Magnetostatics and magnetodynamics

E&M

Chan La-o-vorakiat • (2015/1) • King Mongkut's University of Technology Thonburi

---

Last Revision: September 25, 2015

Table of Contents

1 Charge in Motion and Current **1**

2 Magnetic Induction **2**

3 Ampere's Law **5**

4 Vector Potential **6**

5 Lorentz Force **8**

6 EMF and Flux Rule **9**

7 Faraday's Law **12**

8 Maxwell's Displacement Current **13**

9 Maxwell's Equations in Vacuum **13**

10 Homework **14**

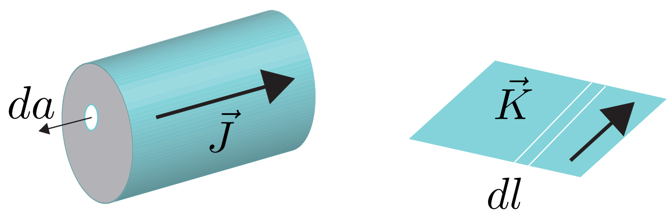


Figure 1.1

1 Charge in Motion and Current

- Charge creates electric field. Charge in motion (current) creates magnetic field.
- Current comes in various shapes (Fig. 2.1): **current density**

$$I = \int \vec{J} \cdot d\vec{a}$$

  Surface current density

$$I = \int \vec{K} \cdot d\vec{l}$$

- The conversion is done by dimension analysis.

$$q\vec{v} \propto \vec{I}dl \propto \vec{K}da \propto \vec{J}d\tau \tag{1.1}$$

Example 1.1. What is current density $\vec{K}$ (in cylindrical coordinate) of a rotating spehere of radius a with angular velocity ω and uniform surface charge density σ (Fig. 1.2)?

- Continuity equation of current follows the conservation of charge a the following. The amount of current leaving an area $d\vec{a}$ is $dI = \vec{J} \cdot d\vec{a}$. For a given volume the total current leaving is

$$I_{leaving} = \oint \vec{J} \cdot d\vec{a}.$$

  But since charge must be conserved, the total charge inside the volume must also be reduced by the same amount:

$$I_{leaving} = \oint \vec{J} \cdot d\vec{a} = -\int \frac{d\rho}{dt} d\tau$$

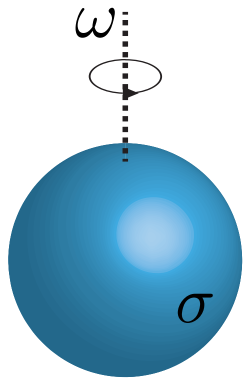


Figure 1.2

where $d\tau$ is a volumn integral (total charge inside the volume is $Q = \int \rho d\tau$). With the help of divergent theorem,

$$I_{leaving} = \oint \vec{J} \cdot d\vec{a} = \int \vec{\nabla} \cdot \vec{J} d\tau = -\int \frac{d\rho}{dt} d\tau.$$

The last equality will results in

Theorem 1.1 (Continuity Equation)**.**

$$\nabla \cdot \vec{J} = -\frac{d\rho}{dt}.$$

- This allows us to define a condition for magnetostatics. Charge and current should be time independent.

Lemma 1.1 (Condition for Steady Current)**.**

$$\nabla \cdot \vec{J} = 0.$$

2 Magnetic Induction and Biot-Savart Law

Theorem 2.1 (Biot-Savart Law for a Line Current)**.**

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l'} \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l'} \times \vec{\mathscr{r}}}{\mathscr{r}^3},$$

where $\vec{\mathscr{r}}$ is a vector from the current element $\vec{r'}$ to the position $\vec{r}$ where we want to calculate the field or $\vec{\mathscr{r}} = \vec{r} - \vec{r'}$. For other types of current sources (eqn. 1.1).

