



Example 6: Calculate magnetic field from a rotating sphere of radius a
Solution: We know earlier that the current density is

$$\vec{K} = \omega a \sin \theta' \hat{\phi};$$

therefore the vector potential is simply

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}}{r} da' = \frac{\mu_0 \omega a}{4\pi} \int \frac{\sin \theta' \hat{\phi}}{r} da', \quad (1)$$

where $\hat{\phi} = -\sin \phi' \hat{x} + \cos \phi' \hat{y}$ in Cartesian coordinate.

The tricky part here is to do the integral (Physics are done). We could ‘simplify’ the step a bit with the help of spherical harmonics.

First, we can expand $\frac{1}{r}$ as

$$\frac{1}{r} = \sum_{m,l} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{l,m}(\theta, \phi) Y_{l,m}^*(\theta', \phi'), \quad (2)$$

where $r_{<}$ is the lesser of r or $r' = a$. In the lecture, I have shown you the field *inside*, so $r_{<} = r$ and $r_{>} = a$. However, if you want to calculate the field outside the sphere then $r_{<} = a$ and $r_{>} = r$ (**Jackson 3.70**).

Second, the $\sin \theta' \cos \phi'$ and $\sin \theta' \sin \phi'$ terms that we are interested are related to spherical harmonics $Y_{1,1}(\theta', \phi')$ and $Y_{1,-1}(\theta', \phi')$.

$$Y_{1,1}(\theta', \phi') = -\frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta' e^{i\phi'}$$

$$Y_{1,-1}(\theta', \phi') = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta' e^{-i\phi'}.$$

And with the help of simple trigonometry,

$$\sin \theta' \cos \phi' = -\sqrt{\frac{2\pi}{3}} (Y_{1,1} - Y_{1,-1}),$$

$$\sin \theta' \sin \phi' = i\sqrt{\frac{2\pi}{3}} (Y_{1,1} + Y_{1,-1}).$$

We then can do the integral on both x and y components, but the procedures are almost identical. I’ll only show you the x. From Equation 1,

$$A_x = \frac{\mu_0 \omega a}{4\pi} \int \frac{-\sin \theta' \sin \phi'}{r} da'.$$

With the help of Equation 2,

$$A_x = \frac{\mu_0 \omega a}{4\pi} \int \sum_{m,l} \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{l,m}(\theta, \phi) Y_{l,m}^*(\theta', \phi') \frac{1}{i} \sqrt{\frac{2\pi}{3}} (Y_{1,1}(\theta', \phi') + Y_{1,-1}(\theta', \phi')) da'.$$

Here is the central idea, we take advantage of the orthonormality of spherical harmonics.

$$\int Y_{l,m}(\theta', \phi') Y_{l',m'}^*(\theta', \phi') d\Omega' = \delta_{l,l'} \delta_{m,m'},$$

where $d\Omega'$ is the solid angle defined as $\sin \theta' d\theta' d\phi'$. The unmatched indexes results in zero of integral (**Jackson 3.55**).

With this A_x is reduced to only two terms with $l = 1$ and $m = \pm 1$.

$$A_x = \frac{\mu_0 \omega a}{3} \frac{r_{<}}{r_{>}^2} \frac{1}{i} \sqrt{\frac{2\pi}{3}} (Y_{1,1}(\theta, \phi) + Y_{1,-1}(\theta, \phi)) a^2.$$

or

$$A_x = -\frac{\mu_0 \omega a^3}{3} \frac{r_{<}}{r_{>}^2} \sin \theta \sin \phi.$$

[**Note:** the angular dependency is almost the same as Equation 1 but just without prime.]

If you repeat the same thing for y, then you will see that

$$\vec{A} = \frac{\mu_0 \omega a^3}{3} \frac{r_{<}}{r_{>}^2} \sin \theta \hat{\phi}.$$

We will consider two cases:

1. $r < a$,

$$\vec{A} = \frac{\mu_0 \omega r a}{3} \sin \theta \hat{\phi}.$$

It's easier in this case to operate in the cylindrical coordinate or $\rho = r \sin \theta$.

$$\vec{A} = \frac{\mu_0 \omega \rho a}{3} \hat{\phi}.$$

And by picking a proper formula from the front-cover table, you would get

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{\rho} \frac{\partial \rho A_\phi}{\partial \rho} \hat{z} = \frac{2\mu_0 \omega a}{3} \hat{z}$$

2. $r > a$,

$$\vec{A} = \frac{\mu_0 \omega a^4}{3r^2} \sin \theta \hat{\phi}.$$

Again do the curl, we arrive at

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r \sin \theta} \frac{\partial (A_\phi \sin \theta)}{\partial \theta} \hat{r} - \frac{1}{r} \frac{\partial (r A_\phi)}{\partial r} \hat{\theta} = \frac{\mu_0 \omega a^4}{3r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta})$$

The final expression is equivalent to the field from a magnetic dipole (**Griffiths 5.86 to be discussed in our third session**) with moment of

$$m = \frac{4\pi \omega a^4}{3}$$

or in a more insightful formation:

$$m = \left(\frac{4\pi a^3}{3}\right)(\omega a),$$

where the first term is a volume of the sphere and the latter is “magnetization.” Magnetic moment is magnetization per unit volume. That’s the bottom line. Well, see it yourself in our third session.