> 1) | ((i & 1) << K);
46     }
47     FFT(a, N, 1);
48     FFT(b, N, 1);
49     for(int i = 0; i < N; ++i) c[i] = a[i] * b[i];
50     FFT(c, N, -1);
51 }

```

2.9 多项式求逆开根

```

1 //bzoj 小朋友与二叉树
2 const int mod = 998244353;
3 const int phi = 998244352;
4 const int inv2 = 499122177;
5 const int maxn = (1 << 18) + 5;
6 const int G = 3;
7 int rev[maxn], Tmp[maxn], bi[maxn], f[maxn], g[maxn];
8 void NTT(int *a, int n, int type){
9     int i, j, k, w, wn, pa, pb, K = 0;
10    for(K = 0; (1 << K) < n; ++K); --K;
11    for(int i = 1; i < n; ++i){
12        rev[i] = (rev[i >> 1] >> 1) | ((i & 1) << K);
13        if(i > rev[i]) swap(a[i], a[rev[i]]);
14    }
15    for(k = 2; k <= n; k <= 1){
16        wn = Pow(G, (type * phi / k % phi + phi) % phi, mod);
17        for(j = 0; j < n; j += k){
18            w = 1;
19            for(i = 0; i < (k >> 1); ++i, w = 1LL * w * wn % mod){
20                pa = a[i + j];
21                pb = 1LL * w * a[i + j + (k >> 1)] % mod;
22                a[i + j] = (pa + pb) % mod;
23                a[i + j + (k >> 1)] = (pa - pb) % mod;
24            }
25        }
26    }
27    if(type == -1){
28        int inv = Pow(n, phi - 1, mod);

```

```

29         for(int i = 0; i < n; ++i) a[i] = 1LL * a[i] * inv % mod;
30     }
31 }
32 void Inv(int *a, int *b, int n){
33     if(n == 1){
34         b[0] = Pow(a[0], phi - 1, mod);
35         return;
36     }
37     Inv(a, b, n >> 1);
38     for(int i = 0; i < n; ++i) Tmp[i] = a[i];
39     for(int i = n; i < (n << 1); ++i) Tmp[i] = 0;
40     NTT(Tmp, n << 1, 1);
41     NTT(b, n << 1, 1);
42     for(int i = 0; i < (n << 1); ++i){
43         b[i] = 1LL * b[i] * (2 - 1LL * b[i] * Tmp[i] % mod) % mod;
44     }
45     NTT(b, n << 1, -1);
46     for(int i = n; i < (n << 1); ++i) b[i] = 0;
47 }
48
49 inline void Sqrt(int *a, int *b, int n){
50     if(n == 1){
51         b[0] = (int)sqrt(a[0]);
52         return;
53     }
54     Sqrt(a, b, n >> 1);
55     for(int i = 0; i < n; ++i) bi[i] = 0;
56     for(int i = n; i < (n << 1); ++i) Tmp[i] = 0;
57     Inv(b, bi, n);
58     for(int i = 0; i < n; ++i) Tmp[i] = a[i];
59     for(int i = n; i < (n << 1); ++i) Tmp[i] = 0;
60     NTT(Tmp, n << 1, 1);
61     NTT(b, n << 1, 1);
62     NTT(bi, n << 1, 1);
63     for(int i = 0; i < (n << 1); ++i) b[i] = 1LL * inv2 * (b[i] + 1LL * bi[i] * Tmp[i] % mod)
        ↪ % mod;
64     NTT(b, n << 1, -1);
65     for(int i = n; i < (n << 1); ++i) b[i] = 0;
66 }
67 int main(){
68     int N, n, m;
69     scanf("%d%d", &m, &n); n++;
70     for(int i = 1, x; i <= m; ++i){
71         scanf("%d", &x);
72         if(x < n) f[x] = mod - 4;
73     }
74     f[0] = 1;
75
76     for(N = 1; N < n; N <= 1);

```

```

77     Sqrt(f, g, N);
78     g[0]++;
79     memset(f, 0, sizeof(f));
80     Inv(g, f, N);
81     for(int i = 1; i < n; ++i) printf("%d\n", ((f[i] << 1) % mod + mod) % mod);
82     return 0;
83 }

```

2.10 分治 NTT

```

1  //bzoj 3456 城市规划
2  const int mod = 1004535809;
3  const int phi = 1004535808;
4  const int maxn = (1 << 19) + 5;
5  const int G = 3;
6
7  int rev[maxn];
8  void NTT(int *a, int n, int type){
9      int i, j, k, w, wn, pa, pb;
10     for(int i = 1; i < n; ++i){
11         if(i > rev[i]) swap(a[i], a[rev[i]]);
12     }
13     for(k = 2; k <= n; k <= 1){
14         wn = Pow(G, (type * phi / k % phi + phi) % phi);
15         for(j = 0; j < n; j += k){
16             w = 1;
17             for(i = 0; i < (k >> 1); ++i, w = 1LL * w * wn % mod){
18                 pa = a[i + j];
19                 pb = 1LL * w * a[i + j + (k >> 1)] % mod;
20                 a[i + j] = (pa + pb) % mod;
21                 a[i + j + (k >> 1)] = (pa - pb) % mod;
22             }
23         }
24     }
25     if(type == -1){
26         int inv = Pow(n, phi - 1);
27         for(int i = 0; i < n; ++i) a[i] = 1LL * a[i] * inv % mod;
28     }
29 }
30
31 int fac[maxn], inv[maxn], c2[maxn], f[maxn], g[maxn], A[maxn], B[maxn], C[maxn];
32 void solve(int l, int r){
33     if(l == r){
34         f[l] = (c2[l] - 1LL * fac[l - 1] * f[l] % mod + mod) % mod;
35         return;
36     }
37     int mid = (l + r) >> 1;
38     solve(l, mid);
39     int L = mid - 1 + 1, R = r - 1 + 1;

```

```

40  int n = 1, K = 0;
41  for(n = 1, K = 0; n < L + R - 1; n <= 1) ++K; --K;
42  for(int i = 0; i < n; ++i) A[i] = B[i] = C[i] = 0;
43  for(int i = 1; i < n; ++i){
44      rev[i] = (rev[i >> 1] >> 1) | ((i & 1) << K);
45  }
46  for(int i = 1; i <= mid; ++i) A[i - 1] = 1LL * f[i] * inv[i - 1] % mod;
47  for(int i = 1; i < R; ++i) B[i] = g[i];
48  NTT(A, n, 1);
49  NTT(B, n, 1);
50  for(int i = 0; i < n; ++i) C[i] = 1LL * A[i] * B[i] % mod;
51  NTT(C, n, -1);
52  for(int i = mid + 1; i <= r; ++i){
53      f[i] = (f[i] + C[i - 1]) % mod;
54  }
55  solve(mid + 1, r);
56 }
57
58 int main(){
59     int n; scanf("%d", &n);
60     fac[0] = 1;
61     for(int i = 1; i <= n; ++i){
62         fac[i] = 1LL * fac[i - 1] * i % mod;
63         int tmp = 1LL * i * (i - 1) / 2 % (mod - 1);
64         c2[i] = Pow(2, tmp);
65     }
66     inv[n] = Pow(fac[n], mod - 2);
67     for(int i = n - 1; i >= 0; --i){
68         inv[i] = 1LL * inv[i + 1] * (i + 1) % mod;
69     }
70     for(int i = 1; i <= n; ++i){
71         g[i] = 1LL * c2[i] * inv[i] % mod;
72     }
73     solve(1, n);
74     printf("%d\n", f[n]);
75     return 0;
76 }

```

2.11 多项式求根

```

1  const double eps=1e-12;
2  double a[10][10];
3  typedef vector<double> vd;
4  int sgn(double x) { return x < -eps ? -1 : x > eps; }
5  double mypow(double x, int num){
6      double ans=1.0;
7      for(int i=1; i<=num; ++i) ans*=x;
8      return ans;
9  }

```

```

10 double f(int n,double x){
11     double ans=0;
12     for(int i=n;i>=0;--i)ans+=a[n][i]*mypow(x,i);
13     return ans;
14 }
15 double getRoot(int n,double l,double r){
16     if(sgn(f(n,l))==0)return l;
17     if(sgn(f(n,r))==0)return r;
18     double temp;
19     if(sgn(f(n,l))>0)temp=-1;else temp=1;
20     double m;
21     for(int i=1;i<=10000;++i){
22         m=(l+r)/2;
23         double mid=f(n,m);
24         if(sgn(mid)==0){
25             return m;
26         }
27         if(mid*temp<0)l=m;else r=m;
28     }
29     return (l+r)/2;
30 }
31 vd did(int n){
32     vd ret;
33     if(n==1){
34         ret.push_back(-1e10);
35         ret.push_back(-a[n][0]/a[n][1]);
36         ret.push_back(1e10);
37         return ret;
38     }
39     vd mid=did(n-1);
40     ret.push_back(-1e10);
41     for(int i=0;i+1<mid.size();++i){
42         int t1=sgn(f(n,mid[i])),t2=sgn(f(n,mid[i+1]));
43         if(t1*t2>0)continue;
44         ret.push_back(getRoot(n,mid[i],mid[i+1]));
45     }
46     ret.push_back(1e10);
47     return ret;
48 }
49 int main(){
50     int n; scanf("%d",&n);
51     for(int i=n;i>=0;--i){
52         scanf("%lf",&a[n][i]);
53     }
54     for(int i=n-1;i>=0;--i)
55         for(int j=0;j<=i;++j)a[i][j]=a[i+1][j+1]*(j+1);
56     vd ans=did(n);
57     sort(ans.begin(),ans.end());
58     for(int i=1;i+1<ans.size();++i)printf("%.10f\n",ans[i]);

```

```

59     return 0;
60 }

```

2.12 快速求逆

```

1  long long inverse(const long long &x, const long long &mod) {
2      if (x == 1) {
3          return 1;
4      } else {
5          return (mod - mod / x) * inverse(mod % x, mod) % mod;
6      }
7  }

```

2.13 魔幻多项式

快速傅里叶变换

注意事项：请实现复数类 `Complex`，并注意快速傅里叶变换精度较差，建议使用快速数论变换。

```

1  int prepare(int n) {
2      int len = 1;
3      for (; len <= 2 * n; len <= 1);
4      for (int i = 0; i < len; i++) {
5          e[0][i] = Complex(cos(2 * pi * i / len), sin(2 * pi * i / len));
6          e[1][i] = Complex(cos(2 * pi * i / len), -sin(2 * pi * i / len));
7      }
8      return len;
9  }
10 void DFT(Complex *a, int n, int f) {
11     for (int i = 0, j = 0; i < n; i++) {
12         if (i > j) std::swap(a[i], a[j]);
13         for (int t = n >> 1; (j ^= t) < t; t >>= 1);
14     }
15     for (int i = 2; i <= n; i <= 1)
16         for (int j = 0; j < n; j += i)
17             for (int k = 0; k < (i >> 1); k++) {
18                 Complex A = a[j + k];
19                 Complex B = e[f][n / i * k] * a[j + k + (i >> 1)];
20                 a[j + k] = A + B;
21                 a[j + k + (i >> 1)] = A - B;
22             }
23     if (f == 1) {
24         for (int i = 0; i < n; i++)
25             a[i].a /= n;
26     }
27 }

```

光速数论变换

注意事项：MOD 应该为一个特殊的质数 $2^n + 1$ 且 n 应该要足够大，PRT 为这个质数的原根。


```

1 // meminit(A, L, r) 是将数组 A 的 [L, r) 清 0。
2 // memcpy(target, source, L, r) 是将 source 的 [L, r) 复制到 target 的 [L, r)
3 #define meminit(A, L, r) memset(A + (L), 0, sizeof(*A) * ((r) - (L)))
4 #define memcpy(B, A, L, r) memcpy(B, A + (L), sizeof(*A) * ((r) - (L)))
5 void DFT(int *a, int n, int f) { // 封闭形式, 常数小 (107 跑 2.23 秒)
6     for (register int i = 0, j = 0; i < n; i++) {
7         if (i > j) std::swap(a[i], a[j]);
8         for (register int t = n >> 1; (j ^= t) < t; t >>= 1);
9     }
10    for (register int i = 2; i <= n; i <= 1) {
11        static int exp[MAXN];
12        exp[0] = 1; exp[1] = fpm(PRT, (MOD - 1) / i);
13        if (f == 1) exp[1] = fpm(exp[1], MOD - 2);
14        for (register int k = 2; k < (i >> 1); k++) {
15            exp[k] = 1ll * exp[k - 1] * exp[1] % MOD;
16        }
17        for (register int j = 0; j < n; j += i) {
18            for (register int k = 0; k < (i >> 1); k++) {
19                register int &pA = a[j + k], &pB = a[j + k + (i >> 1)];
20                register int A = pA, B = 1ll * pB * exp[k] % MOD;
21                pA = (A + B) % MOD;
22                pB = (A - B + MOD) % MOD;
23            }
24        }
25    }
26    if (f == 1) {
27        register int rev = fpm(n, MOD - 2, MOD);
28        for (register int i = 0; i < n; i++) {
29            a[i] = 1ll * a[i] * rev % MOD;
30        }
31    }
32 }
33 // 在不写高精度的情况下合并 FFT 所得结果对 MOD 取模后的答案
34 // 值得注意的是, 这个东西不能最后再合并, 而是应该每做一次多项式乘法就 CRT 一次
35 int CRT(int *a) {
36     static int x[3];
37     for (int i = 0; i < 3; i++) {
38         x[i] = a[i];
39         for (int j = 0; j < i; j++) {
40             int t = (x[i] - x[j] + FFT[i] -> MOD) % FFT[i] -> MOD;
41             if (t < 0) t += FFT[i] -> MOD;
42             x[i] = 1LL * t * inv[j][i] % FFT[i] -> MOD;
43         }
44     }
45     int sum = 1, ret = x[0] % MOD;
46     for (int i = 1; i < 3; i++) {
47         sum = 1LL * sum * FFT[i] -> MOD % MOD;
48         ret += 1LL * x[i] * sum % MOD;
49         if (ret >= MOD) ret -= MOD;

```

```

50     }
51     return ret;
52 }
53 for (int i = 0; i < 3; i++) // inv 数组的预处理过程, inverse(x, p) 表示求 x 在 p 下逆元
54     for (int j = 0; j < 3; j++)
55         inv[i][j] = inverse(FFT[i] -> MOD, FFT[j] -> MOD);

```

牛顿迭代法

问题描述：给出多项式 $G(x)$ ，求解多项式 $F(x)$ 满足：

$$G(F(x)) \equiv 0 \pmod{x^n}$$

答案只需要精确到 $F(x) \bmod x^n$ 即可。

实现原理：考虑倍增，假设有：

$$G(F_t(x)) \equiv 0 \pmod{x^t}$$

对 $G(F_{t+1}(x))$ 在模 x^{2t} 意义下进行 Taylor 展开：

$$G(F_{t+1}(x)) \equiv G(F_t(x)) + \frac{G'(F_t(x))}{1!}(F_{t+1}(x) - F_t(x)) \pmod{x^{2t}}$$

那么就有：

$$F_{t+1}(x) \equiv F_t(x) - \frac{G(F_t(x))}{G'(F_t(x))} \pmod{x^{2t}}$$

注意事项： $G(F(x))$ 的常数项系数必然为 0，这个可以作为求解的初始条件；

多项式求逆

原理：令 $G(x) = x * A - 1$ （其中 A 是一个多项式系数），根据牛顿迭代法有：

$$\begin{aligned} F_{t+1}(x) &\equiv F_t(x) - \frac{F_t(x) * A(x) - 1}{A(x)} \\ &\equiv 2F_t(x) - F_t(x)^2 * A(x) \pmod{x^{2t}} \end{aligned}$$

注意事项：

1. $F(x)$ 的常数项系数必然不为 0，否则没有逆元；
2. 复杂度是 $O(n \log n)$ 但是常数比较大（ 10^5 大概需要 0.3 秒左右）；
3. 传入的两个数组必须不同，但传入的次数界没有要是 2 的次幂；

```

1 void getInv(int *a, int *b, int n) {
2     static int tmp[MAXN];
3     b[0] = fpm(a[0], MOD - 2, MOD);
4     for (int c = 2, M = 1; c < (n << 1); c <= 1) {
5         for (; M <= 3 * (c - 1); M <= 1);
6         meminit(b, c, M);
7         meminit(tmp, c, M);
8         memcpy(tmp, a, 0, c);

```

```

9      DFT(tmp, M, 0);
10     DFT(b, M, 0);
11     for (int i = 0; i < M; i++) {
12         b[i] = 1ll * b[i] * (2ll - 1ll * tmp[i] * b[i] % MOD + MOD) % MOD;
13     }
14     DFT(b, M, 1);
15     meminit(b, c, M);
16 }
17 }

```

多项式取指数和对数

作用：给出一个多项式 $A(x)$ ，求一个多项式 $F(x)$ 满足 $e^A(x) - F(x) \equiv 0 \pmod{x^n}$ 。

原理：令 $G(x) = \ln x - A$ （其中 A 是一个多项式系数），根据牛顿迭代法有：

$$F_{t+1}(x) \equiv F_t(x) - F_t(x)(\ln F_t(x) - A(x)) \pmod{x^{2t}}$$

求 $\ln F_t(x)$ 可以用先求导再积分的办法，即：

$$\ln A(x) = \int \frac{F'(x)}{F(x)} dx$$

多项式的求导和积分可以在 $O(n)$ 的时间内完成，因此总复杂度为 $O(n \log n)$ 。

应用：加速多项式快速幂。

注意事项：

1. 进行 \log 的多项式必须保证常数项系数为 1，否则必须要先求出 $\log a[0]$ 是多少；
2. 传入的两个数组必须不同，但传入的次数界没有必要是 2 的次幂；
3. 常数比较大， 10^5 的数据求指数和对数分别需要 0.37s 和 0.85s 左右的时间，注意这里 `memset` 几乎不占用时。

```

1 void getDiff(int *a, int *b, int n) { // 多项式取微分
2     for (int i = 0; i + 1 < n; i++) {
3         b[i] = 1ll * (i + 1) * a[i + 1] % MOD;
4     }
5     b[n - 1] = 0;
6 }
7 void getInt(int *a, int *b, int n) { // 多项式取积分，积分常数为 0
8     static int inv[MAXN];
9     inv[1] = 1;
10    for (int i = 2; i < n; i++) {
11        inv[i] = 1ll * (MOD - MOD / i) * inv[MOD % i] % MOD;
12    }
13    b[0] = 0;
14    for (int i = 1; i < n; i++) {
15        b[i] = 1ll * a[i - 1] * inv[i] % MOD;
16    }
17 }

```

```

18 void getLn(int *a, int *b, int n) {
19     static int inv[MAXN], d[MAXN];
20     int M = 1;
21     for (; M <= 2 * (n - 1); M <= 1);
22     getInv(a, inv, n);
23     getDiff(a, d, n);
24     meminit(d, n, M);
25     meminit(inv, n, M);
26     DFT(d, M, 0); DFT(inv, M, 0);
27     for (int i = 0; i < M; i++) {
28         d[i] = 1ll * d[i] * inv[i] % MOD;
29     }
30     DFT(d, M, 1);
31     getInt(d, b, n);
32 }
33 void getExp(int *a, int *b, int n) {
34     static int ln[MAXN], tmp[MAXN];
35     b[0] = 1;
36     for (int c = 2, M = 1; c < (n << 1); c <= 1) {
37         for (; M <= 2 * (c - 1); M <= 1);
38         int bound = std::min(c, n);
39         memcpy(tmp, a, 0, bound);
40         meminit(tmp, bound, M);
41         meminit(b, c, M);
42         getLn(b, ln, c);
43         meminit(ln, c, M);
44         DFT(b, M, 0);
45         DFT(tmp, M, 0);
46         DFT(ln, M, 0);
47         for (int i = 0; i < M; i++) {
48             b[i] = 1ll * b[i] * (1ll - ln[i] + tmp[i] + MOD) % MOD;
49         }
50         DFT(b, M, 1);
51         meminit(b, c, M);
52     }
53 }

```

多项式除法

作用：给出两个多项式 $A(x)$ 和 $B(x)$ ，求两个多项式 $D(x)$ 和 $R(x)$ 满足：

$$A(x) \equiv D(x)B(x) + R(x) \pmod{x^n}$$

注意事项：

1. 常数比较大概为 6 倍 FFT 的时间，即大约 10^5 的数据 0.07s 左右；
2. 传入两个多项式的次数界，没有必要是 2 的次幂，但是要保证除数多项式不为 0。

```

1 void divide(int n, int m, int *a, int *b, int *d, int *r) {
    ↪ // n、m 分别为多项式 A (被除数) 和 B (除数) 的次数界
2     static int M, tA[MAXN], tB[MAXN], inv[MAXN], tD[MAXN];
3     for (; n > 0 && a[n - 1] == 0; n--);
4     for (; m > 0 && b[m - 1] == 0; m--);
5     for (int i = 0; i < n; i++) tA[i] = a[n - i - 1];
6     for (int i = 0; i < m; i++) tB[i] = b[m - i - 1];
7     for (M = 1; M <= n - m + 1; M <<= 1);
8     meminit(tB, m, M);
9     getInv(tB, inv, M);
10    for (M = 1; M <= 2 * (n - m + 1); M <<= 1);
11    meminit(inv, n - m + 1, M);
12    meminit(tA, n - m + 1, M);
13    DFT(inv, M, 0);
14    DFT(tA, M, 0);
15    for (int i = 0; i < M; i++) {
16        d[i] = 1ll * inv[i] * tA[i] % MOD;
17    }
18    DFT(d, M, 1);
19    std::reverse(d, d + n - m + 1);
20    for (M = 1; M <= n; M <<= 1);
21    memcpy(tB, b, 0, m); meminit(tB, m, M);
22    memcpy(tD, d, 0, n - m + 1); meminit(tD, n - m + 1, M);
23    DFT(tD, M, 0);
24    DFT(tB, M, 0);
25    for (int i = 0; i < M; i++) {
26        r[i] = 1ll * tD[i] * tB[i] % MOD;
27    }
28    DFT(r, M, 1);
29    meminit(r, n, M);
30    for (int i = 0; i < n; i++) {
31        r[i] = (a[i] - r[i] + MOD) % MOD;
32    }
33 }

```

3 数据结构

3.1 平衡的二叉查找树

3.1.1 Treap

```

1 struct Node {
2     Node *lc, *rc;
3     int r, v, cnt, sz;
4     Node() {}
5     Node(int _v) {
6         v = _v;
7         sz = 0;
8         cnt = 1;

```

```

9      r = rand();
10     lc = rc = NULL;
11 }
12 void update();
13 };
14
15 inline int size(Node *o) {
16     return o ? o->sz : 0;
17 }
18 void Node::update() {
19     sz = cnt + size(lc) + size(rc);
20 }
21 void rotate_l(Node *&o) {
22     Node *k = o->rc;
23     o->rc = k->lc;
24     k->lc = o;
25     o->update();
26     k->update();
27     o = k;
28 }
29 void rotate_r(Node *&o) {
30     Node *k = o->lc;
31     o->lc = k->rc;
32     k->rc = o;
33     o->update();
34     k->update();
35     o = k;
36 }
37 void insert(Node *&o, int v) {
38     if(o == NULL) {
39         o = new Node(v);
40     } else if(v < o->v) {
41         insert(o->lc, v);
42         if(o->lc->r < o->r)
43             rotate_r(o);
44     } else if(v > o->v) {
45         insert(o->rc, v);
46         if(o->rc->r < o->r)
47             rotate_l(o);
48     } else o->cnt++;
49     o->update();
50 }
51 void remove(Node *&o, int v) { // 删除前请确保 v 存在
52     if(v < o->v) {
53         remove(o->lc, v);
54     } else if(v > o->v) {
55         remove(o->rc, v);
56     } else if(o->cnt == 1) {
57         if(o->lc == NULL) {

```

```

58     Node *k = o;
59     o = o->rc;
60     delete k;
61 } else if(o->rc == NULL) {
62     Node *k = o;
63     o = o->lc;
64     delete k;
65 } else {
66     if(o->lc->r < o->rc->r) {
67         rotate_l(o);
68         remove(o->lc, v);
69     } else {
70         rotate_r(o);
71         remove(o->rc, v);
72     }
73 }
74 } else o->cnt--;
75 if(o) o->update();
76 }
77 void merge(Node *&to, Node *from) {
78     if(from == NULL) return;
79     merge(to, from->lc);
80     for(int i = 0; i < from->cnt; ++i)
81         insert(to, from->v);
82     merge(to, from->rc);
83 }
84 void del_tree(Node *&o) {
85     if(o == NULL) return;
86     del_tree(o->lc);
87     del_tree(o->rc);
88     delete o;
89     o = NULL;
90 }
91 int get_rank(Node *o, int v) { // 查询小于等于 v 的个数
92     if(o == NULL) return 0;
93     if(v < o->v) return get_rank(o->lc, v);
94     else if(v == o->v) return size(o->lc) + o->cnt;
95     else return size(o->lc) + o->cnt + get_rank(o->rc, v);
96 }

```

3.1.2 Splay

```

1  struct Node {
2      int fa, c[2];
3      int val, sum, sz;
4      int rev, tag, inv;
5      int lmx, lmn, rmx, rmn;
6  } T[N];
7  int root;
8

```

```

9  #define fa(o) T[o].fa
10 #define lc(o) T[o].c[0]
11 #define rc(o) T[o].c[1]
12 #define sz(o) T[o].sz
13 #define val(o) T[o].val
14 #define sum(o) T[o].sum
15 #define rev(o) T[o].rev
16 #define tag(o) T[o].tag
17 #define inv(o) T[o].inv
18 #define lmx(o) T[o].lmx
19 #define rmx(o) T[o].rmx
20 #define lmn(o) T[o].lmn
21 #define rmn(o) T[o].rmn
22 #define ctype(x) (x == rc(fa(x)))
23
24 inline void setc(int x, int y, const bool d) { if(x) fa(x) = y; if(y) T[y].c[d] = x; }
25 void set_tag(int o, int v) {
26     if(o == 0) return;
27     rev(o) = inv(o) = 0;
28     val(o) = tag(o) = v;
29     sum(o) = v * sz(o);
30     lmx(o) = lmn(o) = rmx(o) = rmn(o) = 0;
31     relax(lmn(o), sum(o)); tense(lmx(o), sum(o));
32     relax(rmn(o), sum(o)); tense(rmx(o), sum(o));
33 }
34 void set_inv(int o) {
35     if(o == 0) return;
36     inv(o) ^= 1; sum(o) *= -1; val(o) *= -1;
37     swap(lmn(o), lmx(o)); lmn(o) *= -1; lmx(o) *= -1;
38     swap(rmn(o), rmx(o)); rmn(o) *= -1; rmx(o) *= -1;
39 }
40 void set_rev(int o) {
41     if(o == 0) return;
42     rev(o) ^= 1;
43     swap(lc(o), rc(o));
44     swap(lmx(o), rmx(o));
45     swap(lmn(o), rmn(o));
46 }
47 inline void pushdown(int o) {
48     if(tag(o)) set_tag(lc(o), tag(o)), set_tag(rc(o), tag(o)), tag(o) = 0;
49     if(rev(o)) set_rev(lc(o)), set_rev(rc(o)), rev(o) = 0;
50     if(inv(o)) set_inv(lc(o)), set_inv(rc(o)), inv(o) = 0;
51 }
52 inline void update(int o) {
53     if(o == 0) return;
54     sz(o) = sz(lc(o)) + sz(rc(o)) + 1;
55     sum(o) = sum(lc(o)) + sum(rc(o)) + val(o);
56     lmx(o) = lmn(o) = rmx(o) = rmn(o) = 0;
57     relax(lmn(o), lmn(lc(o))); relax(lmn(o), sum(lc(o)) + val(o) + lmn(rc(o)));

```



```

58     tense(lmx(o), lmx(lc(o))); tense(lmx(o), sum(lc(o)) + val(o) + lmx(rc(o)));
59     relax(rmn(o), rmn(rc(o))); relax(rmn(o), sum(rc(o)) + val(o) + rmn(lc(o)));
60     tense(rmx(o), rmx(rc(o))); tense(rmx(o), sum(rc(o)) + val(o) + rmx(lc(o)));
61 }
62 inline void rotate(int o) {
63     int p = fa(o), mark = ctype(o);
64     if(fa(p)) setc(o, fa(p), ctype(p));
65     else fa(o) = 0, root = o;
66     setc(T[o].c[mark ^ 1], p, mark);
67     setc(p, o, mark ^ 1);
68     update(p); update(o);
69 }
70 void splay(int x, int rt = 0) {
71     static int q[N], top;
72     q[top = 1] = x;
73     for(int i = x; i != rt; i = fa(i))
74         q[++top] = fa(i);
75     while(top) pushdown(q[top--]);
76     while(fa(x) != rt) {
77         if(fa(fa(x)) != rt) {
78             if(ctype(x) == ctype(fa(x)))
79                 rotate(fa(x));
80             else rotate(x);
81         }
82         rotate(x);
83     }
84 }
85 int find(int rank) {
86     int x = root;
87     while(true) {
88         pushdown(x);
89         int s = sz(lc(x)) + 1;
90         if(rank < s) x = lc(x);
91         else if(rank == s) return x;
92         else rank -= s, x = rc(x);
93     }
94 }
95 void init() {
96     for(int i = 1; i <= n; ++i) {
97         sz(i) = 1;
98         if(src[i] == '(') {
99             val(i) = sum(i) = 1;
100             lmx(i) = rmx(i) = 1;
101             lmn(i) = rmn(i) = 0;
102         } else {
103             val(i) = sum(i) = -1;
104             lmx(i) = rmx(i) = 0;
105             lmn(i) = rmn(i) = -1;
106         }

```

```

107     }
108     for(int i = 1; i <= n; ++i) {
109         setc(i, i + 1, 0);
110         update(i + 1);
111     }
112     root = n + 1;
113 }
114 inline int interval(int l, int r) {
115     int x, y;
116     splay(x = find(r + 1));
117     if(l - 1) {
118         splay(y = find(l - 1), x);
119         return rc(y);
120     } else {
121         return lc(x);
122     }
123 }

```

3.2 坚固的数据结构

3.2.1 坚固的平衡树

```

1  #define sz(x) (x?x->siz:0)
2  struct node{
3      int siz,key;
4      LL val,sum;
5      LL mu,a,d;
6      node *c[2],*f;
7      void split(int ned,node *&p,node *&q);
8      node* rz(){
9          sum=val;siz=1;
10         if(c[0])sum+=c[0]->sum,siz+=c[0]->siz;
11         if(c[1])sum+=c[1]->sum,siz+=c[1]->siz;
12         return this;
13     }
14     void make(LL _mu,LL _a,LL _d){
15         sum=sum*_mu+_a*siz+_d*siz*(siz-1)/2;
16         val=val*_mu+_a*_d*sz(c[0]);
17         mu=_mu;a=a*_mu+_a;d=d*_mu+_d;
18     }
19     void pd(){
20         if(mu==1&&a==0&&d==0)return;
21         if(c[0])c[0]->make(mu,a,d);
22         if(c[1])c[1]->make(mu,a+d*d*sz(c[0]),d);
23         mu=1;a=d=0;
24     }
25     node(){mu=1;}
26 }nd[maxn*2],*root;
27 node *merge(node *p,node *q){
28     if(!p||!q)return p?p->rz():(q?q->rz():0);

```

```

29     p->pd();q->pd();
30     if(p->key<q->key){
31         p->c[1]=merge(p->c[1],q);
32         return p->rz();
33     }else{
34         q->c[0]=merge(p,q->c[0]);
35         return q->rz();
36     }
37 }
38 void node::split(int ned,node *&p,node *&q){
39     if(!ned){p=0;q=this;return;}
40     if(ned==siz){p=this;q=0;return;}
41     pd();
42     if(sz(c[0])>=ned){
43         c[0]->split(ned,p,q);c[0]=0;rz();
44         q=merge(q,this);
45     }else{
46         c[1]->split(ned-sz(c[0])-1,p,q);c[1]=0;rz();
47         p=merge(this,p);
48     }
49 }
50 int main(){
51     for(int i=1;i<=n;i++){
52         nd[i].val=in();
53         nd[i].key=rand();
54         nd[i].rz();
55         root=merge(root,nd+i);
56     }
57 }

```

3.2.2 坚固的左偏树

```

1  int merge(int a, int b)
2  {
3      if(a == 0) return b;
4      if(b == 0) return a;
5      if(v[a] < v[b]) swap(a, b);
6      rc[a] = merge(rc[a], b);
7      if(rc[a]) fa[rc[a]] = a;
8      if(d[lc[a]] < d[rc[a]]) swap(lc[a], rc[a]);
9      d[a] = rc[a] ? d[rc[a]] + 1 : 0;
10     return a;
11 }

```

3.3 树上的魔术师

3.3.1 轻重树链剖分

```

1  int n;
2

```

```

3  vector<int> g[N];
4
5  int son[N], top[N];
6  int dep[N], sz[N], fa[N];
7  int dfn[N], pos[N], segn;
8
9  void dfs(int x, int p) {
10     sz[x] = 1;
11     fa[x] = p;
12     son[x] = -1;
13     dep[x] = dep[p] + 1;
14     for(auto y: g[x]) {
15         if(y == p) continue;
16         dfs(y, x);
17         sz[x] += sz[y];
18         if(son[x] == -1 || sz[son[x]] < sz[y])
19             son[x] = y;
20     }
21 }
22 void DFS(int x, int tp) {
23     top[x] = tp;
24     dfn[pos[x] = ++segn] = x;
25     if(~son[x]) DFS(son[x], tp);
26     for(auto y: g[x])
27         if(y != fa[x] && y != son[x])
28             DFS(y, y);
29 }
30 void modify(int x, int y, int c)
31 {
32     qv = c;
33     while(top[x] != top[y])
34     {
35         if(dep[top[x]] < dep[top[y]])
36             swap(x, y);
37         ql = pos[top[x]]; qr = pos[x];
38         seg_modify(1, 1, segn);
39         x = fa[top[x]];
40     }
41     if(dep[x] > dep[y]) swap(x, y);
42     ql = pos[x]; qr = pos[y];
43     seg_modify(1, 1, segn);
44 }
45 bool flag_res;
46 void seg_query(int o, int l, int r) {
47     if(ql <= l && r <= qr) {
48         if(flag_res)
49             res = res + val[o];
50         else {
51             res = val[o];

```

```

52     flag_res = true;
53 }
54 return;
55 }
56 pushdown(o, l, r);
57 if(q1 <= mid) seg_query(Lc);
58 if(mid < qr) seg_query(Rc);
59 }
60 int query(int x, int y) // 点操作版
61 // 边操作每个点对应其父边（根除外）
62 {
63     bool flag_pre = false;
64     bool flag_suf = false;
65     while(top[x] != top[y])
66     {
67         flag_res = false;
68         if(dep[top[x]] > dep[top[y]])
69         {
70             q1 = pos[top[x]]; qr = pos[x];
71             seg_query(1, 1, segn);
72             if(flag_pre)
73                 pre = pre + reverse(res);
74             else
75             {
76                 pre = reverse(res);
77                 flag_pre = true;
78             }
79             x = fa[top[x]];
80         }
81         else
82         {
83             q1 = pos[top[y]]; qr = pos[y];
84             seg_query(1, 1, segn);
85             if(flag_suf)
86                 suf = res + suf;
87             else
88             {
89                 suf = res;
90                 flag_suf = true;
91             }
92             y = fa[top[y]];
93         }
94     }
95     flag_res = false;
96     if(dep[x] < dep[y])
97     {
98         q1 = pos[x]; qr = pos[y];
99         seg_query(1, 1, segn);
100    }

```

```

101     else
102     {
103         ql = pos[y]; qr = pos[x];
104         seg_query(1, 1, segn);
105         res = reverse(res);
106     }
107     if(flag_pre) res = pre + res;
108     if(flag_suf) res = res + suf;
109     return res.mx;
110 }

```

3.3.2 lct

