Electromagnetic Wave

E&M

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1 Poynting Vector and Energy in the Field

- Electric field and magnetic field really carriers energy.
- The energy density from electric field ($\vec{E}$) and magnetic flux density ($\vec{B}$) is

$$u_{em} = \frac{1}{2}(\epsilon_0 \vec{E} \cdot \vec{E} + \frac{1}{\mu_0} \vec{B} \cdot \vec{B})$$

- **Derivation: Poynting Theorem**

Theorem 1.1 (Poynting Theorem: The energy is conserve)**.**

$$\frac{\partial(u_{em} + u_{mech})}{\partial t} = -\vec{\nabla} \cdot \vec{S}$$

where $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \vec{E} \times \vec{H}$

- $\vec{S}$ represents the energy flux i.e. the rate of energy flow per area.
- The analogy is the current density: $\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}$.

2 Electromagnetic Wave

- The most important application of Maxwell's equation is the discovery of electromagnetic wave.
- Start with Maxwell's equation in free space ($\rho = \vec{J} = 0$)

$$\nabla \cdot \vec{E} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Example 2.1 (Derive the Electromagnetic Wave Equation)**.**

- Without Maxwell's fix to the Ampere's Law there would be no EM wave.
- There are many solutions to the equation. To list a fews,

$$\vec{E} = f(k_x x - \omega t)\hat{j},$$
$$\vec{E} = f(k_x x + \omega t)\hat{j},$$
$$\vec{E} = f(k_x x - k_y y - \omega t)\hat{k}.$$

- But not all satisfies the Maxwell's equations: for example

$$\vec{E} \neq f(k_x x - \omega t)\hat{i},$$

.

- The general equation for a plane wave is

$$\vec{E} = E_0 \exp i(\vec{k} \cdot \vec{r} - \omega t)\hat{n}.$$

$\hat{n}$ is called polarization vector. It's simply the direction of electric field. $\vec{k} = \frac{2\pi}{\lambda}$ is the wave vector, which tells the direction of wave propagation.

- The constraints are

$$\hat{n} \cdot \hat{k} = 0.$$

- The change in $\vec{E}$ would result in $\vec{B}$; hence $\vec{B} = \frac{1}{c}\hat{k} \times \vec{E}$

3 Energy, Intensity and Power of EM Wave

- Poynting vetor is parallel to $\vec{k}$

$$\vec{S} = \vec{E} \times \vec{B} = \sqrt{\frac{\epsilon_0}{\mu_0}}\vec{E} \times (\hat{k} \times \vec{E}) = \sqrt{\frac{\epsilon_0}{\mu_0}}E^2\vec{k}$$

- Intensity is the time average of power per area. It, therefore, can be defined as the time-average of Poynting vector

$$I = \langle S \rangle = \sqrt{\frac{\epsilon_0}{\mu_0}}\frac{E_0^2}{2}$$

- Intensity is proportional to the square of E-field (of B-field).

- But it's easier experimentally to measure power (e.g. by measuring heat generated). But you need to measure the beam size first.

Example 3.1 (Pentawatt Laser). A group from The University of Texas at Austin could produces a petawatt laser (10^{15} W). But this petawatt comes as burst of energy with 170 ps pulse duration and the spot size of 100 μm. Estimate
(1) The energy per pulse of this laser
(2) Peak electric field and compare with the electric field inside Hydrogen atom.
(3) Number of photons per pulse.

4 EM Wave inside Matter

4.1 Refractive Index

- Let's go back one step by starting with Maxwell's equation inside matter. But now will assume that the material is linear ($\vec{D} = \varepsilon\vec{E}$, $\vec{B} = \mu\vec{H}$) but non-conductive ($\vec{J} = 0$)

$$\begin{aligned} \nabla \cdot \vec{D} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial B}{\partial t} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{H} &= \frac{\partial \vec{D}}{\partial t}. \end{aligned}$$

Example 4.1 (Electromagnetic Wave Equation inside Matter)**.**

- The speed of light inside batter becomes

$$v = \frac{c}{n},$$

where $n = c\sqrt{\epsilon\mu}$ is refractive index.

4.2 Complex Refractive Index

- When the sample is conductive (see Homework), the dielectric constant and conductivity become complex

$$\tilde{\epsilon} = \epsilon + \frac{i\sigma}{\omega} = \epsilon_0 + \frac{i\tilde{\sigma}}{\omega},$$

- Dielectric constant $\tilde{\varepsilon}$: real part $\epsilon_1 = \epsilon$ and imaginary part $\epsilon_2 = \frac{\sigma}{\omega}$.
- Conductivity $\tilde{\sigma}$: real part $\sigma_1 = \sigma$ and imaginary part $\sigma_2 = (\epsilon_0 - \epsilon)\omega$.
  (The need for ϵ_0 will be cleared in a second)
- The complex dielectric constant opens for the complex refractive index

$$\tilde{n} = n + ik = c\sqrt{\mu\tilde{\epsilon}} = c\sqrt{\mu(\epsilon - \frac{i\sigma}{\omega})}$$

  (You shouldn't be confused the negative refractive index k and the wave vector $\vec{k}$ or its components, k_x, k_y, k_z
- Or in terms of wave vector

$$\vec{k} = \frac{\omega\tilde{n}}{c}\hat{k},$$

  or

$$\vec{E} \propto \exp\left(\frac{i\omega n x}{c}\right)\exp\left(\frac{-\omega k x}{c}\right),$$

  which suggests that the n governs the velocity of the wave (phase velocity) but k governs the loss of the electric field amplitude.

