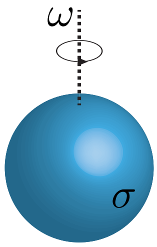

Example 6: Calculate magnetic field from a rotating sphere of radius a
Solution: We know earlier that the current density is

$$\vec{K} = \sigma\omega a \sin\theta' \hat{\phi};$$

therefore the vector potential is simply

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}}{\mathscr{r}} da' = \frac{\mu_0 \sigma\omega a}{4\pi} \int \frac{\sin\theta' \hat{\phi}}{\mathscr{r}} da', \tag{1}$$

where $\hat{\phi} = -\sin\phi' \hat{x} + \cos\phi' \hat{y}$ in Cartesian coordinate.

The tricky part here is to do the integral (Physics are done). We could 'simplify' the step a bit with the help of spherical harmonics.

First, we can expand $\frac{1}{\mathscr{r}}$ as

$$\frac{1}{\mathscr{r}} = \sum_{m,l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{l,m}(\theta, \phi) Y^*_{l,m}(\theta', \phi'), \tag{2}$$

where $r_<$ is the lesser of r or $r' = a$. In the lecture, I have shown you the field *inside*, so $r_< = r$ and $r_> = a$. However, if you want to calculate the field outside the sphere then $r_< = a$ and $r_> = r$ (**Jackson 3.70**).

Second, the $\sin\theta' \cos\phi'$ and $\sin\theta' \sin\phi'$ terms that we are interested are related to spherical harmonics $Y_{1,1}(\theta', \phi')$ and $Y_{1,-1}(\theta', \phi')$.

$$Y_{1,1}(\theta', \phi') = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta' e^{i\phi'}$$

$$Y_{1,-1}(\theta', \phi') = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \sin\theta' e^{-i\phi'}.$$

And with the help of simple trigonometry,

$$\sin\theta' \cos\phi' = -\sqrt{\frac{2\pi}{3}} (Y_{1,1} - Y_{1,-1}),$$

$$\sin\theta' \sin\phi' = i\sqrt{\frac{2\pi}{3}} (Y_{1,1} + Y_{1,-1}).$$

We then can do the integral on both x and y components, but the procedures are almost identical. I'll only show you the x. From Equation 1,

$$A_x = \frac{\mu_0 \sigma\omega a}{4\pi} \int \frac{-\sin\theta' \sin\phi'}{\mathscr{r}} da'.$$

With the help of Equation 2,

$$A_x = \frac{\mu_0 \sigma \omega a}{4\pi} \int \sum_{m,l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{l,m}(\theta,\phi) Y_{l,m}^*(\theta',\phi') \frac{1}{i} \sqrt{\frac{2\pi}{3}} (Y_{1,1}(\theta',\phi') + Y_{1,-1}(\theta',\phi')) da'.$$

Here is the central idea, we take advantage of the orthonormality of spherical harmonics.

$$\int Y_{l,m}(\theta',\phi') Y_{l',m'}^*(\theta',\phi') d\Omega' = \delta_{l,l'} \delta_{m,m'},$$

where $d\Omega'$ is the solid angle defined as $\sin\theta' d\theta' d\phi'$. The unmatched indexes results in zero of integral (**Jackson 3.55**).

With this A_x is reduced to only two terms with $l = 1$ and $m = \pm 1$.

$$A_x = \frac{\mu_0 \sigma \omega a}{3} \frac{r_<}{r_>^2} \frac{1}{i} \sqrt{\frac{2\pi}{3}} (Y_{1,1}(\theta,\phi) + Y_{1,-1}(\theta,\phi)) a^2.$$

or

$$A_x = -\frac{\mu_0 \sigma \omega a^3}{3} \frac{r_<}{r_>^2} \sin\theta \sin\phi.$$

[**Note:** the angular dependency is almost the same as Equation 1 but just without prime.]

If you repeat the same thing for y, then you will see that

$$\vec{A} = \frac{\mu_0 \sigma \omega a^3}{3} \frac{r_<}{r_>^2} \sin\theta \hat{\phi}.$$

We will consider two cases:

1. r < a,

$$\vec{A} = \frac{\mu_0 \sigma \omega r a}{3} \sin\theta \hat{\phi}.$$

It's easier in this case to operate in the cylindrical coordinate or $\rho = r\sin\theta$.

$$\vec{A} = \frac{\mu_0 \sigma \omega \rho a}{3} \hat{\phi}.$$

And by picking a proper formula from the front-cover table, you would get

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{\rho} \frac{\partial \rho A_\phi}{\partial \rho} = \frac{2\mu_0 \sigma \omega a}{3} \hat{z}$$

2. r > a,

$$\vec{A} = \frac{\mu_0 \sigma \omega a^4}{3r^2} \sin\theta \hat{\phi}.$$

Again do the curl, we arrive at

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta} \frac{\partial (A_\phi \sin\theta)}{\partial \theta} \hat{r} - \frac{1}{r} \frac{\partial (r A_\phi)}{\partial r} \hat{\theta} = \frac{\mu_0 \sigma \omega a^4}{3r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta})$$

.

The final expression is equivalent to the field from a magnetic dipole (**Griffiths 5.86 to be discussed in our third session**) with moment of

$$m = \frac{4\pi \sigma \omega a^4}{3}$$

or in a more insightful formation:

$$m = (\frac{4\pi a^3}{3})(\sigma\omega a),$$

where the first term is a volume of the sphere and the latter is "magnetization." Magnetic moment is magnetization per unit volume. That's the bottom line. Well, see it yourself in our third session.