1.

(a)

$$\int (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint \vec{v}\cdot d\vec{l}$$

$$\vec{v} = T\vec{C}$$

$$\int (\vec{\nabla}\times T\vec{C})\cdot d\vec{a} = \oint T\vec{C}\cdot d\vec{l}$$

use $\vec{\nabla}\times T\vec{C} = T(\cancelto{0}{\vec{\nabla}\times\vec{C}}) - \vec{C}\times\vec{\nabla}T$

$$= -\vec{C}\times\vec{\nabla}T$$

$$-\int (\vec{C}\times\vec{\nabla}T)\cdot d\vec{a} = \oint T\vec{C}\cdot d\vec{l}$$

use $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

$$-\int \underline{\vec{C}}\cdot(\vec{\nabla}T\times d\vec{a}) = \oint \underline{\vec{C}}\cdot(T\,d\vec{l})$$

$$\boxed{\int \vec{\nabla}T\times d\vec{a} = -\oint T\,d\vec{l}}$$

(b) set $T = \vec{C}\cdot\vec{r}$

$$\int \vec{\nabla}(\vec{C}\cdot\vec{r})\times d\vec{a} = -\oint (\vec{C}\cdot\vec{r})\,d\vec{l}$$

use $\vec{\nabla}(\vec{C}\cdot\vec{r}) = \vec{C}\times(\cancelto{0}{\vec{\nabla}\times\vec{r}}) + \vec{r}\times(\cancelto{0}{\vec{\nabla}\times\vec{C}}) + \cancelto{0}{(\vec{r}\cdot\vec{\nabla})\vec{C}} + (\vec{C}\cdot\vec{\nabla})\vec{r}$

$\vec{\nabla}\times\vec{r} = 0$ $\qquad$ $\vec{C}$ = constant

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = 0$$

$$\int (\vec{C}\cdot\vec{\nabla})\vec{r}\times d\vec{a} = -\oint (\vec{C}\cdot\vec{r})\,d\vec{l}$$

calculate $(\vec{C}\cdot\vec{\nabla})\vec{r} = (C_x\frac{\partial}{\partial x} + C_y\frac{\partial}{\partial y} + C_z\frac{\partial}{\partial z})(x\hat{i} + y\hat{j} + z\hat{k})$

$$= C_x\hat{i} + C_y\hat{j} + C_z\hat{k} = \vec{C}$$

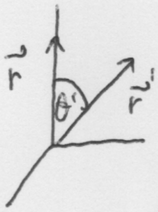
$$\int \vec{c} \times d\vec{a} = -\oint (\vec{c}\cdot\vec{r})\, d\vec{l}$$

$$\vec{c} \times \int d\vec{a} = -\oint (\vec{c}\cdot\vec{r})\, d\vec{l}$$

$$\vec{c} \times \vec{a} = -\oint (\vec{c}\cdot\vec{r})\, d\vec{l}$$

$$\oint (\vec{c}\cdot\vec{r})\, d\vec{l} = \vec{a} \times \vec{c}$$

(c)



$$\vec{r}\cdot\vec{r}' = r r' \cos\theta' \longrightarrow \hat{r}\cdot\vec{r}' = r'\cos\theta'$$

$$I \oint r'\cos\theta'\, d\vec{l}' = I \oint (\hat{r}\cdot\vec{r}')\, d\vec{l}'$$

(d)

$$\vec{c} = I\hat{r} \qquad \vec{r} = \vec{r}'$$

$$\oint I\hat{r}\cdot\vec{r}'\, d\vec{l}' = I\vec{a} \times \hat{r}$$

$$I \oint r'\cos\theta'\, d\vec{l}' = I\vec{a} \times \hat{r}$$

$$= \vec{m} \times \hat{r}$$

2. $\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$

$$= \cancel{\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}} \qquad \vec{m} = m\cos\theta\,\hat{r} - m\sin\theta\,\hat{\theta}$$

$$\vec{m}\times\hat{r} = -m\sin\theta\,(\hat{\theta}\times\hat{r}) = m\sin\theta\,\hat{\phi}$$

$$= \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\,\hat{\phi}$$

(b) $\vec{B} = \vec{\nabla}\times\vec{A}$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\hat{\theta}$$

$$= \cancel{\frac{\mu_0}{4\pi}}\frac{m}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\right)\hat{r} - \cancel{\frac{\mu_0}{4\pi}}\frac{m\sin\theta}{r}\frac{\partial}{\partial r}\left(\frac{r}{r^2}\right)\hat{\theta}$$

$$= \frac{\mu_0}{4\pi}m\left[\left(\frac{2\sin\theta\cos\theta}{r^3\sin\theta}\right)\hat{r} - \frac{\sin\theta}{r}\frac{(-1)}{r^2}\hat{\theta}\right]$$

$$= \frac{\mu_0 m}{4\pi r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

(c) $\vec{m} = m\cos\theta\,\hat{r} + m\sin\theta\,\hat{\theta}$

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left[3m\cos\theta\,\hat{r} - m\cos\theta\,\hat{r} + m\sin\theta\,\hat{\theta}\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left[2m\cos\theta\,\hat{r} + m\sin\theta\,\hat{\theta}\right]$$