  Example 4.2 (Beer-Lambert's Law).

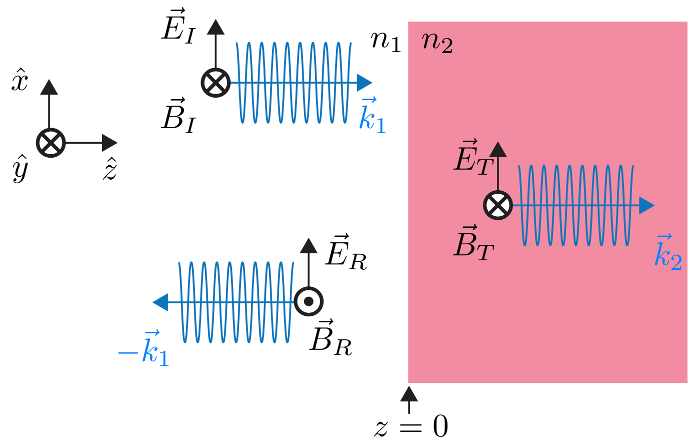

Figure 5.1

5 EM Wave and Interface

5.1 EM Boundary Conditions

- The four Maxwell's equations result also in four boundary conditions.

$$\begin{aligned} D_1^{\perp} - D_2^{\perp} &= \rho_f \qquad (\qquad\qquad\qquad\qquad) \\ \vec{E}_1^{||} &= \vec{E}_2^{||} \qquad (\qquad\qquad\qquad\qquad) \\ B_1^{\perp} B_2^{\perp} &= 0 \qquad (\qquad\qquad\qquad\qquad) \\ \vec{H}_1^{||} - \vec{H}_2^{||} &= \vec{K}_f \times \hat{n} \qquad (\qquad\qquad\qquad\qquad). \end{aligned}$$

5.2 Reflection and Transmission from an Interface

Consider the non-magnetic interface located at $z = 0$ of two linear media.
The incident EM wave has the wave function of

$$\vec{E}_I = E_{oI} \exp i(k_1 z - \omega t)\hat{i}$$

$$\vec{B}_I = \frac{1}{v_1} E_{oI} \exp i(k_1 z - \omega t)\hat{j}$$

.
This give rises to the reflected wave

$$\vec{E}_R = E_{oR} \exp i(-k_1 z - \omega t)\hat{i}$$

$$\vec{B}_R = -\frac{1}{v_1} E_{oR} \exp i(-k_1 z - \omega t)\hat{j}$$

and the transmiited wave

$$\vec{E}_T = E_{oT} \exp i(k_2 z - \omega t)\hat{i}$$

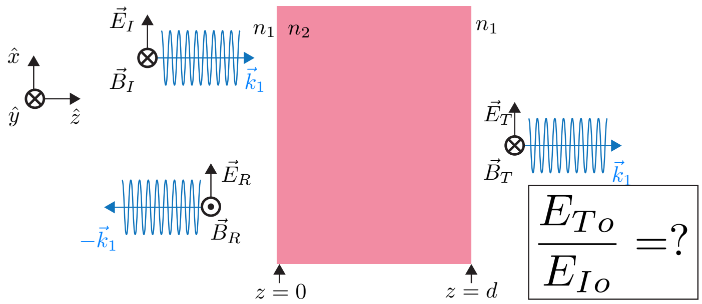

Figure 5.2

$$\vec{B}_T = \frac{1}{v_2} E_{oI} \exp i(k_2 z - \omega t)\hat{j}$$

.
We have to match the boundary for $\vec{E}^{\parallel}$ at z=0

$$E_{oI} + E_{oR} = E_{oT}$$

and for $\vec{H}^{\parallel}$

$$\frac{1}{v_1} E_{oI} - \frac{1}{v_1} E_{oR} = \frac{1}{v_2} E_{oT}$$

Put these two equations together we would get

Theorem 5.1 (Amplitude Reflection and Transmission Coefficients)**.**

$$r = \frac{E_{0R}}{E_{0I}} = \frac{n_2 - n_1}{n_1 + n_2}$$

$$t = \frac{E_{0T}}{E_{0I}} = \frac{2n_1}{n_1 + n_2}$$

And since intensity is $\propto \sqrt{\epsilon} E^2 = nE^2$, we could calculated

Theorem 5.2 (Intensity Reflection and Transmission Coefficients)**.**

$$R = |\frac{E_{0R}}{E_{0I}}|^2 = |\frac{n_2 - n_1}{n_1 + n_2}|^2$$

$$T = \frac{n_2}{n_1} |\frac{E_{0T}}{E_{0I}}|^2 = \frac{4n_1 n_2}{|n_1 + n_2|^2}$$