```

1  struct LCT
2  {
3      int fa[N], c[N][2], rev[N], sz[N], szi[N]; // sz 表示子树大小, szi 表示虚儿子子树大小
4
5      void update(int o) {
6          sz[o] = sz[c[o][0]] + sz[c[o][1]] + szi[o] + 1;
7      }
8      void pushdown(int o) {
9          if(rev[o]) {
10             rev[o] = 0;
11             rev[c[o][0]] ^= 1;
12             rev[c[o][1]] ^= 1;
13             swap(c[o][0], c[o][1]);
14         }
15     }
16     bool ch(int o) {
17         return o == c[fa[o]][1];
18     }
19     bool isroot(int o) {
20         return c[fa[o]][0] != o && c[fa[o]][1] != o;
21     }
22     void setc(int x, int y, bool d) {
23         if(x) fa[x] = y;
24         if(y) c[y][d] = x;
25     }
26     void rotate(int x) {
27         if(isroot(x)) return;
28         int p = fa[x], d = ch(x);
29         if(isroot(p)) fa[x] = fa[p];
30         else setc(x, fa[p], ch(p));
31         setc(c[x][d^1], p, d);
32         setc(p, x, d^1);
33         update(p);
34         update(x);
35     }
36     void splay(int x) {
37         static int q[N], top;

```

```

38     int y = q[top = 1] = x;
39     while(!isroot(y)) q[++top] = y = fa[y];
40     while(top) pushdown(q[top--]);
41     while(!isroot(x)) {
42         if(!isroot(fa[x]))
43             rotate(ch(fa[x]) == ch(x) ? fa[x] : x);
44         rotate(x);
45     }
46 }
47 void access(int x) {
48     for(int y = 0; x; y = x, x = fa[x]) {
49         splay(x);
50         sz[x] += sz[c[x][1]] - sz[y];
51         c[x][1] = y;
52         update(x);
53     }
54 }
55 void makeroot(int x) {
56     access(x), splay(x), rev[x] ^= 1;
57 }
58 void link(int x, int y) { // 注意 makeroot(y)
59     makeroot(x), makeroot(y), fa[x] = y, sz[x] += sz[y], splay(x);
60 }
61 void cut(int x, int y) {
62     makeroot(x);
63     access(y);
64     splay(y);
65     c[y][0] = fa[x] = 0;
66 }
67 };

```

3.4 RMQ

```

1  for (int i = 1; i <= n; i++)
2      Log[i] = int(log2(i));
3
4  for (int i = 1; i <= n; i++)
5      Rmq[i][0] = i;
6
7  for (int k = 1; (1 << k) <= n; k++)
8      for (int i = 1; i + (1 << k) - 1 <= n; i++){
9          int x = Rmq[i][k - 1], y = Rmq[i + (1 << (k - 1))][k - 1];
10         if (a[x] < a[y])
11             Rmq[i][k] = x;
12         else
13             Rmq[i][k] = y;
14     }
15
16 int Smallest(int l, int r){

```

```

17     int k = Log[r - 1 + 1];
18
19     int x = Rmq[l][k];
20     int y = Rmq[r - (1 << k) + 1][k];
21
22     if (a[x] < a[y]) return x;
23     else return y;
24 }

```

3.5 可持久化线段树

```

1  struct node1 {
2      int L, R, Lson, Rson, Sum;
3  } tree[N * 40];
4  int root[N], a[N], b[N];
5  int tot, n, m;
6  int Real[N];
7  int Same(int x) {
8      ++tot;
9      tree[tot] = tree[x];
10     return tot;
11 }
12 int build(int L, int R) {
13     ++tot;
14     tree[tot].L = L;
15     tree[tot].R = R;
16     tree[tot].Lson = tree[tot].Rson = tree[tot].Sum = 0;
17     if (L == R) return tot;
18     int s = tot;
19     int mid = (L + R) >> 1;
20     tree[s].Lson = build(L, mid);
21     tree[s].Rson = build(mid + 1, R);
22     return s;
23 }
24 int Ask(int Lst, int Cur, int L, int R, int k) {
25     if (L == R) return L;
26     int Mid = (L + R) >> 1;
27     int Left = tree[tree[Cur].Lson].Sum - tree[tree[Lst].Lson].Sum;
28     if (Left >= k) return Ask(tree[Lst].Lson, tree[Cur].Lson, L, Mid, k);
29     k -= Left;
30     return Ask(tree[Lst].Rson, tree[Cur].Rson, Mid + 1, R, k);
31 }
32 int Add(int Lst, int pos) {
33     int root = Same(Lst);
34     tree[root].Sum++;
35     if (tree[root].L == tree[root].R) return root;
36     int mid = (tree[root].L + tree[root].R) >> 1;
37     if (pos <= mid) tree[root].Lson = Add(tree[root].Lson, pos);
38     else tree[root].Rson = Add(tree[root].Rson, pos);

```



```

39     return root;
40 }
41 int main() {
42     scanf("%d%d", &n, &m);
43     int up = 0;
44     for (int i = 1; i <= n; i++){
45         scanf("%d", &a[i]);
46         b[i] = a[i];
47     }
48     sort(b + 1, b + n + 1);
49     up = unique(b + 1, b + n + 1) - b - 1;
50     for (int i = 1; i <= n; i++){
51         int tmp = lower_bound(b + 1, b + up + 1, a[i]) - b;
52         Real[tmp] = a[i];
53         a[i] = tmp;
54     }
55     tot = 0;
56     root[0] = build(1, up);
57     for (int i = 1; i <= n; i++){
58         root[i] = Add(root[i - 1], a[i]);
59     }
60     for (int i = 1; i <= m; i++){
61         int u, v, w;
62         scanf("%d%d%d", &u, &v, &w);
63         printf("%d\n", Real[Ask(root[u - 1], root[v], 1, up, w)]);
64     }
65     return 0;
66 }

```

3.6 可持久化 Trie

```

1  int Pre[N];
2  int n, q, Len, cnt, Lstans;
3  char s[N];
4  int First[N], Last[N];
5  int Root[N];
6  int Trie_tot;
7  struct node{
8      int To[30];
9      int Lst;
10 }Trie[N];
11 int tot;
12 struct node1{
13     int L, R, Lson, Rson, Sum;
14 }tree[N * 25];
15 int Build(int L, int R){
16     ++tot;
17     tree[tot].L = L;
18     tree[tot].R = R;

```

```

19     tree[tot].Lson = tree[tot].Rson = tree[tot].Sum = 0;
20     if (L == R) return tot;
21     int s = tot;
22     int mid = (L + R) >> 1;
23     tree[s].Lson = Build(L, mid);
24     tree[s].Rson = Build(mid + 1, R);
25     return s;
26 }
27 int Same(int x){
28     ++tot;
29     tree[tot] = tree[x];
30     return tot;
31 }
32 int Add(int Lst, int pos){
33     int s = Same(Lst);
34     tree[s].Sum++;
35     if (tree[s].L == tree[s].R) return s;
36     int Mid = (tree[s].L + tree[s].R) >> 1;
37     if (pos <= Mid) tree[s].Lson = Add(tree[Lst].Lson, pos);
38     else tree[s].Rson = Add(tree[Lst].Rson, pos);
39     return s;
40 }
41
42 int Ask(int Lst, int Cur, int L, int R, int pos){
43     if (L >= pos) return 0;
44     if (R < pos) return tree[Cur].Sum - tree[Lst].Sum;
45     int Mid = (L + R) >> 1;
46     int Ret = Ask(tree[Lst].Lson, tree[Cur].Lson, L, Mid, pos);
47     Ret += Ask(tree[Lst].Rson, tree[Cur].Rson, Mid + 1, R, pos);
48     return Ret;
49 }
50
51 int main(){
52     while (scanf("%d", &n) == 1){
53         for (int i = 1; i <= Trie_tot; i++){
54             for (int j = 1; j <= 26; j++){
55                 Trie[i].To[j] = 0;
56                 Trie[i].Lst = 0;
57             }
58             Trie_tot = 1;
59             cnt = 0;
60             for (int ii = 1; ii <= n; ii++){
61                 scanf("%s", s + 1);
62                 Len = strlen(s + 1);
63                 int Cur = 1;
64                 First[ii] = cnt + 1;
65                 for (int i = 1; i <= Len; i++){
66                     int ch = s[i] - 'a' + 1;
67                     if (Trie[Cur].To[ch] == 0){

```

```

68         ++Trie_tot;
69         Trie[Cur].To[ch] = Trie_tot;
70     }
71     Cur = Trie[Cur].To[ch];
72     Pre[++cnt] = Trie[Cur].Lst;
73     Trie[Cur].Lst = ii;
74 }
75 Last[ii] = cnt;
76 }
77 tot = 0;
78 Root[0] = Build(0, n);
79 for (int i = 1; i <= cnt; i++){
80     Root[i] = Add(Root[i - 1], Pre[i]);
81 }
82 Lstans = 0;
83 scanf("%d", &q);
84 for (int ii = 1; ii <= q; ii++){
85     int L, R;
86     scanf("%d%d", &L, &R);
87     L = (L + Lstans) % n + 1;
88     R = (R + Lstans) % n + 1;
89     if (L > R) swap(L, R);
90     int Ret = Ask(Root[First[L] - 1], Root[Last[R]], 0, n, L);
91     printf("%d\n", Ret);
92     Lstans = Ret;
93 }
94 }
95 return 0;
96 }

```

3.7 k-d 树

```

1  struct Point{
2      int data[MAXK], id;
3  }p[MAXN];
4
5  struct KdNode{
6      int l, r;
7      Point p, dmin, dmax;
8      KdNode() {}
9      KdNode(const Point &rhs) : l(0), r(0), p(rhs), dmin(rhs), dmax(rhs) {}
10     inline void merge(const KdNode &rhs) {
11         for (register int i = 0; i < k; i++) {
12             dmin.data[i] = std::min(dmin.data[i], rhs.dmin.data[i]);
13             dmax.data[i] = std::max(dmax.data[i], rhs.dmax.data[i]);
14         }
15     }
16     inline long long getMinDist(const Point &rhs) const {
17         register long long ret = 0;

```

```

18     for (register int i = 0; i < k; i++) {
19         if (dmin.data[i] <= rhs.data[i] && rhs.data[i] <= dmax.data[i]) continue;
20         ret += std::min(1ll * (dmin.data[i] - rhs.data[i]) * (dmin.data[i] - rhs.data[i]),
21             1ll * (dmax.data[i] - rhs.data[i]) * (dmax.data[i] - rhs.data[i]));
22     }
23     return ret;
24 }
25 long long getMaxDist(const Point &rhs) {
26     long long ret = 0;
27     for (register int i = 0; i < k; i++) {
28         int tmp = std::max(std::abs(dmin.data[i] - rhs.data[i]),
29             std::abs(dmax.data[i] - rhs.data[i]));
30         ret += 1ll * tmp * tmp;
31     }
32     return ret;
33 }
34 }tree[MAXN * 4];
35
36 struct Result{
37     long long dist;
38     Point d;
39     Result() {}
40     Result(const long long &dist, const Point &d) : dist(dist), d(d) {}
41     bool operator >(const Result &rhs) const {
42         return dist > rhs.dist || (dist == rhs.dist && d.id < rhs.d.id);
43     }
44     bool operator <(const Result &rhs) const {
45         return dist < rhs.dist || (dist == rhs.dist && d.id > rhs.d.id);
46     }
47 };
48
49 inline long long sqrdist(const Point &a, const Point &b) {
50     register long long ret = 0;
51     for (register int i = 0; i < k; i++) {
52         ret += 1ll * (a.data[i] - b.data[i]) * (a.data[i] - b.data[i]);
53     }
54     return ret;
55 }
56
57 inline int alloc() {
58     size++;
59     tree[size].l = tree[size].r = 0;
60     return size;
61 }
62
63 void build(const int &depth, int &rt, const int &l, const int &r) {
64     if (l > r) return;
65     register int middle = l + r >> 1;
66     std::nth_element(p + l, p + middle, p + r + 1,

```

```

67     [=](const Point &a, const Point &b){return a.data[depth] < b.data[depth];};
68     tree[rt = alloc()] = KdNode(p[middle]);
69     if (l == r) return;
70     build((depth + 1) % k, tree[rt].l, l, middle - 1);
71     build((depth + 1) % k, tree[rt].r, middle + 1, r);
72     if (tree[rt].l) tree[rt].merge(tree[tree[rt].l]);
73     if (tree[rt].r) tree[rt].merge(tree[tree[rt].r]);
74 }
75
76 std::priority_queue<Result, std::vector<Result>, std::greater<Result> > heap;
77
78 void getMinKth(const int &depth, const int &rt, const int &m, const Point &d) {
79     ↪ // 求 K 近点
80     Result tmp = Result(sqrdist(tree[rt].p, d), tree[rt].p);
81     if ((int)heap.size() < m) {
82         heap.push(tmp);
83     } else if (tmp < heap.top()) {
84         heap.pop();
85         heap.push(tmp);
86     }
87     int x = tree[rt].l, y = tree[rt].r;
88     if (x != 0 && y != 0 && sqrdist(d, tree[x].p) > sqrdist(d, tree[y].p)) std::swap(x, y);
89     if (x != 0 && ((int)heap.size() < m || tree[x].getMinDist(d) < heap.top().dist)) {
90         getMinKth((depth + 1) % k, x, m, d);
91     }
92     if (y != 0 && ((int)heap.size() < m || tree[y].getMinDist(d) < heap.top().dist)) {
93         getMinKth((depth + 1) % k, y, m, d);
94     }
95 }
96
97 void getMaxKth(const int &depth, const int &rt, const int &m, const Point &d) {
98     ↪ // 求 K 远点
99     Result tmp = Result(sqrdist(tree[rt].p, d), tree[rt].p);
100    if ((int)heap.size() < m) {
101        heap.push(tmp);
102    } else if (tmp > heap.top()) {
103        heap.pop();
104        heap.push(tmp);
105    }
106    int x = tree[rt].l, y = tree[rt].r;
107    if (x != 0 && y != 0 && sqrdist(d, tree[x].p) < sqrdist(d, tree[y].p)) std::swap(x, y);
108    if (x != 0 && ((int)heap.size() < m || tree[x].getMaxDist(d) >= heap.top().dist)) {
109        ↪ // 这里的 >= 是因为在距离相等的时候需要按照 id 排序
110        getMaxKth((depth + 1) % k, x, m, d);
111    }
112    if (y != 0 && ((int)heap.size() < m || tree[y].getMaxDist(d) >= heap.top().dist)) {
113        getMaxKth((depth + 1) % k, y, m, d);
114    }
115 }

```

3.8 莫队算法

```

1  // uva12345
2  // single testcase
3  #include <bits/stdc++.h>
4
5  using namespace std;
6
7  const int N = 50005;
8  const int MAXV = 1e6 + 5;
9
10 int n, m;
11 int a[N];
12 int bid[N];
13 int qn, modn;
14
15 struct Query
16 {
17     int l, r, last, i;
18
19     friend bool operator < (const Query &a, const Query &b)
20     {
21         if(bid[a.l] != bid[b.l])
22             return bid[a.l] < bid[b.l];
23         if(bid[a.r] != bid[b.r])
24             return bid[a.r] < bid[b.r];
25         return a.last < b.last;
26     }
27 } q[N];
28
29 struct Modify
30 {
31     int x, to, from;
32 } mod[N];
33
34 void input()
35 {
36     scanf("%d%d", &n, &m);
37     for(int i = 1; i <= n; i++)
38         scanf("%d", &a[i]);
39     for(int i = 1; i <= m; i++)
40     {
41         int x, y;
42         static char op[2];
43         scanf("%s%d%d", op, &x, &y);
44         x++;
45         if(op[0] == 'Q')
46         {
47             ++qn;
48             q[qn] = (Query){x, y, modn, qn};

```

```

49     }
50     else if(op[0] == 'M')
51     {
52         mod[++modn] = (Modify){x, y, a[x]};
53         a[x] = y;
54     }
55     else assert(false);
56 }
57 int BS = (int)pow(n + 0.5, 2.0 / 3.0);
58 for(int i = 1; i <= n; i++)
59     bid[i] = (i + BS - 1) / BS;
60 }
61
62 int ans[N];
63 int curans;
64 int cnt[MAXV];
65 bool state[N];
66
67 void modify(int x)
68 {
69     int &c = cnt[a[x]];
70     curans -= !!c;
71     c += (state[x] ^= 1) ? 1 : -1;
72     curans += !!c;
73 }
74
75 void change(int i, bool type)
76 {
77     int x = mod[i].x;
78     int newv = type ? mod[i].to : mod[i].from;
79     if(state[x])
80     {
81         modify(x);
82         a[x] = newv;
83         modify(x);
84     }
85     else
86     {
87         a[x] = newv;
88     }
89 }
90
91 void solve()
92 {
93     sort(q + 1, q + qn + 1);
94     int l = 1, r = 0, now = modn;
95     for(int i = 1; i <= qn; i++)
96     {
97         while(q[i].last < now) change(now--, 0);

```

```

98     while(q[i].last > now) change(++now, 1);
99     while(q[i].l < 1) modify(--l);
100    while(q[i].l > 1) modify(l++);
101    while(q[i].r < r) modify(r--);
102    while(q[i].r > r) modify(++r);
103    ans[q[i].i] = curans;
104 }
105 for(int i = 1; i <= qn; i++)
106     printf("%d\n", ans[i]);
107 }
108
109 int main()
110 {
111     input();
112     solve();
113     return 0;
114 }

```

3.9 树上莫队

```

1  #include <bits/stdc++.h>
2
3  using namespace std;
4
5  const int N = 40005;
6  const int M = 100005;
7  const int LOGN = 17;
8
9  int n, m;
10 int w[N];
11 vector<int> g[N];
12 int bid[N << 1];
13
14 struct Query
15 {
16     int l, r, extra, i;
17     friend bool operator < (const Query &a, const Query &b)
18     {
19         if(bid[a.l] != bid[b.l])
20             return bid[a.l] < bid[b.l];
21         return a.r < b.r;
22     }
23 } q[M];
24
25 void input()
26 {
27     vector<int> vs;
28     scanf("%d%d", &n, &m);
29     for(int i = 1; i <= n; i++)

```



```

30 {
31     scanf("%d", &w[i]);
32     vs.push_back(w[i]);
33 }
34 sort(vs.begin(), vs.end());
35 vs.resize(unique(vs.begin(), vs.end()) - vs.begin());
36 for(int i = 1; i <= n; i++)
37     w[i] = lower_bound(vs.begin(), vs.end(), w[i]) - vs.begin() + 1;
38 for(int i = 2; i <= n; i++)
39 {
40     int a, b;
41     scanf("%d%d", &a, &b);
42     g[a].push_back(b);
43     g[b].push_back(a);
44 }
45 for(int i = 1; i <= m; i++)
46 {
47     scanf("%d%d", &q[i].l, &q[i].r);
48     q[i].i = i;
49 }
50 }
51
52 int dfs_clock;
53 int st[N], ed[N];
54 int fa[N][LOGN], dep[N];
55 int col[N << 1];
56 int id[N << 1];
57
58 void dfs(int x, int p)
59 {
60     col[st[x] = ++dfs_clock] = w[x];
61     id[st[x]] = x;
62     fa[x][0] = p; dep[x] = dep[p] + 1;
63     for(int i = 0; fa[x][i]; i++)
64         fa[x][i + 1] = fa[fa[x][i]][i];
65     for(auto y: g[x])
66         if(y != p)
67             dfs(y, x);
68     col[ed[x] = ++dfs_clock] = w[x];
69     id[ed[x]] = x;
70 }
71
72 int lca(int x, int y)
73 {
74     if(dep[x] < dep[y]) swap(x, y);
75     for(int i = LOGN - 1; i >= 0; i--)
76         if(dep[fa[x][i]] >= dep[y])
77             x = fa[x][i];
78     if(x == y) return x;

```

```

79     for(int i = LOGN - 1; i >= 0; i--)
80         if(fa[x][i] != fa[y][i])
81             x = fa[x][i], y = fa[y][i];
82     return fa[x][0];
83 }
84
85 void prepare()
86 {
87     dfs_clock = 0;
88     dfs(1, 0);
89     int BS = (int)sqrt(dfs_clock + 0.5);
90     for(int i = 1; i <= dfs_clock; i++)
91         bid[i] = (i + BS - 1) / BS;
92     for(int i = 1; i <= m; i++)
93     {
94         int a = q[i].l;
95         int b = q[i].r;
96         int c = lca(a, b);
97         if(st[a] > st[b]) swap(a, b);
98         if(c == a)
99         {
100             q[i].l = st[a];
101             q[i].r = st[b];
102             q[i].extra = 0;
103         }
104         else
105         {
106             q[i].l = ed[a];
107             q[i].r = st[b];
108             q[i].extra = c;
109         }
110     }
111     sort(q + 1, q + m + 1);
112 }
113
114 int curans;
115 int ans[M];
116 int cnt[N];
117 bool state[N];
118
119 void rev(int x)
120 {
121     int &c = cnt[col[x]];
122     curans -= !!c;
123     c += (state[id[x]] ^ 1) ? 1 : -1;
124     curans += !!c;
125 }
126
127 void solve()

```

```

128 {
129     prepare();
130     curans = 0;
131     memset(cnt, 0, sizeof(cnt));
132     memset(state, 0, sizeof(state));
133
134     int l = 1, r = 0;
135     for(int i = 1; i <= m; i++)
136     {
137         while(l < q[i].l) rev(l++);
138         while(l > q[i].l) rev(--l);
139         while(r < q[i].r) rev(++r);
140         while(r > q[i].r) rev(r--);
141         if(q[i].extra) rev(st[q[i].extra]);
142         ans[q[i].i] = curans;
143         if(q[i].extra) rev(st[q[i].extra]);
144     }
145     for(int i = 1; i <= m; i++)
146         printf("%d\n", ans[i]);
147 }
148
149 int main()
150 {
151     input();
152     solve();
153     return 0;
154 }

```

3.10 树状数组 kth

```

1 int find(int k){
2     int cnt=0,ans=0;
3     for(int i=22;i>=0;i--){
4         ans+=(1<<i);
5         if(ans>n || cnt+d[ans]>=k)ans-=(1<<i);
6         else cnt+=d[ans];
7     }
8     return ans+1;
9 }

```

3.11 虚树

```

1 int a[maxn*2],sta[maxn*2];
2 int top=0,k;
3 void build(){
4     top=0;
5     sort(a,a+k,bydfn);
6     k=unique(a,a+k)-a;
7     sta[top++]=1;_n=k;

```

```

8     for(int i=0;i<k;i++){
9         int LCA=lca(a[i],sta[top-1]);
10        while(dep[LCA]<dep[sta[top-1]]){
11            if(dep[LCA]>=dep[sta[top-2]]){
12                add_edge(LCA,sta[--top]);
13                if(sta[top-1]!=LCA)sta[top++]=LCA;
14                break;
15            }add_edge(sta[top-2],sta[top-1]);top--;
16        }if(sta[top-1]!=a[i])sta[top++]=a[i];
17    }
18    while(top>1)
19        add_edge(sta[top-2],sta[top-1]),top--;
20    for(int i=0;i<k;i++)inr[a[i]]=1;
21 }

```

3.12 点分治 (zky)

```

1  int siz[maxn],f[maxn],dep[maxn],cant[maxn],root,All,d[maxn];
2  void makert(int u,int fa){
3      siz[u]=1;f[u]=0;
4      for(int i=0;i<G[u].size();i++){
5          edge e=G[u][i];
6          if(e.v!=fa&&!cant[e.v]){
7              dep[e.v]=dep[u]+1;
8              makert(e.v,u);
9              siz[u]+=siz[e.v];
10             f[u]=max(f[u],siz[e.v]);
11         }
12     }f[u]=max(f[u],All-f[u]);
13     if(f[root]>f[u])root=u;
14 }
15 void dfs(int u,int fa){
16     //Gain data
17     for(int i=0;i<G[u].size();i++){
18         edge e=G[u][i];
19         if(e.v==fa||cant[e.v])continue;
20         d[e.v]=d[u]+e.w;
21         dfs(e.v,u);
22     }
23 }
24 void calc(int u){
25     d[u]=0;
26     for(int i=0;i<G[u].size();i++){
27         edge e=G[u][i];
28         if(cant[e.v])continue;
29         d[e.v]=e.w;
30         dfs(e.v,u);
31     }
32 }

```

```

33 }
34 void solve(int u){
35     calc(u);cant[u]=1;
36     for(int i=0;i<G[u].size();i++){
37         edge e=G[u][i];
38         if(cant[e.v])continue;
39         All=siz[e.v];
40         f[root=0]=n+1;
41         makert(e.v,0);
42         solve(root);
43     }
44 }
45 All=n
46 f[root=0]=n+1;
47 makert(1,1);
48 solve(root);

```

4 图论

4.1 点双连通分量

```

1 //边的编号从 2 开始
2 void Targan(int x, int last){
3     int ch = 0;
4     dfn[x] = low[x] = ++dfsclock;
5     for(int i = he[x], v; i; i = e[i].nx){
6         if(i == last)continue;
7         v = e[i].v;
8         if(!dfn[v]){
9             sta[++stm] = e[i], ch++;
10            Targan(v, i ^ 1);
11            low[x] = min(low[x], low[v]);
12            if(low[v] >= dfn[x]){
13                iscut[x] = 1;
14                bcc[++bccnt].clear();
15                while(1){
16                    Edge t = sta[stm--];
17                    if(bccno[t.u] != bccnt)bcc[bccnt].push_back(t.u), bccno[t.u] = bccnt;
18                    if(bccno[t.v] != bccnt)bcc[bccnt].push_back(t.v), bccno[t.v] = bccnt;
19                    if(t.u == u && t.v == v)break;
20                }
21            }
22        }
23        else if(dfn[v] < dfn[x]){
24            sta[++stm] = e[i];
25            low[x] = min(low[x], dfn[v]);
26        }
27    }
28    if(last == 0 && ch == 1)iscut[u] = 0;

```

```
29 }
```

4.2 点双连通分量 (1yx)

```
1  #define SZ(x) ((int)x.size())
2
3  const int N = 400005; // N 开 2 倍点数, 因为新树会加入最多 n 个新点
4  const int M = 200005;
5
6  vector<int> g[N];
7
8  int bccno[N], bcc_cnt;
9  vector<int> bcc[N];
10 bool iscut[N];
11
12 struct Edge {
13     int u, v;
14 } stk[M << 2];
15 int top; // 注意栈大小为边数 4 倍
16 int dfn[N], low[N], dfs_clock;
17
18 void dfs(int x, int fa)
19 {
20     low[x] = dfn[x] = ++dfs_clock;
21     int child = 0;
22     for(int i = 0; i < SZ(g[x]); i++) {
23         int y = g[x][i];
24         if(!dfn[y]) {
25             child++;
26             stk[++top] = (Edge){x, y};
27             dfs(y, x);
28             low[x] = min(low[x], low[y]);
29             if(low[y] >= dfn[x]) {
30                 iscut[x] = true;
31                 bcc[++bcc_cnt].clear();
32                 for(;;) {
33                     Edge e = stk[top--];
34                     if(bccno[e.u] != bcc_cnt) { bcc[bcc_cnt].push_back(e.u); bccno[e.u] = bcc_cnt; }
35                     if(bccno[e.v] != bcc_cnt) { bcc[bcc_cnt].push_back(e.v); bccno[e.v] = bcc_cnt; }
36                     if(e.u == x && e.v == y) break;
37                 }
38             }
39         } else if(y != fa && dfn[y] < dfn[x]) {
40             stk[++top] = (Edge){x, y};
41             low[x] = min(low[x], dfn[y]);
42         }
43     }
44     if(fa == 0 && child == 1) iscut[x] = false;
45 }
```

```

46
47 void find_bcc() // 求点双联通分量, 需要时手动 1 到 n 清空, 1-based
48 {
49     memset(dfn, 0, sizeof(dfn));
50     memset(iscut, 0, sizeof(iscut));
51     memset(bccno, 0, sizeof(bccno));
52     dfs_clock = bcc_cnt = 0;
53     for(int i = 1; i <= n; i++)
54         if(!dfn[i])
55             dfs(i, 0);
56 }
57
58 vector<int> G[N];
59
60 void prepare() { // 建出缩点后的树
61     for(int i = 1; i <= n + bcc_cnt; i++)
62         G[i].clear();
63     for(int i = 1; i <= bcc_cnt; i++) {
64         int x = i + n;
65         for(int j = 0; j < SZ(bcc[i]); j++) {
66             int y = bcc[i][j];
67             G[x].push_back(y);
68             G[y].push_back(x);
69         }
70     }
71 }

```

4.3 Hungary 求最大匹配

```

1  int n, m, stamp;
2  int match[N], visit[N];
3
4  bool dfs(int x) {
5      for (int i = 0; i < (int)edge[x].size(); ++i) {
6          int y = edge[x][i];
7          if (visit[y] != stamp) {
8              visit[y] = stamp;
9              if (match[y] == -1 || dfs(match[y])) {
10                 match[y] = x;
11                 return true;
12             }
13         }
14     }
15     return false;
16 }
17
18 int solve() {
19     std::fill(match, match + m, -1);
20     int answer = 0;

```

```

21     for (int i = 0; i < n; ++i) {
22         stamp++;
23         answer += dfs(i);
24     }
25     return answer;
26 }

```

4.4 Hopcroft-Karp 求最大匹配

```

1  int matchx[N], matchy[N], level[N];
2
3  bool dfs(int x) {
4      for (int i = 0; i < (int)edge[x].size(); ++i) {
5          int y = edge[x][i];
6          int w = matchy[y];
7          if (w == -1 || level[x] + 1 == level[w] && dfs(w)) {
8              matchx[x] = y;
9              matchy[y] = x;
10             return true;
11         }
12     }
13     level[x] = -1;
14     return false;
15 }
16
17 int solve() {
18     std::fill(matchx, matchx + n, -1);
19     std::fill(matchy, matchy + m, -1);
20     for (int answer = 0; ; ) {
21         std::vector<int> queue;
22         for (int i = 0; i < n; ++i) {
23             if (matchx[i] == -1) {
24                 level[i] = 0;
25                 queue.push_back(i);
26             } else {
27                 level[i] = -1;
28             }
29         }
30         for (int head = 0; head < (int)queue.size(); ++head) {
31             int x = queue[head];
32             for (int i = 0; i < (int)edge[x].size(); ++i) {
33                 int y = edge[x][i];
34                 int w = matchy[y];
35                 if (w != -1 && level[w] < 0) {
36                     level[w] = level[x] + 1;
37                     queue.push_back(w);
38                 }
39             }
40         }

```



```

41     int delta = 0;
42     for (int i = 0; i < n; ++i) {
43         if (matchx[i] == -1 && dfs(i)) {
44             delta++;
45         }
46     }
47     if (delta == 0) {
48         return answer;
49     } else {
50         answer += delta;
51     }
52 }
53 }

```

4.5 KM 带权匹配

注意事项：最小权完美匹配，复杂度为 $\mathcal{O}(|V|^3)$ 。

```

1  int DFS(int x){
2      visx[x] = 1;
3      for (int y = 1; y <= ny; y++){
4          if (visy[y]) continue;
5          int t = lx[x] + ly[y] - w[x][y];
6          if (t == 0) {
7              visy[y] = 1;
8              if (link[y] == -1 || DFS(link[y])){
9                  link[y] = x;
10                 return 1;
11             }
12         }
13         else slack[y] = min(slack[y], t);
14     }
15     return 0;
16 }
17 int KM(){
18     int i, j;
19     memset(link, -1, sizeof(link));
20     memset(ly, 0, sizeof(ly));
21     for (i = 1; i <= nx; i++){
22         for (j = 1, lx[i] = -inf; j <= ny; j++){
23             lx[i] = max(lx[i], w[i][j]);
24         }
25         for (int x = 1; x <= nx; x++){
26             for (i = 1; i <= ny; i++) slack[i] = inf;
27             while (true) {
28                 memset(visx, 0, sizeof(visx));
29                 memset(visy, 0, sizeof(visy));
30                 if (DFS(x)) break;
31                 int d = inf;
32                 for (i = 1; i <= ny; i++){
33                     if (!visy[i] && d > slack[i]) d = slack[i];

```

```

33         for (i = 1; i <= nx; i++)
34             if (visx[i]) lx[i] -= d;
35         for (i = 1; i <= ny; i++)
36             if (visy[i]) ly[i] += d;
37             else slack[i] -= d;
38     }
39 }
40 int res = 0;
41 for (i = 1; i <= ny; i++)
42     if (link[i] > -1) res += w[link[i]][i];
43 return res;
44 }

```

4.6 稀疏图最大流

注意事项：适用于比较稀疏的一般图。

```

1  int Maxflow_Isap(int s, int t, int n) {
2      std::fill(pre + 1, pre + n + 1, 0);
3      std::fill(d + 1, d + n + 1, 0);
4      std::fill(gap + 1, gap + n + 1, 0);
5      for (int i = 1; i <= n; i++) cur[i] = h[i];
6      gap[0] = n;
7      int u = pre[s] = s, v, maxflow = 0;
8      while (d[s] < n) {
9          v = n + 1;
10         for (int i = cur[u]; i; i = e[i].next)
11             if (e[i].flow && d[u] == d[e[i].node] + 1) {
12                 v = e[i].node; cur[u] = i; break;
13             }
14         if (v <= n) {
15             pre[v] = u; u = v;
16             if (v == t) {
17                 int dflow = INF, p = t; u = s;
18                 while (p != s) {
19                     p = pre[p];
20                     dflow = std::min(dflow, e[cur[p]].flow);
21                 }
22                 maxflow += dflow; p = t;
23                 while (p != s) {
24                     p = pre[p];
25                     e[cur[p]].flow -= dflow;
26                     e[e[cur[p]].opp].flow += dflow;
27                 }
28             }
29         }
30         else {
31             int mindist = n + 1;
32             for (int i = h[u]; i; i = e[i].next)
33                 if (e[i].flow && mindist > d[e[i].node]) {

```

```

34         mindist = d[e[i].node]; cur[u] = i;
35     }
36     if (!--gap[d[u]]) return maxflow;
37     gap[d[u] = mindist + 1]++; u = pre[u];
38 }
39 }
40 return maxflow;
41 }

```

4.7 稀疏图费用流

```

1  struct EdgeList {
2      int size;
3      int last[N];
4      int succ[M], other[M], flow[M], cost[M];
5      void clear(int n) {
6          size = 0;
7          std::fill(last, last + n, -1);
8      }
9      void add(int x, int y, int c, int w) {
10         succ[size] = last[x];
11         last[x] = size;
12         other[size] = y;
13         flow[size] = c;
14         cost[size++] = w;
15     }
16 } e;
17
18 int n, source, target;
19 int prev[N];
20
21 void add(int x, int y, int c, int w) {
22     e.add(x, y, c, w);
23     e.add(y, x, 0, -w);
24 }
25
26 bool augment() {
27     static int dist[N], occur[N];
28     std::vector<int> queue;
29     std::fill(dist, dist + n, INT_MAX);
30     std::fill(occur, occur + n, 0);
31     dist[source] = 0;
32     occur[source] = true;
33     queue.push_back(source);
34     for (int head = 0; head < (int)queue.size(); ++head) {
35         int x = queue[head];
36         for (int i = e.last[x]; ~i; i = e.succ[i]) {
37             int y = e.other[i];
38             if (e.flow[i] && dist[y] > dist[x] + e.cost[i]) {

```

```

39         dist[y] = dist[x] + e.cost[i];
40         prev[y] = i;
41         if (!occur[y]) {
42             occur[y] = true;
43             queue.push_back(y);
44         }
45     }
46 }
47 occur[x] = false;
48 }
49 return dist[target] < INT_MAX;
50 }
51
52 std::pair<int, int> solve() {
53     std::pair<int, int> answer = std::make_pair(0, 0);
54     while (augment()) {
55         int number = INT_MAX;
56         for (int i = target; i != source; i = e.other[prev[i] ^ 1]) {
57             number = std::min(number, e.flow[prev[i]]);
58         }
59         answer.first += number;
60         for (int i = target; i != source; i = e.other[prev[i] ^ 1]) {
61             e.flow[prev[i]] -= number;
62             e.flow[prev[i] ^ 1] += number;
63             answer.second += number * e.cost[prev[i]];
64         }
65     }
66     return answer;
67 }

```

4.8 稠密图费用流