Theorem 2.2 (Biot-Savart Law for Surface Current)**.**

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}(\vec{r'}) \times \vec{\mathscr{r}}}{\mathscr{r}^3} da'$$

Theorem 2.3 (Biot-Savart Law for Volumetric current)**.**

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'}) \times \vec{\mathscr{r}}}{\mathscr{r}^3} d\tau'$$

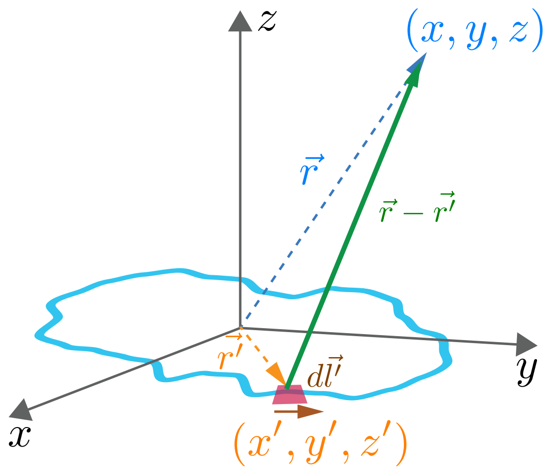


Figure 2.1: Current loop produces magnetic induction $\vec{B}$ via Biot Savart law.

Name	Symbol	Unit (SI)	Unit (CGS)
Magnetic Induction	$\vec{B}$	Tesla	Gauss
Magnetic Field	$\vec{H}$	Ampere/Meter	Oested

Table 1: Units comparision of $\vec{B}$ and $\vec{H}$

- Where there is no

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{q\vec{v} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

- $\vec{B}$ is magnetic induction. $\vec{H}$ is magnetic field (discussed later on). They are not always the same but can have the same numerical value in vacuum (in CGS unit i.e. 1 G = 1 Oe).
- The units of $\vec{B}$ and $\vec{H}$ are quite confusing (Table 1).
- The prime $'$ is very important. It represents the coordinate of the current source. It is independent from the un prime coordinate system. Any derivative of a prime position with respect to the unprime position is **zero** (e.g. $\frac{\partial x}{\partial x'} = 0$).
- Biot-Savart law is all you need to calculate magnetic induction (in a sense, this week lesson is done). But it usually leads to a very messy mathematics due to the cross product and an inverse cube dependent ($\propto \frac{1}{|\vec{r}-\vec{r'}|^3}$). In practice Biot-Savart law is only practicable for only handful of model systems. You will learn more about other approaches next.

Example 2.1. Find magnetic induction from a long straight section of current.

Example 2.2. Show that

$$\nabla \cdot \vec{B} = 0$$

3 Ampere's Law

Theorem 3.1 (Ampere's Law in Differential Form)**.**

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

derivation

- Please be reminded that we are discussing magnetostatics. Maxwell had generalized this law by adding another term to account for a non-static case.
- Ampere law can be written in the form of integral. It can be derived by use of Stroke's theorem

$$\int (\nabla \times \vec{B}) \cdot d\vec{a} = \int \mu_0 \vec{J} \cdot d\vec{a}$$

Lemma 3.1 (Ampere's Law in integral Form)**.**

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

- Ampere's law is useful when you know partially about the answer but it's only limited to very few cases with very high symmetry. To list them all

  1. infinite wires
  2. infinite planes
  3. solenoids
  4. toroid

  But still it is very handy when you want to recall some formula quickly.

Example 3.1. Find magnetic induction from an infinitely long wire carrying current I.

4 Vector Potential

Another approach to calculate magnetic induction is to use vector potential. Since the divergence of $\vec{B}$ is zero (Ex. 2.2), we can always write $\vec{B}$ as a curl of a vector potential $\vec{A}$.

Definition 4.1 (Vector Potential)**.**

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

This is due to the basic property of divergent $\vec{\nabla}\cdot$ where the divergent of curl is always zero.

Example 4.1. Show that one possible way find a vector potential is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'})}{\mathscr{r}} d\tau'$$

- From this definition, $\vec{A}(\vec{r})$ is parallel to the current.
- Vector potential is a continuous function.
- The calculation of $\vec{A}$ is simpler than Biot-Savart law. There is no cross product and it is proportional to $1/r$.