3. (a)

$$\vec{M} = \chi \vec{H}$$

$$= \chi(\vec{H}_d + \vec{H}')$$

use $\vec{H}_d = -N\vec{M}$

$$= \chi(-N\vec{M} + \vec{H}')$$

use $\vec{M} = \chi'\vec{H}'$

$$\cancel{\vec{M}}(1 + \chi N) = \chi \vec{H}' = \chi \frac{\cancel{\vec{M}}}{\chi'}$$

$$\chi' = \frac{\chi}{1 + \chi N}$$

(b) $\cancel{N = \frac{1}{3}}$

large χ ; $\chi' = \frac{1}{N}$

$$N = \frac{1}{3}$$

$$\chi' = 3$$

(c)

$$\vec{M} = \chi'\vec{H}'$$

$$\vec{M}_s = 3\vec{H}'$$

$$\vec{H}' = \frac{\vec{M}_s}{3}$$ ← required field to saturate $\vec{M}$

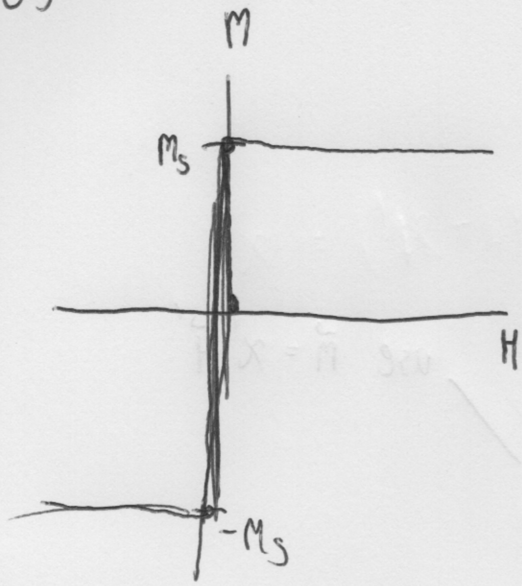
(d) $\vec{H} = \vec{H}' + \vec{H}_d$

$$= \vec{H}' - N\vec{M}$$

$$= \frac{\vec{M}}{\chi'} - N\vec{M} = \frac{\vec{M}}{3} - \frac{\vec{M}}{3} = 0$$

$\chi' = 3$ (from b) $\quad N = \frac{1}{3}$ (sphere)

(e)



4. (a)

$$g\mu_B \frac{\partial}{\partial y} \ln \sum_i e^{-m_J y} = \frac{g\mu_B \sum_{-J}^{J} (-m_J) e^{-m_J y}}{\sum_{-J}^{J} e^{-m_J y}}$$

$$= \cancel{g\mu_B} \frac{-\sum g\mu_B m_J e^{-m_J y}}{\sum e^{-m_J y}}$$

$$\langle m_z \rangle = \frac{\sum_{-J}^{J} (m_z)_i \, e^{-m_J y}}{\sum_{-J}^{J} e^{-m_J y}}$$

(b)

$$\sum_{-J}^{\infty} e^{-m_J y} = \frac{e^{+Jy}}{1-e^{-y}}$$

$$\sum_{J+1}^{\infty} e^{-m_J y} = \frac{e^{-(J+1)y}}{1-e^{-y}}$$

$$\sum_{-J}^{J} e^{-m_J y} = -\sum_{J+1}^{\infty} e^{-m_J y} + \sum_{-J}^{\infty} e^{-m_J y}$$

$$= \frac{e^{+Jy} - e^{-(J+1)y}}{1-e^{-y}}$$

$$= \frac{e^{+(J+\frac{1}{2})y} - e^{-(J+\frac{1}{2})y}}{e^{y/2} - e^{-y/2}}$$

$$= \frac{\sinh(J+\frac{1}{2})y}{\sinh y/2}$$

(c)

$$\frac{\partial}{\partial y} \ln \sum_{-J}^{J} e^{-m_J y} = \frac{\partial}{\partial y} \ln \left(\frac{\sinh(J+\frac{1}{2})y}{\sinh y/2} \right)$$