```

1 namespace zkw{
2     struct eglist{
3         int other[maxM], succ[maxM], last[maxM], cap[maxM], cost[maxM], sum;
4         void clear() {
5             memset(last, -1, sizeof last);
6             sum = 0;
7         }
8         void _addEdge(int a,int b,int c,int d) {
9             other[sum] = b, succ[sum] = last[a], last[a] = sum, cost[sum] = d, cap[sum++] = c;
10        }
11        void addEdge(int a,int b,int c,int d) {
12            _addEdge(a, b, c, d);
13            _addEdge(b, a, 0, -d);
14        }
15    }e;
16
17    int n, m, S, T, tot, totFlow, totCost;

```

```

18  int dis[maxN], slack[maxN], visit[maxN], cur[maxN];
19
20  int modlable() {
21      int delta = inf;
22      for (int i = 1; i <= T; ++i) {
23          if (!visit[i] && slack[i] < delta)
24              delta = slack[i];
25          slack[i] = inf;
26          // cur[i] = e.last[i];
27      }
28      if (delta == inf)
29          return 1;
30      for (int i = 1; i <= T; ++i)
31          if (visit[i]) dis[i] += delta;
32      return 0;
33  }
34
35  int dfs(int x, int flow) {
36      if (x == T) {
37          totFlow += flow;
38          totCost += flow * (dis[S] - dis[T]);
39          return flow;
40      }
41      visit[x] = 1;
42      int left = flow;
43      for (int i = e.last[x]; ~i; i = e.succ[i])
44          if (e.cap[i] > 0 && !visit[e.other[i]]) {
45              int y = e.other[i];
46              if (dis[y] + e.cost[i] == dis[x]) {
47                  int delta = dfs(y, std::min(left, e.cap[i]));
48                  e.cap[i] -= delta;
49                  e.cap[i ^ 1] += delta;
50                  left -= delta;
51                  if (!left) {visit[x] = 0; return flow;}
52              } else {
53                  slack[y] = std::min(slack[y], dis[y] + e.cost[i] - dis[x]);
54              }
55          }
56      return flow - left;
57  }
58
59  std::pair<int, int> minC() {
60      totFlow = totCost = 0;
61      std::fill(dis + 1, dis + T + 1, 0);
62      for (int i = 1; i <= T; ++i) cur[i] = e.last[i];
63      do {
64          do {
65              std::fill(visit + 1, visit + T + 1, 0);
66              while(dfs(S, inf));

```

```

67     }while(!modlable());
68     return std::make_pair(totFlow, totCost);
69 }
70 }

```

4.9 2-SAT 问题

```

1  int stamp, comps, top;
2  int dfn[N], low[N], comp[N], stack[N];
3
4  void add(int x, int a, int y, int b) {
5      edge[x << 1 | a].push_back(y << 1 | b);
6  }
7
8  void tarjan(int x) {
9      dfn[x] = low[x] = ++stamp;
10     stack[top++] = x;
11     for (int i = 0; i < (int)edge[x].size(); ++i) {
12         int y = edge[x][i];
13         if (!dfn[y]) {
14             tarjan(y);
15             low[x] = std::min(low[x], low[y]);
16         } else if (!comp[y]) {
17             low[x] = std::min(low[x], dfn[y]);
18         }
19     }
20     if (low[x] == dfn[x]) {
21         comps++;
22         do {
23             int y = stack[--top];
24             comp[y] = comps;
25         } while (stack[top] != x);
26     }
27 }
28
29 bool solve() {
30     int counter = n + n + 1;
31     stamp = top = comps = 0;
32     std::fill(dfn, dfn + counter, 0);
33     std::fill(comp, comp + counter, 0);
34     for (int i = 0; i < counter; ++i) {
35         if (!dfn[i]) {
36             tarjan(i);
37         }
38     }
39     for (int i = 0; i < n; ++i) {
40         if (comp[i << 1] == comp[i << 1 | 1]) {
41             return false;
42         }

```

```

43         answer[i] = (comp[i << 1 | 1] < comp[i << 1]);
44     }
45     return true;
46 }

```

4.10 有根树的同构

```

1  const unsigned long long MAGIC = 4423;
2
3  unsigned long long magic[N];
4  std::pair<unsigned long long, int> hash[N];
5
6  void solve(int root) {
7      magic[0] = 1;
8      for (int i = 1; i <= n; ++i) {
9          magic[i] = magic[i - 1] * MAGIC;
10     }
11     std::vector<int> queue;
12     queue.push_back(root);
13     for (int head = 0; head < (int)queue.size(); ++head) {
14         int x = queue[head];
15         for (int i = 0; i < (int)son[x].size(); ++i) {
16             int y = son[x][i];
17             queue.push_back(y);
18         }
19     }
20     for (int index = n - 1; index >= 0; --index) {
21         int x = queue[index];
22         hash[x] = std::make_pair(0, 0);
23
24         std::vector<std::pair<unsigned long long, int> > value;
25         for (int i = 0; i < (int)son[x].size(); ++i) {
26             int y = son[x][i];
27             value.push_back(hash[y]);
28         }
29         std::sort(value.begin(), value.end());
30
31         hash[x].first = hash[x].first * magic[1] + 37;
32         hash[x].second++;
33         for (int i = 0; i < (int)value.size(); ++i) {
34             hash[x].first = hash[x].first * magic[value[i].second] + value[i].first;
35             hash[x].second += value[i].second;
36         }
37         hash[x].first = hash[x].first * magic[1] + 41;
38         hash[x].second++;
39     }
40 }

```

4.11 Dominator Tree

```

1  #include <bits/stdc++.h>
2  using namespace std;
3
4  const int MAXN = 50101;
5  const int MAXM = 110101;
6
7  class Edge
8  {public:
9      int size;
10     int begin[MAXN], dest[MAXM], next[MAXM];
11     void clear(int n)
12     {
13         size = 0;
14         fill(begin, begin + n, -1);
15     }
16     Edge(int n = MAXN)
17     {
18         clear(n);
19     }
20     void add_edge(int u, int v)
21     {
22         dest[size] = v;
23         next[size] = begin[u];
24         begin[u] = size++;
25     }
26 };
27
28 class dominator
29 {public:
30     int dfn[MAXN], sdom[MAXN], idom[MAXN], id[MAXN], f[MAXN], fa[MAXN], smin[MAXN], stamp;
31
32     void predfs(int x, const Edge &succ)
33     {
34         id[dfn[x] = stamp++] = x;
35         for(int i = succ.begin[x]; ~i; i = succ.next[i])
36         {
37             int y = succ.dest[i];
38             if(dfn[y] < 0)
39             {
40                 f[y] = x;
41                 predfs(y, succ);
42             }
43         }
44     }
45     int getfa(int x)
46     {
47         if(fa[x] == x)
48             return x;

```



```

49     int ret = getfa(fa[x]);
50     if(dfn[sdom[smin[fa[x]]]] < dfn[sdom[smin[x]]])
51         smin[x] = smin[fa[x]];
52     return fa[x] = ret;
53 }
54 void solve(int s, int n, const Edge &succ)
55 {
56     fill(dfn, dfn + n, -1);
57     fill(idom, idom + n, -1);
58     static Edge pred, tmp;
59     pred.clear(n);
60     for(int i = 0; i < n; ++i)
61         for(int j = succ.begin[i]; ~j; j = succ.next[j])
62             pred.add_edge(succ.dest[j], i);
63     stamp = 0;
64     tmp.clear(n);
65     predfs(s, succ);
66     for(int i = 0; i < stamp; ++i)
67         fa[id[i]] = smin[id[i]] = id[i];
68     for(int o = stamp - 1; o >= 0; --o)
69     {
70         int x = id[o];
71         if(o)
72         {
73             sdom[x] = f[x];
74             for(int i = pred.begin[x]; ~i; i = pred.next[i])
75             {
76                 int p = pred.dest[i];
77                 if(dfn[p] < 0)
78                     continue;
79                 if(dfn[p] > dfn[x])
80                 {
81                     getfa(p);
82                     p = sdom[smin[p]];
83                 }
84                 if(dfn[sdom[x]] > dfn[p])
85                     sdom[x] = p;
86             }
87             tmp.add_edge(sdom[x], x);
88         }
89         while(~tmp.begin[x])
90         {
91             int y = tmp.dest[tmp.begin[x]];
92             tmp.begin[x] = tmp.next[tmp.begin[x]];
93             getfa(y);
94             if(x != sdom[smin[y]])
95                 idom[y] = smin[y];
96             else
97                 idom[y] = x;

```

```

98     }
99     for(int i = succ.begin[x]; ~i; i = succ.next[i])
100         if(f[succ.dest[i]] == x)
101             fa[succ.dest[i]] = x;
102     }
103     idom[s] = s;
104     for(int i = 1; i < stamp; ++i)
105     {
106         int x = id[i];
107         if(idom[x] != sdom[x])
108             idom[x] = idom[idom[x]];
109     }
110 }
111 };
112
113 int ans[MAXN];
114
115 Edge e;
116 dominator dom1;
117
118 int dfs(int x)
119 {
120     if(dom1.idom[x] <= 0)
121         return 0;
122     if(ans[x] > 0)
123         return ans[x];
124     if(dom1.idom[x] == x)
125         return ans[x] = x;
126     return ans[x] = x + dfs(dom1.idom[x]);
127 }
128
129 int main()
130 {
131     int n, m;
132     while(scanf("%d%d", &n, &m) == 2)
133     {
134         e.clear(n + 1);
135         fill(ans, ans + n + 1, 0);
136         for(int i = 0; i < m; ++i)
137         {
138             int u, v;
139             scanf("%d%d", &u, &v);
140             e.add_edge(u, v);
141         }
142         dom1.solve(n, n + 1, e);
143         for(int i = 1; i <= n; ++i)
144             printf("%d%c", dfs(i), " \n"[i == n]);
145     }
146     return 0;

```

147 }

4.12 哈密尔顿回路 (ORE 性质的图)

```

1  int left[N], right[N], next[N], last[N];
2
3  void cover(int x) {
4      left[right[x]] = left[x];
5      right[left[x]] = right[x];
6  }
7
8  int adjacent(int x) {
9      for (int i = right[0]; i <= n; i = right[i]) {
10         if (graph[x][i]) {
11             return i;
12         }
13     }
14     return 0;
15 }
16
17 std::vector<int> solve() {
18     for (int i = 1; i <= n; ++i) {
19         left[i] = i - 1;
20         right[i] = i + 1;
21     }
22     int head, tail;
23     for (int i = 2; i <= n; ++i) {
24         if (graph[1][i]) {
25             head = 1;
26             tail = i;
27             cover(head);
28             cover(tail);
29             next[head] = tail;
30             break;
31         }
32     }
33     while (true) {
34         int x;
35         while (x = adjacent(head)) {
36             next[x] = head;
37             head = x;
38             cover(head);
39         }
40         while (x = adjacent(tail)) {
41             next[tail] = x;
42             tail = x;
43             cover(tail);
44         }
45         if (!graph[head][tail]) {

```

```

46     for (int i = head, j; i != tail; i = next[i]) {
47         if (graph[head][next[i]] && graph[tail][i]) {
48             for (j = head; j != i; j = next[j]) {
49                 last[next[j]] = j;
50             }
51             j = next[head];
52             next[head] = next[i];
53             next[tail] = i;
54             tail = j;
55             for (j = i; j != head; j = last[j]) {
56                 next[j] = last[j];
57             }
58             break;
59         }
60     }
61 }
62 next[tail] = head;
63 if (right[0] > n) {
64     break;
65 }
66 for (int i = head; i != tail; i = next[i]) {
67     if (adjacent(i)) {
68         head = next[i];
69         tail = i;
70         next[tail] = 0;
71         break;
72     }
73 }
74 }
75 std::vector<int> answer;
76 for (int i = head; ; i = next[i]) {
77     if (i == 1) {
78         answer.push_back(i);
79         for (int j = next[i]; j != i; j = next[j]) {
80             answer.push_back(j);
81         }
82         answer.push_back(i);
83         break;
84     }
85     if (i == tail) {
86         break;
87     }
88 }
89 return answer;
90 }

```

4.13 无向图最小割

```

1  int node[N], dist[N];
2  bool visit[N];
3
4  int solve(int n) {
5      int answer = INT_MAX;
6      for (int i = 0; i < n; ++i) {
7          node[i] = i;
8      }
9      while (n > 1) {
10         int max = 1;
11         for (int i = 0; i < n; ++i) {
12             dist[node[i]] = graph[node[0]][node[i]];
13             if (dist[node[i]] > dist[node[max]]) {
14                 max = i;
15             }
16         }
17         int prev = 0;
18         memset(visit, 0, sizeof(visit));
19         visit[node[0]] = true;
20         for (int i = 1; i < n; ++i) {
21             if (i == n - 1) {
22                 answer = std::min(answer, dist[node[max]]);
23                 for (int k = 0; k < n; ++k) {
24                     graph[node[k]][node[prev]] =
25                         (graph[node[prev]][node[k]] += graph[node[k]][node[max]]);
26                 }
27                 node[max] = node[--n];
28             }
29             visit[node[max]] = true;
30             prev = max;
31             max = -1;
32             for (int j = 1; j < n; ++j) {
33                 if (!visit[node[j]]) {
34                     dist[node[j]] += graph[node[prev]][node[j]];
35                     if (max == -1 || dist[node[max]] < dist[node[j]]) {
36                         max = j;
37                     }
38                 }
39             }
40         }
41     }
42     return answer;
43 }

```

4.14 弦图判定

```

1  int n, m, first[1001], l, next[2000001], where[2000001], f[1001], a[1001], c[1001], L[1001],
    ↪ R[1001],
2  v[1001], idx[1001], pos[1001];
3  bool b[1001][1001];
4
5  inline void makelist(int x, int y){
6      where[++l] = y;
7      next[l] = first[x];
8      first[x] = l;
9  }
10
11 bool cmp(const int &x, const int &y){
12     return(idx[x] < idx[y]);
13 }
14
15 int main(){
16     for (;;)
17     {
18         n = read(); m = read();
19         if (!n && !m) return 0;
20         memset(first, 0, sizeof(first)); l = 0;
21         memset(b, false, sizeof(b));
22         for (int i = 1; i <= m; i++)
23         {
24             int x = read(), y = read();
25             if (x != y && !b[x][y])
26             {
27                 b[x][y] = true; b[y][x] = true;
28                 makelist(x, y); makelist(y, x);
29             }
30         }
31         memset(f, 0, sizeof(f));
32         memset(L, 0, sizeof(L));
33         memset(R, 255, sizeof(R));
34         L[0] = 1; R[0] = n;
35         for (int i = 1; i <= n; i++) c[i] = i, pos[i] = i;
36         memset(idx, 0, sizeof(idx));
37         memset(v, 0, sizeof(v));
38         for (int i = n; i; --i)
39         {
40             int now = c[i];
41             R[f[now]]--;
42             if (R[f[now]] < L[f[now]]) R[f[now]] = -1;
43             idx[now] = i; v[i] = now;
44             for (int x = first[now]; x; x = next[x])
45                 if (!idx[where[x]])
46                 {
47                     swap(c[pos[where[x]]], c[R[f[where[x]]]]);

```

```

48         pos[c[pos[where[x]]]] = pos[where[x]];
49         pos[where[x]] = R[f[where[x]]];
50         L[f[where[x]] + 1] = R[f[where[x]]]--;
51         if (R[f[where[x]]] < L[f[where[x]]]) R[f[where[x]]] = -1;
52         if (R[f[where[x]] + 1] == -1)
53             R[f[where[x]] + 1] = L[f[where[x]] + 1];
54         ++f[where[x]];
55     }
56 }
57 bool ok = true;
58 //v 是完美消除序列.
59 for (int i = 1; i <= n && ok; i++)
60 {
61     int cnt = 0;
62     for (int x = first[v[i]]; x; x = next[x])
63         if (idx[where[x]] > i) c[++cnt] = where[x];
64     sort(c + 1, c + cnt + 1, cmp);
65     bool can = true;
66     for (int j = 2; j <= cnt; j++)
67         if (!b[c[1]][c[j]])
68         {
69             ok = false;
70             break;
71         }
72 }
73 if (ok) printf("Perfect\n");
74 else printf("Imperfect\n");
75 printf("\n");
76 }
77 }

```

4.15 弦图求最大团

```

1  int n, m, first[100001], next[2000001], where[2000001], l, L[100001], R[100001], c[100001],
    ↪ f[100001],
2  pos[100001], idx[100001], v[100001], ans;
3
4  inline void makelist(int x, int y){
5      where[++l] = y;
6      next[l] = first[x];
7      first[x] = l;
8  }
9
10 int read(){
11     char ch;
12     for (ch = getchar(); ch < '0' || ch > '9'; ch = getchar());
13     int cnt = 0;
14     for (; ch >= '0' && ch <= '9'; ch = getchar()) cnt = cnt * 10 + ch - '0';
15     return(cnt);

```

```

16 }
17
18 int main(){
19     //freopen("1006.in", "r", stdin);
20     //freopen("1006.out", "w", stdout);
21     memset(first, 0, sizeof(first)); l = 0;
22     n = read(); m = read();
23     for (int i = 1; i <= m; i++)
24     {
25         int x, y;
26         x = read(); y = read();
27         makelist(x, y); makelist(y, x);
28     }
29     memset(L, 0, sizeof(L));
30     memset(R, 255, sizeof(R));
31     memset(f, 0, sizeof(f));
32     memset(idx, 0, sizeof(idx));
33     for (int i = 1; i <= n; i++) c[i] = i, pos[i] = i;
34     L[0] = 1; R[0] = n; ans = 0;
35     for (int i = n; i; --i)
36     {
37         int now = c[i], cnt = 1;
38         idx[now] = i; v[i] = now;
39         if (--R[f[now]] < L[f[now]]) R[f[now]] = -1;
40         for (int x = first[now]; x; x = next[x])
41             if (!idx[where[x]])
42             {
43                 swap(c[pos[where[x]]], c[R[f[where[x]]]]);
44                 pos[c[pos[where[x]]]] = pos[where[x]];
45                 pos[where[x]] = R[f[where[x]]];
46                 L[f[where[x]] + 1] = R[f[where[x]]]--;
47                 if (R[f[where[x]]] < L[f[where[x]]]) R[f[where[x]]] = -1;
48                 if (R[f[where[x]] + 1] == -1) R[f[where[x]] + 1] = L[f[where[x]] + 1];
49                 ++f[where[x]];
50             }
51         else ++cnt;
52         ans = max(ans, cnt);
53     }
54     printf("%d\n", ans);
55 }

```

4.16 最大团搜索

```

1 // mc[i] 代表只用 i-n 号点的答案
2 // g 代表连通性
3 void dfs(int size) {
4     int i, j, k;
5     if (len[size] == 0) {
6         if (size > ans) {

```



```

7      ans = size;
8      found = true;
9  }
10     return;
11 }
12 for (k = 0; k < len[size] && !found; k++) {
13     if (size + len[size] - k <= ans) break;
14     i = list[size][k];
15     if (size + mc[i] <= ans) break;
16     for (j = k + 1, len[size + 1] = 0; j < len[size]; j++)
17         if (g[i][list[size][j]]) list[size + 1][len[size + 1]++] = list[size][j];
18     dfs(size + 1);
19     if (found) {
20         prin[size + 1] = i;
21     }
22 }
23 }
24 void work() {
25     int i, j;
26     mc[n] = ans = 1;
27     ansi = 1;
28     for (i = n - 1; i; i--) {
29         found = false;
30         len[1] = 0;
31         for (j = i + 1; j <= n; j++) if (g[i][j]) list[1][len[1]++] = j;
32         dfs(1);
33         mc[i] = ans;
34         if (found) prin[1] = i;
35     }
36 }
37 void print() {
38     printf("%d\n", ans);
39     for (int i = 1; i < ans; i++) printf("%d ", prin[i]);
40     printf("%d\n", prin[ans]);
41 }

```

4.17 极大团计数

```

1  bool g[N][N];
2  int ne[N], ce[N], list[N][N], ans;
3  void dfs(int size) {
4      if (ans > 1000) return;
5      int i, j, k, t, cnt, best = 0;
6      bool bb;
7      if (ne[size] == ce[size]) {
8          if (ce[size] == 0) ++ans;
9          return;
10     }
11     for (t = 0, i = 1; i <= ne[size]; ++i) {

```

```

12     for (cnt = 0, j = ne[size] + 1; j <= ce[size]; ++j)
13         if (!g[list[size][i]][list[size][j]]) ++cnt;
14     if (t == 0 || cnt < best) t = i, best = cnt;
15 }
16 if (t && best <= 0) return;
17 for (k = ne[size] + 1; k <= ce[size]; ++k) {
18     if (t > 0) {
19         for (i = k; i <= ce[size]; ++i)
20             if (!g[list[size][t]][list[size][i]]) break;
21         swap(list[size][k], list[size][i]);
22     }
23     i = list[size][k];
24     ne[size + 1] = ce[size + 1] = 0;
25     for (j = 1; j < k; ++j)
26         if (g[i][list[size][j]])
27             list[size + 1][++ne[size + 1]] = list[size][j];
28     for (ce[size + 1] = ne[size + 1], j = k + 1; j <= ce[size]; ++j)
29         if (g[i][list[size][j]]) list[size + 1][++ce[size + 1]] = list[size][j];
30     dfs(size + 1);
31     ++ne[size];
32     --best;
33     for (j = k + 1, cnt = 0; j <= ce[size]; ++j)
34         if (!g[i][list[size][j]]) ++cnt;
35     if (t == 0 || cnt < best) t = k, best = cnt;
36     if (t && best <= 0) break;
37 }
38 }
39 int main(){
40     int n, m;
41     while (scanf("%d%d", &n, &m) == 2) {
42         for (int i = 1; i <= n; ++i)
43             for (int j = 1; j <= n; ++j)
44                 g[i][j] = false;
45         while (m--) {
46             int x, y;
47             scanf("%d%d", &x, &y);
48             g[x][y] = g[y][x] = true;
49         }
50         ne[0] = 0;
51         ce[0] = 0;
52         for (int i = 1; i <= n; ++i)
53             list[0][++ce[0]] = i;
54         ans = 0;
55         dfs(0);
56         if (ans > 1000) puts("Too many maximal sets of friends.");
57         else printf("%d\n", ans);
58     }
59     return 0;
60 }

```

4.18 最小树形图

```

1  int n, m, used[N], pass[N], eg[N], more, queue[N];
2  double g[N][N];
3
4  void combine(int id, double &sum) {
5      int tot = 0, from, i, j, k;
6      for (; id != 0 && !pass[id]; id = eg[id]) {
7          queue[tot++] = id;
8          pass[id] = 1;
9      }
10
11     for (from = 0; from < tot && queue[from] != id; from++);
12     if (from == tot) return;
13     more = 1;
14     for (i = from; i < tot; i++) {
15         sum += g[eg[queue[i]]][queue[i]];
16         if (i != from) {
17             used[queue[i]] = 1;
18             for (j = 1; j <= n; j++) if (!used[j]) {
19                 if (g[queue[i]][j] < g[id][j]) g[id][j] = g[queue[i]][j];
20             }
21         }
22     }
23
24     for (i = 1; i <= n; i++) if (!used[i] && i != id) {
25         for (j = from; j < tot; j++) {
26             k = queue[j];
27             if (g[i][id] > g[i][k] - g[eg[k]][k]) g[i][id] = g[i][k] - g[eg[k]][k];
28         }
29     }
30 }
31
32 double mdst(int root) {
33     int i, j, k;
34     double sum = 0;
35     memset(used, 0, sizeof(used));
36     for (more = 1; more; ) {
37         more = 0;
38         memset(eg, 0, sizeof(eg));
39         for (i = 1; i <= n; i++) if (!used[i] && i != root) {
40             for (j = 1, k = 0; j <= n; j++) if (!used[j] && i != j)
41                 if (k == 0 || g[j][i] < g[k][i]) k = j;
42             eg[i] = k;
43         }
44
45         memset(pass, 0, sizeof(pass));
46         for (i = 1; i <= n; i++) if (!used[i] && !pass[i] && i != root) combine(i, sum);
47     }
48 }

```

```

49     for (i = 1; i <= n; i++) if (!used[i] && i != root) sum += g[eg[i]][i];
50     return sum;
51 }

```

4.19 带花树

```

1  int match[N], belong[N], next[N], mark[N], visit[N];
2  std::vector<int> queue;
3
4  int find(int x) {
5      if (belong[x] != x) {
6          belong[x] = find(belong[x]);
7      }
8      return belong[x];
9  }
10
11 void merge(int x, int y) {
12     x = find(x);
13     y = find(y);
14     if (x != y) {
15         belong[x] = y;
16     }
17 }
18
19 int lca(int x, int y) {
20     static int stamp = 0;
21     stamp++;
22     while (true) {
23         if (x != -1) {
24             x = find(x);
25             if (visit[x] == stamp) {
26                 return x;
27             }
28             visit[x] = stamp;
29             if (match[x] != -1) {
30                 x = next[match[x]];
31             } else {
32                 x = -1;
33             }
34         }
35         std::swap(x, y);
36     }
37 }
38
39 void group(int a, int p) {
40     while (a != p) {
41         int b = match[a], c = next[b];
42         if (find(c) != p) {
43             next[c] = b;

```

```

44     }
45     if (mark[b] == 2) {
46         mark[b] = 1;
47         queue.push_back(b);
48     }
49     if (mark[c] == 2) {
50         mark[c] = 1;
51         queue.push_back(c);
52     }
53     merge(a, b);
54     merge(b, c);
55     a = c;
56 }
57 }
58
59 void augment(int source) {
60     queue.clear();
61     for (int i = 0; i < n; ++i) {
62         next[i] = visit[i] = -1;
63         belong[i] = i;
64         mark[i] = 0;
65     }
66     mark[source] = 1;
67     queue.push_back(source);
68     for (int head = 0; head < (int)queue.size() && match[source] == -1; ++head) {
69         int x = queue[head];
70         for (int i = 0; i < (int)edge[x].size(); ++i) {
71             int y = edge[x][i];
72             if (match[x] == y || find(x) == find(y) || mark[y] == 2) {
73                 continue;
74             }
75             if (mark[y] == 1) {
76                 int r = lca(x, y);
77                 if (find(x) != r) {
78                     next[x] = y;
79                 }
80                 if (find(y) != r) {
81                     next[y] = x;
82                 }
83                 group(x, r);
84                 group(y, r);
85             } else if (match[y] == -1) {
86                 next[y] = x;
87                 for (int u = y; u != -1; ) {
88                     int v = next[u];
89                     int mv = match[v];
90                     match[v] = u;
91                     match[u] = v;
92                     u = mv;

```

```

93         }
94         break;
95     } else {
96         next[y] = x;
97         mark[y] = 2;
98         mark[match[y]] = 1;
99         queue.push_back(match[y]);
100     }
101 }
102 }
103 }
104
105 int solve() {
106     std::fill(match, match + n, -1);
107     for (int i = 0; i < n; ++i) {
108         if (match[i] == -1) {
109             augment(i);
110         }
111     }
112     int answer = 0;
113     for (int i = 0; i < n; ++i) {
114         answer += (match[i] != -1);
115     }
116     return answer;
117 }

```

4.20 度限制生成树

```

1  int n, m, S, K, ans, cnt, Best[N], fa[N], FE[N];
2  int f[N], p[M], t[M], c[M], o, Cost[N];
3  bool u[M], d[M];
4  pair<int, int> MinCost[N];
5  struct Edge {
6      int a, b, c;
7      bool operator < (const Edge & E) const { return c < E.c; }
8  }E[M];
9  vector<int> SE;
10 inline int F(int x) {
11     return fa[x] == x ? x : fa[x] = F(fa[x]);
12 }
13 inline void AddEdge(int a, int b, int C) {
14     p[++o] = b; c[o] = C;
15     t[o] = f[a]; f[a] = o;
16 }
17 void dfs(int i, int father) {
18     fa[i] = father;
19     if (father == S) Best[i] = -1;
20     else {
21         Best[i] = i;

```

```

22     if (~Best[father] && Cost[Best[father]] > Cost[i]) Best[i] = Best[father];
23 }
24 for (int j = f[i]; j; j = t[j])
25     if (!d[j] && p[j] != father) {
26         Cost[p[j]] = c[j];
27         FE[p[j]] = j;
28         dfs(p[j], i);
29     }
30 }
31 inline bool Kruskal() {
32     cnt = n - 1, ans = 0; o = 1;
33     for (int i = 1; i <= n; i++) fa[i] = i, f[i] = 0;
34     sort(E + 1, E + m + 1);
35     for (int i = 1; i <= m; i++) {
36         if (E[i].b == S) swap(E[i].a, E[i].b);
37         if (E[i].a != S && F(E[i].a) != F(E[i].b)) {
38             fa[F(E[i].a)] = F(E[i].b);
39             ans += E[i].c;
40             cnt --;
41             u[i] = true;
42             AddEdge(E[i].a, E[i].b, E[i].c);
43             AddEdge(E[i].b, E[i].a, E[i].c);
44         }
45     }
46     for (int i = 1; i <= n; i++) MinCost[i] = make_pair(INF, INF);
47     for (int i = 1; i <= m; i++)
48         if (E[i].a == S) {
49             SE.push_back(i);
50             MinCost[F(E[i].b)] = min(MinCost[F(E[i].b)], make_pair(E[i].c, i));
51         }
52     int dif = 0;
53     for (int i = 1; i <= n; i++)
54         if (i != S && fa[i] == i) {
55             if (MinCost[i].second == INF) return false;
56             if (++dif > K) return false;
57             dfs(E[MinCost[i].second].b, S);
58             u[MinCost[i].second] = true;
59             ans += MinCost[i].first;
60         }
61     return true;
62 }
63 bool Solve() {
64     memset(d, false, sizeof d);
65     memset(u, false, sizeof u);
66     if (!Kruskal()) return false;
67     for (int i = cnt + 1; i <= K && i <= n; i++) {
68         int MinD = INF, MinID = -1;
69         for (int j = (int) SE.size() - 1; j >= 0; j--)
70             if (u[SE[j]])

```

```

71     SE.erase(SE.begin() + j);
72     for (int j = 0; j < (int) SE.size(); j++) {
73         int tmp = E[SE[j]].c - Cost[Best[E[SE[j]].b]];
74         if (tmp < MinD) {
75             MinD = tmp;
76             MinID = SE[j];
77         }
78     }
79     if (MinID == -1) return true;
80     if (MinD >= 0) break;
81     ans += MinD;
82     u[MinID] = true;
83     d[FE[Best[E[MinID].b]]] = d[FE[Best[E[MinID].b]] ^ 1] = true;
84     dfs(E[MinID].b, S);
85 }
86 return true;
87 }
88 int main(){
89     Solve();
90     return 0;
91 }

```

4.21 圆方树

```

1 //圆点：原图中的点，方点：每个点双连通分量新添加的点。
2 //nodecnt 从 n+1 开始作为方点的编号
3 void Targan(int x, int last){
4     dfn[x] = low[x] = ++dfsclock;
5     sta[++stm] = x;
6     for(int i = he[x], v; i; i = e[i].nx){
7         if(i == last)continue;
8         v = G.e[i].v;
9         if(!dfn[v]){
10             Targan(v, i ^ 1);
11             low[x] = min(low[x], low[v]);
12             if(low[v] >= dfn[x]){
13                 addedge(++nodecnt, x);
14                 while(stm && sta[stm] != v){
15                     addedge(nodecnt, sta[stm--]);
16                 }
17                 addedge(nodecnt, sta[stm--]);
18             }
19         }
20         else low[x] = min(low[x], dfn[v]);
21     }
22 }

```


5 字符串

5.1 KMP

```

1 void Gnext(){
2     for(int i = 2, j; a[i] != '\0'; ++i){
3         j = nxt[i - 1];
4         while(j && a[j + 1] != a[i]) j = nxt[j];
5         if(a[j + 1] == a[i]) j++;
6         nxt[i] = j;
7     }
8 }
9 int MP(){
10     int j = 0, res = 0;
11     for(int i = 1; b[i] != '\0'; ++i){
12         while(j && a[j + 1] != b[i]) j = nxt[j];
13         if(a[j + 1] == b[i]) j++;
14         if(a[j + 1] == '\0'){
15             res++, j = nxt[j];
16         }
17     }
18     return res;
19 }

```

5.2 EXKMP

```

1 //求字符串 b[0, n) 的每个后缀和 a[0, m) 的最长公共前缀。
2 //将字符串翻转后可以求回文串。
3 void ExtendedKmp(int n, int m){
4     int i, j, k;
5     for(j = 0; j + 1 < m && a[j] == a[j + 1]; ++j);
6     nxt[1] = j; k = 1;
7     for(i = 2; i < m; ++i){
8         int pos = k + nxt[k], len = nxt[i - k];
9         if(i + len < pos) nxt[i] = len;
10        else {
11            for(j = max(0, pos - i); i + j < m && a[j] == a[i + j]; ++j);
12            nxt[i] = j; k = i;
13        }
14    }
15    for(j = 0; j < m && j < n && a[j] == b[j]; ++j);
16    f[0] = j; k = 0;
17    for(i = 1; i < n; ++i){
18        int pos = k + f[k], len = nxt[i - k];
19        if(i + len < pos) f[i] = len;
20        else {
21            for(j = max(0, pos - i); j < m && i + j < n && a[j] == b[i + j]; ++j);
22            f[i] = j; k = i;
23        }
24    }

```

```

25 }

1 //z[i] 表示 s[i..n-1] 和 s[0..n-1] 的最长公共前缀
2 void exkmp(char *s, int n, int *z) {
3     memset(z, 0, sizeof(z[0]) * n);
4     for (int i = 1, x = 0, y = 0; i < n; ++i) {
5         if (i <= y) z[i] = min(y - i, z[i - x]);
6         while (i + z[i] < n && s[i + z[i]] == s[z[i]]) z[i]++;
7         if (y <= i + z[i]) x = i, y = i + z[i];
8     }
9     z[0] = n;
10 }

```

5.3 AC 自动机

```

1 void Insert(){
2     int p = 0;
3     for(int i = 0, c; str[i] != '\0'; ++i){
4         c = str[i] - 'a';
5         if(!ch[p][c]) ch[p][c] = ++nodecnt;
6         p = ch[p][c];
7     }
8     val[p] = 1;
9 }
10 void Build(){
11     int h = 1, t = 0, p, u;
12     for(int c = 0; c < 26; ++c){
13         p = ch[0][c];
14         if(p) fail[p] = 0, Q[++t] = p;
15     }
16     while(h <= t){
17         u = Q[h++];
18         for(int c = 0; c < 26; ++c){
19             p = ch[u][c];
20             if(!p) ch[u][c] = ch[fail[u]][c];
21             else{
22                 fail[p] = ch[fail[u]][c];
23                 Q[++t] = p;
24             }
25         }
26     }
27 }

```

5.4 SAM

```

1 int n, K, flag, nodecnt;
2 char str[maxn];
3 int siz[maxn], Ws[maxn], Q[maxn]; ll sum[maxn];
4 struct Node{
5     int nx[26], len, root;

```

```

6  }T[maxn];
7  inline void Init(){
8      nodecnt = 0;
9      T[0].root = -1, T[0].len = 0;
10 }
11 inline int Extend(int p, int c){
12     int np = ++nodecnt;
13     T[np].len = T[p].len + 1, siz[np] = 1;
14     for(;p != -1 && !T[p].nx[c];p = T[p].root)T[p].nx[c] = np;
15     if(p == -1)T[np].root = 0;
16     else{
17         int q = T[p].nx[c];
18         if(T[q].len == T[p].len + 1)T[np].root = q;
19         else{
20             int nq = ++nodecnt;
21             T[nq] = T[q];T[nq].len = T[p].len + 1;
22             for(;p != -1 && T[p].nx[c] == q;p = T[p].root)T[p].nx[c] = nq;
23             T[q].root = T[np].root = nq;
24         }
25     }
26     return np;
27 }
28 void Dfs(int x){
29     if(K <= siz[x])return;
30     K -= siz[x];
31     for(int c = 0;c < 26;++c){
32         if(T[x].nx[c]){
33             if(K <= sum[T[x].nx[c]]){
34                 putchar('a' + c);
35                 Dfs(T[x].nx[c]);
36                 break;
37             }
38             else K -= sum[T[x].nx[c]];
39         }
40     }
41 }
42 int main(){
43     scanf("%s", str);n = strlen(str);
44     scanf("%d%d", &flag, &K);
45     Init();
46     for(int i = 0, last = 0;i < n;++i)last = Extend(last, str[i] - 'a');
47     for(int i = 1;i <= nodecnt;++i)Ws[T[i].len]++;
48     for(int i = 1;i <= n;++i)Ws[i] += Ws[i - 1];
49     for(int i = nodecnt;i > 0;--i)Q[Ws[T[i].len]--] = i;
50     for(int i = nodecnt, x;i > 0;--i){
51         x = Q[i];
52         if(!flag)siz[x] = 1;
53         else siz[T[x].root] += siz[x];
54     }

```