Example 4.2. Calculate magnetic field from a rotating sphere of radius a (Fig. 1.2)

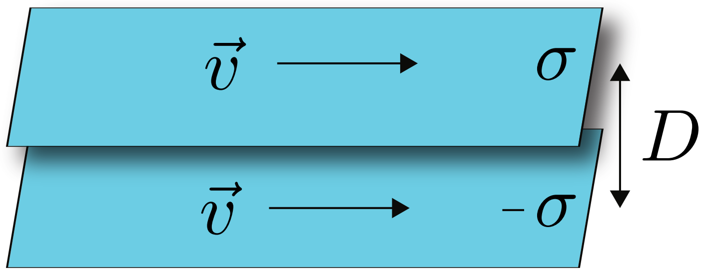


Figure 5.1

5 Lorentz Force

Theorem 5.1 (Force on a Moving Charge q due to Electric and Magnetic induction)**.**

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}))$$

Example 5.1. Consider two infinite charge surfaces with opposite charge density of σ and $-\sigma$. The surfaces are separated by distance D and they move at the same direction with the same velocity v (Fig 5.1). Find
(a) electric force on the top plate due to the bottom plate
(b) magnetic force on the top plate due to the bottom plater
(c) velocity to balance both forces

6 Electromotive Force and Flux Rule

- We talked about a current loop as a circulation of charge, but never talk about the mechanism to the motion of charge.
- Electromotive force $\mathscr{E}$ is related to a force on a charge that goes around the loop. But it is a force per unit charge, $\vec{f} = \frac{\vec{F}}{q}$.
- And it's not a force but line integral of force around the loop.

$$\mathscr{E} = \oint \vec{f} \cdot d\vec{l}$$

- $\vec{f}$ can be anything e.g magnetic ($\vec{f} = \vec{v} \times \vec{B}$) or electric ($\vec{f} = \vec{E}$).

Example 6.1. What is the electromotive force of a a sqare current loop when pulling across the area with magnetic field to the empty space at a constant velocity $\vec{v}$.

- Flux rule links electromotive force to the change in magnetic flux.
- Consider an arbitrary current loop covering the area of $\vec{a}(t)$ at time t. This loop moves to another position at time dt later with the velocity $\vec{v}$ (Fig 6.1a).

  The change in magnetic flux from time t to $t + dt$ is

$$d\Phi(t) = \Phi(t + dt) - \Phi(t) = \vec{B} \cdot [\vec{a}(t + dt) - \vec{a}(t)]$$

  By considering Fig 6.1, $\vec{a}(t + dt)$ is $\vec{a}(t)$ plus a sidewall,

$$\vec{a}(t + dt) - \vec{a}(t) = \oint (\vec{v} \times d\vec{l})dt$$

  The flux change is then

$$d\Phi(t) = \oint \vec{B} \cdot (\vec{v} \times d\vec{l})dt$$

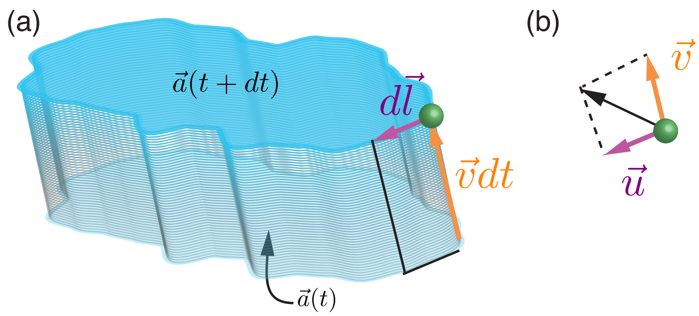


Figure 6.1

Use the relationship $\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{C}\cdot(\vec{B}\times\vec{A})$,

$$\frac{d\Phi(t)}{dt} = -\oint d\vec{l}\cdot(\vec{v}\times\vec{B}).$$

Next, we will consider magnetic force on the charge to calculate EMF. It turns out that the velocity of the charge is the sum of two contributions:

$$\vec{w} = \vec{v} + \vec{u},$$

where $\vec{u}$ is the charge velocity along the current loop and $\vec{u}$ is due to the motion of the loop (Fig 6.1b); therefore,

$$\frac{d\Phi(t)}{dt} = -\oint d\vec{l}\cdot(\vec{w}-\vec{u})\times\vec{B}.$$

But we know that $\vec{u}$ should be along the line of current loop so it is parallel to $d\vec{l}$. Hence finally,

$$\frac{d\Phi(t)}{dt} = -\oint d\vec{l}\cdot(\vec{w}\times\vec{B}) = -\mathscr{E}.$$