Example 5.1 (Transmission Through a Dielectric Slab)**.** To simplify the math a bit, let the slab sit in the vacuum (set $n_1 = 1$ and $n_2 = n$) (Fig. 5.2)

5.3 Transfer-Matrix Formalism

- What if we have more than two interfaces? The math presented Ex.5.1 and Homework 2 should be overly complicated.
- We could systematize the analysis with the help of transfer-matrix formalism.
- Back to basic, the E and B fields inside a material could be written as the following

$$E_{\parallel} = E_{+} + E_{-}$$

$$B_{\parallel} = B_{+} + B_{-} = \frac{E_{+}}{v} - \frac{E_{-}}{v}$$

  or in matrix format

$$\begin{bmatrix} E_{\parallel} \\ B_{\parallel} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{v} & -\frac{1}{v} \end{bmatrix} \begin{bmatrix} E_{+} \\ E_{-} \end{bmatrix}. \tag{5.1}$$

- If you could refer back to the section 5.2 (also Fig.5.1), you could see that
  1. $E_{+} = E_I$, $E_{-} = E_R$ and $B_{+} = B_I$, $B_{-} = B_R$ for the left medium
  2. $E_{+} =$, $E_{-} =$ and $B_{+} =$, $B_{-} =$ for the right medium.
- There are two choices of basis. One is $E_{\parallel}$ and $B_{\parallel}$; the other is E_{+} and E_{-}. Both are good at the right situation.
- Across the interface (from no primed to primed), $E_{\parallel}$ and $B_{\parallel}$ are good since they have a very simple boundary conditions:

$$E_{\parallel} = E'_{\parallel}$$

$$B_{\|} = B'_{\|}$$

or in matrix format

$$\begin{bmatrix} E_{\|} \\ B_{\|} \end{bmatrix} = \begin{bmatrix} E'_{\|} \\ B'_{\|} \end{bmatrix}. \tag{5.2}$$

- But E_+ and E_+ are easier if you consider the propagation through space. The wave would simply gain the phase factor e^{ik_0nd} when (traveling from no primed to primed position).

$$E'_+ = e^{ik_0nd}E_+$$

$$E'_- = e^{-ik_0nd}E_-$$

or in matrix format

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \begin{bmatrix} e^{-ik_0nd} & 0 \\ 0 & e^{ik_0nd} \end{bmatrix} \begin{bmatrix} E'_+ \\ E'_- \end{bmatrix}. \tag{5.3}$$

- By use of equation 5.1 and 5.2, we could derive the interface-crossing matrix with the $+/-$ basis.

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{t} \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix} \begin{bmatrix} E'_+ \\ E'_- \end{bmatrix}. \tag{5.4}$$

  r and t are amplitude reflection and transmission coefficients (Theorem 5.1)

- Equation 5.4 and 5.3 are all we need for the calculation of transmission or reflection through any number of interfaces.

Example 5.2. Derive Equation 5.4

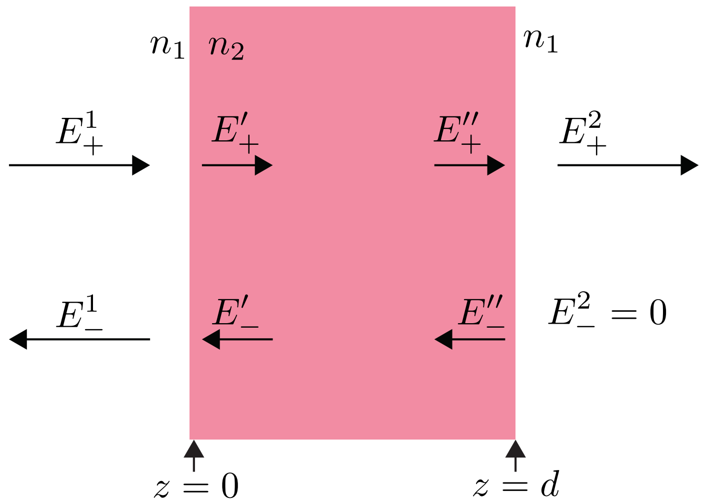

Figure 5.3

Example 5.3 (Transmission Through a Dielectric Slab (revisited)).

6 Models of Material Conductivities

- So far we have represent materials as $\tilde{n}$, this is boring. In reality we talks about conductors or semiconductors but not a slab of whatever with complex index of $\tilde{n}$.

6.1 Free Electrons: Drude Model

- Electrons are treated as free carriers.
- Equation of motion due to the presence of electric field and the drag force due to the scattering with other electrons is

$$m\frac{d\vec{v}}{dt} + \frac{m\vec{v}}{\tau} = e\vec{E}_0 \exp(-i\omega t)$$

- The second term is a damping term. The parameter τ is called "scattering time". The inverse $\Gamma = \frac{1}{\tau}$ is "scattering rate". Without the term, electron would be accelerated forever and you won't reach the equilibrium. By assuming a harmonic solution for velocity $\vec{v} = \vec{v}_0 \exp(-i\omega t)$, we could show that the velocity becomes

$$\vec{v_0} = \frac{e\vec{E_0}}{\frac{m}{\tau} - im\omega}$$

From the Ohm's law $\vec{J_0} = \tilde{\sigma}\vec{E_0} = Ne\vec{v}_0$ with N as a carrier density, we finally derive Drude conductivity.