```

55     siz[0] = 0;
56     for(int i = nodecnt, x; i >= 0; --i){
57         x = Q[i];
58         sum[x] = siz[x];
59         for(int c = 0; c < 26; ++c){
60             if(T[x].nx[c]) sum[x] += sum[T[x].nx[c]];
61         }
62     }
63     if(sum[0] < K) puts("-1");
64     else Dfs(0);
65     return 0;
66 }

```

广义 SAM

```

1  const int maxn = 100010;
2  int nodecnt, n, m;
3  int A[maxn];
4  struct Node{
5      int len, root, nx[10];
6  }T[maxn*40];
7
8  void Init(){
9      nodecnt = 0;
10     T[0].len = 0; T[0].root = -1;
11 }
12 int Extend(int p, int c){
13     int np = ++nodecnt;
14     T[np].len = T[p].len + 1;
15     for(; p != -1 && !T[p].nx[c]; p = T[p].root) T[p].nx[c] = np;
16     if(p == -1) T[np].root = 0;
17     else {
18         int q = T[p].nx[c];
19         if(T[q].len == T[p].len + 1) T[np].root = q;
20         else {
21             int nw = ++nodecnt;
22             T[nw] = T[q];
23             T[nw].len = T[p].len + 1;
24             for(; p != -1 && T[p].nx[c] == q; p = T[p].root) T[p].nx[c] = nw;
25             T[q].root = T[np].root = nw;
26         }
27     }return np;
28 }
29
30 int ed, he[maxn], ind[maxn];
31 struct Edge{
32     int v, nx;
33     Edge(int _v, int _nx){v = _v; nx = _nx;}
34     Edge(){};

```

```

35 }e[maxn<<1];
36 void Add(int u,int v){
37     e[++ed] = Edge(v,he[u]);he[u] = ed;
38 }
39 void Dfs(int x,int fa,int last){
40     last = Extend(last,A[x]);
41     for(int i = he[x];i;i = e[i].nx){
42         int v = e[i].v;
43         if(v == fa)continue;
44         Dfs(v,x,last);
45     }
46 }
47 int main(){
48     freopen("zjoi15_substring.in","r",stdin);
49     freopen("zjoi15_substring.out","w",stdout);
50     scanf("%d%d",&n,&m);
51     for(int i = 1;i <= n;++i)scanf("%d",&A[i]);
52     for(int i = 1,u,v;i < n;++i){
53         scanf("%d%d",&u,&v);
54         Add(u,v);Add(v,u);
55         ind[u]++;ind[v]++;
56     }
57     Init();
58     for(int i = 1;i <= n;++i){
59         if(ind[i] == 1)Dfs(i,0,0);
60     }
61     ll ans = 0;
62     for(int i = 1;i <= nodecnt;++i){
63         ans += 1LL*(T[i].len - T[T[i].root].len);
64     }
65     printf("%lld",ans);
66     // while(1);
67     return 0;
68 }

```

5.5 后缀数组

```

1  int Len;
2  int sa[maxn], Rank[maxn], h[maxn], r[maxn], wa[maxn << 1], wb[maxn << 1], Ws[maxn];
3  bool cmp(int *y, int a, int b, int len){
4      return y[a] == y[b] && y[a + len] == y[b + len];
5  }
6  void Da(int n, int m){
7      int i, j, p, *x = wa, *y = wb;
8      for(i = 0;i < m;++i)Ws[i] = 0;
9      for(i = 0;i < n;++i)Ws[x[i] = r[i]]++;
10     for(i = 1;i < m;++i)Ws[i] += Ws[i - 1];
11     for(i = n - 1;i >= 0;--i)sa[--Ws[x[i]]] = i;
12     for(j = 1, p = 0;p < n;j <= 1, m = p){

```

```

13     for(p = 0, i = n - j; i < n; ++i) y[p++] = i;
14     for(i = 0; i < n; ++i){
15         if(sa[i] >= j) y[p++] = sa[i] - j;
16     }
17     for(i = 0; i < m; ++i) ws[i] = 0;
18     for(i = 0; i < n; ++i) ws[x[y[i]]]++;
19     for(i = 1; i < m; ++i) ws[i] += ws[i - 1];
20     for(i = n - 1; i >= 0; --i) sa[--ws[x[y[i]]]] = y[i];
21     for(swap(x, y), i = 1, p = 1, x[sa[0]] = 0; i < n; ++i){
22         x[sa[i]] = cmp(y, sa[i - 1], sa[i], j) ? p - 1 : p++;
23     }
24 }
25 }
26 void Calheight(int n){
27     int i, j, k = 0;
28     for(i = 1; i <= n; ++i) Rank[sa[i]] = i;
29     for(i = 0; i < n; h[Rank[i++]] = k){
30         for(k > 0 ? k-- : 0, j = sa[Rank[i] - 1]; r[i + k] == r[j + k]; ++k);
31     }
32 }
33 int Log[maxn], f[maxn << 1][20];
34 void ST(int n){
35     Log[1] = 0;
36     for(int i = 2; i <= n; ++i){
37         Log[i] = Log[i - 1];
38         if((1 << (Log[i] + 1)) == i) Log[i]++;
39     }
40     memset(f, 0x3f, sizeof(f));
41     for(int i = 1; i <= n; ++i) f[i][0] = h[i];
42     for(int j = 1; (1 << j) <= n; ++j){
43         for(int i = 1; i <= n; ++i){
44             f[i][j] = min(f[i][j - 1], f[i + (1 << (j - 1))][j - 1]);
45         }
46     }
47 }
48 int LCP(int x, int y){
49     if(x == y) return Len - x;
50     x = Rank[x], y = Rank[y];
51     if(x > y) swap(x, y); ++x;
52     int len = y - x + 1;
53     return min(f[x][Log[len]], f[y - (1 << Log[len]) + 1][Log[len]]);
54 }

```

5.6 回文自动机

```

1  /*
2  Palindrome Automaton
3  本质不同的回文子串的个数 = 自动机节点个数 - 2。
4  siz[x] 表示 x 节点代表的回文串在整个字符串中的出现次数。

```

```

5  Apio2014 Palindrome
6  */
7  struct Node{
8      int len, fail, siz, nx[26];
9  }T[maxn];
10 void Init(){
11     nodecnt = 1;
12     T[0].len = 0, T[0].fail = 1;
13     T[1].len = -1;
14 }
15 int Extend(int p, int c, int len){
16     for(;str[len - T[p].len - 1] != str[len];p = T[p].fail);
17     if(!T[p].nx[c]){
18         int np = ++nodecnt, x;
19         for(x = T[p].fail;str[len - T[x].len - 1] != str[len];x = T[x].fail);
20         T[np].fail = T[x].nx[c];
21         T[p].nx[c] = np;
22         T[np].len = T[p].len + 2;
23     }
24     T[T[p].nx[c]].siz++;
25     return T[p].nx[c];
26 }
27 int main(){
28     Init();
29     scanf("%s", str + 1);
30     for(int i = 1, last = 0;str[i] != '\0';++i){
31         last = Extend(last, str[i] - 'a', i);
32     }
33     ll ans = 0;
34     for(int i = nodecnt;i >= 2;--i){
35         T[T[i].fail].siz += T[i].siz;
36         ans = max(ans, 1LL * T[i].siz * T[i].len);
37     }
38 }

```

5.7 Manacher

```

1  void Manacher(int n){
2      for(int i = n;i >= 1;--i){
3          if(i & 1)str[i] = '#';
4          else str[i] = str[i >> 1];
5      }
6      str[0] = '$';str[n + 1] = '*';
7      for(int i = 1, mx = 0, pos = 0;i <= n;++i){
8          d[i] = i < mx ? min(d[pos*2 - i], mx - i) : 1;
9          while(str[i - d[i]] == str[i + d[i]])d[i]++;
10         if(i + d[i] > mx)mx = i + d[i], pos = i;
11     }
12 }

```

5.8 循环串的最小表示

注意事项：0-Based 算法，请注意下标。

```
1  #include <bits/stdc++.h>
2  using namespace std;
3  const int N = 100100;
4  char s[N];
5  /*
6  int work1(int *a, int n){//输出最靠左的最小表示
7      for(int i = 0; i < n; i++)
8          a[i + n] = a[i];
9      int pos = 0;
10     for(int i = 1, k; i < n;){
11         for(k = 0; k < n && a[pos + k] == a[i + k]; k++);
12         if(k < n && a[i + k] < a[pos + k]){
13             int t = pos;
14             pos = i;
15             i = max(i + 1, t + k + 1);
16         }
17         else{
18             i += k + 1;
19         }
20     }
21     return pos;
22 }
23
24 int work2(int *a, int n){//输出最靠右的最小表示，待验，谨慎使用
25     for(int i = 0; i < n; i++)
26         a[i + n] = a[i];
27     int pos = 0;
28     for(int i = 1, k; i < n;){
29         for(k = 0; k < n && a[pos + k] == a[i + k]; k++);
30         if(k == n){
31             pos = i;
32             i++;
33             continue;
34         }
35         if(k < n && a[i + k] < a[pos + k]){
36             int t = pos;
37             pos = i;
38             i = max(i + 1, t + k + 1);
39         }
40         else{
41             i += k + 1;
42         }
43     }
44     return pos;
45 }
46 */
```



```

47 int getmin(char *s, int n){// 0-base
48     int i = 0, j = 1, k = 0;
49     while(i < n && j < n && k < n){
50         int x = i + k;
51         int y = j + k;
52         if(x >= n) x -= n;
53         if(y >= n) y -= n;
54         if(s[x] == s[y])
55             k++;
56         else{
57             if(s[x] > s[y])
58                 i += k + 1;
59             else
60                 j += k + 1;
61             if(i == j)
62                 j++;
63             k = 0;
64         }
65     }
66     return min(i, j);
67 }
68
69 int main(){
70     int T;
71     scanf("%d", &T);
72     while(T--){
73         int n;
74         scanf("%d", &n);
75         scanf("%s", s);
76         printf("%d\n", getmin(s, n));
77     }
78 }

```

6 计算几何

6.1 二维几何

```

1 struct Point {
2     Point rotate(const double ang) { // 逆时针旋转 ang 弧度
3         return Point(cos(ang) * x - sin(ang) * y, cos(ang) * y + sin(ang) * x);
4     }
5     Point turn90() { // 逆时针旋转 90 度
6         return Point(-y, x);
7     }
8 };
9 Point isLL(const Line &l1, const Line &l2) {
10     double s1 = det(l2.b - l2.a, l1.a - l2.a),
11         s2 = -det(l2.b - l2.a, l1.b - l2.a);
12     return (l1.a * s2 + l1.b * s1) / (s1 + s2);

```

```

13 }
14 bool onSeg(const Line &l, const Point &p) { // 点在线段上
15     return sign(det(p - l.a, l.b - l.a)) == 0 && sign(dot(p - l.a, p - l.b)) <= 0;
16 }
17 Point projection(const Line &l, const Point &p) { // 点到直线投影
18     return l.a + (l.b - l.a) * (dot(p - l.a, l.b - l.a) / (l.b - l.a).len2());
19 }
20 double disToLine(const Line &l, const Point &p) {
21     return abs(det(p - l.a, l.b - l.a) / (l.b - l.a).len());
22 }
23 double disToSeg(const Line &l, const Point &p) { // 点到线段距离
24     return sign(dot(p - l.a, l.b - l.a)) * sign(dot(p - l.b, l.a - l.b)) == 1 ?
25         disToLine(l, p) : min((p - l.a).len(), (p - l.b).len());
26 }
27 Point symmetryPoint(const Point a, const Point b) { // 点 b 关于点 a 的中心对称点
28     return a + a - b;
29 }
30 Point reflection(const Line &l, const Point &p) { // 点关于直线的对称点
31     return symmetryPoint(projection(l, p), p);
32 }
33 // 求圆与直线的交点
34 bool isCL(Circle a, Line l, Point &p1, Point &p2) {
35     double x = dot(l.a - a.o, l.b - l.a),
36         y = (l.b - l.a).len2(),
37         d = x * x - y * ((l.a - a.o).len2() - a.r * a.r);
38     if (sign(d) < 0) return false;
39     d = max(d, 0.0);
40     Point p = l.a - ((l.b - l.a) * (x / y)), delta = (l.b - l.a) * (sqrt(d) / y);
41     p1 = p + delta, p2 = p - delta;
42     return true;
43 }
44 // 求圆与圆的交面积
45 double areaCC(const Circle &c1, const Circle &c2) {
46     double d = (c1.o - c2.o).len();
47     if (sign(d - (c1.r + c2.r)) >= 0) {
48         return 0;
49     }
50     if (sign(d - abs(c1.r - c2.r)) <= 0) {
51         double r = min(c1.r, c2.r);
52         return r * r * PI;
53     }
54     double x = (d * d + c1.r * c1.r - c2.r * c2.r) / (2 * d),
55         t1 = acos(x / c1.r), t2 = acos((d - x) / c2.r);
56     return c1.r * c1.r * t1 + c2.r * c2.r * t2 - d * c1.r * sin(t1);
57 }
58 // 求圆与圆的交点, 注意调用前要先判定重圆
59 bool isCC(Circle a, Circle b, Point &p1, Point &p2) {
60     double s1 = (a.o - b.o).len();
61     if (sign(s1 - a.r - b.r) > 0 || sign(s1 - abs(a.r - b.r)) < 0) return false;

```

```

62     double s2 = (a.r * a.r - b.r * b.r) / s1;
63     double aa = (s1 + s2) * 0.5, bb = (s1 - s2) * 0.5;
64     Point o = (b.o - a.o) * (aa / (aa + bb)) + a.o;
65     Point delta = (b.o - a.o).unit().turn90() * newSqrt(a.r * a.r - aa * aa);
66     p1 = o + delta, p2 = o - delta;
67     return true;
68 }
69 // 求点到圆的切点, 按关于点的顺时针方向返回两个点
70 bool tanCP(const Circle &c, const Point &p0, Point &p1, Point &p2) {
71     double x = (p0 - c.o).len2(), d = x - c.r * c.r;
72     if (d < EPS) return false; // 点在圆上认为没有切点
73     Point p = (p0 - c.o) * (c.r * c.r / x);
74     Point delta = ((p0 - c.o) * (-c.r * sqrt(d) / x)).turn90();
75     p1 = c.o + p + delta;
76     p2 = c.o + p - delta;
77     return true;
78 }
79 // 求圆到圆的外共切线, 按关于 c1.o 的顺时针方向返回两条线
80 vector<Line> extanCC(const Circle &c1, const Circle &c2) {
81     vector<Line> ret;
82     if (sign(c1.r - c2.r) == 0) {
83         Point dir = c2.o - c1.o;
84         dir = (dir * (c1.r / dir.len())).turn90();
85         ret.push_back(Line(c1.o + dir, c2.o + dir));
86         ret.push_back(Line(c1.o - dir, c2.o - dir));
87     } else {
88         Point p = (c1.o * -c2.r + c2.o * c1.r) / (c1.r - c2.r);
89         Point p1, p2, q1, q2;
90         if (tanCP(c1, p, p1, p2) && tanCP(c2, p, q1, q2)) {
91             if (c1.r < c2.r) swap(p1, p2), swap(q1, q2);
92             ret.push_back(Line(p1, q1));
93             ret.push_back(Line(p2, q2));
94         }
95     }
96     return ret;
97 }
98 // 求圆到圆的内共切线, 按关于 c1.o 的顺时针方向返回两条线
99 vector<Line> intanCC(const Circle &c1, const Circle &c2) {
100     vector<Line> ret;
101     Point p = (c1.o * c2.r + c2.o * c1.r) / (c1.r + c2.r);
102     Point p1, p2, q1, q2;
103     if (tanCP(c1, p, p1, p2) && tanCP(c2, p, q1, q2)) { // 两圆相切认为没有切线
104         ret.push_back(Line(p1, q1));
105         ret.push_back(Line(p2, q2));
106     }
107     return ret;
108 }
109 bool contain(vector<Point> polygon, Point p) { // 判断点 p 是否被多边形包含, 包括落在边界上
110     int ret = 0, n = polygon.size();

```

```

111     for(int i = 0; i < n; ++ i) {
112         Point u = polygon[i], v = polygon[(i + 1) % n];
113         if (onSeg(Line(u, v), p)) return true;
114         if (sign(u.y - v.y) <= 0) swap(u, v);
115         if (sign(p.y - u.y) > 0 || sign(p.y - v.y) <= 0) continue;
116         ret += sign(det(p, v, u)) > 0;
117     }
118     return ret & 1;
119 }
120 vector<Point> convexCut(const vector<Point>&ps, Line l) {
121     ↪ // 用半平面 (q1,q2) 的逆时针方向去切凸多边形
122     vector<Point> qs;
123     int n = ps.size();
124     for (int i = 0; i < n; ++i) {
125         Point p1 = ps[i], p2 = ps[(i + 1) % n];
126         int d1 = sign(det(l.a, l.b, p1)), d2 = sign(det(l.a, l.b, p2));
127         if (d1 >= 0) qs.push_back(p1);
128         if (d1 * d2 < 0) qs.push_back(isLL(Line(p1, p2), l));
129     }
130     return qs;
131 }
132 vector<Point> convexHull(vector<Point> ps) { // 求点集 ps 组成的凸包
133     int n = ps.size(); if (n <= 1) return ps;
134     sort(ps.begin(), ps.end());
135     vector<Point> qs;
136     for (int i = 0; i < n; qs.push_back(ps[i++]))
137         while (qs.size() > 1 && sign(det(qs[qs.size()-2], qs.back(), ps[i])) <= 0) qs.pop_back();
138     for (int i = n - 2, t = qs.size(); i >= 0; qs.push_back(ps[i--]))
139         while ((int)qs.size() > t && sign(det(qs[(int)qs.size()-2], qs.back(), ps[i])) <= 0)
140             ↪ qs.pop_back();
141     qs.pop_back(); return qs;
142 }

```

6.2 阿波罗尼茨圆

```

1 struct Point {
2     Point rotate(const double ang) { // 逆时针旋转 ang 弧度
3         return Point(cos(ang) * x - sin(ang) * y, cos(ang) * y + sin(ang) * x);
4     }
5     Point turn90() { // 逆时针旋转 90 度
6         return Point(-y, x);
7     }
8 };
9 Point isLL(const Line &l1, const Line &l2) {
10     double s1 = det(l2.b - l2.a, l1.a - l2.a),
11            s2 = -det(l2.b - l2.a, l1.b - l2.a);
12     return (l1.a * s2 + l1.b * s1) / (s1 + s2);
13 }
14 bool onSeg(const Line &l, const Point &p) { // 点在线段上

```

```

15     return sign(det(p - l.a, l.b - l.a)) == 0 && sign(dot(p - l.a, p - l.b)) <= 0;
16 }
17 Point projection(const Line &l, const Point &p) { // 点到直线投影
18     return l.a + (l.b - l.a) * (dot(p - l.a, l.b - l.a) / (l.b - l.a).len2());
19 }
20 double disToLine(const Line &l, const Point &p) {
21     return abs(det(p - l.a, l.b - l.a) / (l.b - l.a).len());
22 }
23 double disToSeg(const Line &l, const Point &p) { // 点到线段距离
24     return sign(dot(p - l.a, l.b - l.a)) * sign(dot(p - l.b, l.a - l.b)) == 1 ?
25         disToLine(l, p) : min((p - l.a).len(), (p - l.b).len());
26 }
27 Point symmetryPoint(const Point a, const Point b) { // 点 b 关于点 a 的中心对称点
28     return a + a - b;
29 }
30 Point reflection(const Line &l, const Point &p) { // 点关于直线的对称点
31     return symmetryPoint(projection(l, p), p);
32 }
33 // 求圆与直线的交点
34 bool isCl(Circle a, Line l, Point &p1, Point &p2) {
35     double x = dot(l.a - a.o, l.b - l.a),
36         y = (l.b - l.a).len2(),
37         d = x * x - y * ((l.a - a.o).len2() - a.r * a.r);
38     if (sign(d) < 0) return false;
39     d = max(d, 0.0);
40     Point p = l.a - ((l.b - l.a) * (x / y)), delta = (l.b - l.a) * (sqrt(d) / y);
41     p1 = p + delta, p2 = p - delta;
42     return true;
43 }
44 // 求圆与圆的交面积
45 double areaCC(const Circle &c1, const Circle &c2) {
46     double d = (c1.o - c2.o).len();
47     if (sign(d - (c1.r + c2.r)) >= 0) {
48         return 0;
49     }
50     if (sign(d - abs(c1.r - c2.r)) <= 0) {
51         double r = min(c1.r, c2.r);
52         return r * r * PI;
53     }
54     double x = (d * d + c1.r * c1.r - c2.r * c2.r) / (2 * d),
55         t1 = acos(x / c1.r), t2 = acos((d - x) / c2.r);
56     return c1.r * c1.r * t1 + c2.r * c2.r * t2 - d * c1.r * sin(t1);
57 }
58 // 求圆与圆的交点, 注意调用前要先判定重圆
59 bool isCC(Circle a, Circle b, Point &p1, Point &p2) {
60     double s1 = (a.o - b.o).len();
61     if (sign(s1 - a.r - b.r) > 0 || sign(s1 - abs(a.r - b.r)) < 0) return false;
62     double s2 = (a.r * a.r - b.r * b.r) / s1;
63     double aa = (s1 + s2) * 0.5, bb = (s1 - s2) * 0.5;

```

```

64   Point o = (b.o - a.o) * (aa / (aa + bb)) + a.o;
65   Point delta = (b.o - a.o).unit().turn90() * newSqrt(a.r * a.r - aa * aa);
66   p1 = o + delta, p2 = o - delta;
67   return true;
68 }
69 // 求点到圆的切点, 按关于点的顺时针方向返回两个点
70 bool tanCP(const Circle &c, const Point &p0, Point &p1, Point &p2) {
71     double x = (p0 - c.o).len2(), d = x - c.r * c.r;
72     if (d < EPS) return false; // 点在圆上认为没有切点
73     Point p = (p0 - c.o) * (c.r * c.r / x);
74     Point delta = ((p0 - c.o) * (-c.r * sqrt(d) / x)).turn90();
75     p1 = c.o + p + delta;
76     p2 = c.o + p - delta;
77     return true;
78 }
79 // 求圆到圆的外共切线, 按关于 c1.o 的顺时针方向返回两条线
80 vector<Line> extanCC(const Circle &c1, const Circle &c2) {
81     vector<Line> ret;
82     if (sign(c1.r - c2.r) == 0) {
83         Point dir = c2.o - c1.o;
84         dir = (dir * (c1.r / dir.len())).turn90();
85         ret.push_back(Line(c1.o + dir, c2.o + dir));
86         ret.push_back(Line(c1.o - dir, c2.o - dir));
87     } else {
88         Point p = (c1.o * -c2.r + c2.o * c1.r) / (c1.r - c2.r);
89         Point p1, p2, q1, q2;
90         if (tanCP(c1, p, p1, p2) && tanCP(c2, p, q1, q2)) {
91             if (c1.r < c2.r) swap(p1, p2), swap(q1, q2);
92             ret.push_back(Line(p1, q1));
93             ret.push_back(Line(p2, q2));
94         }
95     }
96     return ret;
97 }
98 // 求圆到圆的内共切线, 按关于 c1.o 的顺时针方向返回两条线
99 vector<Line> intanCC(const Circle &c1, const Circle &c2) {
100     vector<Line> ret;
101     Point p = (c1.o * c2.r + c2.o * c1.r) / (c1.r + c2.r);
102     Point p1, p2, q1, q2;
103     if (tanCP(c1, p, p1, p2) && tanCP(c2, p, q1, q2)) { // 两圆相切认为没有切线
104         ret.push_back(Line(p1, q1));
105         ret.push_back(Line(p2, q2));
106     }
107     return ret;
108 }
109 bool contain(vector<Point> polygon, Point p) { // 判断点 p 是否被多边形包含, 包括落在边界上
110     int ret = 0, n = polygon.size();
111     for(int i = 0; i < n; ++ i) {
112         Point u = polygon[i], v = polygon[(i + 1) % n];

```

```

113     if (onSeg(Line(u, v), p)) return true;
114     if (sign(u.y - v.y) <= 0) swap(u, v);
115     if (sign(p.y - u.y) > 0 || sign(p.y - v.y) <= 0) continue;
116     ret += sign(det(p, v, u)) > 0;
117 }
118 return ret & 1;
119 }
120 vector<Point> convexCut(const vector<Point>&ps, Line l) {
    ↪ // 用半平面 (q1,q2) 的逆时针方向去切凸多边形
121     vector<Point> qs;
122     int n = ps.size();
123     for (int i = 0; i < n; ++i) {
124         Point p1 = ps[i], p2 = ps[(i + 1) % n];
125         int d1 = sign(det(l.a, l.b, p1)), d2 = sign(det(l.a, l.b, p2));
126         if (d1 >= 0) qs.push_back(p1);
127         if (d1 * d2 < 0) qs.push_back(isLL(Line(p1, p2), l));
128     }
129     return qs;
130 }
131 vector<Point> convexHull(vector<Point> ps) { // 求点集 ps 组成的凸包
132     int n = ps.size(); if (n <= 1) return ps;
133     sort(ps.begin(), ps.end());
134     vector<Point> qs;
135     for (int i = 0; i < n; qs.push_back(ps[i++]))
136         while (qs.size() > 1 && sign(det(qs[qs.size()-2], qs.back(), ps[i])) <= 0) qs.pop_back();
137     for (int i = n - 2, t = qs.size(); i >= 0; qs.push_back(ps[i--]))
138         while ((int)qs.size() > t && sign(det(qs[(int)qs.size()-2], qs.back(), ps[i])) <= 0)
139             ↪ qs.pop_back();
140     qs.pop_back(); return qs;
141 }

```

6.3 三角形与圆交

```

1 // 反三角函数要在 [-1, 1] 中, sqrt 要与 0 取 max 别忘了取正负
2 // 改成周长请用注释, res1 为直线长度, res2 为弧线长度
3 // 多边形与圆求交时, 相切精度比较差
4 D areaCT(P pa, P pb, D r) { //, D & res1, D & res2) {
5     if (pa.len() < pb.len()) swap(pa, pb);
6     if (sign(pb.len()) == 0) return 0;
7     ↪ // if (sign(pb.len()) == 0) { res1 += min(r, pa.len()); return; }
8     D a = pb.len(), b = pa.len(), c = (pb - pa).len();
9     D sinB = fabs(pb * (pb - pa)), cosB = pb % (pb - pa), area = fabs(pa * pb);
10    D S, B = atan2(sinB, cosB), C = atan2(area, pa % pb);
11    sinB /= a * c; cosB /= a * c;
12    if (a > r) {
13        S = C / 2 * r * r; D h = area / c; //res2 += -1 * sgn * C * r; D h = area / c;
14        if (h < r && B < pi / 2) {
15            //res2 -= -1 * sgn * 2 * acos(max((D)-1., min((D)1., h / r))) * r;
16            //res1 += 2 * sqrt(max((D)0., r * r - h * h));

```

```

16         S -= (acos(max((D)-1., min((D)1., h / r))) * r * r - h * sqrt(max((D)0., r * r
    ↪ - h * h)));
17     }
18     } else if (b > r) {
19         D theta = pi - B - asin(max((D)-1., min((D)1., sinB / r * a)));
20         S = a * r * sin(theta) / 2 + (C - theta) / 2 * r * r;
21         //res2 += -1 * sgn * (C - theta) * r;
22         //res1 += sqrt(max((D)0., r * r + a * a - 2 * r * a * cos(theta)));
23     } else S = area / 2; //res1 += (pb - pa).len();
24     return S;
25 }

```

6.4 圆并

```

1  struct Event {
2      Point p;
3      double ang;
4      int delta;
5      Event (Point p = Point(0, 0), double ang = 0, double delta = 0) : p(p), ang(ang),
    ↪ delta(delta) {}
6  };
7  bool operator < (const Event &a, const Event &b) {
8      return a.ang < b.ang;
9  }
10 void addEvent(const Circle &a, const Circle &b, vector<Event> &evt, int &cnt) {
11     double d2 = (a.o - b.o).len2(),
12         dRatio = ((a.r - b.r) * (a.r + b.r) / d2 + 1) / 2,
13         pRatio = sqrt(-(d2 - sqr(a.r - b.r)) * (d2 - sqr(a.r + b.r)) / (d2 * d2 * 4));
14     Point d = b.o - a.o, p = d.rotate(PI / 2),
15         q0 = a.o + d * dRatio + p * pRatio,
16         q1 = a.o + d * dRatio - p * pRatio;
17     double ang0 = (q0 - a.o).ang(),
18         ang1 = (q1 - a.o).ang();
19     evt.push_back(Event(q1, ang1, 1));
20     evt.push_back(Event(q0, ang0, -1));
21     cnt += ang1 > ang0;
22 }
23 bool issame(const Circle &a, const Circle &b) { return sign((a.o - b.o).len()) == 0 &&
    ↪ sign(a.r - b.r) == 0; }
24 bool overlap(const Circle &a, const Circle &b) { return sign(a.r - b.r - (a.o - b.o).len())
    ↪ >= 0; }
25 bool intersect(const Circle &a, const Circle &b) { return sign((a.o - b.o).len() - a.r -
    ↪ b.r) < 0; }
26 int C;
27 Circle c[N];
28 double area[N];
29 void solve() {
30     memset(area, 0, sizeof(double) * (C + 1));
31     for (int i = 0; i < C; ++i) {

```



```

32     int cnt = 1;
33     vector<Event> evt;
34     for (int j = 0; j < i; ++j) if (issame(c[i], c[j])) ++cnt;
35     for (int j = 0; j < C; ++j)
36         if (j != i && !issame(c[i], c[j]) && overlap(c[j], c[i]))
37             ++cnt;
38     for (int j = 0; j < C; ++j) {
39         if (j != i && !overlap(c[j], c[i]) && !overlap(c[i], c[j]) && intersect(c[i], c[j]))
40             addEvent(c[i], c[j], evt, cnt);
41         if (evt.size() == 0) {
42             area[cnt] += PI * c[i].r * c[i].r;
43         } else {
44             sort(evt.begin(), evt.end());
45             evt.push_back(evt.front());
46             for (int j = 0; j + 1 < (int)evt.size(); ++j) {
47                 cnt += evt[j].delta;
48                 area[cnt] += det(evt[j].p, evt[j + 1].p) / 2;
49                 double ang = evt[j + 1].ang - evt[j].ang;
50                 if (ang < 0) ang += PI * 2;
51                 area[cnt] += ang * c[i].r * c[i].r / 2 - sin(ang) * c[i].r * c[i].r / 2;
52             }
53         }
54     }
55 }

```

6.5 整数半平面交

```

1  typedef __int128 J; // 坐标 /1e9/ 就要用 int128 来判断
2  struct Line {
3      bool include(P a) const { return (a - s) * d >= 0; } // 严格去掉 =
4      bool include(Line a, Line b) const {
5          J l1(a.d * b.d);
6          if(!l1) return true;
7          J x(l1 * (a.s.x - s.x)), y(l1 * (a.s.y - s.y));
8          J l2((b.s - a.s) * b.d);
9          x += l2 * a.d.x; y += l2 * a.d.y;
10         J res(x * d.y - y * d.x);
11         return l1 > 0 ? res >= 0 : res <= 0; // 严格去掉 =
12     }
13 };
14 bool HPI(vector<Line> v) { // 返回 v 中每个射线的右侧的交是否非空
15     sort(v.begin(), v.end()); // 按方向排极角序
16     { // 同方向取最严格的一个
17         vector<Line> t; int n(v.size());
18         for(int i = 0, j; i < n; i = j) {
19             LL mx(-9e18); int mxi;
20             for(j = i; j < n && v[i].d * v[j].d == 0; j++) {
21                 LL tmp(v[j].s * v[i].d);
22                 if(tmp > mx)

```

```

23         mx = tmp, mxi = j;
24     }
25     t.push_back(v[mxi]);
26 }
27 swap(v, t);
28 }
29 deque<Line> res;
30 bool emp(false);
31 for(auto i : v) {
32     if(res.size() == 1) {
33         if(res[0].d * i.d == 0 && !i.include(res[0].s)) {
34             res.pop_back();
35             emp = true;
36         }
37     } else if(res.size() >= 2) {
38         while(res.size() >= 2u && !i.include(res.back(), res[res.size() - 2])) {
39             if(i.d * res[res.size() - 2].d == 0 || !res.back().include(i, res[res.size() - 2]))
40                 ↪ {
41                 emp = true;
42                 break;
43             }
44             res.pop_back();
45         }
46         while(res.size() >= 2u && !i.include(res[0], res[1])) res.pop_front();
47         if(emp) break;
48         res.push_back(i);
49     }
50     while (res.size() > 2u && !res[0].include(res.back(), res[res.size() - 2]))
51         ↪ res.pop_back();
52     return !emp; // emp: 是否为空, res 按顺序即为半平面交
53 }

```

6.6 半平面交

```

1 struct Point {
2     int quad() const { return sign(y) == 1 || (sign(y) == 0 && sign(x) >= 0); }
3 };
4 struct Line {
5     bool include(const Point &p) const { return sign(det(b - a, p - a)) > 0; }
6     Line push() const { // 将半平面向外推 eps
7         const double eps = 1e-6;
8         Point delta = (b - a).turn90().norm() * eps;
9         return Line(a - delta, b - delta);
10    }
11 };
12 bool sameDir(const Line &l0, const Line &l1) { return parallel(l0, l1) && sign(dot(l0.b -
13     ↪ l0.a, l1.b - l1.a)) == 1; }
14 bool operator < (const Point &a, const Point &b) {

```

```

14     if (a.quad() != b.quad()) {
15         return a.quad() < b.quad();
16     } else {
17         return sign(det(a, b)) > 0;
18     }
19 }
20 bool operator < (const Line &l0, const Line &l1) {
21     if (sameDir(l0, l1)) {
22         return l1.include(l0.a);
23     } else {
24         return (l0.b - l0.a) < (l1.b - l1.a);
25     }
26 }
27 bool check(const Line &u, const Line &v, const Line &w) { return w.include(intersect(u,
↪ v)); }
28 vector<Point> intersection(vector<Line> &l) {
29     sort(l.begin(), l.end());
30     deque<Line> q;
31     for (int i = 0; i < (int)l.size(); ++i) {
32         if (i && sameDir(l[i], l[i - 1])) {
33             continue;
34         }
35         while (q.size() > 1 && !check(q[q.size() - 2], q[q.size() - 1], l[i])) q.pop_back();
36         while (q.size() > 1 && !check(q[1], q[0], l[i])) q.pop_front();
37         q.push_back(l[i]);
38     }
39     while (q.size() > 2 && !check(q[q.size() - 2], q[q.size() - 1], q[0])) q.pop_back();
40     while (q.size() > 2 && !check(q[1], q[0], q[q.size() - 1])) q.pop_front();
41     vector<Point> ret;
42     for (int i = 0; i < (int)q.size(); ++i) ret.push_back(intersect(q[i], q[(i + 1) %
↪ q.size()]));
43     return ret;
44 }

```

6.7 三角形

```

1 Point fermat(const Point& a, const Point& b, const Point& c) {
2     double ab((b - a).len()), bc((b - c).len()), ca((c - a).len());
3     double cosa(dot(b - a, c - a) / ab / ca);
4     double cosb(dot(a - b, c - b) / ab / bc);
5     double cosc(dot(b - c, a - c) / ca / bc);
6     Point mid; double sq3(sqrt(3) / 2);
7     if(sgn(det(b - a, c - a)) < 0) swap(b, c);
8     if(sgn(cosa + 0.5) < 0) mid = a;
9     else if(sgn(cosb + 0.5) < 0) mid = b;
10    else if(sgn(cosc + 0.5) < 0) mid = c;
11    else mid = isLL(Line(a, c + (b - c).rot(sq3) - a), Line(c, b + (a - b).rot(sq3) - c));
12    return mid;
13    // mid 为三角形 abc 费马点, 要求 abc 非退化

```

