Derivation of Vector Potential Due to Magnetization: Bound Currents $\vec{J_b}$ and $\vec{K_b}$

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Since the vector potential due to a magnetic moment $\vec{m}$ is

$$d\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{\mathscr{r}}}{\mathscr{r}^2}.$$

Therefore total vector potential becomes

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{M}\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

Rewrite the equation with the help of $\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = -\vec{\nabla}\frac{1}{\mathscr{r}} = \vec{\nabla}'\frac{1}{\mathscr{r}}$ (See *).
With this

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\vec{M}(\vec{r'})\times\vec{\nabla}'\frac{1}{\mathscr{r}}d\tau'.$$

By use of the vector identity

$$\vec{\nabla}'\times(f\vec{A}) = f(\vec{\nabla}'\times\vec{A}) - \vec{A}\times(\vec{\nabla}'f),$$

we could rearrange the terms such that

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\nabla\times\vec{M}}{\mathscr{r}}d\tau' - \frac{\mu_0}{4\pi}\oint\vec{\nabla}'\times\frac{\vec{M}}{\mathscr{r}}d\tau'.$$

The second term reduces to an area integral (See ** and Griffiths Prob 1.60b):

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\nabla\times\vec{M}}{\mathscr{r}}d\tau' + \frac{\mu_0}{4\pi}\oint\frac{\vec{M}\times d\vec{a'}}{\mathscr{r}}.$$

By comparing the formular to the expression for the vector potential due to volumn current and surface current, we could see that having magnetization is equivalent to having current sources $\vec{J_b} = \nabla\times\vec{M}$ and $\vec{K_b} = \vec{M}\times d\hat{n}$.

(*) Simple to derive it in Cartesian coordinate (xyz), since

$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}.$$

Insert into the definition of gradient,

$$\vec{\nabla}\frac{1}{\mathscr{r}} = \frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\hat{x}$$
$$+\frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\hat{y}$$
$$+\frac{\partial}{\partial x}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\hat{z}.$$

The differentiation would result in

$$\vec{\nabla}\frac{1}{\mathscr{r}} = \frac{x'-x}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{\frac{3}{2}}}\hat{x}$$
$$+\frac{y'-y}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{\frac{3}{2}}}\hat{y}$$
$$+\frac{z'-z}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{\frac{3}{2}}}\hat{z}. \tag{0.1}$$

or

$$\vec{\nabla}\frac{1}{\mathscr{r}} = -\frac{\vec{\mathscr{r}}}{\mathscr{r}^{3}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^{2}}.$$

Similarly for $\vec{\nabla}'\frac{1}{\mathscr{r}}$, the differentiations are with respect to x', y', z', instead. This result in another factor of -1.

$$\vec{\nabla}'\frac{1}{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{\mathscr{r}^{3}} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^{2}}.$$

(**) From a divergence theorem,

$$\int \vec{\nabla}\cdot\vec{v}d\tau' = \oint \vec{v}\cdot d\vec{a}'.$$

By setting $\vec{v}\to\vec{v}\times\vec{c}$, where $\vec{c}$ is a constant. The left hand side becomes

$$\vec{\nabla}\cdot(\vec{v}\times\vec{c}) = \vec{c}\cdot(\vec{\nabla}\times\vec{v}) - \vec{v}\cdot(\vec{\nabla}\times\vec{c}) = \vec{c}\cdot(\vec{\nabla}\times\vec{v}),$$

since $\vec{c}$ is a constant.
On the right hand side,

$$(\vec{v}\times\vec{c})\cdot d\vec{a}' = \vec{c}\cdot(d\vec{a}'\times\vec{v}) = -\vec{c}\cdot(\vec{v}\times d\vec{a}'),$$

using the vector product property:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = -\vec{C}\cdot(\vec{B}\times\vec{A}).$$

By combining things together,

$$\int \vec{c}\cdot(\vec{\nabla}\times\vec{v})d\tau' = \oint -\vec{c}\cdot(\vec{v}\times d\vec{a}')$$

or

$$\int (\vec{\nabla} \times \vec{v}) d\tau' = -\oint \vec{v} \times d\vec{a}'.$$

The setting of $\vec{v} = \frac{\vec{M}}{\mathscr{r}}$ would result in the desired result.