Theorem 6.1 (Drude Model for Free Electron Conductivity)**.**

$$\tilde{\sigma} = \frac{Ne^2\tau}{m(1 - i\omega\tau)}$$

$$\tilde{\epsilon} = \epsilon_0 + \frac{iNe^2\tau}{\omega m(1 - i\omega\tau)}$$

- Drude model allows us to introduce carrier mobility μ defined as $\mu = \frac{e\tau}{m}$.
- **DC limit ($\omega = 0$):** $\tilde{\sigma} = \frac{Ne^2\tau}{m} = e\mu N$. So by measing a DC conductivity (e.g. by use of a four-point probe setup), we could measure the carrier mobility μ if the carrier density N is known. This is the essence of the transport measurement.
- **Low-freq limit ($\omega\tau \ll 0$):** if carrier density is large enough, the material will a be perfect reflector.

- **High-freq limit ($\omega\tau \gg 0$):** free carriers do not contribute to conductivity at high-frequency. The reflection becomes weak.

- **Plasma Frequency:** defined as
$$\omega_p^2 = \frac{Ne^2}{m\epsilon_0},$$
so the Drude becomes more compact
$$\tilde{\sigma} = \frac{\epsilon_0 \omega_p^2}{\Gamma - i\omega}$$

- plasma frequency governs where ϵ is zero causing the transition from being reflective metal to transparent dielectic. This is also due to the reduction of k to zero and the convergence of n to 1.

6.2 Bounded Electrons: Lorentz Model

- Next the electrons are no longer free, it is trapped by a potential.

- In this case we could approximate as a mass-on-spring system. The equation of motion is
$$m\frac{d^2\vec{r}}{dt^2} + \frac{m}{\tau}\frac{d\vec{r}}{dt} + m\omega_0^2\vec{r} = e\vec{E}_0 \exp\left(-i\omega t\right)$$

- This simple model is good for many purposes like inter-band transition in semiconductors, phonon mode vibrations, or excitonic transition.

- Similar to the derivation of the Drude model, it could be shown that
$$\vec{r} = \frac{e\vec{E}}{(\omega_0^2 - \omega^2) - i\omega/\tau}$$
and
$$\tilde{\sigma} = \frac{\epsilon_0 \omega_p^2 \omega}{\left[i(\omega_0^2 - \omega^2) + \omega\Gamma\right]}$$
(Details are for Homework)