```

14     length = (mid - a).len() + (mid - b).len() + (mid - c).len();
15     // 以下求法仅在三角形三个角均小于 120 度时, 可以求出 ans 为费马点到 abc 三点距离和
16     length = (a - c - (b - c).rot(sq3)).len();
17 }
18 Point inCenter(const Point &A, const Point &B, const Point &C) { // 内心
19     double a = (B - C).len(), b = (C - A).len(), c = (A - B).len(),
20         s = fabs(det(B - A, C - A)), r = s / p;
21     return (A * a + B * b + C * c) / (a + b + c); // 偏心则将对应点前两个加号改为减号
22 }
23 Point circumCenter(const Point &a, const Point &b, const Point &c) { // 外心
24     Point bb = b - a, cc = c - a;
25     double db = bb.len2(), dc = cc.len2(), d = 2 * det(bb, cc);
26     return a - Point(bb.y * dc - cc.y * db, cc.x * db - bb.x * dc) / d;
27 }
28 Point othroCenter(const Point &a, const Point &b, const Point &c) { // 垂心
29     Point ba = b - a, ca = c - a, bc = b - c;
30     double Y = ba.y * ca.y * bc.y,
31         A = ca.x * ba.y - ba.x * ca.y,
32         x0 = (Y + ca.x * ba.y * b.x - ba.x * ca.y * c.x) / A,
33         y0 = -ba.x * (x0 - c.x) / ba.y + ca.y;
34     return Point(x0, y0);
35 }

```

6.8 经纬度求球面最短距离

```

1 double sphereDis(double lon1, double lat1, double lon2, double lat2, double R) {
2     return R * acos(cos(lat1) * cos(lat2) * cos(lon1 - lon2) + sin(lat1) * sin(lat2));
3 }

```

6.9 长方体表面两点最短距离

```

1 int r;
2 void turn(int i, int j, int x, int y, int z, int x0, int y0, int L, int W, int H) {
3     if (z==0) { int R = x*x+y*y; if (R<r) r=R;
4     } else {
5         if(i>=0 && i< 2) turn(i+1, j, x0+L+z, y, x0+L-x, x0+L, y0, H, W, L);
6         if(j>=0 && j< 2) turn(i, j+1, x, y0+W+z, y0+W-y, x0, y0+W, L, H, W);
7         if(i<=0 && i>-2) turn(i-1, j, x0-z, y, x-x0, x0-H, y0, H, W, L);
8         if(j<=0 && j>-2) turn(i, j-1, x, y0-z, y-y0, x0, y0-H, L, H, W);
9     }
10 }
11 int main(){
12     int L, H, W, x1, y1, z1, x2, y2, z2;
13     cin >> L >> W >> H >> x1 >> y1 >> z1 >> x2 >> y2 >> z2;
14     if (z1!=0 && z1!=H) if (y1==0 || y1==W)
15         swap(y1,z1), std::swap(y2,z2), std::swap(W,H);
16     else swap(x1,z1), std::swap(x2,z2), std::swap(L,H);
17     if (z1==H) z1=0, z2=H-z2;
18     r=0x3fffffff;

```

```

19     turn(0,0,x2-x1,y2-y1,z2,-x1,-y1,L,W,H);
20     cout<<r<<endl;
21 }

```

6.10 点到凸包切线

```

1  P lb(P x, vector<P> & v, int le, int ri, int sg) {
2      if (le > ri) le = ri;
3      int s(le), t(ri);
4      while (le != ri) {
5          int mid((le + ri) / 2);
6          if (sign((v[mid] - x) * (v[mid + 1] - v[mid]))) == sg)
7              le = mid + 1; else ri = mid;
8      }
9      return x - v[le]; // le 即为下标, 按需返回
10 }
11 // v[0] 为顺时针上凸壳, v[1] 为顺时针下凸壳, 均允许起始两个点横坐标相同
12 // 返回值为真代表严格在凸包外, 顺时针旋转在 d1 方向先碰到凸包
13 bool getTan(P x, vector<P> * v, P & d1, P & d2) {
14     if (x.x < v[0][0].x) {
15         d1 = lb(x, v[0], 0, sz(v[0]) - 1, 1);
16         d2 = lb(x, v[1], 0, sz(v[1]) - 1, -1);
17         return true;
18     } else if (x.x > v[0].back().x) {
19         d1 = lb(x, v[1], 0, sz(v[1]) - 1, 1);
20         d2 = lb(x, v[0], 0, sz(v[0]) - 1, -1);
21         return true;
22     } else {
23         for(int d(0); d < 2; d++) {
24             int id(lower_bound(v[d].begin(), v[d].end(), x,
25                 [&](const P & a, const P & b) {
26                     return d == 0 ? a < b : b < a;
27                 }) - v[d].begin());
28             if (id && (id == sz(v[d]) || (v[d][id - 1] - x) * (v[d][id] - x) > 0)) {
29                 d1 = lb(x, v[d], id, sz(v[d]) - 1, 1);
30                 d2 = lb(x, v[d], 0, id, -1);
31                 return true;
32             }
33         }
34     }
35     return false;
36 }

```

6.11 直线与凸包的交点

```

1  // a 是顺时针凸包, i1 为 x 最小的点, j1 为 x 最大的点 需保证 j1 > i1
2  // n 是凸包上的点数, a 需复制多份或写循环数组类
3  int lowerBound(int le, int ri, const P & dir) {
4      while (le < ri) {

```

```

5     int mid((le + ri) / 2);
6     if (sign((a[mid + 1] - a[mid]) * dir) <= 0) {
7         le = mid + 1;
8     } else ri = mid;
9 }
10 return le;
11 }
12 int boundLower(int le, int ri, const P & s, const P & t) {
13     while (le < ri) {
14         int mid((le + ri + 1) / 2);
15         if (sign((a[mid] - s) * (t - s)) <= 0) {
16             le = mid;
17         } else ri = mid - 1;
18     }
19     return le;
20 }
21 void calc(P s, P t) {
22     if(t < s) swap(t, s);
23     int i3(lowerBound(i1, j1, t - s)); // 和上凸包的切点
24     int j3(lowerBound(j1, i1 + n, s - t)); // 和下凸包的切点
25     int i4(boundLower(i3, j3, s, t));
26     // ↪ // 如果有交则是右侧的交点, 与 a[i4]~a[i4+1] 相交 要判断是否有交的话 就手动 check 一下
27     int j4(boundLower(j3, i3 + n, t, s)); // 如果有交左侧的交点, 与 a[j4]~a[j4+1] 相交
28     // 返回的下标不一定在 [0 ~ n-1] 内
29 }

```

6.12 平面最近点对

```

1 struct Data { double x, y; };
2 double sqr(double x) { return x * x; }
3 double dis(Data a, Data b) { return sqrt(sqr(a.x - b.x) + sqr(a.y - b.y)); }
4 int n; Data p[N], q[N];
5 double solve(int l, int r) {
6     if(l == r) return 1e18;
7     if(l + 1 == r) return dis(p[l], p[r]);
8     int m = (l + r) / 2;
9     double d = min(solve(l, m), solve(m + 1, r));
10    int qt = 0;
11    for(int i = l; i <= r; i++)
12        if(fabs(p[m].x - p[i].x) <= d)
13            q[++qt] = p[i];
14    sort(q + 1, q + qt + 1, [&](const Data &a, const Data &b) { return a.y < b.y; });
15    for(int i = 1; i <= qt; i++) {
16        for(int j = i + 1; j <= qt; j++) {
17            if(q[j].y - q[i].y >= d) break;
18            d = min(d, dis(q[i], q[j]));
19        }
20    }
21    return d;

```

22 }

6.13 三维几何

```

1 Point3D det(const Point3D &a, const Point3D &b) {
2     return Point3D(a.y * b.z - a.z * b.y, a.z * b.x - a.x * b.z, a.x * b.y - a.y * b.x);
3 }
4 // 平面法向量 : 平面上两个向量叉积 点共平面 : 平面上一点与之的向量点积法向量为 0
5 // 点在线段 ( 直线 ) 上 : 共线且两边点积非正
6 // 点在三角形内 ( 不包含边界, 需再判断是与某条边共线 )
7 bool pointInTri(const Point3D &a, const Point3D &b, const Point3D &c, const Point3D &p) {
8     return sign(det(a - b, a - c).len() - det(p - a, p - b).len() - det(p - b, p - c).len() -
9         ↪ det(p - c, p - a).len()) == 0;
10 }
11 // 共平面的两点是否在这平面上一条直线的同侧
12 bool sameSide(const Point3D &a, const Point3D &b, const Point3D &p0, const Point3D &p1) {
13     return sign(dot(det(a - b, p0 - b), det(a - b, p1 - b))) > 0;
14 }
15 // 两点在平面同侧 : 点积法向量符号相同 两直线平行 / 垂直 : 同二维
16 // 平面平行 / 垂直 : 判断法向量 线面垂直 : 法向量和直线平行
17 // 判断空间线段是否相交 : 四点共面两线段不平行相互在异侧
18 // 线段和三角形是否相交 : 线段在三角形平面不同侧 三角形任意两点在线段和第三点组成的平面的不同侧
19 Point3D intersection(const Point3D &a0, const Point3D &b0, const Point3D &a1, const Point3D
20     ↪ &b1) { // 求空间直线交点
21     double t = ((a0.x - a1.x) * (a1.y - b1.y) - (a0.y - a1.y) * (a1.x - b1.x)) / ((a0.x -
22         ↪ b0.x) * (a1.y - b1.y) - (a0.y - b0.y) * (a1.x - b1.x));
23     return a0 + (b0 - a0) * t;
24 }
25 Point3D intersection(const Point3D &a, const Point3D &b, const Point3D &c, const Point3D
26     ↪ &l0, const Point3D &l1) { // 求平面和直线的交点
27     Point3D p = pVec(a, b, c); // 平面法向量
28     double t = (p.x * (a.x - l0.x) + p.y * (a.y - l0.y) + p.z * (a.z - l0.z)) / (p.x * (l1.x
29         ↪ - l0.x) + p.y * (l1.y - l0.y) + p.z * (l1.z - l0.z));
30     return l0 + (l1 - l0) * t;
31 }
32 // 求平面交线 : 取不平行的一条直线的一个交点, 以及法向量叉积得到直线方向
33 // 点到直线距离 : 叉积得到三角形的面积除以底边 点到平面距离 : 点积法向量
34 // 直线间距离 : 平行时随便取一点求距离, 否则叉积方向向量得到方向点积计算长度
35 // 直线夹角 : 点积 平面夹角 : 法向量点积
36 // 三维向量旋转操作 (绕向量 s 旋转 ang 角度), 对于右手系 s 指向观察者时逆时针
37 void rotate(const Point3D &s, double ang) {
38     double l = s.len(), x = s.x / l, y = s.y / l, z = s.z / l, sinA = sin(ang), cosA =
39         ↪ cos(ang);
40     double p[4][4] = {CosA + (1 - CosA) * x * x, (1 - CosA) * x * y - SinA * z, (1 - CosA) * x
41         ↪ * z + SinA * y, 0,
42         (1 - CosA) * y * x + SinA * z, CosA + (1 - CosA) * y * y, (1 - CosA) * y * z - SinA *
43         ↪ x, 0,
44         (1 - CosA) * z * x - SinA * y, (1 - CosA) * z * y + SinA * x, CosA + (1 - CosA) * z *
45         ↪ z, 0,

```

```

37     0, 0, 0, 1 };
38 }
39 // 计算版 : 把需要旋转的向量按照 s 分解, 做二维旋转, 再回到三维

```

7 其他

7.1 DP 优化

7.1.1 决策单调性

```

1  //Decision Monotony
2  //分治写法
3  ll Calc(int j, int i){
4      return f[la][j] + A[j] * (A[i] - A[j]);
5  }
6  void solve(int l, int r, int pl, int pr){
7      if(l > r)return;
8      int mid = (l + r) >> 1, pm = pr;
9      f[no][mid] = -inf;
10     for(int i = pl; i <= pr && i < mid; ++i){
11         if(Calc(i, mid) > f[no][mid]){
12             f[no][mid] = Calc(i, mid);
13             pm = i;
14         }
15     }
16     solve(l, mid - 1, pl, pm);
17     solve(mid + 1, r, pm, pr);
18 }
19
20 //二分写法
21 ll Calc(int fl, int j, int i){
22     return f[f1][j] + A[j] * (A[i] - A[j]);
23 }
24 void solve(){
25     int la = 1, no = 0;
26     for(int k = 1; k <= K; ++k){
27         int h = 1, t = 0;
28         la ^= 1, no ^= 1;
29         Q[++t] = Node(k, k + 1, n);
30         for(int i = k + 1; i <= n; ++i){
31             if(h <= t && Q[h].l > Q[h].r)h++;
32             Q[h].l++;
33             f[no][i] = Calc(la, Q[h].p, i);
34             if(h <= t && Calc(la, i, n) <= Calc(la, Q[h].p, n))continue;
35             while(h <= t && Calc(la, i, Q[t].l) >= Calc(la, Q[t].p, Q[t].l))t--;
36             if(h <= t){
37                 int l = Q[t].l, r = Q[t].r, mid;
38                 while(l <= r){
39                     mid = (l + r) >> 1;

```



```

40         if(Calc(la, i, mid) > Calc(la, Q[t].p, mid))r = mid - 1;
41         else l = mid + 1;
42     }
43     Q[t].r = l - 1;
44     Q[++t] = Node(i, l, n);
45 }
46 else Q[++t] = Node(i, i + 1, n);
47 }
48 }
49 printf("%lld\n", f[no][n]);
50 }

```

7.1.2 斜率优化

```

1 //Hdu 3507 Print Article
2 //根据小于或大于判断维护斜率递增还是递减。
3 //根据递增或递减判断答案取队首还是队尾。
4 int n, m, Q[maxn];
5 ll f[maxn], sum[maxn];
6 ll Calc(int j, int i){
7     return f[j] + (sum[i] - sum[j]) * (sum[i] - sum[j]) + m;
8 }
9 double Slope(int j, int k){
10     if(sum[j] == sum[k])return 1e20;
11     return (double)(f[j] + 1.0 * sum[j] * sum[j] - f[k] - 1.0 * sum[k] * sum[k]) / (sum[j] -
    ↪ sum[k]);
12 }
13 void DP(){
14     int h = 1, t = 0;
15     Q[++t] = 0;
16     for(int i = 1; i <= n; ++i){
17         while(h < t && Slope(Q[h], Q[h + 1]) <= 2.0 * sum[i])h++;
18         f[i] = Calc(Q[h], i);
19         while(h < t && Slope(Q[t - 1], Q[t]) >= Slope(Q[t], i))t--;
20         Q[++t] = i;
21     }
22     printf("%lld\n", f[n]);
23 }

```

7.2 斯坦纳树

```

1 void spfa(int state) {
2     while (!q.empty()) {
3         int now = q.front(); q.pop();
4         for (auto v: g[now]) {
5             double len = d[now][v];
6             if (f[v][st[v] | state] > f[now][state] + len) {
7                 f[v][st[v] | state] = f[now][state] + len;
8                 if ((st[v] | state) != state || vis[v][state])
9                     continue;

```

```

10         vis[v][state] = true;
11         q.push(v);
12     }
13 }
14     vis[now][state] = false;
15 }
16 }
17
18 double solve() {
19     int endst = 1 << k;
20
21     for (int i = 1; i <= n; i++)
22         for (int j = 0; j < endst; j++)
23             f[i][j] = 1e18;
24
25     for (int i = 1; i <= n; i++) f[i][st[i]] = 0;
26     for (int j = 1; j < endst; j++) {
27         for (int i = 1; i <= n; i++) {
28             if (!st[i] || (st[i] & j)) {
29                 for (int sub = (j-1)&j; sub; sub = (sub-1) & j) {
30                     int x = sub | st[i], y = (j - sub) | st[i];
31                     update(f[i][j], f[i][x] + f[i][y]);
32                 }
33             }
34             if (f[i][j] < 1e18) {
35                 q.push(i);
36                 vis[i][j] = true;
37             }
38         }
39         spfa(j);
40     }
41
42     double ans = 1e18;
43     for (int i = 1; i <= n; i++) {
44         update(ans, f[i][endst - 1]);
45     }
46     return ans;
47 }

```

7.3 无敌的读入优化

```

1 namespace Reader {
2     const int L = (1 << 20) + 5;
3     char buffer[L], *S, *T;
4     __inline bool getchar(char &ch) {
5         if (S == T) {
6             T = (S = buffer) + fread(buffer, 1, L, stdin);
7             if (S == T) {
8                 ch = EOF;

```

```

9         return false;
10    }
11 }
12 ch = *S ++;
13 return true;
14 }
15 __inline bool getint(int &x) {
16     char ch;
17     for (; getchar(ch) && (ch < '0' || ch > '9'); );
18     if (ch == EOF) return false;
19     x = ch - '0';
20     for (; getchar(ch), ch >= '0' && ch <= '9'; )
21         x = x * 10 + ch - '0';
22     return true;
23 }
24 }
25 Reader::getint(x);
26 Reader::getint(y);

```

7.4 最小树形图

```

1  const int maxn=1100;
2
3  int n,m , g[maxn][maxn] , used[maxn] , pass[maxn] , eg[maxn] , more , queue[maxn];
4
5  void combine (int id , int &sum ) {
6      int tot = 0 , from , i , j , k ;
7      for ( ; id!=0 && !pass[ id ] ; id=eg[id] ) {
8          queue[tot++]=id ; pass[id]=1;
9      }
10     for ( from=0; from<tot && queue[from]!=id ; from++);
11     if ( from==tot ) return ;
12     more = 1 ;
13     for ( i=from ; i<tot ; i++) {
14         sum+=g[eg[queue[i]]][queue[i]] ;
15         if ( i!=from ) {
16             used[queue[i]]=1;
17             for ( j = 1 ; j <= n ; j++) if ( !used[j] )
18                 if ( g[queue[i]][j]<g[id][j] ) g[id][j]=g[queue[i]][j] ;
19         }
20     }
21     for ( i=1; i<=n ; i++) if ( !used[i] && i!=id ) {
22         for ( j=from ; j<tot ; j++){
23             k=queue[j];
24             if ( g[i][id]>g[i][k]-g[eg[k]][k] ) g[i][id]=g[i][k]-g[eg[k]][k];
25         }
26     }
27 }
28

```

```

29 int mdst( int root ) { // return the total length of MDST
30     int i , j , k , sum = 0 ;
31     memset ( used , 0 , sizeof ( used ) ) ;
32     for ( more =1; more ; ) {
33         more = 0 ;
34         memset (eg,0,sizeof(eg)) ;
35         for ( i=1 ; i <= n ; i ++ ) if ( !used[i] && i!=root ) {
36             for ( j=1 , k=0 ; j <= n ; j ++ ) if ( !used[j] && i!=j )
37                 if ( k==0 || g[j][i] < g[k][i] ) k=j ;
38             eg[i] = k ;
39         }
40         memset(pass,0,sizeof(pass));
41         for ( i=1; i<=n ; i++) if ( !used[i] && !pass[i] && i!= root ) combine ( i , sum ) ;
42     }
43     for ( i =1; i<=n ; i ++ ) if ( !used[i] && i!= root ) sum+=g[eg[i]][i];
44     return sum ;
45 }

```

7.5 DLX

```

1  int n,m,K;
2  struct DLX{
3      int L[maxn],R[maxn],U[maxn],D[maxn];
4      int sz,col[maxn],row[maxn],s[maxn],H[maxn];
5      bool vis[233];
6      int ans[maxn],cnt;
7      void init(int m){
8          for(int i=0;i<=m;i++){
9              L[i]=i-1;R[i]=i+1;
10             U[i]=D[i]=i;s[i]=0;
11         }
12         memset(H,-1,sizeof H);
13         L[0]=m;R[m]=0;sz=m+1;
14     }
15     void Link(int r,int c){
16         U[sz]=c;D[sz]=D[c];U[D[c]]=sz;D[c]=sz;
17         if(H[r]<0)H[r]=L[sz]=R[sz]=sz;
18         else{
19             L[sz]=H[r];R[sz]=R[H[r]];
20             L[R[H[r]]]=sz;R[H[r]]=sz;
21         }
22         s[c]++;col[sz]=c;row[sz]=r;sz++;
23     }
24     void remove(int c){
25         for(int i=D[c];i!=c;i=D[i])
26             L[R[i]]=L[i],R[L[i]]=R[i];
27     }
28     void resume(int c){
29         for(int i=U[c];i!=c;i=U[i])

```

```

30     L[R[i]]=R[L[i]]=i;
31 }
32 int A(){
33     int res=0;
34     memset(vis,0,sizeof vis);
35     for(int i=R[0];i;i=R[i])if(!vis[i]){
36         vis[i]=1;res++;
37         for(int j=D[i];j!=i;j=D[j])
38             for(int k=R[j];k!=j;k=R[k])
39                 vis[col[k]]=1;
40     }
41     return res;
42 }
43 void dfs(int d,int &ans){
44     if(R[0]==0){ans=min(ans,d);return;}
45     if(d+A()>=ans)return;
46     int tmp=23333,c;
47     for(int i=R[0];i;i=R[i])
48         if(tmp>s[i])tmp=s[i],c=i;
49     for(int i=D[c];i!=c;i=D[i]){
50         remove(i);
51         for(int j=R[i];j!=i;j=R[j])remove(j);
52         dfs(d+1,ans);
53         for(int j=L[i];j!=i;j=L[j])resume(j);
54         resume(i);
55     }
56 }
57 void del(int c){//exactly cover
58     L[R[c]]=L[c];R[L[c]]=R[c];
59     for(int i=D[c];i!=c;i=D[i])
60         for(int j=R[i];j!=i;j=R[j])
61             U[D[j]]=U[j],D[U[j]]=D[j],--s[col[j]]];
62 }
63 void add(int c){ //exactly cover
64     R[L[c]]=L[R[c]]=c;
65     for(int i=U[c];i!=c;i=U[i])
66         for(int j=L[i];j!=i;j=L[j])
67             ++s[col[U[D[j]]=D[U[j]]=j]];
68 }
69 bool dfs2(int k){//exactly cover
70     if(!R[0]){
71         cnt=k;return 1;
72     }
73     int c=R[0];
74     for(int i=R[0];i;i=R[i])
75         if(s[c]>s[i])c=i;
76     del(c);
77     for(int i=D[c];i!=c;i=D[i]){
78         for(int j=R[i];j!=i;j=R[j])

```

```

79         del(col[j]);
80         ans[k]=row[i];if(dfs2(k+1))return true;
81     for(int j=L[i];j!=i;j=L[j])
82         add(col[j]);
83     }
84     add(c);
85     return 0;
86 }
87 }dlx;
88 int main(){
89     dlx.init(n);
90     for(int i=1;i<=m;i++)
91         for(int j=1;j<=n;j++)
92             if(dis(station[i],city[j])<mid-eps)
93                 dlx.Link(i,j);
94     dlx.dfs(0,ans);
95 }

```

7.6 插头 DP

```

1  int n,m,l;
2  struct L{
3      int d[11];
4      int& operator[](int x){return d[x];}
5      int mc(int x){
6          int an=1;
7          if(d[x]==1){
8              for(x++;x<1;x++)if(d[x]){
9                  an=an+(d[x]==1?1:-1);
10                 if(!an)return x;
11             }
12         }else{
13             for(x--;x>=0;x--)if(d[x]){
14                 an=an+(d[x]==2?1:-1);
15                 if(!an)return x;
16             }
17         }
18     }
19     int h(){int an=0;for(int i=1-1;i>=0;i--)an=an*3+d[i];return an;}
20     L s(int x,int y){
21         L S=*this;
22         S[x]=y;return S;
23     }
24     L operator>>(int _){
25         L S=*this;
26         for(int i=1-1;i>=1;i--)S[i]=S[i-1];
27         S[0]=0;return S;
28     }
29 };

```

```

30 struct Int{
31     int len;
32     int a[40];
33     Int(){len=1;memset(a,0,sizeof a);}
34     Int operator+=(const Int &o){
35         int l=max(len,o.len);
36         for(int i=0;i<l;i++){
37             a[i]=a[i]+o.a[i];
38         }
39         for(int i=0;i<l;i++){
40             a[i+1]+=a[i]/10,a[i]%=10;
41         }
42         if(a[l])l++;len=l;
43         return *this;
44     }
45     void print(){
46         for(int i=len-1;i>=0;i--)
47             printf("%d",a[i]);
48         puts("");
49     }
50 };
51 struct hashtable{
52     int sz;
53     int tab[177147];
54     Int w[177147];
55     L s[177147];
56     hashtable(){memset(tab,-1,sizeof tab);}
57     void cl(){
58         for(int i=0;i<sz;i++)tab[s[i].h()]=-1;
59         sz=0;
60     }
61     Int& operator[](L S){
62         int h=S.h();
63         if(tab[h]==-1)tab[h]=sz,s[sz]=S,w[sz]=Int(),sz++;
64         return w[tab[h]];
65     }
66 }f[2];
67 bool check(L S){
68     int cn1=0,cn2=0;
69     for(int i=0;i<l;i++){
70         cn1+=S[i]==1;
71         cn2+=S[i]==2;
72     }return cn1==1&&cn2==1;
73 }
74 int main(){
75     Int One;One.a[0]=1;
76     scanf("%d%d",&n,&m);if(n<m)swap(n,m);l=m+1;
77     if(n==1||m==1){puts("1");return 0;}
78     int cur=0;f[cur].cl();
79     for(int i=1;i<=n;i++){
80         for(int j=1;j<=m;j++){

```

```

79         if(i==1&&j==1){
80             f[cur][L().s(0,1).s(1,2)]+=One;
81             continue;
82         }
83         cur^=1;f[cur].cl();
84         for(int k=0;k<f[!cur].sz;k++){
85             L S=f[!cur].s[k];Int w=f[!cur][S];
86             int d1=S[j-1],d2=S[j];
87             if(d1==0&&d2==0){
88                 if(i!=n&&j!=m)f[cur][S.s(j-1,1).s(j,2)]+=w;
89             }else
90             if(d1==0||d2==0){
91                 if(i!=n)f[cur][S.s(j-1,d1|d2).s(j,0)]+=w;
92                 if(j!=m)f[cur][S.s(j-1,0).s(j,d1|d2)]+=w;
93             }else
94             if(d1==1&&d2==2){
95                 if(i==n&&j==m&&check(S))
96                     (w+=w).print();
97             }else
98             if(d1==2&&d2==1){
99                 f[cur][S.s(j-1,0).s(j,0)]+=w;
100             }else
101             if((d1==1&&d2==1)|| (d1==2&&d2==2)){
102                 int m1=S.mc(j),m2=S.mc(j-1);
103                 f[cur][S.s(j-1,0).s(j,0).s(m1,1).s(m2,2)]+=w;
104             }
105         }
106     }
107     cur^=1;f[cur].cl();
108     for(int k=0;k<f[!cur].sz;k++){
109         L S=f[!cur].s[k];Int w=f[!cur][S];
110         f[cur][S>>1]=w;
111     }
112 }
113 return 0;
114 }

```

7.7 某年某月某日是星期几

```

1  int solve(int year, int month, int day) {
2      int answer;
3      if (month == 1 || month == 2) {
4          month += 12;
5          year--;
6      }
7      if ((year < 1752) || (year == 1752 && month < 9) ||
8          (year == 1752 && month == 9 && day < 3)) {
9          answer = (day + 2 * month + 3 * (month + 1) / 5 + year + year / 4 + 5) % 7;
10     } else {

```



```

11         answer = (day + 2 * month + 3 * (month + 1) / 5 + year + year / 4
12                 - year / 100 + year / 400) % 7;
13     }
14     return answer;
15 }

```

7.8 枚举大小为 k 的子集

使用条件: $k > 0$

```

1 void solve(int n, int k) {
2     for (int comb = (1 << k) - 1; comb < (1 << n); ) {
3         // ...
4         int x = comb & -comb, y = comb + x;
5         comb = (((comb & ~y) / x) >> 1) | y;
6     }
7 }

```

7.9 环状最长公共子串

```

1 int n, a[N << 1], b[N << 1];
2
3 bool has(int i, int j) {
4     return a[(i - 1) % n] == b[(j - 1) % n];
5 }
6
7 const int DELTA[3][2] = {{0, -1}, {-1, -1}, {-1, 0}};
8
9 int from[N][N];
10
11 int solve() {
12     memset(from, 0, sizeof(from));
13     int ret = 0;
14     for (int i = 1; i <= 2 * n; ++i) {
15         from[i][0] = 2;
16         int left = 0, up = 0;
17         for (int j = 1; j <= n; ++j) {
18             int upleft = up + 1 + !!from[i - 1][j];
19             if (!has(i, j)) {
20                 upleft = INT_MIN;
21             }
22             int max = std::max(left, std::max(upleft, up));
23             if (left == max) {
24                 from[i][j] = 0;
25             } else if (upleft == max) {
26                 from[i][j] = 1;
27             } else {
28                 from[i][j] = 2;
29             }
30             left = max;

```

```

31     }
32     if (i >= n) {
33         int count = 0;
34         for (int x = i, y = n; y; ) {
35             int t = from[x][y];
36             count += t == 1;
37             x += DELTA[t][0];
38             y += DELTA[t][1];
39         }
40         ret = std::max(ret, count);
41         int x = i - n + 1;
42         from[x][0] = 0;
43         int y = 0;
44         while (y <= n && from[x][y] == 0) {
45             y++;
46         }
47         for (; x <= i; ++x) {
48             from[x][y] = 0;
49             if (x == i) {
50                 break;
51             }
52             for (; y <= n; ++y) {
53                 if (from[x + 1][y] == 2) {
54                     break;
55                 }
56                 if (y + 1 <= n && from[x + 1][y + 1] == 1) {
57                     y++;
58                     break;
59                 }
60             }
61         }
62     }
63 }
64 return ret;
65 }

```

7.10 LLMOD

```

1 LL multiplyMod(LL a, LL b, LL P) { // `需要保证 a 和 b 非负`
2     LL t = (a * b - LL((long double)a / P * b + 1e-3) * P) % P;
3     return t < 0 : t + P : t;
4 }

```

7.11 STL 内存清空

```

1 template <typename T>
2 __inline void clear(T& container) {
3     container.clear(); // 或者删除了一堆元素
4     T(container).swap(container);

```

```
5 }
```

7.12 开栈

```
1 register char *_sp __asm__("rsp");
2 int main() {
3     const int size = 400 << 20; //400MB
4     static char *sys, *mine(new char[size] + size - 4096);
5     sys = _sp; _sp = mine; _main(); _sp = sys;
6 }
```

7.13 32-bit/64-bit 随机素数

| 32-bit | 64-bit |
|------------|---------------------|
| 73550053 | 1249292846855685773 |
| 148898719 | 1701750434419805569 |
| 189560747 | 3605499878424114901 |
| 459874703 | 5648316673387803781 |
| 1202316001 | 6125342570814357977 |
| 1431183547 | 6215155308775851301 |
| 1438011109 | 6294606778040623451 |
| 1538762023 | 6347330550446020547 |
| 1557944263 | 7429632924303725207 |
| 1981315913 | 8524720079480389849 |

8 vimrc

```
1 set ruler
2 set number
3 set smartindent
4 set autoindent
5 set tabstop=4
6 set softtabstop=4
7 set shiftwidth=4
8 set hlsearch
9 set incsearch
10 set autoread
11 set backspace=2
12 set mouse=a
13
14 syntax on
15
16 nmap <C-A> ggVG
17 vmap <C-C> "+y
18
19 filetype plugin indent on
20
```

```

21 autocmd FileType cpp set cindent
22 autocmd FileType cpp map <F9> :w <CR> :!g++ % -o %< -g -std=c++11 -Wall -Wextra
    ↪ -Wconversion && size %< <CR>
23 autocmd FileType cpp map <C-F9> :!g++ % -o %< -std=c++11 -O2 && size %< <CR>
24 autocmd FileType cpp map <F8> :!time ./%< < %<.in <CR>
25 autocmd FileType cpp map <F5> :!time ./%< <CR>
26
27 map <F3> :vnew %<.in <CR>
28 map <F4> :!gedit % <CR>

```

9 常用结论

9.1 上下界网络流

$B(u, v)$ 表示边 (u, v) 流量的下界, $C(u, v)$ 表示边 (u, v) 流量的上界, $F(u, v)$ 表示边 (u, v) 的流量。设 $G(u, v) = F(u, v) - B(u, v)$, 显然有

$$0 \leq G(u, v) \leq C(u, v) - B(u, v)$$

无源汇的上下界可行流

建立超级源点 S^* 和超级汇点 T^* , 对于原图每条边 (u, v) 在新网络中连如下三条边: $S^* \rightarrow v$, 容量为 $B(u, v)$; $u \rightarrow T^*$, 容量为 $B(u, v)$; $u \rightarrow v$, 容量为 $C(u, v) - B(u, v)$ 。最后求新网络的最大流, 判断从超级源点 S^* 出发的边是否都满流即可, 边 (u, v) 的最终解中的实际流量为 $G(u, v) + B(u, v)$ 。

有源汇的上下界可行流

从汇点 T 到源点 S 连一条上界为 ∞ , 下界为 0 的边。按照无源汇的上下界可行流一样做即可, 流量即为 $T \rightarrow S$ 边上的流量。

有源汇的上下界最大流

1. 在有源汇的上下界可行流中, 从汇点 T 到源点 S 的边改为连一条上界为 ∞ , 下届为 x 的边。 x 满足二分性质, 找到最大的 x 使得新网络存在无源汇的上下界可行流即为原图的最大流。
2. 从汇点 T 到源点 S 连一条上界为 ∞ , 下界为 0 的边, 变成无源汇的网络。按照无源汇的上下界可行流的方法, 建立超级源点 S^* 和超级汇点 T^* , 求一遍 $S^* \rightarrow T^*$ 的最大流, 再将 从汇点 T 到源点 S 的这条边拆掉, 求一次 $S \rightarrow T$ 的最大流即可。

有源汇的上下界最小流

1. 在有源汇的上下界可行流中, 从汇点 T 到源点 S 的边改为连一条上界为 x , 下界为 0 的边。 x 满足二分性质, 找到最小的 x 使得新网络存在无源汇的上下界可行流即为原图的最小流。

2. 按照无源汇的上下界可行流的方法，建立超级源点 S^* 与超级汇点 T^* ，求一遍 $S^* \rightarrow T^*$ 的最大流，但是注意这一次不加上汇点 T 到源点 S 的这条边，即不使之改为无源汇的网络去求解。求完后，再加上那条汇点 T 到源点 S 上界 ∞ 的边。因为这条边下界为 0，所以 S^* ， T^* 无影响，再直接求一次 $S^* \rightarrow T^*$ 的最大流。若超级源点 S^* 出发的边全部满流，则 $T \rightarrow S$ 边上的流量即为原图的最小流，否则无解。

9.2 上下界费用流

设汇 t ，源 s ，超级源 S ，超级汇 T ，本质是每条边的下界为 1，上界为 MAX ，跑一遍有源汇的上下界最小费用最小流。（因为上界无穷大，所以只要满足所有下界的最小费用最小流）

1. 对每个点 x ：从 x 到 t 连一条费用为 0，流量为 MAX 的边，表示可以任意停止当前的剧情（接下来的剧情从更优的路径去走，画个样例就知道了）
2. 对于每一条边权为 z 的边 $x \rightarrow y$ ：
 - 从 S 到 y 连一条流量为 1，费用为 z 的边，代表这条边至少要被走一次。
 - 从 x 到 y 连一条流量为 MAX ，费用为 z 的边，代表这条边除了至少走的一次之外还可以随便走。
 - 从 x 到 T 连一条流量为 1，费用为 0 的边。（注意是每一条 $x \rightarrow y$ 的边都连，或者你可以记下 x 的出边数 K_x ，连一次流量为 K_x ，费用为 0 的边）。

建完图后从 S 到 T 跑一遍费用流，即可。（当前跑出来的就是满足上下界的最小费用最小流了）

9.3 弦图相关

1. 团数 \leq 色数，弦图团数 = 色数
2. 设 $\text{next}(v)$ 表示 $N(v)$ 中最前的点。令 w^* 表示所有满足 $A \in B$ 的 w 中最后的一个点，判断 $v \cup N(v)$ 是否为极大团，只需判断是否存在一个 w ，满足 $\text{Next}(w) = v$ 且 $|N(v)| + 1 \leq |N(w)|$ 即可。
3. 最小染色：完美消除序列从后往前依次给每个点染色，给每个点染上可以染的最小的颜色
4. 最大独立集：完美消除序列从前往后能选就选
5. 弦图最大独立集数 = 最小团覆盖数，最小团覆盖：设最大独立集为 $\{p_1, p_2, \dots, p_t\}$ ，则 $\{p_1 \cup N(p_1), \dots, p_t \cup N(p_t)\}$ 为最小团覆盖

10 常见错误

1. 数组或者变量类型开错，例如将 `double` 开成 `int`；
2. 函数忘记返回返回值；
3. 初始化数组没有初始化完全；
4. 对空间限制判断不足导致 MLE；
5. 对于重边未注意，

6. 对于 0、1base 未弄清楚，用混
7. map 的赋值问题 (`dis[] = find(dis[])`)
8. 输出格式
9. 置换的正反顺序写错

11 测试列表

1. 检测评测机是否开 O2;
2. 检测 `__int128` 以及 `__float128` 是否能够使用;
3. 检测是否能够使用 C++11;
4. 检测是否能够使用 Ext Lib;
5. 检测程序运行所能使用的内存大小;
6. 检测程序运行所能使用的栈大小;
7. 检测是否有代码长度限制;
8. 检测是否能够正常返 Runtime Error (assertion、return 1、空指针);
9. 检测是否有 MLE 的结果;
10. 查清楚厕所方位和打印机方位;

12 Java

12.1 Java Hints