Theorem 6.1 (Flux Rule)**.**

$$\mathscr{E} = -\frac{\partial\Phi}{\partial t}$$

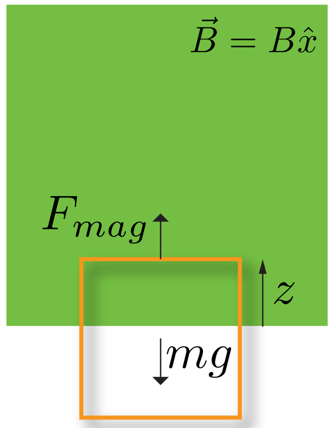


Figure 6.2

Example 6.2. A square copper loop is placed in a uniform field $\vec{B}$ and allowed to fall under gravity. Find the loop velocity as a function of time as a function of copper density (ρ = 8.96 g/cm^2) and conductivity (σ =1/ 16.78 (n Ω·m)$^{-1}$).

7 Faraday's Law

- Moving a magnet away from a current loop also induces current.
- A similar law to flux rules holds i.e. $\mathscr{E} = -\frac{\partial \Phi}{\partial t}$
- EMF due to electric field is $\oint \vec{E} \cdot d\vec{l}$; therefore

Theorem 7.1 (Faraday's Law in the integral form)**.**

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi}{\partial t} = \oint -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

By use of a Stroke's Theorem

Theorem 7.2 (Faraday's Law in Differential Form)**.**

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

- Faraday's law explains that the change in magnetic field results in electric field.
- It's in the same form of Ampere's law (Eq. 3.1), so this suggests that we can use the same trick as the ampere to calculate the induced electric field.

Example 7.1. A solenoid of radius a with the initial magnetic induction B_i pointing straight up. What is the induced electric field at the radius b outside when the solenoid is switched off.

8 Maxwell's Displacement Current

Maxwell's realized that there is some problems in the Ampere's law (theorem 1.1). The divergence of it is always zero:

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J} = 0.$$

For steady state, this is ok. However, in general, this is not always the case due to the continuity equation (Theorem 1.1) would results in the term $-\frac{\partial \rho}{\partial t}$. The way to fix is to modify Ampere's law to introduce a displacement current.

Theorem 8.1 (Maxwell's Modified Ampere's Law)**.**

$$\nabla \times \vec{B} = \mu_0(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t})$$

The additional term is called displacement current density $\vec{J_D}$.

Definition 8.1.

$$\vec{J_D} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Example 8.1. Take the divergent of our new Ampere's law.
Hint You need help from Gauss's law ($\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$)

9 Maxwell's Equations in Vacuum

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad (\qquad\qquad\qquad\qquad)$$

$$\nabla \times \vec{E} = \frac{\partial B}{\partial t} \qquad (\qquad\qquad\qquad\qquad)$$

$$\nabla \cdot \vec{B} = 0 \qquad (\qquad\qquad\qquad\qquad)$$

$$\nabla \times \vec{B} = \mu_0(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) \qquad (\qquad\qquad\qquad\qquad)$$