- **DC limit ($\omega = 0$):**

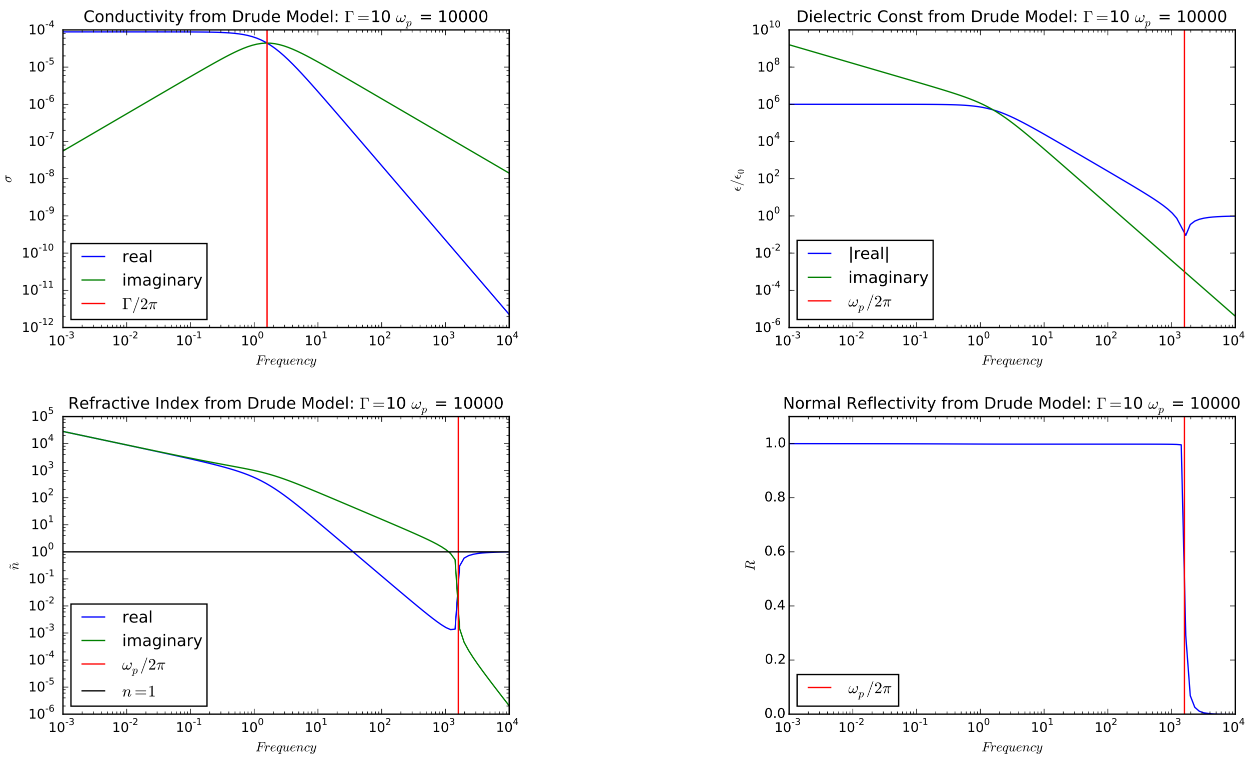

Figure 6.1: Material Parameters according to Drude Model

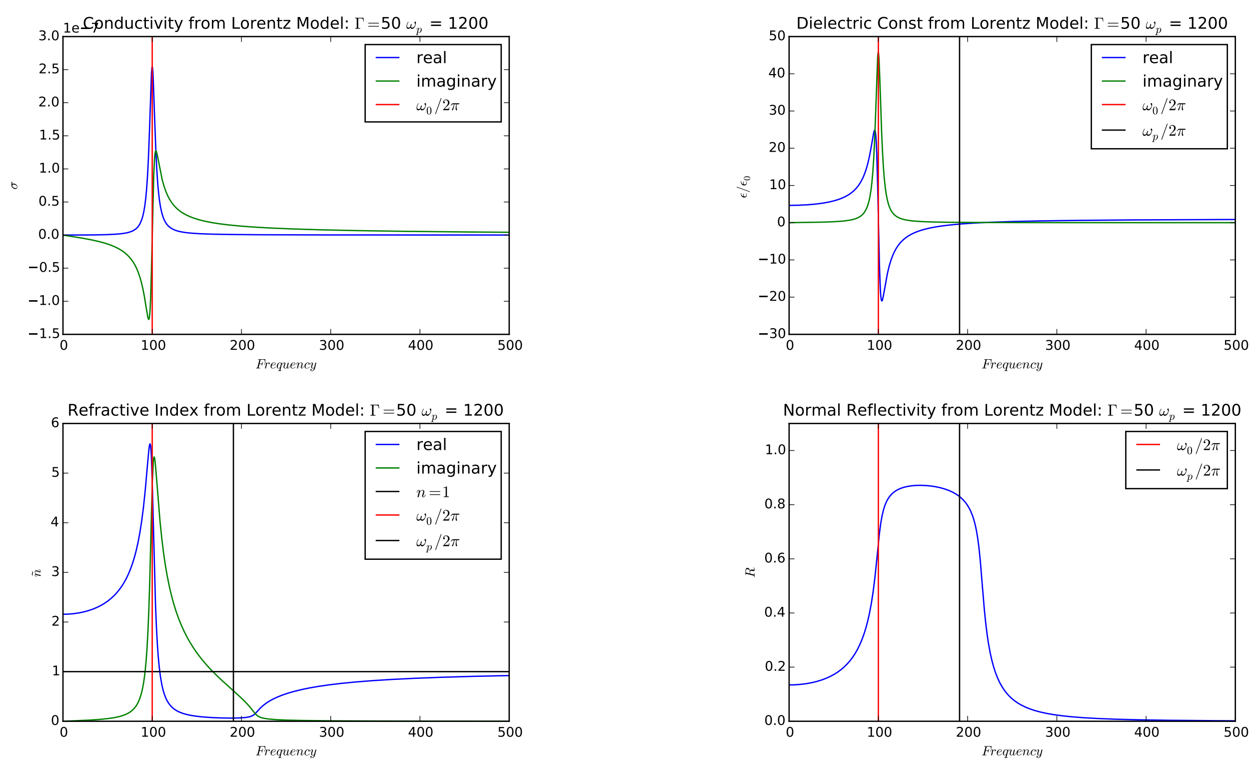

Figure 6.2: Material Parameters according to Lorentz Model

- **Low-freq limit ($\omega\tau \ll 0$):**

7 Homework

Homework 1 (EM Wave in Conductive Materials)**. (10 points)** Let's begin with a set of equations for a conductive medium.

$$\nabla \cdot \vec{D} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial B}{\partial t}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}.$$

(a) (4 points) With the help of Ohm's law ($\vec{J} = \sigma \vec{E}$), we could introduce conductivity in to Maxwell-Ampere Law. And by assuming that the wave function is harmoncic i.e. $\vec{E} \propto \exp(-\omega t)$ we could transform the Maxwell-Ampere Law to

$$\nabla \times \vec{B} = \mu(\sigma + i\omega\varepsilon)\vec{E}.$$

(b) (2 points) Compare with the derivation of EM wave in non-conductive material (Example 4.1), we could map out the complex dielectric constant $\tilde{\varepsilon} = \varepsilon - \frac{i\sigma}{\omega}$.
(c) (4 points) What is n and k in terms of σ and ε.
Hint:

