Moo's Question: Why a gradient of potential is the negative of electric field?

Answer: It is a consequence of the fundamental property of the line integral of **an electrostatic** electric field $\vec{E}$ that

$$\Phi = \int_i^f \vec{E} \cdot d\vec{s}$$

is independent of any path from point i to point f but depends only on the final (x_f, y_f, z_f) and initial (x_i, y_i, z_i) positions or $\Phi(x_f, y_f, z_f, x_i, y_i, z_i)$.

It turns out that $\Phi(x_f, y_f, z_f, x_i, y_i, z_i)$ can be only in the form of

$$\Phi = \mathcal{V}(x_f, y_f, z_f) - \mathcal{V}(x_i, y_i, z_i). \tag{1}$$

Why?

Because if we split the line integral into two sections with m as an itermediate point:

$$\Phi = \int_i^f \vec{E} \cdot d\vec{s} = \int_m^f \vec{E} \cdot d\vec{s} + \int_i^m \vec{E} \cdot d\vec{s},$$

it is necessary that Φ follows Eqn. 1, because otherwise Φ would depend on the intermediate position m:

$$\Phi = [\mathcal{V}(x_f, y_f, z_f) - \mathcal{V}(x_m, y_m, z_m)] + [\mathcal{V}(x_m, y_m, z_m) - \mathcal{V}(x_i, y_i, z_i)] = \mathcal{V}(x_f, y_f, z_f) - \mathcal{V}(x_i, y_i, z_i).$$

(Try other possibilities like $\Phi = \mathcal{V}(x_f, y_f, z_f) + \mathcal{V}(x_i, y_i, z_i)$.)

Therefore,

$$\mathcal{V}(x_f, y_f, z_f) - \mathcal{V}(x_i, y_i, z_i) = \int_i^f \vec{E} \cdot d\vec{s}. \tag{2}$$

Next, to have the result of $\mathcal{V}(x_f, y_f, z_f) - \mathcal{V}(x_i, y_i, z_i)$ from any integral, you need to write that it the form of

$$\mathcal{V}(x_f, y_f, z_f) - \mathcal{V}(x_i, y_i, z_i) = \int_i^f d\mathcal{V}$$

And since $\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$ and $d\vec{s} = dx\hat{x} + dy\hat{y} + dz\hat{z}$, we can relate

$$\vec{E} \cdot d\vec{s} = E_x dx + E_y dy + E_z dz = d\mathcal{V}.$$

We write $d\mathcal{V}$ in a partial differentiation form.

$$d\mathcal{V} = \frac{\partial \mathcal{V}}{\partial x} dx + \frac{\partial \mathcal{V}}{\partial y} dy + \frac{\partial \mathcal{V}}{\partial z} dz.$$

So, we found that

$$\frac{\partial \mathcal{V}}{\partial x} = E_x, \frac{\partial \mathcal{V}}{\partial y} = E_y, \frac{\partial \mathcal{V}}{\partial z} = E_z. \tag{3}$$

Finally, we would like to define the electric potential such that it has the same sign as the potential energy $U = qV$ (so that electric field points toward the lowered potential direction not the other way around). We need to replace

$$\mathcal{V} \to -V$$

.

The last part was shown in the lecture.