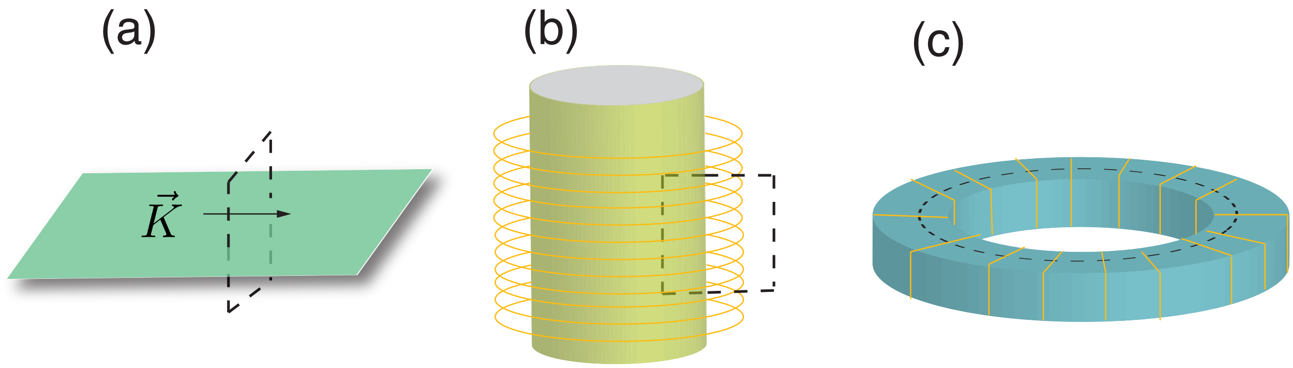
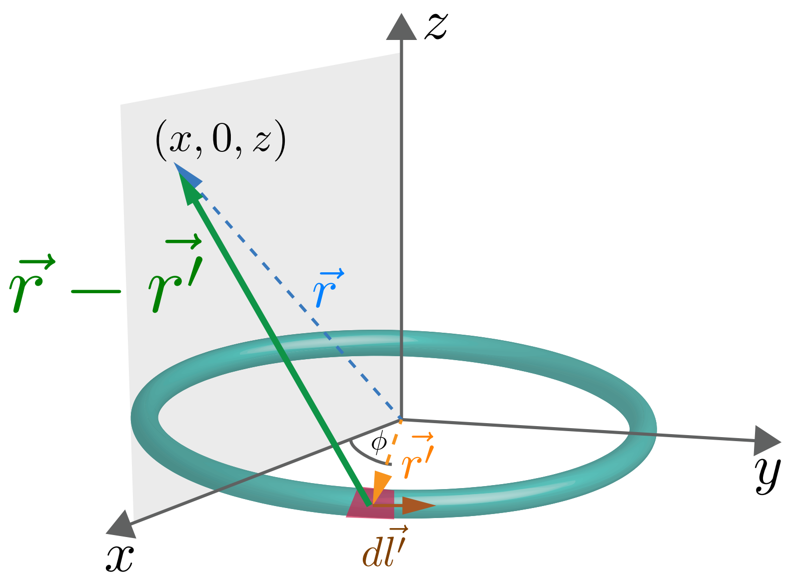


Figure 10.1

10 Homework

Homework 1. (6 points) Use Ampere's Law to derive the magnetic inductions of the following scenarios: (a) on top of an infinite uniform surface current $\vec{K} = K\hat{x}$, (b) inside a solenoid with n turns of wire per length, (c) inside a toriod with N turnings of wire.



Homework 2 (Vector Potential from a Current Loop)**. (15 points)**
(a) Think for a moment about the symmetry. You should notice that we still have a polar symmetry (along ϕ direction). Therefore, we could simplify our problem by calculating a vector potential only at the a point (x,0,z) on xz plane (no writing required. Just really make sure that you understand it.)
(b) (5 points) Given the loop has a radius of a, show that the position vector from the current loop element can be written as

$$\vec{\imath} = \vec{r} - \vec{r'} = (x - a\cos\phi, -a\sin\phi, z),$$
$$\imath = \sqrt{x^2 + a^2 + z^2 - 2xa\cos\phi}.$$

And the current element is

$$d\vec{l} = (-a\sin\phi d\phi, a\cos\phi d\phi, 0).$$

and write down the integral for vector potential
(c) (3 points) Perform the integral and show that the x-component of the vector potential is zero.
(d) (7 points) Here is the most tricky part. Show that the vector potential can be written as

$$\vec{A}(x, 0, z) = \frac{\mu I}{\pi k}\sqrt{\frac{a}{x}}[(1 - \frac{k^2}{2})K(k) - E(k)]\hat{y}$$

where $K(k) = \int_0^{\pi/2} \frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}$ and $E(k) = \int_0^{\pi/2} \sqrt{1 - k^2\sin^2\beta}d\beta$. $K(k)$ and $E(k)$ are called elliptic integral of the first kind and second kind, respectively.

Hints: Use the following steps:

1. Change the variable to $\phi = \pi + 2\beta$.
2. With the help of the symmetry of the integrand, the limit of integral is reduced to half from 0 to $\frac{\pi}{2}$ and an extra factor of 2.
3. Make a substitution $k^2 = \frac{4ax}{(a+x)^2+z^2}$.