1. $\tilde{n}^2 = c^2 \tilde{\varepsilon} \mu$
2. $\tilde{n} = n + ik$

Answer

$$n = c\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{1 + \sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}}$$

$$k = c\sqrt{\frac{\mu\varepsilon}{2}}\sqrt{-1 + \sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}}$$

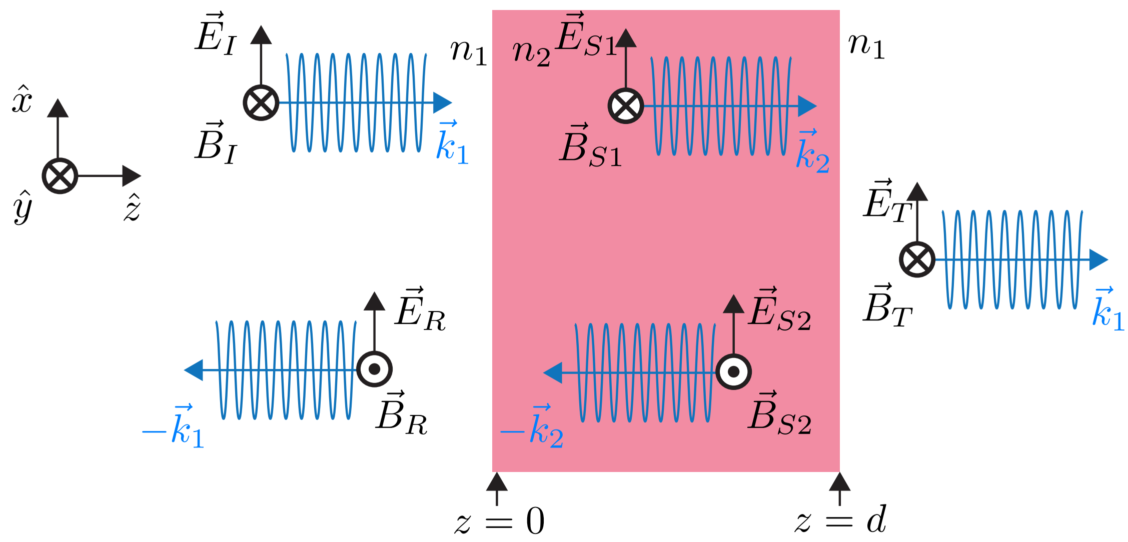

Figure 7.1

Homework 2 (Transmission from a slab of Dielectric (third time)). **(15 points)** (a) (10 points) By matching the boundary conditions, rederive the expression for the transmission coefficient. To simpify the math, let's again set $n_1 = 1$ and $n_2 = n$

I will start with those wave functions for you.

Incident Wave:

$$\vec{E}_I = E_I \exp i(k_o z - \omega t)\hat{i}$$

$$\vec{B}_I = \frac{1}{c} E_I \exp i(k_o z - \omega t)\hat{j}$$

Reflected Wave:

$$\vec{E}_R = E_R \exp i(-k_o z - \omega t)\hat{i}$$

$$\vec{B}_R = -\frac{1}{c} E_R \exp i(-k_o z - \omega t)\hat{j}$$

Internal Wave 1:

$$\vec{E}_{S1} = E_{S1} \exp i(kz - \omega t)\hat{i}$$

$$\vec{B}_{S1} = \frac{1}{v} E_{S1} \exp i(kz - \omega t)\hat{j}$$

Internal Wave 2:

$$\vec{E}_{S2} = E_{S2} \exp i(-kz - \omega t)\hat{i}$$

$$\vec{B}_{S2} = -\frac{1}{v} E_{S2} \exp i(-kz - \omega t)\hat{j}$$

Transmitted Wave:

$$\vec{E}_T = E_T \exp i(k_o z - \omega t)\hat{i}$$

$$\vec{B}_T = \frac{1}{c} E_T \exp i(k_o z - \omega t)\hat{j}$$

You have to match the boundaries at two location at $z = 0$ and $z = d$.

Answer:

$$\frac{E_T}{E_I} = \frac{4n}{(1+n)^2} e^{ik_o(n-1)d} \frac{1}{1 - \frac{(n-1)^2}{(n+1)^2} e^{2ik_o nd}}$$

(b) (5 points) Notice that the final answer is not the same as the result from infinite summation that $\frac{E_T}{E_I} \propto e^{ik_o nd}$ but ours is $e^{ik_o(n-1)d}$. And you might realize that our latest result is indeed funny. If you set $n = 1$ You would expect that $\frac{E_T}{E_I}$ should be $e^{ik_o d}$. Just like a wave that propagate to a free space of thickness d.

What wrong with our derivation? How to fix it?