$$\cancel{= \frac{\sinh y/2}{\sinh(J+\frac{1}{2})y} \left[\frac{\sinh y/2 \, \cosh(J+\frac{1}{2})y}{\cosh y/2} - \sin \right.}$$

$$= \frac{\cancel{\sinh \frac{y}{2}}}{\sinh(J+\frac{1}{2})y}\left[\frac{(J+\frac{1}{2})\cancel{\sinh \frac{y}{2}} \cosh(J+\frac{1}{2})y - \frac{1}{2}\cancel{\sinh(J+\frac{1}{2})y} \cosh \frac{y}{2}}{\sinh^2 \frac{y}{2}}\right.$$

$$= (J+\tfrac{1}{2}) \cancel{\sinh \tfrac{y}{2}} \coth (J+\tfrac{1}{2})y - \tfrac{1}{2} \coth \tfrac{y}{2}$$

set $x = Jy$

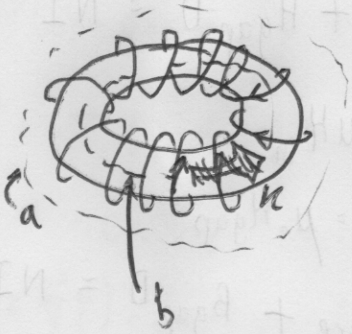
$$= (J+\tfrac{1}{2}) \coth (J+\tfrac{1}{2})\frac{x}{J} - \frac{1}{2} \coth\left(\frac{x}{2J}\right)$$

$$\langle m_z \rangle = g\mu_B J \left[\left(\frac{2J+1}{2J}\right)\coth \frac{(2J+1)x}{2J} - \frac{1}{2J} \coth \frac{x}{2J}\right]$$

5.

(a.) ~~sole~~ toroid

or folded solenoid

(b.) 

$\vec{K} = \vec{M} \times \hat{n}$

$K = M$

outside toroid (loop a)

$\oint \vec{B} \cdot d\ell = \mu_0 I_{enc} = 0$

$B = 0$

(c) inside the toroid (loop b)

$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$

$B \, 2\pi r = \underbrace{\mu_0 N I}_{\text{due to electrical current}} + \mu_0 \underbrace{M 2\pi a}$

$K\ell = (M)(\underbrace{2\pi a}_{\text{inner radius}})$

$\vec{B} = \left(\frac{\mu_0 N I}{2\pi r} + \frac{\mu_0 M a}{r} \right) \hat{\phi}$

~~(d)~~

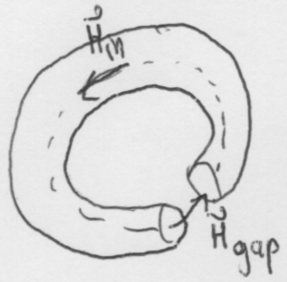
$\vec{B} = \mu_0 (\vec{H} + \vec{M})$

$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} = \left(\frac{NI}{2\pi r} + \frac{Ma}{r} - M \right) \hat{\phi}$

if $a \approx r$

$\vec{H} = \frac{NI}{2\pi r} \hat{\phi}$

(d) using $\vec{\nabla}\times\vec{H} = \vec{J}_f$



$$\downarrow$$

$$\oint \vec{H}\cdot d\vec{l} = I_{enc}$$

$$H_{in} l_{core} + H_{gap} D = NI$$

since $B_{in} = \mu H_{in}$

$$B_{gap} = \mu_0 H_{gap}$$

$$\frac{B_{in}}{\mu} l_{core} + \frac{B_{gap}}{\mu_0} D = NI$$

use $\vec{\nabla}\cdot\vec{B} = 0 \longrightarrow \Phi_{out} = \Phi_{in}$

$\oint \vec{B}\cdot d\vec{a} = 0$ $\quad B_{in} = B_{gap}$

$$B_{gap}\left(\frac{l_{core}}{\mu} + \frac{D}{\mu_0}\right) = NI$$

$$\mu_0 H_{gap}\left(\frac{l_{core}}{\mu} + \frac{D}{\mu_0}\right) = NI$$

$$\cancel{H_{gap} = \frac{NI\mu_0}{\frac{(\mu_0 l_{core} + D\mu)}{\mu\mu_0}}}$$

$$H_{gap}\left(l_{core}\frac{\mu_0}{\mu} + D\right) = NI$$

$$\cancel{H_{gap} = \frac{\mu_0 NI}{\frac{l_{core}}{\mu} + \frac{D}{\mu_0}}} =$$

$$H_{gap} = \frac{NI}{l_{core}\frac{\mu_0}{\mu} + D}$$