```

1  import java.util.*;
2  import java.math.*;
3  import java.io.*;
4
5  public class Main{
6      static class Task{
7          void solve(int testId, InputReader cin, PrintWriter cout) {
8              // Write down the code you want
9          }
10     };
11
12     public static void main(String args[]) {
13         InputStream inputStream = System.in;
14         OutputStream outputStream = System.out;
15         InputReader in = new InputReader(inputStream);
16         PrintWriter out = new PrintWriter(outputStream);
17         TaskA solver = new TaskA();
18         solver.solve(1, in, out);
19         out.close();
20     }

```

```

21
22 static class InputReader {
23     public BufferedReader reader;
24     public StringTokenizer tokenizer;
25
26     public InputReader(InputStream stream) {
27         reader = new BufferedReader(new InputStreamReader(stream), 32768);
28         tokenizer = null;
29     }
30
31     public String next() {
32         while (tokenizer == null || !tokenizer.hasMoreTokens()) {
33             try {
34                 tokenizer = new StringTokenizer(reader.readLine());
35             } catch (IOException e) {
36                 throw new RuntimeException(e);
37             }
38         }
39         return tokenizer.nextToken();
40     }
41
42     public int nextInt() {
43         return Integer.parseInt(next());
44     }
45
46 }
47 };
48 // Arrays
49 int a[];
50 .fill(a[, int fromIndex, int toIndex],val); | .sort(a[, int fromIndex, int toIndex])
51 // String
52 String s;
53 .charAt(int i); | compareTo(String) | compareToIgnoreCase () | contains(String) |
54 length () | substring(int l, int len)
55 // BigInteger
56 .abs() | .add() | bitLength () | subtract () | divide () | remainder () |
57   ↳ divideAndRemainder () | modPow(b, c) |
58 pow(int) | multiply () | compareTo () |
59 gcd() | intValue () | longValue () | isProbablePrime(int c) (1 - 1/2^c) |
60 nextProbablePrime () | shiftLeft(int) | valueOf ()
61 // BigDecimal
62 .ROUND_CEILING | ROUND_DOWN_FLOOR | ROUND_HALF_DOWN | ROUND_HALF_EVEN | ROUND_HALF_UP |
63   ↳ ROUND_UP
64 .divide(BigDecimal b, int scale , int round_mode) | doubleValue () | movePointLeft(int) |
65   ↳ pow(int) |
66 setScale(int scale , int round_mode) | stripTrailingZeros ()
67     BigDecimal.setScale()方法用于格式化小数点
68     setScale(1)表示保留一位小数,默认用四舍五入方式
69     setScale(1,BigDecimal.ROUND_DOWN)直接删除多余的小数位,如 2.35会变成 2.3

```

```

67 setScale(1,BigDecimal.ROUND_UP)进位处理, 2.35变成 2.4
68 setScale(1,BigDecimal.ROUND_HALF_UP)四舍五入, 2.35变成 2.4
69 setScaler(1,BigDecimal.ROUND_HALF_DOWN)四舍五入, 2.35变成 2.3, 如果是 5 则向下舍
70 setScaler(1,BigDecimal.ROUND_CEILING)接近正无穷大的舍入
71 setScaler(1,BigDecimal.ROUND_FLOOR)接近负无穷大的舍入, 数字>0和 ROUND_UP 作用一样, 数字<0和 ROUND_DOWN 作用一样
72 setScaler(1,BigDecimal.ROUND_HALF_EVEN)向最接近的数字舍入, 如果与两个相邻数字的距离相等, 则向相邻的偶数舍入。
73 // StringBuilder
74 StringBuilder sb = new StringBuilder ();
75 sb.append(elem) | out.println(sb)
76 // TODO Java STL 的使用方法以及上面这些方法的检验

```

12.2 样例代码

```

1  import java.io.*;
2  import java.math.*;
3  import java.util.*;
4
5  public static class edge implements Comparable<edge>{
6      public int u,v,w;
7      public int compareTo(edge e){
8          return w-e.w;
9      }
10 }
11 public static class cmp implements Comparator<edge>{
12     public int compare(edge a,edge b){
13         if(a.w<b.w)return 1;
14         if(a.w>b.w)return -1;
15         return 0;
16     }
17 }
18
19 public class Main{
20     public static long max(long a,long b){
21         if(a>b)return a;
22         return b;
23     }
24     public static long Calc(long A, int x){
25         long Ret = (A / (1L << x)) * (1L << (x - 1));
26
27         Ret += max(A % (1L << x) - (1L << (x - 1)) + 1, 0L);
28
29         return Ret;
30     }
31     private static class InputReader {
32
33         public BufferedReader rea;
34         public StringTokenizer tok;
35
36         public InputReader(InputStream stream) {

```



```

37         rea = new BufferedReader(new InputStreamReader(stream), 32768);
38         tok = null;
39     }
40
41     public String next() {
42         while (tok == null || !tok.hasMoreTokens()) {
43             try {
44                 tok = new StringTokenizer(re.readLine());
45             } catch (IOException e) {
46                 throw new RuntimeException(e);
47             }
48         }
49         return tok.nextToken();
50     }
51
52     public int nextInt() {
53         return Integer.parseInt(next());
54     }
55
56     public long nextLong() {
57         return Long.parseLong(next());
58     }
59 }
60
61 public static void main(String arg[]){
62     InputReader cin = new InputReader(System.in);
63     //Scanner cin = new Scanner(System.in);
64     int N = 70;
65
66     long k[] = new long[N];
67     int n;
68
69     while(true){
70         n=cin.nextInt();
71         if (n == 0) break;
72
73         for (int i = 1; i <= n; i++){
74             k[i]=cin.nextLong();
75             //System.out.println(k[i]);
76         }
77
78         long Len;
79         BigInteger Sum;
80         long A, B;
81         long AnsA = -1, AnsB = -1;
82         int Ans = 0;
83
84         for (int i = -1; i <= 1; i++){
85             Len = k[1] * 2 + i;

```

```

86
87     if(Len<=0)continue;
88
89     Sum = BigInteger.ZERO;
90     for (int j = 1; j <= n; j++){
91         Sum = Sum.add(BigInteger.valueOf(1L << (j - 1))
92             ↳ .multiply(BigInteger.valueOf(k[j])));
93     }
94     //System.out.println(Sum);
95
96     long x = Len;
97
98     if ((Sum.multiply(BigInteger.valueOf(2))) .mod
99     ↳ (BigInteger.valueOf(Len)).compareTo(BigInteger.ZERO) > 0) continue;
100
101     long y =
102     ↳ Sum.multiply(BigInteger.valueOf(2)).divide(BigInteger.valueOf(Len)).longValue();
103     if ((y - x + 1) % 2 > 0) continue;
104     A = (y - x + 1) / 2;
105     if ((x + y - 1) % 2 > 0) continue;
106     B = (x + y - 1) / 2;
107
108     if (A < 1 || A > (long)1e18) continue;
109     if (B < 1 || B > (long)1e18) continue;
110
111     int flag = 1;
112
113     long Cnt;
114     long Cnt_B;
115     long Cnt_A;
116
117     for (int j = 1; j <= n; j++){
118         Cnt_B = Calc(B, j);
119         Cnt_A = Calc(A - 1, j);
120
121         if (Cnt_B - Cnt_A != k[j]){
122             flag = 0;
123             break;
124         }
125     }
126
127     if (flag==1){
128         Ans++;
129         //printf("%lld %lld\n", A, B);
130         //System.out.println(A+" "+B);
131         AnsA = A;
132         AnsB = B;
133     }
134 }

```

```

132
133     if (Ans == 0) System.out.println("None");//puts("None");
134     else
135     if (Ans == 1)
136         System.out.println(AnsA+" "+AnsB);//cout << AnsA << ' ' << AnsB << endl;
137     else
138         System.out.println("Many");//puts("Many");
139 }
140
141 }
142 }

```

12.3 样例代码 2

```

1  import java.io.*;
2  import java.math.*;
3  import java.lang.*;
4  import java.util.*;
5
6  public class Main {
7      static final int MAXN = 21;
8      static final int MAXM = 110;
9      static final int oo = 214000000;
10
11     public static int read(Scanner cin) {
12         char x = cin.next().charAt(0);
13         /* while(x < 'A' || x > 'Z')
14             x = cin.next().charAt(0);*/
15         return x - 'L' + 1;
16     }
17
18     private static boolean contain(int s, int i) {
19         return ((s >> i) & 1) == 1;
20     }
21
22     public static void main(String args[]) {
23         Scanner cin = new Scanner(System.in);
24         try {
25             cin = new Scanner(new FileInputStream("exclusive.in"));
26         } catch (FileNotFoundException e) {
27             System.exit(0);
28         }
29         PrintWriter out = new PrintWriter(System.out);
30         try {
31             out = new PrintWriter("exclusive.out", "UTF-8");
32         } catch (FileNotFoundException e) {
33             System.exit(0);
34         } catch (UnsupportedEncodingException e) {
35             System.exit(0);

```

```
36     }
37
38     int m, n = 15;
39     m = cin.nextInt();
40     int ea[] = new int [MAXM], eb[] = new int [MAXM];
41     boolean hs[][] = new boolean [MAXN][MAXN];
42
43
44     out.printf("%d\n", f[MAXS - 1] - 2);
45     for(int i = 1; i <= m; ++i) {
46         if(bel[ea[i]] < bel[eb[i]])
47             out.printf("%c %c\n", ea[i] + 'K', eb[i] + 'K');
48         else
49             out.printf("%c %c\n", eb[i] + 'K', ea[i] + 'K');
50     }
51     out.close();
52 }
53 }
```

[PREV CLASS](#) [NEXT CLASS](#) [FRAMES](#) [NO FRAMES](#) [ALL CLASSES](#)[SUMMARY: NESTED](#) | [FIELD](#) | [CONSTR](#) | [METHOD](#) [DETAIL: FIELD](#) | [CONSTR](#) | [METHOD](#)[compact1](#), [compact2](#), [compact3](#)[java.math](#)

Class BigInteger

[java.lang.Object](#)[java.lang.Number](#)[java.math.BigInteger](#)

All Implemented Interfaces:

[Serializable](#), [Comparable<BigInteger>](#)

```
public class BigInteger
    extends Number
    implements Comparable<BigInteger>
```

Immutable arbitrary-precision integers. All operations behave as if `BigInteger`s were represented in two's-complement notation (like Java's primitive integer types). `BigInteger` provides analogues to all of Java's primitive integer operators, and all relevant methods from `java.lang.Math`. Additionally, `BigInteger` provides operations for modular arithmetic, GCD calculation, primality testing, prime generation, bit manipulation, and a few other miscellaneous operations.

Semantics of arithmetic operations exactly mimic those of Java's integer arithmetic operators, as defined in *The Java Language Specification*. For example, division by zero throws an `ArithmeticException`, and division of a negative by a positive yields a negative (or zero) remainder. All of the details in the Spec concerning overflow are ignored, as `BigInteger`s are made as large as necessary to accommodate the results of an operation.

Semantics of shift operations extend those of Java's shift operators to allow for negative shift distances. A right-shift with a negative shift distance results in a left shift, and vice-versa. The unsigned right shift operator (`>>>`) is omitted, as this operation makes little sense in combination with the "infinite word size" abstraction provided by this class.

Semantics of bitwise logical operations exactly mimic those of Java's bitwise integer operators. The binary operators (`and`, `or`, `xor`) implicitly perform sign extension on the shorter of the two operands prior to performing the operation.

Comparison operations perform signed integer comparisons, analogous to those performed by Java's relational and equality operators.

Modular arithmetic operations are provided to compute residues, perform exponentiation, and compute multiplicative inverses. These methods always return a non-negative result, between 0 and (modulus - 1), inclusive.

Bit operations operate on a single bit of the two's-complement representation of their operand. If necessary, the operand is sign-extended so that it contains the designated bit. None of the single-bit operations can produce a `BigInteger` with a different sign from the `BigInteger` being operated on, as they affect only a single bit, and the "infinite word size" abstraction provided by this class ensures that there are infinitely many "virtual sign bits"

preceding each BigInteger.

For the sake of brevity and clarity, pseudo-code is used throughout the descriptions of BigInteger methods. The pseudo-code expression `(i + j)` is shorthand for "a BigInteger whose value is that of the BigInteger `i` plus that of the BigInteger `j`." The pseudo-code expression `(i == j)` is shorthand for "true if and only if the BigInteger `i` represents the same value as the BigInteger `j`." Other pseudo-code expressions are interpreted similarly.

All methods and constructors in this class throw `NullPointerException` when passed a null object reference for any input parameter. BigInteger must support values in the range `-2Integer.MAX_VALUE` (exclusive) to `+2Integer.MAX_VALUE` (exclusive) and may support values outside of that range. The range of probable prime values is limited and may be less than the full supported positive range of BigInteger. The range must be at least 1 to `25000000000`.

Implementation Note:

BigInteger constructors and operations throw `ArithmeticException` when the result is out of the supported range of `-2Integer.MAX_VALUE` (exclusive) to `+2Integer.MAX_VALUE` (exclusive).

Since:

JDK1.1

See Also:

[BigDecimal](#), [Serialized Form](#)

Field Summary

Fields

| Modifier and Type | Field and Description |
|-----------------------------------|--|
| static BigInteger | ONE The BigInteger constant one. |
| static BigInteger | TEN The BigInteger constant ten. |
| static BigInteger | ZERO The BigInteger constant zero. |

Constructor Summary

Constructors

| Constructor and Description |
|---|
| BigInteger (byte[] val) Translates a byte array containing the two's-complement binary representation of a BigInteger into a BigInteger. |
| BigInteger (int signum, byte[] magnitude) Translates the sign-magnitude representation of a BigInteger into a BigInteger. |

BigInteger(int bitLength, int certainty, **Random** rnd)

Constructs a randomly generated positive BigInteger that is probably prime, with the specified bitLength.

BigInteger(int numBits, **Random** rnd)

Constructs a randomly generated BigInteger, uniformly distributed over the range 0 to ($2^{\text{numBits}} - 1$), inclusive.

BigInteger(String val)

Translates the decimal String representation of a BigInteger into a BigInteger.

BigInteger(String val, int radix)

Translates the String representation of a BigInteger in the specified radix into a BigInteger.

Method Summary

All Methods **Static Methods** **Instance Methods** **Concrete Methods**

| Modifier and Type | Method and Description |
|-------------------|--|
| BigInteger | abs() Returns a BigInteger whose value is the absolute value of this BigInteger. |
| BigInteger | add(BigInteger val) Returns a BigInteger whose value is (<code>this + val</code>). |
| BigInteger | and(BigInteger val) Returns a BigInteger whose value is (<code>this & val</code>). |
| BigInteger | andNot(BigInteger val) Returns a BigInteger whose value is (<code>this & ~val</code>). |
| int | bitCount() Returns the number of bits in the two's complement representation of this BigInteger that differ from its sign bit. |
| int | bitLength() Returns the number of bits in the minimal two's-complement representation of this BigInteger, <i>excluding</i> a sign bit. |
| byte | byteValueExact() Converts this BigInteger to a byte, checking for lost information. |
| BigInteger | clearBit(int n) Returns a BigInteger whose value is equivalent to this BigInteger with the designated bit cleared. |
| int | compareTo(BigInteger val) Compares this BigInteger with the specified BigInteger. |
| BigInteger | divide(BigInteger val) Returns the BigInteger that is the quotient of this BigInteger divided by the specified BigInteger. |

| | |
|---------------------------|---|
| | Returns a <code>BigInteger</code> whose value is <code>(this / val)</code> . |
| <code>BigInteger[]</code> | <code>divideAndRemainder(BigInteger val)</code> Returns an array of two <code>BigInteger</code> s containing <code>(this / val)</code> followed by <code>(this % val)</code> . |
| <code>double</code> | <code>doubleValue()</code> Converts this <code>BigInteger</code> to a <code>double</code> . |
| <code>boolean</code> | <code>equals(Object x)</code> Compares this <code>BigInteger</code> with the specified <code>Object</code> for equality. |
| <code>BigInteger</code> | <code>flipBit(int n)</code> Returns a <code>BigInteger</code> whose value is equivalent to this <code>BigInteger</code> with the designated bit flipped. |
| <code>float</code> | <code>floatValue()</code> Converts this <code>BigInteger</code> to a <code>float</code> . |
| <code>BigInteger</code> | <code>gcd(BigInteger val)</code> Returns a <code>BigInteger</code> whose value is the greatest common divisor of <code>abs(this)</code> and <code>abs(val)</code> . |
| <code>int</code> | <code>getLowestSetBit()</code> Returns the index of the rightmost (lowest-order) one bit in this <code>BigInteger</code> (the number of zero bits to the right of the rightmost one bit). |
| <code>int</code> | <code>hashCode()</code> Returns the hash code for this <code>BigInteger</code> . |
| <code>int</code> | <code>intValue()</code> Converts this <code>BigInteger</code> to an <code>int</code> . |
| <code>int</code> | <code>intValueExact()</code> Converts this <code>BigInteger</code> to an <code>int</code> , checking for lost information. |
| <code>boolean</code> | <code>isProbablePrime(int certainty)</code> Returns <code>true</code> if this <code>BigInteger</code> is probably prime, <code>false</code> if it's definitely composite. |
| <code>long</code> | <code>longValue()</code> Converts this <code>BigInteger</code> to a <code>long</code> . |
| <code>long</code> | <code>longValueExact()</code> Converts this <code>BigInteger</code> to a <code>long</code> , checking for lost information. |
| <code>BigInteger</code> | <code>max(BigInteger val)</code> Returns the maximum of this <code>BigInteger</code> and <code>val</code> . |
| <code>BigInteger</code> | <code>min(BigInteger val)</code> Returns the minimum of this <code>BigInteger</code> and <code>val</code> . |
| <code>BigInteger</code> | <code>mod(BigInteger m)</code> Returns a <code>BigInteger</code> whose value is <code>(this mod m)</code> . |

| | |
|--------------------------|--|
| BigInteger | modInverse(BigInteger m) Returns a BigInteger whose value is $(\text{this}^{-1} \bmod m)$. |
| BigInteger | modPow(BigInteger exponent, BigInteger m) Returns a BigInteger whose value is $(\text{this}^{\text{exponent}} \bmod m)$. |
| BigInteger | multiply(BigInteger val) Returns a BigInteger whose value is $(\text{this} * \text{val})$. |
| BigInteger | negate() Returns a BigInteger whose value is $(-\text{this})$. |
| BigInteger | nextProbablePrime() Returns the first integer greater than this BigInteger that is probably prime. |
| BigInteger | not() Returns a BigInteger whose value is $(\sim \text{this})$. |
| BigInteger | or(BigInteger val) Returns a BigInteger whose value is $(\text{this} \text{val})$. |
| BigInteger | pow(int exponent) Returns a BigInteger whose value is $(\text{this}^{\text{exponent}})$. |
| static BigInteger | probablePrime(int bitLength, Random rnd) Returns a positive BigInteger that is probably prime, with the specified bitLength. |
| BigInteger | remainder(BigInteger val) Returns a BigInteger whose value is $(\text{this} \% \text{val})$. |
| BigInteger | setBit(int n) Returns a BigInteger whose value is equivalent to this BigInteger with the designated bit set. |
| BigInteger | shiftLeft(int n) Returns a BigInteger whose value is $(\text{this} \ll n)$. |
| BigInteger | shiftRight(int n) Returns a BigInteger whose value is $(\text{this} \gg n)$. |
| short | shortValueExact() Converts this BigInteger to a short, checking for lost information. |
| int | signum() Returns the signum function of this BigInteger. |
| BigInteger | subtract(BigInteger val) Returns a BigInteger whose value is $(\text{this} - \text{val})$. |
| boolean | testBit(int n) Returns true if and only if the designated bit is set. |
| byte[] | toByteArray() Returns a byte array containing the two's complement binary representation of the BigInteger value. |

Returns a byte array containing the two's-complement representation of this BigInteger.

String

toString()

Returns the decimal String representation of this BigInteger.

String

toString(int radix)

Returns the String representation of this BigInteger in the given radix.

static **BigInteger** **valueOf(long val)**

Returns a BigInteger whose value is equal to that of the specified long.

BigInteger

xor(BigInteger val)

Returns a BigInteger whose value is (this ^ val).

Methods inherited from class java.lang.Number

byteValue, shortValue

Methods inherited from class java.lang.Object

clone, finalize, getClass, notify, notifyAll, wait, wait, wait

Field Detail

ZERO

public static final **BigInteger** ZERO

The BigInteger constant zero.

Since:

1.2

ONE

public static final **BigInteger** ONE

The BigInteger constant one.

Since:

1.2

TEN

public static final **BigInteger** TEN

The BigInteger constant ten.

[PREV CLASS](#) [NEXT CLASS](#) [FRAMES](#) [NO FRAMES](#) [ALL CLASSES](#)[SUMMARY: NESTED](#) | [FIELD](#) | [CONSTR](#) | [METHOD](#) [DETAIL: FIELD](#) | [CONSTR](#) | [METHOD](#)

compact1, compact2, compact3

java.util

Class `TreeMap<K,V>`

java.lang.Object

java.util.AbstractMap<K,V>

java.util.TreeMap<K,V>

Type Parameters:

K - the type of keys maintained by this map

V - the type of mapped values

All Implemented Interfaces:

`Serializable`, `Cloneable`, `Map<K,V>`, `NavigableMap<K,V>`, `SortedMap<K,V>`

```
public class TreeMap<K,V>
extends AbstractMap<K,V>
implements NavigableMap<K,V>, Cloneable, Serializable
```

A Red-Black tree based `NavigableMap` implementation. The map is sorted according to the **natural ordering** of its keys, or by a `Comparator` provided at map creation time, depending on which constructor is used.

This implementation provides guaranteed log(n) time cost for the `containsKey`, `get`, `put` and `remove` operations. Algorithms are adaptations of those in Cormen, Leiserson, and Rivest's *Introduction to Algorithms*.

Note that the ordering maintained by a tree map, like any sorted map, and whether or not an explicit comparator is provided, must be *consistent with equals* if this sorted map is to correctly implement the `Map` interface. (See `Comparable` or `Comparator` for a precise definition of *consistent with equals*.) This is so because the `Map` interface is defined in terms of the `equals` operation, but a sorted map performs all key comparisons using its `compareTo` (or `compare`) method, so two keys that are deemed equal by this method are, from the standpoint of the sorted map, equal. The behavior of a sorted map is well-defined even if its ordering is inconsistent with `equals`; it just fails to obey the general contract of the `Map` interface.

Note that this implementation is not synchronized. If multiple threads access a map concurrently, and at least one of the threads modifies the map structurally, it *must* be synchronized externally. (A structural modification is any operation that adds or deletes one or more mappings; merely changing the value associated with an existing key is not a structural modification.) This is typically accomplished by synchronizing on some object that naturally encapsulates the map. If no such object exists, the map should be "wrapped" using the `Collections.synchronizedSortedMap` method. This is best done at creation time, to prevent accidental unsynchronized access to the map:

```
SortedMap m = Collections.synchronizedSortedMap(new TreeMap(...));
```

The iterators returned by the `iterator` method of the collections returned by all of this

class's "collection view methods" are *fail-fast*: if the map is structurally modified at any time after the iterator is created, in any way except through the iterator's own `remove` method, the iterator will throw a `ConcurrentModificationException`. Thus, in the face of concurrent modification, the iterator fails quickly and cleanly, rather than risking arbitrary, non-deterministic behavior at an undetermined time in the future.

Note that the fail-fast behavior of an iterator cannot be guaranteed as it is, generally speaking, impossible to make any hard guarantees in the presence of unsynchronized concurrent modification. Fail-fast iterators throw `ConcurrentModificationException` on a best-effort basis. Therefore, it would be wrong to write a program that depended on this exception for its correctness: *the fail-fast behavior of iterators should be used only to detect bugs*.

All `Map.Entry` pairs returned by methods in this class and its views represent snapshots of mappings at the time they were produced. They do **not** support the `Entry.setValue` method. (Note however that it is possible to change mappings in the associated map using `put`.)

This class is a member of the [Java Collections Framework](#).

Since:

1.2

See Also:

[Map](#), [HashMap](#), [Hashtable](#), [Comparable](#), [Comparator](#), [Collection](#), [Serialized Form](#)

Nested Class Summary

Nested classes/interfaces inherited from class `java.util.AbstractMap`

`AbstractMap.SimpleEntry<K,V>`, `AbstractMap.SimpleImmutableEntry<K,V>`

Constructor Summary

Constructors

Constructor and Description

`TreeMap()`

Constructs a new, empty tree map, using the natural ordering of its keys.

`TreeMap(Comparator<? super K> comparator)`

Constructs a new, empty tree map, ordered according to the given comparator.

`TreeMap(Map<? extends K,? extends V> m)`

Constructs a new tree map containing the same mappings as the given map, ordered according to the *natural ordering* of its keys.

`TreeMap(SortedMap<K,? extends V> m)`

Constructs a new tree map containing the same mappings and using the same ordering as the specified sorted map.

Method Summary

| Modifier and Type | Method and Description |
|---|--|
| Map.Entry<K, V> | ceilingEntry(K key) Returns a key-value mapping associated with the least key greater than or equal to the given key, or null if there is no such key. |
| K | ceilingKey(K key) Returns the least key greater than or equal to the given key, or null if there is no such key. |
| void | clear() Removes all of the mappings from this map. |
| Object | clone() Returns a shallow copy of this TreeMap instance. |
| Comparator<? super K> | comparator() Returns the comparator used to order the keys in this map, or null if this map uses the natural ordering of its keys. |
| boolean | containsKey(Object key) Returns true if this map contains a mapping for the specified key. |
| boolean | containsValue(Object value) Returns true if this map maps one or more keys to the specified value. |
| NavigableSet<K> | descendingKeySet() Returns a reverse order NavigableSet view of the keys contained in this map. |
| NavigableMap<K, V> | descendingMap() Returns a reverse order view of the mappings contained in this map. |
| Set<Map.Entry<K, V>> | entrySet() Returns a Set view of the mappings contained in this map. |
| Map.Entry<K, V> | firstEntry() Returns a key-value mapping associated with the least key in this map, or null if the map is empty. |
| K | firstKey() Returns the first (lowest) key currently in this map. |
| Map.Entry<K, V> | floorEntry(K key) Returns a key-value mapping associated with the greatest key less than or equal to the given key, or null if there is no such key. |
| K | floorKey(K key) Returns the greatest key less than or equal to the given key, or null if there is no such key. |

or null if there is no such key.

void

forEach(**BiConsumer**<? super **K**,? super **V**> action)

Performs the given action for each entry in this map until all entries have been processed or the action throws an exception.

V

get(**Object** key)

Returns the value to which the specified key is mapped, or null if this map contains no mapping for the key.

SortedMap<**K**,**V**>

headMap(**K** toKey)

Returns a view of the portion of this map whose keys are strictly less than toKey.

NavigableMap<**K**,**V**>

headMap(**K** toKey, boolean inclusive)

Returns a view of the portion of this map whose keys are less than (or equal to, if inclusive is true) toKey.

Map.Entry<**K**,**V**>

higherEntry(**K** key)

Returns a key-value mapping associated with the least key strictly greater than the given key, or null if there is no such key.

K

higherKey(**K** key)

Returns the least key strictly greater than the given key, or null if there is no such key.

Set<**K**>

keySet()

Returns a **Set** view of the keys contained in this map.

Map.Entry<**K**,**V**>

lastEntry()

Returns a key-value mapping associated with the greatest key in this map, or null if the map is empty.

K

lastKey()

Returns the last (highest) key currently in this map.

Map.Entry<**K**,**V**>

lowerEntry(**K** key)

Returns a key-value mapping associated with the greatest key strictly less than the given key, or null if there is no such key.

K

lowerKey(**K** key)

Returns the greatest key strictly less than the given key, or null if there is no such key.

NavigableSet<**K**>

navigableKeySet()

Returns a **NavigableSet** view of the keys contained in this map.

Map.Entry<**K**,**V**>

pollFirstEntry()

Removes and returns a key-value mapping associated with the least key in this map, or null if the map is empty.

Map.Entry<**K**,**V**>

pollLastEntry()

Removes and returns a key-value mapping associated with the greatest key in this map, or null if the map is empty.

the greatest key in this map, or null if the map is empty.

| | |
|--------------------------------|--|
| V | put(K key, V value) Associates the specified value with the specified key in this map. |
| void | putAll(Map<? extends K,? extends V> map) Copies all of the mappings from the specified map to this map. |
| V | remove(Object key) Removes the mapping for this key from this TreeMap if present. |
| V | replace(K key, V value) Replaces the entry for the specified key only if it is currently mapped to some value. |
| boolean | replace(K key, V oldValue, V newValue) Replaces the entry for the specified key only if currently mapped to the specified value. |
| void | replaceAll(BiFunction<? super K,? super V,? extends V> function) Replaces each entry's value with the result of invoking the given function on that entry until all entries have been processed or the function throws an exception. |
| int | size() Returns the number of key-value mappings in this map. |
| NavigableMap<K,V> | subMap(K fromKey, boolean fromInclusive, K toKey, boolean toInclusive) Returns a view of the portion of this map whose keys range from fromKey to toKey. |
| SortedMap<K,V> | subMap(K fromKey, K toKey) Returns a view of the portion of this map whose keys range from fromKey, inclusive, to toKey, exclusive. |
| SortedMap<K,V> | tailMap(K fromKey) Returns a view of the portion of this map whose keys are greater than or equal to fromKey. |
| NavigableMap<K,V> | tailMap(K fromKey, boolean inclusive) Returns a view of the portion of this map whose keys are greater than (or equal to, if inclusive is true) fromKey. |
| Collection<V> | values() Returns a Collection view of the values contained in this map. |

Methods inherited from class java.util.**AbstractMap**

`equals`, `hashCode`, `isEmpty`, `toString`

13 数学

13.1 常用数学公式

13.1.1 求和公式

1. $\sum_{k=1}^n (2k-1)^2 = \frac{n(4n^2-1)}{3}$
2. $\sum_{k=1}^n k^3 = [\frac{n(n+1)}{2}]^2$
3. $\sum_{k=1}^n (2k-1)^3 = n^2(2n^2-1)$
4. $\sum_{k=1}^n k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
5. $\sum_{k=1}^n k^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$
6. $\sum_{k=1}^n k(k+1) = \frac{n(n+1)(n+2)}{3}$
7. $\sum_{k=1}^n k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$
8. $\sum_{k=1}^n k(k+1)(k+2)(k+3) = \frac{n(n+1)(n+2)(n+3)(n+4)}{5}$

13.1.2 斐波那契数列

1. $fib_0 = 0, fib_1 = 1, fib_n = fib_{n-1} + fib_{n-2}$
2. $fib_{n+2} \cdot fib_n - fib_{n+1}^2 = (-1)^{n+1}$
3. $fib_{-n} = (-1)^{n-1} fib_n$
4. $fib_{n+k} = fib_k \cdot fib_{n+1} + fib_{k-1} \cdot fib_n$
5. $gcd(fib_m, fib_n) = fib_{gcd(m,n)}$
6. $fib_m | fib_n^2 \Leftrightarrow n fib_n | m$

13.1.3 Bernoulli 数

1. 初始化: $B_0(n) = 1$
2. 递推公式:

$$B_m(n) = n^m - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k(n)}{m-k+1}$$

3. 应用:

$$\sum_{k=1}^n k^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} n^{m+1-k}$$

13.1.4 错排公式

1. $D_n = (n-1)(D_{n-2} + D_{n-1})$
2. $D_n = n! \cdot (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!})$

13.1.5 莫比乌斯函数

$$\mu(n) = \begin{cases} 1 & \text{若 } n = 1 \\ (-1)^k & \text{若 } n \text{ 无平方数因子, 且 } n = p_1 p_2 \cdots p_k \\ 0 & \text{若 } n \text{ 有大于1的平方数因数} \end{cases}$$

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{若 } n = 1 \\ 0 & \text{其他情况} \end{cases}$$

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$$

$$g(x) = \sum_{n=1}^{[x]} f\left(\frac{x}{n}\right) \Leftrightarrow f(x) = \sum_{n=1}^{[x]} \mu(n) g\left(\frac{x}{n}\right)$$

13.1.6 伯恩赛德引理

设 G 是一个有限群, 作用在集合 X 上. 对每个 g 属于 G , 令 X^g 表示 X 中在 g 作用下的不动元素, 轨道数 (记作 $|X/G|$) 由如下公式给出:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

13.1.7 五边形数定理

设 $p(n)$ 是 n 的拆分数, 有

$$p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k-1} p\left(n - \frac{k(3k-1)}{2}\right)$$

13.1.8 树的计数

1. 有根树计数: $n+1$ 个结点的有根树的个数为

$$a_{n+1} = \frac{\sum_{j=1}^n j \cdot a_j \cdot S_{n,j}}{n}$$

其中,

$$S_{n,j} = \sum_{i=1}^{n/j} a_{n+1-ij} = S_{n-j,j} + a_{n+1-j}$$

2. 无根树计数: 当 n 为奇数时, n 个结点的无根树的个数为

$$a_n - \sum_{i=1}^{n/2} a_i a_{n-i}$$

当 n 为偶数时, n 个结点的无根树的个数为

$$a_n - \sum_{i=1}^{n/2} a_i a_{n-i} + \frac{1}{2} a_{\frac{n}{2}} (a_{\frac{n}{2}} + 1)$$

3. n 个结点的完全图的生成树个数为

$$n^{n-2}$$

4. 矩阵—树定理: 图 G 由 n 个结点构成, 设 $A[G]$ 为图 G 的邻接矩阵、 $D[G]$ 为图 G 的度数矩阵, 则图 G 的不同生成树的个数为 $C[G] = D[G] - A[G]$ 的任意一个 $n-1$ 阶主子式的行列式值。

13.1.9 欧拉公式

平面图的顶点个数、边数和面的个数有如下关系:

$$V - E + F = C + 1$$

其中, V 是顶点的数目, E 是边的数目, F 是面的数目, C 是组成图形的连通部分的数目。当图是单连通图的时候, 公式简化为:

$$V - E + F = 2$$

13.1.10 皮克定理

给定顶点坐标均是整点 (或正方形格点) 的简单多边形, 其面积 A 和内部格点数目 i 、边上格点数目 b 的关系:

$$A = i + \frac{b}{2} - 1$$

13.1.11 牛顿恒等式

设

$$\prod_{i=1}^n (x - x_i) = a_n + a_{n-1}x + \cdots + a_1x^{n-1} + a_0x^n$$

$$p_k = \sum_{i=1}^n x_i^k$$

则

$$a_0 p_k + a_1 p_{k-1} + \cdots + a_{k-1} p_1 + k a_k = 0$$

特别地, 对于

$$|A - \lambda E| = (-1)^n (a_n + a_{n-1} \lambda + \cdots + a_1 \lambda^{n-1} + a_0 \lambda^n)$$

有

$$p_k = \text{Tr}(A^k)$$

13.2 平面几何公式

13.2.1 三角形

1. 半周长

$$p = \frac{a+b+c}{2}$$

2. 面积

$$S = \frac{a \cdot H_a}{2} = \frac{ab \cdot \sin C}{2} = \sqrt{p(p-a)(p-b)(p-c)}$$

3. 中线

$$M_a = \frac{\sqrt{2(b^2+c^2)-a^2}}{2} = \frac{\sqrt{b^2+c^2+2bc \cdot \cos A}}{2}$$

4. 角平分线

$$T_a = \frac{\sqrt{bc \cdot [(b+c)^2 - a^2]}}{b+c} = \frac{2bc \cos \frac{A}{2}}{b+c}$$

5. 高线

$$H_a = b \sin C = c \sin B = \sqrt{b^2 - \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2}$$

6. 内切圆半径

$$\begin{aligned} r &= \frac{S}{p} = \frac{\arcsin \frac{B}{2} \cdot \sin \frac{C}{2}}{\sin \frac{B+C}{2}} = 4R \cdot \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ &= \sqrt{\frac{(p-a)(p-b)(p-c)}{p}} = p \cdot \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \end{aligned}$$

7. 外接圆半径

$$R = \frac{abc}{4S} = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

13.2.2 四边形

D_1, D_2 为对角线, M 为对角线中点连线, A 为对角线夹角, p 为半周长

$$1. a^2 + b^2 + c^2 + d^2 = D_1^2 + D_2^2 + 4M^2$$

$$2. S = \frac{1}{2} D_1 D_2 \sin A$$

3. 对于圆内接四边形

$$ac + bd = D_1 D_2$$

4. 对于圆内接四边形

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}$$

13.2.3 正 n 边形

R 为外接圆半径, r 为内切圆半径

1. 中心角

$$A = \frac{2\pi}{n}$$

2. 内角

$$C = \frac{n-2}{n}\pi$$

3. 边长

$$a = 2\sqrt{R^2 - r^2} = 2R \cdot \sin \frac{A}{2} = 2r \cdot \tan \frac{A}{2}$$

4. 面积

$$S = \frac{nar}{2} = nr^2 \cdot \tan \frac{A}{2} = \frac{nR^2}{2} \cdot \sin A = \frac{na^2}{4 \cdot \tan \frac{A}{2}}$$

13.2.4 圆

1. 弧长

$$l = rA$$

2. 弦长

$$a = 2\sqrt{2hr - h^2} = 2r \cdot \sin \frac{A}{2}$$

3. 弓形高

$$h = r - \sqrt{r^2 - \frac{a^2}{4}} = r(1 - \cos \frac{A}{2}) = \frac{1}{2} \cdot \arctan \frac{A}{4}$$

4. 扇形面积

$$S_1 = \frac{rl}{2} = \frac{r^2 A}{2}$$

5. 弓形面积

$$S_2 = \frac{rl - a(r - h)}{2} = \frac{r^2}{2}(A - \sin A)$$

13.2.5 棱柱

1. 体积

$$V = Ah$$

A 为底面积, h 为高

2. 侧面积

$$S = lp$$

l 为棱长, p 为直截面周长

3. 全面积

$$T = S + 2A$$

13.2.6 棱锥

1. 体积

$$V = Ah$$

A 为底面积, h 为高

2. 正棱锥侧面积

$$S = lp$$

l 为棱长, p 为直截面周长

3. 正棱锥全面积

$$T = S + 2A$$

13.2.7 棱台

1. 体积

$$V = (A_1 + A_2 + \sqrt{A_1 A_2}) \cdot \frac{h}{3}$$

A_1, A_2 为上下底面积, h 为高

2. 正棱台侧面积

$$S = \frac{p_1 + p_2}{2} l$$

p_1, p_2 为上下底面周长, l 为斜高

3. 正棱台全面积

$$T = S + A_1 + A_2$$

13.2.8 圆柱

1. 侧面积

$$S = 2\pi r h$$

2. 全面积

$$T = 2\pi r(h + r)$$

3. 体积

$$V = \pi r^2 h$$

13.2.9 圆锥

1. 母线

$$l = \sqrt{h^2 + r^2}$$

2. 侧面积

$$S = \pi r l$$

3. 全面积

$$T = \pi r(l + r)$$

4. 体积

$$V = \frac{\pi}{3} r^2 h$$

13.2.10 圆台

1. 母线

$$l = \sqrt{h^2 + (r_1 - r_2)^2}$$

2. 侧面积

$$S = \pi(r_1 + r_2)l$$

3. 全面积

$$T = \pi r_1(l + r_1) + \pi r_2(l + r_2)$$

4. 体积

$$V = \frac{\pi}{3}(r_1^2 + r_2^2 + r_1 r_2)h$$

13.2.11 球

1. 全面积

$$T = 4\pi r^2$$

2. 体积

$$V = \frac{4}{3}\pi r^3$$

13.2.12 球台

1. 侧面积

$$S = 2\pi r h$$

2. 全面积

$$T = \pi(2rh + r_1^2 + r_2^2)$$

3. 体积

$$V = \frac{\pi h[3(r_1^2 + r_2^2) + h^2]}{6}$$

13.2.13 球扇形

1. 全面积

$$T = \pi r(2h + r_0)$$

h 为球冠高, r_0 为球冠底面半径

2. 体积

$$V = \frac{2}{3}\pi r^2 h$$

13.3 积分表

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\begin{aligned} \int \frac{x}{a^2+x^2} dx &= \frac{1}{2} \ln |a^2+x^2| \\ \int \frac{x^2}{a^2+x^2} dx &= x - a \tan^{-1} \frac{x}{a} \\ \int \sqrt{x^2 \pm a^2} dx &= \frac{1}{2} x \sqrt{x^2 \pm a^2} \pm \frac{1}{2} a^2 \ln |x + \sqrt{x^2 \pm a^2}| \\ \int \sqrt{a^2 - x^2} dx &= \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} \\ \int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx &= \frac{1}{2} x \sqrt{x^2 \pm a^2} \mp \frac{1}{2} a^2 \ln |x + \sqrt{x^2 \pm a^2}| \\ \int \frac{1}{\sqrt{x^2 \pm a^2}} dx &= \ln |x + \sqrt{x^2 \pm a^2}| \\ \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \sin^{-1} \frac{x}{a} \\ \int \frac{x}{\sqrt{x^2 \pm a^2}} dx &= \sqrt{x^2 \pm a^2} \\ \int \frac{x}{\sqrt{a^2 - x^2}} dx &= -\sqrt{a^2 - x^2} \\ \int \sqrt{ax^2 + bx + c} dx &= \frac{b+2ax}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac-b^2}{8a^{3/2}} \ln |2ax + b + 2\sqrt{a(ax^2 + bx + c)}| \\ \int x^n e^{ax} dx &= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx \\ \int \sin^2 ax dx &= \frac{x}{2} - \frac{1}{4a} \sin 2ax \\ \int \sin^3 ax dx &= -\frac{3 \cos ax}{4a} + \frac{\cos 3ax}{12a} \\ \int \cos^2 ax dx &= \frac{x}{2} + \frac{\sin 2ax}{4a} \\ \int \cos^3 ax dx &= \frac{3 \sin ax}{4a} + \frac{\sin 3ax}{12a} \\ \int \tan ax dx &= -\frac{1}{a} \ln \cos ax \\ \int \tan^2 ax dx &= -x + \frac{1}{a} \tan ax \\ \int x \cos ax dx &= \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax \\ \int x^2 \cos ax dx &= \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax \\ \int x \sin ax dx &= -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2} \\ \int x^2 \sin ax dx &= \frac{2-a^2 x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} \end{aligned}$$

13.4 立体几何公式

13.4.1 球面三角公式

设 a, b, c 是边长, A, B, C 是所对的二面角, 有余弦定理

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

正弦定理

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

三角形面积是 $A + B + C - \pi$

13.4.2 四面体体积公式

U, V, W, u, v, w 是四面体的 6 条棱, U, V, W 构成三角形, $(U, u), (V, v), (W, w)$ 互为对棱, 则

$$V = \frac{\sqrt{(s-2a)(s-2b)(s-2c)(s-2d)}}{192uvw}$$

其中

$$\left\{ \begin{array}{l} a = \sqrt{xYZ}, \\ b = \sqrt{yZX}, \\ c = \sqrt{zXY}, \\ d = \sqrt{xyz}, \\ s = a + b + c + d, \\ X = (w - U + v)(U + v + w), \\ x = (U - v + w)(v - w + U), \\ Y = (u - V + w)(V + w + u), \\ y = (V - w + u)(w - u + V), \\ Z = (v - W + u)(W + u + v), \\ z = (W - u + v)(u - v + W) \end{array} \right.$$

13.5 博弈游戏

13.6 巴什博弈

1. 只有一堆 n 个物品，两个人轮流从这堆物品中取物，规定每次至少取一个，最多取 m 个。最后取光者得胜。
2. 显然，如果 $n = m + 1$ ，那么由于一次最多只能取 m 个，所以，无论先取者拿走多少个，后取者都能够一次拿走剩余的物品，后者取胜。因此我们发现了如何取胜的法则：如果 $n = \text{Im} + 1 \text{Ir} + s$ ，(r 为任意自然数， $s \leq m$)，那么先取者要拿走 s 个物品，如果后取者拿走 $k(k \leq m)$ 个，那么先取者再拿走 $m + 1 - k$ 个，结果剩下 $(m + 1)(r - 1)$ 个，以后保持这样的取法，那么先取者肯定获胜。总之，要保持给对手留下 $(m + 1)$ 的倍数，就能最后获胜。

13.7 威佐夫博弈

1. 有两堆各若干个物品，两个人轮流从某一堆或同时从两堆中取同样多的物品，规定每次至少取一个，多者不限，最后取光者得胜。
2. 判断一个局势 (a, b) 为奇异局势（必败态）的方法：

$$a_k = [k(1 + \sqrt{5})/2] \text{ } b_k = a_k + k$$

13.8 阶梯博弈

1. 博弈在一列阶梯上进行，每个阶梯上放着自然数个点，两个人进行阶梯博弈，每一步则是将一个阶梯上的若干个点（至少一个）移到前面去，最后没有点可以移动的人输。
2. 解决方法：把所有奇数阶梯看成 N 堆石子，做 **NIM**。（把石子从奇数堆移动到偶数堆可以理解为拿走石子，就相当于几个奇数堆的石子在做 **Nim**）

13.9 图上删边游戏

13.9.1 链的删边游戏

1. 游戏规则：对于一条链，其中一个端点是根，两人轮流删边，脱离根的部分也算被删去，最后没边可删的人输。
2. 做法： $sg[i] = n - dist(i) - 1$ （其中 n 表示总点数， $dist(i)$ 表示离根的距离）

13.9.2 树的删边游戏

1. 游戏规则：对于一棵有根树，两人轮流删边，脱离根的部分也算被删去，没边可删的人输。
2. 做法：叶子结点的 $sg = 0$ ，其他节点的 sg 等于儿子结点的 $sg + 1$ 的异或和。

13.9.3 局部连通图的删边游戏

1. 游戏规则：在一个局部连通图上，两人轮流删边，脱离根的部分也算被删去，没边可删的人输。
局部连通图的构图规则是，在一棵基础树上加边得到，所有形成的环保证不共用边，且只与基础树有一个公共点。
2. 做法：去掉所有的偶环，将所有的奇环变为长度为 1 的链，然后做树的删边游戏。

13.10 常用数学公式

13.11 求和公式

1. $\sum_{k=1}^n (2k-1)^2 = \frac{n(4n^2-1)}{3}$
2. $\sum_{k=1}^n k^3 = [\frac{n(n+1)}{2}]^2$
3. $\sum_{k=1}^n (2k-1)^3 = n^2(2n^2-1)$
4. $\sum_{k=1}^n k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$
5. $\sum_{k=1}^n k^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$
6. $\sum_{k=1}^n k(k+1) = \frac{n(n+1)(n+2)}{3}$
7. $\sum_{k=1}^n k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$
8. $\sum_{k=1}^n k(k+1)(k+2)(k+3) = \frac{n(n+1)(n+2)(n+3)(n+4)}{5}$

13.12 斐波那契数列

1. $fib_0 = 0, fib_1 = 1, fib_n = fib_{n-1} + fib_{n-2}$
2. $fib_{n+2} \cdot fib_n - fib_{n+1}^2 = (-1)^{n+1}$
3. $fib_{-n} = (-1)^{n-1} fib_n$
4. $fib_{n+k} = fib_k \cdot fib_{n+1} + fib_{k-1} \cdot fib_n$

$$5. \gcd(\text{fib}_m, \text{fib}_n) = \text{fib}_{\gcd(m, n)}$$

$$6. \text{fib}_m | \text{fib}_n^2 \Leftrightarrow n \text{fib}_n | m$$

13.13 错排公式

$$1. D_n = (n-1)(D_{n-2} - D_{n-1})$$

$$2. D_n = n! \cdot (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!})$$

13.14 莫比乌斯函数

$$\mu(n) = \begin{cases} 1 & \text{若 } n = 1 \\ (-1)^k & \text{若 } n \text{ 无平方数因子, 且 } n = p_1 p_2 \dots p_k \\ 0 & \text{若 } n \text{ 有大于 } 1 \text{ 的平方数因数} \end{cases}$$

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{若 } n = 1 \\ 0 & \text{其他情况} \end{cases}$$

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$$

$$g(x) = \sum_{n=1}^{[x]} f\left(\frac{x}{n}\right) \Leftrightarrow f(x) = \sum_{n=1}^{[x]} \mu(n) g\left(\frac{x}{n}\right)$$

13.15 Burnside 引理

设 G 是一个有限群, 作用在集合 X 上. 对每个 g 属于 G , 令 X^g 表示 X 中在 g 作用下的不动元素, 轨道数 (记作 $|X/G|$) 由如下公式给出:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

13.16 五边形数定理

设 $p(n)$ 是 n 的拆分数, 有

$$p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k-1} p\left(n - \frac{k(3k-1)}{2}\right)$$

13.17 树的计数

1. 有根树计数: $n+1$ 个结点的有根树的个数为

$$a_{n+1} = \frac{\sum_{j=1}^n j \cdot a_j \cdot S_{n,j}}{n}$$

其中,

$$S_{n,j} = \sum_{i=1}^{n/j} a_{n+1-ij} = S_{n-j,j} + a_{n+1-j}$$

2. 无根树计数: 当 n 为奇数时, n 个结点的无根树的个数为

$$a_n - \sum_{i=1}^{n/2} a_i a_{n-i}$$

当 n 为偶数时, n 个结点的无根树的个数为

$$a_n - \sum_{i=1}^{n/2} a_i a_{n-i} + \frac{1}{2} a_{\frac{n}{2}} (a_{\frac{n}{2}} + 1)$$

3. n 个结点的完全图的生成树个数为

$$n^{n-2}$$

4. 矩阵—树定理: 图 G 由 n 个结点构成, 设 $A[G]$ 为图 G 的邻接矩阵、 $D[G]$ 为图 G 的度数矩阵, 则图 G 的不同生成树的个数为 $C[G] = D[G] - A[G]$ 的任意一个 $n-1$ 阶主子式的行列式值。

13.18 欧拉公式

平面图的顶点个数、边数和面的个数有如下关系:

$$V - E + F = C + 1$$

其中, V 是顶点的数目, E 是边的数目, F 是面的数目, C 是组成图形的连通部分的数目。当图是单连通图的时候, 公式简化为:

$$V - E + F = 2$$

13.19 皮克定理

给定顶点坐标均是整点 (或正方形格点) 的简单多边形, 其面积 A 和内部格点数目 i 、边上格点数目 b 的关系:

$$A = i + \frac{b}{2} - 1$$

13.20 牛顿恒等式

设

$$\prod_{i=1}^n (x - x_i) = a_n + a_{n-1}x + \cdots + a_1x^{n-1} + a_0x^n$$

$$p_k = \sum_{i=1}^n x_i^k$$

则

$$a_0 p_k + a_1 p_{k-1} + \cdots + a_{k-1} p_1 + k a_k = 0$$

特别地, 对于

$$|\mathbf{A} - \lambda \mathbf{E}| = (-1)^n (a_n + a_{n-1} \lambda + \cdots + a_1 \lambda^{n-1} + a_0 \lambda^n)$$

有

$$p_k = \text{Tr}(\mathbf{A}^k)$$

14 平面几何公式

14.1 三角形

1. 半周长

$$p = \frac{a + b + c}{2}$$

2. 面积

$$S = \frac{a \cdot H_a}{2} = \frac{ab \cdot \sin C}{2} = \sqrt{p(p-a)(p-b)(p-c)}$$

3. 中线

$$M_a = \frac{\sqrt{2(b^2 + c^2) - a^2}}{2} = \frac{\sqrt{b^2 + c^2 + 2bc \cdot \cos A}}{2}$$

4. 角平分线

$$T_a = \frac{\sqrt{bc \cdot [(b+c)^2 - a^2]}}{b+c} = \frac{2bc \cos \frac{A}{2}}{b+c}$$

5. 高线

$$H_a = b \sin C = c \sin B = \sqrt{b^2 - \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2}$$

6. 内切圆半径

$$\begin{aligned} r &= \frac{S}{p} = \frac{\arcsin \frac{B}{2} \cdot \sin \frac{C}{2}}{\sin \frac{B+C}{2}} = 4R \cdot \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ &= \sqrt{\frac{(p-a)(p-b)(p-c)}{p}} = p \cdot \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} \end{aligned}$$

7. 外接圆半径

$$R = \frac{abc}{4S} = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$$

14.2 四边形

D_1, D_2 为对角线, M 为对角线中点连线, A 为对角线夹角, p 为半周长

$$1. a^2 + b^2 + c^2 + d^2 = D_1^2 + D_2^2 + 4M^2$$

$$2. S = \frac{1}{2} D_1 D_2 \sin A$$

3. 对于圆内接四边形

$$ac + bd = D_1 D_2$$

4. 对于圆内接四边形

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}$$

14.3 正 n 边形

R 为外接圆半径, r 为内切圆半径

1. 中心角

$$A = \frac{2\pi}{n}$$

2. 内角

$$C = \frac{n-2}{n}\pi$$

3. 边长

$$a = 2\sqrt{R^2 - r^2} = 2R \cdot \sin \frac{A}{2} = 2r \cdot \tan \frac{A}{2}$$

4. 面积

$$S = \frac{nar}{2} = nr^2 \cdot \tan \frac{A}{2} = \frac{nR^2}{2} \cdot \sin A = \frac{na^2}{4 \cdot \tan \frac{A}{2}}$$

14.4 圆

1. 弧长

$$l = rA$$

2. 弦长

$$a = 2\sqrt{2hr - h^2} = 2r \cdot \sin \frac{A}{2}$$

3. 弓形高

$$h = r - \sqrt{r^2 - \frac{a^2}{4}} = r(1 - \cos \frac{A}{2}) = \frac{1}{2} \cdot \arctan \frac{A}{4}$$

4. 扇形面积

$$S_1 = \frac{rl}{2} = \frac{r^2 A}{2}$$

5. 弓形面积

$$S_2 = \frac{rl - a(r-h)}{2} = \frac{r^2}{2}(A - \sin A)$$

14.5 棱柱

1. 体积

$$V = Ah$$

A 为底面积, h 为高

2. 侧面积

$$S = lp$$

l 为棱长, p 为直截面周长

3. 全面积

$$T = S + 2A$$

14.6 棱锥

1. 体积

$$V = Ah$$

A 为底面积, h 为高

2. 正棱锥侧面积

$$S = lp$$

l 为棱长, p 为直截面周长

3. 正棱锥全面积

$$T = S + 2A$$

14.7 棱台

1. 体积

$$V = (A_1 + A_2 + \sqrt{A_1 A_2}) \cdot \frac{h}{3}$$

A_1, A_2 为上下底面积, h 为高

2. 正棱台侧面积

$$S = \frac{p_1 + p_2}{2} l$$

p_1, p_2 为上下底面周长, l 为斜高

3. 正棱台全面积

$$T = S + A_1 + A_2$$

14.8 圆柱

1. 侧面积

$$S = 2\pi r h$$

2. 全面积

$$T = 2\pi r(h + r)$$

3. 体积

$$V = \pi r^2 h$$

14.9 圆锥

1. 母线

$$l = \sqrt{h^2 + r^2}$$

2. 侧面积

$$S = \pi r l$$

3. 全面积

$$T = \pi r(l + r)$$

4. 体积

$$V = \frac{\pi}{3} r^2 h$$

14.10 圆台

1. 母线

$$l = \sqrt{h^2 + (r_1 - r_2)^2}$$

2. 侧面积

$$S = \pi(r_1 + r_2)l$$

3. 全面积

$$T = \pi r_1(l + r_1) + \pi r_2(l + r_2)$$

4. 体积

$$V = \frac{\pi}{3} (r_1^2 + r_2^2 + r_1 r_2) h$$

14.11 球

1. 全面积

$$T = 4\pi r^2$$

2. 体积

$$V = \frac{4}{3} \pi r^3$$

14.12 球台

1. 侧面积

$$S = 2\pi r h$$

2. 全面积

$$T = \pi(2rh + r_1^2 + r_2^2)$$

3. 体积

$$V = \frac{\pi h [3(r_1^2 + r_2^2) + h^2]}{6}$$

14.13 球扇形**1. 全面积**

$$T = \pi r(2h + r_0)$$

h 为球冠高, r_0 为球冠底面半径

2. 体积

$$V = \frac{2}{3}\pi r^2 h$$

15 立体几何公式**15.1 球面三角公式**

设 a, b, c 是边长, A, B, C 是所对的二面角, 有余弦定理

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

正弦定理

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

三角形面积是 $A + B + C - \pi$

15.2 四面体体积公式

U, V, W, u, v, w 是四面体的 6 条棱, U, V, W 构成三角形, $(U, u), (V, v), (W, w)$ 互为对棱, 则

$$V = \frac{\sqrt{(s-2a)(s-2b)(s-2c)(s-2d)}}{192uvw}$$

其中

$$\left\{ \begin{array}{l} a = \sqrt{xYZ}, \\ b = \sqrt{yZX}, \\ c = \sqrt{zXY}, \\ d = \sqrt{xyz}, \\ s = a + b + c + d, \\ X = (w - U + v)(U + v + w), \\ x = (U - v + w)(v - w + U), \\ Y = (u - V + w)(V + w + u), \\ y = (V - w + u)(w - u + V), \\ Z = (v - W + u)(W + u + v), \\ z = (W - u + v)(u - v + W) \end{array} \right.$$

16 附录

16.1 NTT 素数及原根列表

| Id | Primes | Primitive Root | Id | Primes | Primitive Root | Id | Primes | Primitive |
|----|-----------|----------------|----|-----------|----------------|-----|------------|-----------|
| 1 | 7340033 | 3 | 38 | 311427073 | 7 | 75 | 786432001 | 7 |
| 2 | 13631489 | 15 | 39 | 330301441 | 22 | 76 | 799014913 | 13 |
| 3 | 23068673 | 3 | 40 | 347078657 | 3 | 77 | 800063489 | 3 |
| 4 | 26214401 | 3 | 41 | 359661569 | 3 | 78 | 802160641 | 11 |
| 5 | 28311553 | 5 | 42 | 361758721 | 29 | 79 | 818937857 | 5 |
| 6 | 69206017 | 5 | 43 | 377487361 | 7 | 80 | 824180737 | 5 |
| 7 | 70254593 | 3 | 44 | 383778817 | 5 | 81 | 833617921 | 13 |
| 8 | 81788929 | 7 | 45 | 387973121 | 6 | 82 | 850395137 | 3 |
| 9 | 101711873 | 3 | 46 | 399507457 | 5 | 83 | 862978049 | 3 |
| 10 | 104857601 | 3 | 47 | 409993217 | 3 | 84 | 880803841 | 26 |
| 11 | 111149057 | 3 | 48 | 415236097 | 5 | 85 | 883949569 | 7 |
| 12 | 113246209 | 7 | 49 | 447741953 | 3 | 86 | 897581057 | 3 |
| 13 | 120586241 | 6 | 50 | 459276289 | 11 | 87 | 899678209 | 7 |
| 14 | 132120577 | 5 | 51 | 463470593 | 3 | 88 | 907018241 | 3 |
| 15 | 136314881 | 3 | 52 | 468713473 | 5 | 89 | 913309697 | 3 |
| 16 | 138412033 | 5 | 53 | 469762049 | 3 | 90 | 918552577 | 5 |
| 17 | 141557761 | 26 | 54 | 493879297 | 10 | 91 | 919601153 | 3 |
| 18 | 147849217 | 5 | 55 | 531628033 | 5 | 92 | 924844033 | 5 |
| 19 | 155189249 | 6 | 56 | 576716801 | 6 | 93 | 925892609 | 3 |
| 20 | 158334977 | 3 | 57 | 581959681 | 11 | 94 | 935329793 | 3 |
| 21 | 163577857 | 23 | 58 | 595591169 | 3 | 95 | 938475521 | 3 |
| 22 | 167772161 | 3 | 59 | 597688321 | 11 | 96 | 940572673 | 7 |
| 23 | 169869313 | 5 | 60 | 605028353 | 3 | 97 | 943718401 | 7 |
| 24 | 185597953 | 5 | 61 | 635437057 | 11 | 98 | 950009857 | 7 |
| 25 | 186646529 | 3 | 62 | 639631361 | 6 | 99 | 957349889 | 6 |
| 26 | 199229441 | 3 | 63 | 645922817 | 3 | 100 | 962592769 | 7 |
| 27 | 204472321 | 19 | 64 | 648019969 | 17 | 101 | 972029953 | 10 |
| 28 | 211812353 | 3 | 65 | 655360001 | 3 | 102 | 975175681 | 17 |
| 29 | 221249537 | 3 | 66 | 666894337 | 5 | 103 | 976224257 | 3 |
| 30 | 230686721 | 6 | 67 | 683671553 | 3 | 104 | 985661441 | 3 |
| 31 | 246415361 | 3 | 68 | 710934529 | 17 | 105 | 998244353 | 3 |
| 32 | 249561089 | 3 | 69 | 715128833 | 3 | 106 | 1004535809 | 3 |
| 33 | 257949697 | 5 | 70 | 718274561 | 3 | 107 | 1007681537 | 3 |
| 34 | 270532609 | 22 | 71 | 740294657 | 3 | 108 | 1012924417 | 5 |
| 35 | 274726913 | 3 | 72 | 745537537 | 5 | 109 | 1045430273 | 3 |
| 36 | 290455553 | 3 | 73 | 754974721 | 11 | 110 | 1051721729 | 6 |
| 37 | 305135617 | 5 | 74 | 770703361 | 11 | 111 | 1053818881 | 7 |

16.2 cheat.pdf

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| Definitions | | Series |
|---|--|--|
| $f(n) = O(g(n))$ | iff \exists positive c, n_0 such that $0 \leq f(n) \leq cg(n) \ \forall n \geq n_0$. | $\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$ |
| $f(n) = \Omega(g(n))$ | iff \exists positive c, n_0 such that $f(n) \geq cg(n) \geq 0 \ \forall n \geq n_0$. | In general: |
| $f(n) = \Theta(g(n))$ | iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. | $\sum_{i=1}^n i^m = \frac{1}{m+1} \left[(n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$ |
| $f(n) = o(g(n))$ | iff $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$. | $\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}.$ |
| $\lim_{n \rightarrow \infty} a_n = a$ | iff $\forall \epsilon > 0, \exists n_0$ such that $ a_n - a < \epsilon, \forall n \geq n_0$. | Geometric series: |
| $\sup S$ | least $b \in \mathbb{R}$ such that $b \geq s, \forall s \in S$. | $\sum_{i=0}^n c^i = \frac{c^{n+1} - 1}{c - 1}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1 - c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1 - c}, \quad c < 1,$ |
| $\inf S$ | greatest $b \in \mathbb{R}$ such that $b \leq s, \forall s \in S$. | $\sum_{i=0}^n ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c - 1)^2}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1 - c)^2}, \quad c < 1.$ |
| $\liminf_{n \rightarrow \infty} a_n$ | $\lim_{n \rightarrow \infty} \inf \{a_i \mid i \geq n, i \in \mathbb{N}\}.$ | Harmonic series: |
| $\limsup_{n \rightarrow \infty} a_n$ | $\lim_{n \rightarrow \infty} \sup \{a_i \mid i \geq n, i \in \mathbb{N}\}.$ | $H_n = \sum_{i=1}^n \frac{1}{i}, \quad \sum_{i=1}^n iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$ |
| $\binom{n}{k}$ | Combinations: Size k sub-sets of a size n set. | $\sum_{i=1}^n H_i = (n+1)H_n - n, \quad \sum_{i=1}^n \binom{i}{m} H_i = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right).$ |
| $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ | Stirling numbers (1st kind): Arrangements of an n element set into k cycles. | 1. $\binom{n}{k} = \frac{n!}{(n-k)!k!},$ 2. $\sum_{k=0}^n \binom{n}{k} = 2^n,$ 3. $\binom{n}{k} = \binom{n}{n-k},$ |
| $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ | Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets. | 4. $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1},$ 5. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$ |
| $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle$ | 1st order Eulerian numbers: Permutations $\pi_1 \pi_2 \dots \pi_n$ on $\{1, 2, \dots, n\}$ with k ascents. | 6. $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k},$ 7. $\sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n},$ |
| $\langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle$ | 2nd order Eulerian numbers. | 8. $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1},$ 9. $\sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n},$ |
| C_n | Catalan Numbers: Binary trees with $n+1$ vertices. | 10. $\binom{n}{k} = (-1)^k \binom{k-n-1}{k},$ 11. $\left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\} = 1,$ |
| 14. $\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix} \right] = (n-1)!,$ 15. $\left[\begin{smallmatrix} n \\ 2 \end{smallmatrix} \right] = (n-1)!H_{n-1},$ 16. $\left[\begin{smallmatrix} n \\ n \end{smallmatrix} \right] = 1,$ 17. $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \geq \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\},$ | | 13. $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\},$ |
| 18. $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = (n-1) \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right] + \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right],$ 19. $\left\{ \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right\} = \left[\begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right] = \binom{n}{2},$ 20. $\sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = n!,$ 21. $C_n = \frac{1}{n+1} \binom{2n}{n},$ | | |
| 22. $\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \rangle = \langle \begin{smallmatrix} n \\ n-1 \end{smallmatrix} \rangle = 1,$ 23. $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle = \langle \begin{smallmatrix} n \\ n-1-k \end{smallmatrix} \rangle,$ 24. $\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle = (k+1) \langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \rangle + (n-k) \langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \rangle,$ | | |
| 25. $\langle \begin{smallmatrix} 0 \\ k \end{smallmatrix} \rangle = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases}$ 26. $\langle \begin{smallmatrix} n \\ 1 \end{smallmatrix} \rangle = 2^n - n - 1,$ 27. $\langle \begin{smallmatrix} n \\ 2 \end{smallmatrix} \rangle = 3^n - (n+1)2^n + \binom{n+1}{2},$ | | |
| 28. $x^n = \sum_{k=0}^n \langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle \binom{x+k}{n},$ 29. $\langle \begin{smallmatrix} n \\ m \end{smallmatrix} \rangle = \sum_{k=0}^m \binom{n+1}{k} (m+1-k)^n (-1)^k,$ 30. $m! \left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} = \sum_{k=0}^n \langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle \binom{k}{n-m},$ | | |
| 31. $\langle \begin{smallmatrix} n \\ m \end{smallmatrix} \rangle = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \binom{n-k}{m} (-1)^{n-k-m} k!,$ 32. $\langle\langle \begin{smallmatrix} n \\ 0 \end{smallmatrix} \rangle\rangle = 1,$ 33. $\langle\langle \begin{smallmatrix} n \\ n \end{smallmatrix} \rangle\rangle = 0 \text{ for } n \neq 0,$ | | |
| 34. $\langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle = (k+1) \langle\langle \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \rangle\rangle + (2n-1-k) \langle\langle \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \rangle\rangle,$ 35. $\sum_{k=0}^n \langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle = \frac{(2n)^n}{2^n},$ | | |
| 36. $\left\{ \begin{smallmatrix} x \\ x-n \end{smallmatrix} \right\} = \sum_{k=0}^n \langle\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \rangle\rangle \binom{x+n-1-k}{2n},$ 37. $\left\{ \begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{smallmatrix} k \\ m \end{smallmatrix} \right\} = \sum_{k=0}^n \left\{ \begin{smallmatrix} k \\ m \end{smallmatrix} \right\} (m+1)^{n-k},$ | | |

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Identities Cont.

$$\begin{aligned}
 38. \quad \binom{n+1}{m+1} &= \sum_k \binom{n}{k} \binom{k}{m} = \sum_{k=0}^n \binom{k}{m} n^{\overline{n-k}} = n! \sum_{k=0}^n \frac{1}{k!} \binom{k}{m}, & 39. \quad \begin{bmatrix} x \\ x-n \end{bmatrix} &= \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{x+k}{2n}, \\
 40. \quad \left\{ \begin{matrix} n \\ m \end{matrix} \right\} &= \sum_k \binom{n}{k} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} (-1)^{n-k}, & 41. \quad \begin{bmatrix} n \\ m \end{bmatrix} &= \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \binom{k}{m} (-1)^{m-k}, \\
 42. \quad \left\{ \begin{matrix} m+n+1 \\ m \end{matrix} \right\} &= \sum_{k=0}^m k \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\}, & 43. \quad \begin{bmatrix} m+n+1 \\ m \end{bmatrix} &= \sum_{k=0}^m k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix}, \\
 44. \quad \binom{n}{m} &= \sum_k \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k}, & 45. \quad (n-m)! \binom{n}{m} &= \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (-1)^{m-k}, \quad \text{for } n \geq m, \\
 46. \quad \left\{ \begin{matrix} n \\ n-m \end{matrix} \right\} &= \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix}, & 47. \quad \begin{bmatrix} n \\ n-m \end{bmatrix} &= \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \left\{ \begin{matrix} m+k \\ k \end{matrix} \right\}, \\
 48. \quad \left\{ \begin{matrix} n \\ \ell+m \end{matrix} \right\} \binom{\ell+m}{\ell} &= \sum_k \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} \left\{ \begin{matrix} n-k \\ m \end{matrix} \right\} \binom{n}{k}, & 49. \quad \begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} &= \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}.