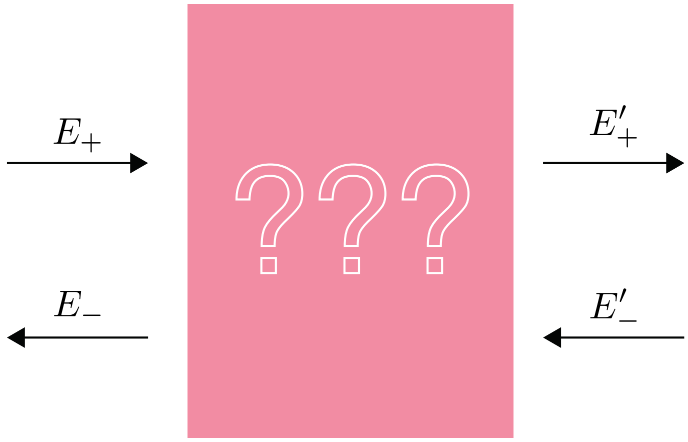

Figure 7.2

Homework 3 (Transfer Matrix)**. (20 points)** For any complicated stacking of dielectric slabs (i.e multilayer structure), if we know the transfer matrix we would know the transmission and reflection coefficient from these slabs. Let's say the transfer matrix is

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E'_+ \\ E'_- \end{bmatrix}. \tag{7.1}$$

(a) (2 points) Show that the transmission coefficient is $t_{multilayer} = \frac{1}{A}$ when the incident beam is from the left hand side only $E'_- = 0$.
(b) (2 points) Similly the reflection coefficient is $r_{multilayer} = \frac{C}{A}$.
(c) (3 points) What are the physical meanings of B and D?
(d) (3 points) Calculate the reflection coefficient of the dielectric slab from the Example 5.2.
Answer:

$$r = \frac{r_1 + r_2 e^{i2k_0 n_2 d}}{1 + r_1 r_2 e^{i2k_0 n_2 d}},$$

where $r_1 = \frac{n_1 - n_2}{n_1 + n_2}$ and $r_2 = \frac{n_2 - n_3}{n_2 + n_3}$.
(e) (5 points) Let's pretend that you're working as an R&D of Top Charoen Optical. You boss want to minimize the reflection of a glasses lens by selecting the right material n_2 with the right thickness d to set r to zero. In this case, n_3 is the refractive index of quartz glass $n_3 \approx 1.5$ and $n_1 \approx 1$.
There is an easy way to set r to zero i.e. to set the thickness to satisfy a "quarter wave" condition by finding the combination of n_2 and d such that $e^{i2k_0 n_2 d} = -1$. Show that $r = 0$, when

$$n_2 = \sqrt{n_1 n_3}.$$

Find the possible thicknesses d in nanometers to coat the n_2 film if you want the suppress the reflection of green light ($\lambda = 500$ nm)
(f) (5 points) Show that instead if you prefer "half wave" condition for thickness $e^{i2k_0 n_2 d} = 1$, there won't be anyway to minimize the reflectivity unless you create a free-standing thin film or $n_1 = n_3$.

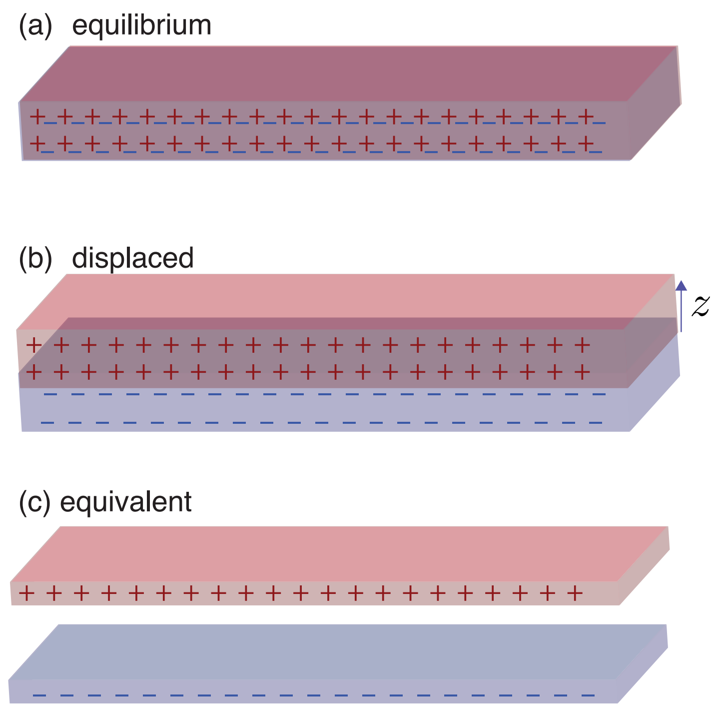

Figure 7.3

Homework 4 (How could plasma frequency get its name)**.** (**10 points**)
A possible picture to visual a slab of metal is to treat it as a combination of electron gas and positive ions located at the lattice sites (Fig 7.3a). Since positive charges balance with the negative, the metal remain neutral.
Now apply a perturbation such that it lifts a slab of electrons upward by a small distance of z. As the The shifted electrons are then equivalent
(a) (4 points) From Gauss's law (back to the begining of the semester), write down the electric field between the layers as a function of density of electrons n and theie charge e. Approximate that the plane is infinitedly large in xy directions. **Answer:**

$$\vec{E} = -\frac{nze}{\epsilon_0}\hat{z}$$

(b) (6 points) Set up the equation of motion due to the force from electric field, calculate the oscillation frequency. Compare it to the plasma frequency from Drude or Lorentz models.