4. Write everything in terms of the elliptic integral.

5. In addition, you need help from the identities of the $E(k)$:

$$\frac{\partial E}{\partial k} = k \int_0^{\pi/2} \frac{1}{(1 - k^2 \sin^2 \beta)^{1/2}} = \frac{E}{k} - \frac{K}{k}.$$

(e) Perform a curl to calculate the magnetic induction. Just kidding. You don't have to do it. The mathematical details is very lengthy. See it yourself. [Schill, R.A., "General relation for the vector magnetic field of a circular current loop: a closer look," Magnetics, IEEE Transactions on , vol.39, no.2, pp.961,967, Mar 2003 doi: 10.1109/TMAG.2003.808597]

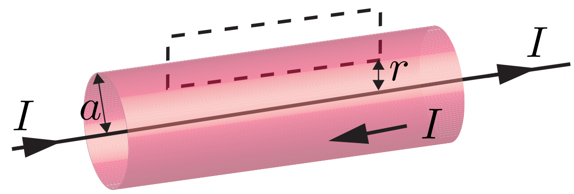


Figure 10.2

Homework 3. (10 points) An alternating current $I = I_0 \cos \omega t$ flows along a straight wire, and returns along a coaxial conducting tube of radius a

(a) (2 points) By use of Ampere's Law, find magnetic induction for both inside an outside of the shell.

(b) (2 points) Find the magnetic flux going through the dashed rectangle loop

(c) (2 points) By symmetry the induced electric field due to the change in magnetic field should be longitudinal. Show that the field at position r is $\vec{E} = \frac{\mu I_0 \omega}{2\pi} \sin \omega t \ln \frac{a}{r} \hat{z}$ by use of a line integral around the loop and the related change in magnetic flux.

(d) (2 points) Calculate displacement current density.

(e) (2 points) Integrate the current density of the total displacement current inside the coaxial tube.

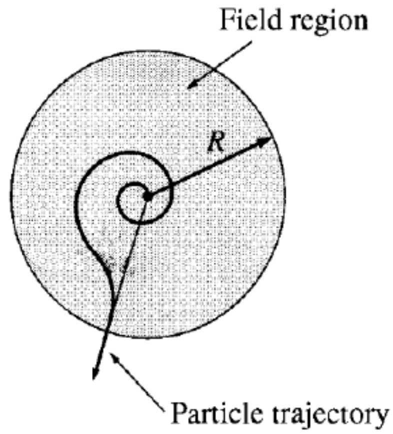


Figure 10.3

Homework 4. (12 points) A circular symmetric magnetic induction pointing perpendicular to the page ($\vec{B} = B(r)\hat{z}$). The field is contained in a circular region of radius R. What is the angular momentum of a charge particles when exiting from the circular area.
(a) (3 points) From Lorentz magnetic force, show that the torque on the emerging point charge q ($\vec{\tau} = \vec{r} \times \vec{F}_{mag}$) is

$$\vec{\tau} = -q\vec{B}(\vec{r} \cdot \vec{v})$$

(b) (5 points) Show that final angular momentum ($\vec{L}$) of exit particle is related to total magnetic flux ($\Phi = \int \vec{B} \cdot d\vec{a} = 2\pi \int_0^R B(r)rdr$)

$$\vec{L} - \vec{L}_0 = \frac{q\Phi}{2\pi}\hat{z},$$

where $\vec{L}_0$ is the initial angular momentum.
Hint: Break down the displacement vector to the particle $d\vec{l} = \vec{v}dt$ into the radial and angular component ($d\vec{l} = dr\hat{r} + d\phi\hat{\phi}$).
(c) (2 points) Explain why when the total flux is zero, any charge starts out from at the center of the field region can exit only on with radial part (i.e. along to $\hat{r}$ direction).
(d) (2 points) In case of uniform field (B = const.), the charge that starts out from the center will have the final angular velocity equal to Larmor frequency

$$\omega = \frac{qB}{2m}.$$

Note: An important implication of this problem is that angular momentum must come from some where. The source of angular momentum must be magnetic induction; although there is nothing moving.