 \end{aligned}$$

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are d_1, \dots, d_n :

$$\sum_{i=1}^n 2^{-d_i} \leq 1,$$

and equality holds only if every internal node has 2 sons.

Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \leq cf(n)$ for large n , then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two.

Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \frac{1}{2} + u_i, \quad u_1 = \frac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$.

Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side “telescope”

$$1(T(n) - 3T(n/2)) = n$$

$$3(T(n/2) - 3T(n/4)) = n/2$$

$$\vdots \quad \vdots \quad \vdots$$

$$3^{\log_2 n - 1} (T(2) - 3T(1)) = 2$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$.

Summing the right side we get

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\frac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$n \sum_{i=0}^{m-1} c^i = n \left(\frac{c^m - 1}{c - 1} \right)$$

$$= 2n(c^{\log_2 n} - 1)$$

$$= 2n(c^{(k-1)\log_2 n} - 1)$$

$$= 2n^k - 2n,$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^i T_j.$$

Subtracting we find

$$T_{i+1} - T_i = 1 + \sum_{j=0}^i T_j - 1 - \sum_{j=0}^{i-1} T_j$$

$$= T_i.$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i .
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i .

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i.$$

We choose $G(x) = \sum_{i \geq 0} x^i g_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

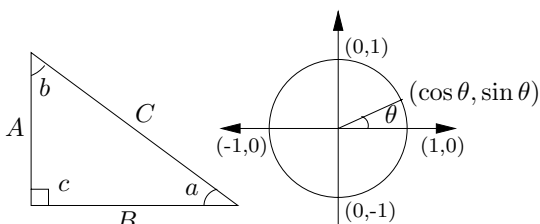
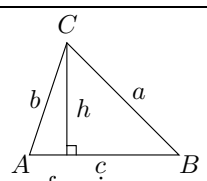
Expand this using partial fractions:

$$\begin{aligned}
 G(x) &= x \left(\frac{2}{1-2x} - \frac{1}{1-x} \right) \\
 &= x \left(2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right) \\
 &= \sum_{i \geq 0} (2^{i+1} - 1) x^{i+1}.
 \end{aligned}$$

So $g_i = 2^i - 1$.

| Theoretical Computer Science Cheat Sheet | | | | | |
|--|---------------|----------------------|---|---|---|
| $\pi \approx 3.14159,$ | | $e \approx 2.71828,$ | $\gamma \approx 0.57721,$ | $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803,$ | $\hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$ |
| i | 2^i | p_i | General | Probability | |
| 1 | 2 | 2 | <p>Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$): $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30},$ $B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$</p> <p>Change of base, quadratic formula: $\log_b x = \frac{\log_a x}{\log_a b}, \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$</p> <p>Euler's number e: $e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$ $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$ $\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$ $\left(1 + \frac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$</p> <p>Harmonic numbers: $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \dots$ $\ln n < H_n < \ln n + 1,$ $H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$</p> <p>Factorial, Stirling's approximation: $1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$ $n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$</p> <p>Ackermann's function and inverse: $a(i, j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$ $\alpha(i) = \min\{j \mid a(j, j) \geq i\}.$</p> | <p>Continuous distributions: If $\Pr[a < X < b] = \int_a^b p(x) dx,$ then p is the probability density function of X. If $\Pr[X < a] = P(a),$ then P is the distribution function of X. If P and p both exist then $P(a) = \int_{-\infty}^a p(x) dx.$</p> <p>Expectation: If X is discrete $E[g(X)] = \sum_x g(x) \Pr[X = x].$</p> <p>If X continuous then $E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x) dx = \int_{-\infty}^{\infty} g(x) dP(x).$</p> <p>Variance, standard deviation: $\text{VAR}[X] = E[X^2] - E[X]^2,$ $\sigma = \sqrt{\text{VAR}[X]}.$</p> <p>For events A and B: $\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$ $\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$ iff A and B are independent. $\Pr[A B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$</p> <p>For random variables X and Y: $E[X \cdot Y] = E[X] \cdot E[Y],$ if X and Y are independent. $E[X + Y] = E[X] + E[Y],$ $E[cX] = c E[X].$</p> <p>Bayes' theorem: $\Pr[A_i B] = \frac{\Pr[B A_i] \Pr[A_i]}{\sum_{j=1}^n \Pr[B A_j] \Pr[A_j]}.$</p> <p>Inclusion-exclusion: $\Pr\left[\bigvee_{i=1}^n X_i\right] = \sum_{i=1}^n \Pr[X_i] +$ $\sum_{k=2}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} \Pr\left[\bigwedge_{j=1}^k X_{i_j}\right].$</p> <p>Moment inequalities: $\Pr[X \geq \lambda E[X]] \leq \frac{1}{\lambda},$ $\Pr\left[X - E[X] \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$</p> <p>Geometric distribution: $\Pr[X = k] = pq^{k-1}, \quad q = 1 - p,$ $E[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$</p> | |
| 2 | 4 | 3 | | | |
| 3 | 8 | 5 | | | |
| 4 | 16 | 7 | | | |
| 5 | 32 | 11 | | | |
| 6 | 64 | 13 | | | |
| 7 | 128 | 17 | | | |
| 8 | 256 | 19 | | | |
| 9 | 512 | 23 | | | |
| 10 | 1,024 | 29 | | | |
| 11 | 2,048 | 31 | | | |
| 12 | 4,096 | 37 | | | |
| 13 | 8,192 | 41 | | | |
| 14 | 16,384 | 43 | | | |
| 15 | 32,768 | 47 | | | |
| 16 | 65,536 | 53 | | | |
| 17 | 131,072 | 59 | | | |
| 18 | 262,144 | 61 | | | |
| 19 | 524,288 | 67 | | | |
| 20 | 1,048,576 | 71 | | | |
| 21 | 2,097,152 | 73 | | | |
| 22 | 4,194,304 | 79 | | | |
| 23 | 8,388,608 | 83 | | | |
| 24 | 16,777,216 | 89 | | | |
| 25 | 33,554,432 | 97 | | | |
| 26 | 67,108,864 | 101 | | | |
| 27 | 134,217,728 | 103 | | | |
| 28 | 268,435,456 | 107 | | | |
| 29 | 536,870,912 | 109 | | | |
| 30 | 1,073,741,824 | 113 | | | |
| 31 | 2,147,483,648 | 127 | | | |
| 32 | 4,294,967,296 | 131 | | | |
| Pascal's Triangle | | | <p>Binomial distribution: $\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \quad q = 1 - p,$ $E[X] = \sum_{k=1}^n k \binom{n}{k} p^k q^{n-k} = np.$</p> <p>Poisson distribution: $\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad E[X] = \lambda.$</p> <p>Normal (Gaussian) distribution: $p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu.$</p> <p>The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is $nH_n.$</p> | | |
| 1 | | | | | |
| 1 1 | | | | | |
| 1 2 1 | | | | | |
| 1 3 3 1 | | | | | |
| 1 4 6 4 1 | | | | | |
| 1 5 10 10 5 1 | | | | | |
| 1 6 15 20 15 6 1 | | | | | |
| 1 7 21 35 35 21 7 1 | | | | | |
| 1 8 28 56 70 56 28 8 1 | | | | | |
| 1 9 36 84 126 126 84 36 9 1 | | | | | |
| 1 10 45 120 210 252 210 120 45 10 1 | | | | | |

Theoretical Computer Science Cheat Sheet

| Trigonometry | Matrices | More Trig. | | | | | | | | | | | | | | | | | | | | | | | | |
|---|--|---|----------------------|---------------|---------------|---|---|---|---|-----------------|---------------|----------------------|----------------------|-----------------|----------------------|----------------------|---|-----------------|----------------------|---------------|------------|-----------------|---|---|----------|---|
| <div></div> <p>Pythagorean theorem: $C^2 = A^2 + B^2.$</p> <p>Definitions:</p> $\sin a = A/C, \quad \cos a = B/C,$ $\csc a = C/A, \quad \sec a = C/B,$ $\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$ <p>Area, radius of inscribed circle:</p> $\frac{1}{2}AB, \quad \frac{AB}{A+B+C}.$ <p>Identities:</p> $\sin x = \frac{1}{\csc x}, \quad \cos x = \frac{1}{\sec x},$ $\tan x = \frac{1}{\cot x}, \quad \sin^2 x + \cos^2 x = 1,$ $1 + \tan^2 x = \sec^2 x, \quad 1 + \cot^2 x = \csc^2 x,$ $\sin x = \cos\left(\frac{\pi}{2} - x\right), \quad \sin x = \sin(\pi - x),$ $\cos x = -\cos(\pi - x), \quad \tan x = \cot\left(\frac{\pi}{2} - x\right),$ $\cot x = -\cot(\pi - x), \quad \csc x = \cot \frac{x}{2} - \cot x,$ $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$ $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$ $\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$ $\sin 2x = 2 \sin x \cos x, \quad \sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$ $\cos 2x = \cos^2 x - \sin^2 x, \quad \cos 2x = 2 \cos^2 x - 1,$ $\cos 2x = 1 - 2 \sin^2 x, \quad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$ $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}, \quad \cot 2x = \frac{\cot^2 x - 1}{2 \cot x},$ $\sin(x + y) \sin(x - y) = \sin^2 x - \sin^2 y,$ $\cos(x + y) \cos(x - y) = \cos^2 x - \sin^2 y.$ <p>Euler's equation:</p> $e^{ix} = \cos x + i \sin x, \quad e^{i\pi} = -1.$ | <p>Multiplication:</p> $C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}.$ <p>Determinants: $\det A \neq 0$ iff A is non-singular.</p> $\det A \cdot B = \det A \cdot \det B,$ $\det A = \sum_{\pi} \prod_{i=1}^n \text{sign}(\pi) a_{i,\pi(i)}.$ <p>2×2 and 3×3 determinant:</p> $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$ $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$ $= aei + bfg + cdh - ceg - fha - ibd.$ <p>Permanents:</p> $\text{perm } A = \sum_{\pi} \prod_{i=1}^n a_{i,\pi(i)}.$ | <div></div> <p>Law of cosines:</p> $c^2 = a^2 + b^2 - 2ab \cos C.$ <p>Area:</p> $A = \frac{1}{2}hc,$ $= \frac{1}{2}ab \sin C,$ $= \frac{c^2 \sin A \sin B}{2 \sin C}.$ <p>Heron's formula:</p> $A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$ $s = \frac{1}{2}(a + b + c),$ $s_a = s - a,$ $s_b = s - b,$ $s_c = s - c.$ <p>More identities:</p> $\sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$ $\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$ $\tan \frac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$ $= \frac{1 - \cos x}{\sin x},$ $= \frac{\sin x}{1 + \cos x},$ $\cot \frac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$ $= \frac{1 + \cos x}{\sin x},$ $= \frac{\sin x}{1 - \cos x},$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i},$ $\cos x = \frac{e^{ix} + e^{-ix}}{2},$ $\tan x = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$ $= -i \frac{e^{2ix} - 1}{e^{2ix} + 1},$ $\sin x = \frac{\sinh ix}{i},$ $\cos x = \cosh ix,$ $\tan x = \frac{\tanh ix}{i}.$ | | | | | | | | | | | | | | | | | | | | | | | | |
| | <p>Hyperbolic Functions</p> <p>Definitions:</p> $\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2},$ $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad \text{csch } x = \frac{1}{\sinh x},$ $\text{sech } x = \frac{1}{\cosh x}, \quad \coth x = \frac{1}{\tanh x}.$ <p>Identities:</p> $\cosh^2 x - \sinh^2 x = 1, \quad \tanh^2 x + \text{sech}^2 x = 1,$ $\coth^2 x - \text{csch}^2 x = 1, \quad \sinh(-x) = -\sinh x,$ $\cosh(-x) = \cosh x, \quad \tanh(-x) = -\tanh x,$ $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y,$ $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y,$ $\sinh 2x = 2 \sinh x \cosh x,$ $\cosh 2x = \cosh^2 x + \sinh^2 x,$ $\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x},$ $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$ $2 \sinh^2 \frac{x}{2} = \cosh x - 1, \quad 2 \cosh^2 \frac{x}{2} = \cosh x + 1.$ | | | | | | | | | | | | | | | | | | | | | | | | | |
| | <table><tr><th>θ</th><th>$\sin \theta$</th><th>$\cos \theta$</th><th>$\tan \theta$</th></tr><tr><td>0</td><td>0</td><td>1</td><td>0</td></tr><tr><td>$\frac{\pi}{6}$</td><td>$\frac{1}{2}$</td><td>$\frac{\sqrt{3}}{2}$</td><td>$\frac{\sqrt{3}}{3}$</td></tr><tr><td>$\frac{\pi}{4}$</td><td>$\frac{\sqrt{2}}{2}$</td><td>$\frac{\sqrt{2}}{2}$</td><td>1</td></tr><tr><td>$\frac{\pi}{3}$</td><td>$\frac{\sqrt{3}}{2}$</td><td>$\frac{1}{2}$</td><td>$\sqrt{3}$</td></tr><tr><td>$\frac{\pi}{2}$</td><td>1</td><td>0</td><td>∞</td></tr></table> | θ | $\sin \theta$ | $\cos \theta$ | $\tan \theta$ | 0 | 0 | 1 | 0 | $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{\pi}{2}$ | 1 | 0 | ∞ | <p>... in mathematics you don't understand things, you just get used to them.</p> <p>- J. von Neumann</p> |
| θ | $\sin \theta$ | $\cos \theta$ | $\tan \theta$ | | | | | | | | | | | | | | | | | | | | | | | |
| 0 | 0 | 1 | 0 | | | | | | | | | | | | | | | | | | | | | | | |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | | | | | | | | | | | | | | | | | | | | | | | |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | | | | | | | | | | | | | | | | | | | | | | | |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | | | | | | | | | | | | | | | | | | | | | | | |
| $\frac{\pi}{2}$ | 1 | 0 | ∞ | | | | | | | | | | | | | | | | | | | | | | | |
| <p>v2.02 ©1994 by Steve Seiden sseiden@acm.org http://www.csc.lsu.edu/~seiden</p> | | | | | | | | | | | | | | | | | | | | | | | | | | |

Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \pmod{m_1}$$

$$\vdots \quad \vdots \quad \vdots$$

$$C \equiv r_n \pmod{m_n}$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x . If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^n p_i^{e_i-1} (p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \pmod{b}.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \pmod{p}.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n - 1)! \equiv -1 \pmod{n}.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n}$$

$$+ O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3}$$

$$+ O\left(\frac{n}{(\ln n)^4}\right).$$

Graph Theory

Definitions:

Loop An edge connecting a vertex to itself.

Directed Each edge has a direction.

Simple Graph with no loops or multi-edges.

Walk A sequence $v_0 e_1 v_1 \dots e_\ell v_\ell$.

Trail A walk with distinct edges.

Path A trail with distinct vertices.

Connected A graph where there exists a path between any two vertices.

Component A maximal connected subgraph.

Tree A connected acyclic graph.

Free tree A tree with no root.

DAG Directed acyclic graph.

Eulerian Graph with a trail visiting each edge exactly once.

Hamiltonian Graph with a cycle visiting each vertex exactly once.

Cut A set of edges whose removal increases the number of components.

Cut-set A minimal cut.

Cut edge A size 1 cut.

k-Connected A graph connected with the removal of any $k - 1$ vertices.

k-Tough $\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq |S|$.

k-Regular A graph where all vertices have degree k .

k-Factor A k -regular spanning subgraph.

Matching A set of edges, no two of which are adjacent.

Clique A set of vertices, all of which are adjacent.

Ind. set A set of vertices, none of which are adjacent.

Vertex cover A set of vertices which cover all edges.

Planar graph A graph which can be embedded in the plane.

Plane graph An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5 .

Notation:

$E(G)$ Edge set

$V(G)$ Vertex set

$c(G)$ Number of components

$G[S]$ Induced subgraph

$\deg(v)$ Degree of v

$\Delta(G)$ Maximum degree

$\delta(G)$ Minimum degree

$\chi(G)$ Chromatic number

$\chi_E(G)$ Edge chromatic number

G^c Complement graph

K_n Complete graph

K_{n_1, n_2} Complete bipartite graph

$r(k, \ell)$ Ramsey number

Geometry

Projective coordinates: triples (x, y, z) , not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian Projective

$$(x, y) \quad (x, y, 1)$$

$$y = mx + b \quad (m, -1, b)$$

$$x = c \quad (1, 0, -c)$$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

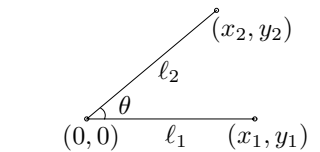
$$[|x_1 - x_0|^p + |y_1 - y_0|^p]^{1/p},$$

$$\lim_{p \rightarrow \infty} [|x_1 - x_0|^p + |y_1 - y_0|^p]^{1/p}.$$

Area of triangle (x_0, y_0) , (x_1, y_1) and (x_2, y_2) :

$$\frac{1}{2} \text{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos \theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{l_1 l_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1) :

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \quad V = \frac{4}{3} \pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.

– Issac Newton

Theoretical Computer Science Cheat Sheet

π

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:

$$\frac{\pi}{4} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \cdots}}}}$$

Gregory's series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Newton's series:

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots \right)$$

Euler's series:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x . We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D , divide N by D , obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of N' is less than that of D . Second, factor $D(x)$. Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[\frac{N(x)}{D(x)} \right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!} \left[\frac{d^k}{dx^k} \left(\frac{N(x)}{D(x)} \right) \right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

$$1. \frac{d(cu)}{dx} = c \frac{du}{dx}, \quad 2. \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \quad 3. \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx},$$

$$4. \frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx}, \quad 5. \frac{d(u/v)}{dx} = \frac{v \left(\frac{du}{dx} \right) - u \left(\frac{dv}{dx} \right)}{v^2}, \quad 6. \frac{d(e^{cu})}{dx} = ce^{cu} \frac{du}{dx},$$

$$7. \frac{d(c^u)}{dx} = (\ln c) c^u \frac{du}{dx}, \quad 8. \frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx},$$

$$9. \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}, \quad 10. \frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx},$$

$$11. \frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}, \quad 12. \frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx},$$

$$13. \frac{d(\sec u)}{dx} = \tan u \sec u \frac{du}{dx}, \quad 14. \frac{d(\csc u)}{dx} = -\cot u \csc u \frac{du}{dx},$$

$$15. \frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad 16. \frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx},$$

$$17. \frac{d(\arctan u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx}, \quad 18. \frac{d(\operatorname{arccot} u)}{dx} = \frac{-1}{1+u^2} \frac{du}{dx},$$

$$19. \frac{d(\operatorname{arcsec} u)}{dx} = \frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad 20. \frac{d(\operatorname{arccsc} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx},$$

$$21. \frac{d(\sinh u)}{dx} = \cosh u \frac{du}{dx}, \quad 22. \frac{d(\cosh u)}{dx} = \sinh u \frac{du}{dx},$$

$$23. \frac{d(\tanh u)}{dx} = \operatorname{sech}^2 u \frac{du}{dx}, \quad 24. \frac{d(\coth u)}{dx} = -\operatorname{csch}^2 u \frac{du}{dx},$$

$$25. \frac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u \frac{du}{dx}, \quad 26. \frac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u \frac{du}{dx},$$

$$27. \frac{d(\operatorname{arcsinh} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}, \quad 28. \frac{d(\operatorname{arccosh} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx},$$

$$29. \frac{d(\operatorname{arctanh} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \quad 30. \frac{d(\operatorname{arcoth} u)}{dx} = \frac{1}{u^2-1} \frac{du}{dx},$$

$$31. \frac{d(\operatorname{arcsech} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad 32. \frac{d(\operatorname{arccsch} u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}} \frac{du}{dx}.$$

Integrals:

$$1. \int cu \, dx = c \int u \, dx, \quad 2. \int (u+v) \, dx = \int u \, dx + \int v \, dx,$$

$$3. \int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1, \quad 4. \int \frac{1}{x} \, dx = \ln x, \quad 5. \int e^x \, dx = e^x,$$

$$6. \int \frac{dx}{1+x^2} = \arctan x, \quad 7. \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx,$$

$$8. \int \sin x \, dx = -\cos x, \quad 9. \int \cos x \, dx = \sin x,$$

$$10. \int \tan x \, dx = -\ln |\cos x|, \quad 11. \int \cot x \, dx = \ln |\cos x|,$$

$$12. \int \sec x \, dx = \ln |\sec x + \tan x|, \quad 13. \int \csc x \, dx = \ln |\csc x + \cot x|,$$

$$14. \int \arcsin \frac{x}{a} \, dx = \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0,$$

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Calculus Cont.

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$
16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$
17. $\int \sin^2(ax) dx = \frac{1}{2a}(ax - \sin(ax) \cos(ax)),$
18. $\int \cos^2(ax) dx = \frac{1}{2a}(ax + \sin(ax) \cos(ax)),$
19. $\int \sec^2 x dx = \tan x,$
20. $\int \csc^2 x dx = -\cot x,$
21. $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx,$
22. $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx,$
23. $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx, \quad n \neq 1,$
24. $\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx, \quad n \neq 1,$
25. $\int \sec^n x dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \quad n \neq 1,$
26. $\int \csc^n x dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx, \quad n \neq 1,$
27. $\int \sinh x dx = \cosh x,$
28. $\int \cosh x dx = \sinh x,$
29. $\int \tanh x dx = \ln |\cosh x|,$
30. $\int \coth x dx = \ln |\sinh x|,$
31. $\int \operatorname{sech} x dx = \arctan \sinh x,$
32. $\int \operatorname{csch} x dx = \ln \left| \tanh \frac{x}{2} \right|,$
33. $\int \sinh^2 x dx = \frac{1}{4} \sinh(2x) - \frac{1}{2} x,$
34. $\int \cosh^2 x dx = \frac{1}{4} \sinh(2x) + \frac{1}{2} x,$
35. $\int \operatorname{sech}^2 x dx = \tanh x,$
36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$
37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|,$
38. $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$
39. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left(x + \sqrt{a^2 + x^2} \right), \quad a > 0,$
40. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$
41. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
43. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$
44. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|,$
45. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}},$
46. $\int \sqrt{a^2 \pm x^2} dx = \frac{x}{2} \sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{a^2 \pm x^2} \right|,$
47. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right|, \quad a > 0,$
48. $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right|,$
49. $\int x \sqrt{a+bx} dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2},$
50. $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{1}{x\sqrt{a+bx}} dx,$
51. $\int \frac{x}{\sqrt{a+bx}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right|, \quad a > 0,$
52. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
53. $\int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2},$
54. $\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
55. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
56. $\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$
57. $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
58. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|,$
59. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$
60. $\int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2},$
61. $\int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right|,$

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Calculus Cont.

$$\begin{aligned}
 62. \int \frac{dx}{x\sqrt{x^2 - a^2}} &= \frac{1}{a} \arccos \frac{a}{|x|}, \quad a > 0, & 63. \int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} &= \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x}, \\
 64. \int \frac{x dx}{\sqrt{x^2 \pm a^2}} &= \sqrt{x^2 \pm a^2}, & 65. \int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx &= \mp \frac{(x^2 + a^2)^{3/2}}{3a^2 x^3}, \\
 66. \int \frac{dx}{ax^2 + bx + c} &= \begin{cases} \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right|, & \text{if } b^2 > 4ac, \\ \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}}, & \text{if } b^2 < 4ac, \end{cases} \\
 67. \int \frac{dx}{\sqrt{ax^2 + bx + c}} &= \begin{cases} \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right|, & \text{if } a > 0, \\ \frac{1}{\sqrt{-a}} \arcsin \frac{-2ax - b}{\sqrt{b^2 - 4ac}}, & \text{if } a < 0, \end{cases} \\
 68. \int \sqrt{ax^2 + bx + c} dx &= \frac{2ax + b}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \\
 69. \int \frac{x dx}{\sqrt{ax^2 + bx + c}} &= \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \\
 70. \int \frac{dx}{x\sqrt{ax^2 + bx + c}} &= \begin{cases} \frac{-1}{\sqrt{c}} \ln \left| \frac{2\sqrt{c}\sqrt{ax^2 + bx + c} + bx + 2c}{x} \right|, & \text{if } c > 0, \\ \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}, & \text{if } c < 0, \end{cases} \\
 71. \int x^3 \sqrt{x^2 + a^2} dx &= \left(\frac{1}{3}x^2 - \frac{2}{15}a^2\right)(x^2 + a^2)^{3/2}, \\
 72. \int x^n \sin(ax) dx &= -\frac{1}{a}x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) dx, \\
 73. \int x^n \cos(ax) dx &= \frac{1}{a}x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx, \\
 74. \int x^n e^{ax} dx &= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx, \\
 75. \int x^n \ln(ax) dx &= x^{n+1} \left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2} \right), \\
 76. \int x^n (\ln ax)^m dx &= \frac{x^{n+1}}{n+1} (\ln ax)^m - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} dx.
 \end{aligned}$$

Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1) - f(x),$$

$$\mathbf{E} f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x) \delta x = F(x) + C.$$

$$\sum_a^b f(x) \delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$$\Delta(cu) = c\Delta u, \quad \Delta(u+v) = \Delta u + \Delta v,$$

$$\Delta(uv) = u\Delta v + \mathbf{E} v \Delta u,$$

$$\Delta(x^n) = nx^{n-1},$$

$$\Delta(H_x) = x^{-1}, \quad \Delta(2^x) = 2^x,$$

$$\Delta(c^x) = (c-1)c^x, \quad \Delta\binom{x}{m} = \binom{x}{m-1}.$$

Sums:

$$\sum cu \delta x = c \sum u \delta x,$$

$$\sum (u+v) \delta x = \sum u \delta x + \sum v \delta x,$$

$$\sum u \Delta v \delta x = uv - \sum \mathbf{E} v \Delta u \delta x,$$

$$\sum x^n \delta x = \frac{x^{n+1}}{n+1}, \quad \sum x^{-1} \delta x = H_x,$$

$$\sum c^x \delta x = \frac{c^x}{c-1}, \quad \sum \binom{x}{m} \delta x = \binom{x}{m+1}.$$

Falling Factorial Powers:

$$x^{\underline{n}} = x(x-1) \cdots (x-n+1), \quad n > 0,$$

$$x^{\underline{0}} = 1,$$

$$x^{\underline{n}} = \frac{1}{(x+1) \cdots (x+|n|)}, \quad n < 0,$$

$$x^{\overline{n+m}} = x^{\overline{n}}(x-m)^{\underline{n}}.$$

Rising Factorial Powers:

$$x^{\overline{n}} = x(x+1) \cdots (x+n-1), \quad n > 0,$$

$$x^{\overline{0}} = 1,$$

$$x^{\overline{n}} = \frac{1}{(x-1) \cdots (x-|n|)}, \quad n < 0,$$

$$x^{\overline{n+m}} = x^{\overline{n}}(x+m)^{\underline{n}}.$$

Conversion:

$$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$$

$$= 1/(x+1)^{-\overline{n}},$$

$$x^{\overline{n}} = (-1)^n (-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$$

$$= 1/(x-1)^{-\underline{n}},$$

$$x^n = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^{n-k} x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] (-1)^{n-k} x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] x^k.$$

| | | | |
|----------------------|---|-----------------------|--|
| $x^1 =$ | $x^{\underline{1}}$ | $=$ | $x^{\overline{1}}$ |
| $x^2 =$ | $x^{\underline{2}} + x^{\underline{1}}$ | $=$ | $x^{\overline{2}} - x^{\overline{1}}$ |
| $x^3 =$ | $x^{\underline{3}} + 3x^{\underline{2}} + x^{\underline{1}}$ | $=$ | $x^{\overline{3}} - 3x^{\overline{2}} + x^{\overline{1}}$ |
| $x^4 =$ | $x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}}$ | $=$ | $x^{\overline{4}} - 6x^{\overline{3}} + 7x^{\overline{2}} - x^{\overline{1}}$ |
| $x^5 =$ | $x^{\underline{5}} + 15x^{\underline{4}} + 25x^{\underline{3}} + 10x^{\underline{2}} + x^{\underline{1}}$ | $=$ | $x^{\overline{5}} - 15x^{\overline{4}} + 25x^{\overline{3}} - 10x^{\overline{2}} + x^{\overline{1}}$ |
| $x^{\overline{1}} =$ | x^1 | $x^{\underline{1}} =$ | x^1 |
| $x^{\overline{2}} =$ | $x^2 + x^1$ | $x^{\underline{2}} =$ | $x^2 - x^1$ |
| $x^{\overline{3}} =$ | $x^3 + 3x^2 + 2x^1$ | $x^{\underline{3}} =$ | $x^3 - 3x^2 + 2x^1$ |
| $x^{\overline{4}} =$ | $x^4 + 6x^3 + 11x^2 + 6x^1$ | $x^{\underline{4}} =$ | $x^4 - 6x^3 + 11x^2 - 6x^1$ |
| $x^{\overline{5}} =$ | $x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1$ | $x^{\underline{5}} =$ | $x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1$ |

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Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \dots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{i=0}^{\infty} x^i, \\ \frac{1}{1-cx} &= 1 + cx + c^2x^2 + c^3x^3 + \dots = \sum_{i=0}^{\infty} c^i x^i, \\ \frac{1}{1-x^n} &= 1 + x^n + x^{2n} + x^{3n} + \dots = \sum_{i=0}^{\infty} x^{ni}, \\ \frac{x}{(1-x)^2} &= x + 2x^2 + 3x^3 + 4x^4 + \dots = \sum_{i=0}^{\infty} ix^i, \\ x^k \frac{d^n}{dx^n} \left(\frac{1}{1-x} \right) &= x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \dots = \sum_{i=0}^{\infty} i^n x^i, \\ e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots = \sum_{i=0}^{\infty} \frac{x^i}{i!}, \\ \ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i}, \\ \ln \frac{1}{1-x} &= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots = \sum_{i=1}^{\infty} \frac{x^i}{i}, \\ \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}, \\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}, \\ \tan^{-1} x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)}, \\ (1+x)^n &= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots = \sum_{i=0}^{\infty} \binom{n}{i} x^i, \\ \frac{1}{(1-x)^{n+1}} &= 1 + (n+1)x + \binom{n+2}{2}x^2 + \dots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i, \\ \frac{x}{e^x - 1} &= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!}, \\ \frac{1}{2x}(1 - \sqrt{1-4x}) &= 1 + x + 2x^2 + 5x^3 + \dots = \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i, \\ \frac{1}{\sqrt{1-4x}} &= 1 + x + 2x^2 + 6x^3 + \dots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i, \\ \frac{1}{\sqrt{1-4x}} \left(\frac{1 - \sqrt{1-4x}}{2x} \right)^n &= 1 + (2+n)x + \binom{4+n}{2}x^2 + \dots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i, \\ \frac{1}{1-x} \ln \frac{1}{1-x} &= x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{25}{12}x^4 + \dots = \sum_{i=1}^{\infty} H_i x^i, \\ \frac{1}{2} \left(\ln \frac{1}{1-x} \right)^2 &= \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \dots = \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i}, \\ \frac{x}{1-x-x^2} &= x + x^2 + 2x^3 + 3x^4 + \dots = \sum_{i=0}^{\infty} F_i x^i, \\ \frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} &= F_n x + F_{2n} x^2 + F_{3n} x^3 + \dots = \sum_{i=0}^{\infty} F_{ni} x^i. \end{aligned}$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$x A'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x) dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^i a_j$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^i a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

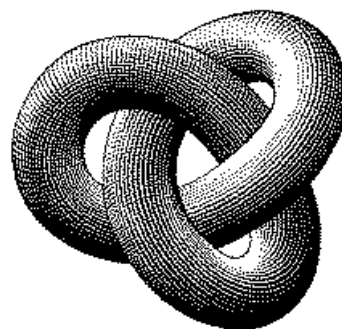
Series

Expansions:

$$\begin{aligned}\frac{1}{(1-x)^{n+1}} \ln \frac{1}{1-x} &= \sum_{i=0}^{\infty} (H_{n+i} - H_n) \binom{n+i}{i} x^i, \\ x^{\overline{n}} &= \sum_{i=0}^{\infty} \left[\begin{matrix} n \\ i \end{matrix} \right] x^i, \\ \left(\ln \frac{1}{1-x} \right)^n &= \sum_{i=0}^{\infty} \left[\begin{matrix} i \\ n \end{matrix} \right] \frac{n! x^i}{i!}, \\ \tan x &= \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i} (2^{2i} - 1) B_{2i} x^{2i-1}}{(2i)!}, \\ \frac{1}{\zeta(x)} &= \sum_{i=1}^{\infty} \frac{\mu(i)}{i^x}, \\ \zeta(x) &= \prod_p \frac{1}{1 - p^{-x}}, \\ \zeta^2(x) &= \sum_{i=1}^{\infty} \frac{d(i)}{x^i} \quad \text{where } d(n) = \sum_{d|n} 1, \\ \zeta(x) \zeta(x-1) &= \sum_{i=1}^{\infty} \frac{S(i)}{x^i} \quad \text{where } S(n) = \sum_{d|n} d, \\ \zeta(2n) &= \frac{2^{2n-1} |B_{2n}|}{(2n)!} \pi^{2n}, \quad n \in \mathbb{N}, \\ \frac{x}{\sin x} &= \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(4^i - 2) B_{2i} x^{2i}}{(2i)!}, \\ \left(\frac{1 - \sqrt{1-4x}}{2x} \right)^n &= \sum_{i=0}^{\infty} \frac{n(2i+n-1)!}{i!(n+i)!} x^i, \\ e^x \sin x &= \sum_{i=1}^{\infty} \frac{2^{i/2} \sin \frac{i\pi}{4}}{i!} x^i, \\ \sqrt{\frac{1 - \sqrt{1-x}}{x}} &= \sum_{i=0}^{\infty} \frac{(4i)!}{16^i \sqrt{2} (2i)! (2i+1)!} x^i, \\ \left(\frac{\arcsin x}{x} \right)^2 &= \sum_{i=0}^{\infty} \frac{4^i i!^2}{(i+1)(2i+1)!} x^{2i}.\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{x} \right)^{\overline{-n}} &= \sum_{i=0}^{\infty} \left\{ \begin{matrix} i \\ n \end{matrix} \right\} x^i, \\ (e^x - 1)^n &= \sum_{i=0}^{\infty} \left\{ \begin{matrix} i \\ n \end{matrix} \right\} \frac{n! x^i}{i!}, \\ x \cot x &= \sum_{i=0}^{\infty} \frac{(-4)^i B_{2i} x^{2i}}{(2i)!}, \\ \zeta(x) &= \sum_{i=1}^{\infty} \frac{1}{i^x}, \\ \frac{\zeta(x-1)}{\zeta(x)} &= \sum_{i=1}^{\infty} \frac{\phi(i)}{i^x},\end{aligned}$$

Escher's Knot



Stieltjes Integration

If G is continuous in the interval $[a, b]$ and F is nondecreasing then

$$\int_a^b G(x) dF(x)$$

exists. If $a \leq b \leq c$ then

$$\int_a^c G(x) dF(x) = \int_a^b G(x) dF(x) + \int_b^c G(x) dF(x).$$

If the integrals involved exist

$$\int_a^b (G(x) + H(x)) dF(x) = \int_a^b G(x) dF(x) + \int_a^b H(x) dF(x),$$

$$\int_a^b G(x) d(F(x) + H(x)) = \int_a^b G(x) dF(x) + \int_a^b G(x) dH(x),$$

$$\int_a^b c \cdot G(x) dF(x) = \int_a^b G(x) d(c \cdot F(x)) = c \int_a^b G(x) dF(x),$$

$$\int_a^b G(x) dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x) dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a, b]$ then

$$\int_a^b G(x) dF(x) = \int_a^b G(x) F'(x) dx.$$

Cramer's Rule

If we have equations:

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$$

$$\vdots \quad \quad \quad \vdots$$

$$a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n$$

Let $A = (a_{i,j})$ and B be the column matrix (b_i) . Then there is a unique solution iff $\det A \neq 0$. Let A_i be A with column i replaced by B . Then

$$x_i = \frac{\det A_i}{\det A}.$$

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)

| | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|
| 00 | 47 | 18 | 76 | 29 | 93 | 85 | 34 | 61 | 52 |
| 86 | 11 | 57 | 28 | 70 | 39 | 94 | 45 | 02 | 63 |
| 95 | 80 | 22 | 67 | 38 | 71 | 49 | 56 | 13 | 04 |
| 59 | 96 | 81 | 33 | 07 | 48 | 72 | 60 | 24 | 15 |
| 73 | 69 | 90 | 82 | 44 | 17 | 58 | 01 | 35 | 26 |
| 68 | 74 | 09 | 91 | 83 | 55 | 27 | 12 | 46 | 30 |
| 37 | 08 | 75 | 19 | 92 | 84 | 66 | 23 | 50 | 41 |
| 14 | 25 | 36 | 40 | 51 | 62 | 03 | 77 | 88 | 99 |
| 21 | 32 | 43 | 54 | 65 | 06 | 10 | 89 | 97 | 78 |
| 42 | 53 | 64 | 05 | 16 | 20 | 31 | 98 | 79 | 87 |

The Fibonacci number system:
Every integer n has a unique representation

$$n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$$

where $k_i \geq k_{i+1} + 2$ for all i ,
 $1 \leq i < m$ and $k_m \geq 2$.

Fibonacci Numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Definitions:

$$F_i = F_{i-1} + F_{i-2}, \quad F_0 = F_1 = 1,$$

$$F_{-i} = (-1)^{i-1} F_i,$$

$$F_i = \frac{1}{\sqrt{5}} \left(\phi^i - \hat{\phi}^i \right),$$

Cassini's identity: for $i > 0$:

$$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n,$$

$$F_{2n} = F_n F_{n+1} + F_{n-1} F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$$