Homework 5 (Faraday Effect)**.** (**Mega Homework 36 points!**) In 1845, Michael Faraday found that once a linear polarized light enters the region where magnetic field is applied, the plane of polarization is rotated. If the polarization vector of the incoming beam is given by

$$\vec{E}_I = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = E_0 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = E_0(\cos\theta\hat{x} + \sin\theta\hat{y}) \tag{7.2}$$

the polarization angle of the outgoing beam are slightly shifted

$$\vec{E}_T = \begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = E_0 \begin{bmatrix} \cos{(\theta+\delta)} \\ \sin{(\theta+\delta)} \end{bmatrix}.$$

This problem will guide you through the process and you will see that the treatment is similar to our discussions of Lorentz and Drude model (Section 6).

(a) (3 points) **Background Math:** We will start by building an infra-structure. First, a linear polarized light presented in Eq. 7.2 is a sum of left and right-handed circular polarized light. Or, instead of writing down the field $\vec{E}_I$ in the basis of $\hat{x}$ and $\hat{y}$, we could write it in $\hat{e}_L$ and $\hat{e}_R$, where

$$\hat{e}_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

and

$$\hat{e}_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

Or,

$$\vec{E}_I = E_+\hat{e}_+ + E_-\hat{e}_-.$$

Show that for $\vec{E}_I = E_x\hat{x} + E_y\hat{y}$,

$$E_+ = \frac{1}{\sqrt{2}}(E_x + iE_y)$$

$$E_- = \frac{1}{\sqrt{2}}(E_x - iE_y).$$

(b) (3 points) Complete the transform matrix,

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -- & -- \\ -- & -- \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}, \tag{7.3}$$

an its inverse for the back transformation matrix,

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -- & -- \\ -- & -- \end{bmatrix} \begin{bmatrix} E_+ \\ E_- \end{bmatrix}. \tag{7.4}$$

Recall your high-school math. Now we have established an easy way to transform the polarization vector written in linear and circular polarized basis.

(c) (5 points) **Physics:** Next, we will deal with physics. We start by writing doown the equation of motion of a point charge under electric field ($\vec{E} = E_x\hat{x} + E_y\hat{y}$). But now there is a twist that we should include the term due to the force from magnetic induction ($\propto \vec{v} \times \vec{B}$). Given that $\vec{B} = B\hat{j}$ show that

$$im\omega v_x = e(E_x + Bv_y)$$

$$im\omega v_y = e(E_y - Bv_x)$$

(d) (5 points) Then we transform these equations from the basis of $\hat{x}, \hat{y}$ into $\hat{e}_+, \hat{e}_-$. Show that

$$v_\pm = \frac{-ieE_\pm}{m(\omega \pm \Omega)},$$

where $\Omega = \frac{eB}{m}$ and $v_\pm = \frac{1}{\sqrt{2}}(v_x \pm iv_y)$.
(e) (10 points) Calculate the refractive index $n_\pm$ in this new basis.
Answers:

$$n_\pm = \sqrt{\epsilon_0 - \frac{\epsilon_0 \omega_p^2}{\omega(\omega \pm \Omega)}},$$

where ω_p is the plasma frequency.
Hints:

1. From the definition of current density: $\vec{J} = \tilde{\sigma}\vec{E}_\pm = Ne\vec{v}_\pm$, find $\tilde{\sigma}$.
2. From $\tilde{\sigma}$, calculate $\tilde{\epsilon}$ and from $\tilde{\epsilon}$ calculate $\tilde{n}$.

(f) (10 points) Results so far suggest that the propagation of light in the $\hat{e}_+, \hat{e}_-$ basis is very simple. The field component along $\hat{e}_+$, will gain the phase factor $e^{ik_0 n_+ d}$ when travelling through the Faraday material with thickness d. The same is true for $-$. However, look at the beginning. The incident beam was written in the $\hat{x}, \hat{y}$ basis. This suggest that the strategy for calculating Faraday's rotation angle δ is the following:

1. Transform $\vec{E}_I$ (Eq. 7.2) to circular polarization basis using your result from Eq. 7.3.
2. Let them beam propagate through the medium by multiplying a phase factor: $e^{ik_0 n_+ d}$ and $e^{ik_0 n_- d}$ with the proper phase components.
3. Back transform to result in $\vec{E}_T$ your result from Eq. 7.4.

Calculate $\vec{E}_T$.
Answer:

$$\vec{E}_T = \begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = E_0 e^{ik_0 \bar{n} d} \begin{bmatrix} \cos(\theta + \delta) \\ \sin(\theta + \delta) \end{bmatrix},$$

where $\bar{n} = \frac{n_+ + n_-}{2}$ and $\delta = \frac{k_0 d(n_+ - n_-)}{2}$. The latter is our Faraday rotation angle.