Mid 1

Ruizhe Zhao

1. [40 points] Consider a stable plant with input-output behavior:

$$u(k) \longrightarrow \boxed{\frac{2+b_1z^{-1}}{1+a_1z^{-1}+a_2z^{-2}}} \longrightarrow y(k+1)$$

where $|b_1| < 2$.

(a) [4 points] Obtain the equations of recursive least squares (RLS), with a forgetting factor $\lambda = 0.999$, to estimate the plant parameters.

$$y(k+1) = -a_1 y(k) - a_2 y(k-1) + 2u(k) + b_1 u(k-1)$$

$$= \underbrace{[a_1 \quad a_2 \quad b_1]}_{\theta^T} \underbrace{\begin{bmatrix} -y(k) \\ -y(k-1) \\ u(k-1) \end{bmatrix}}_{\phi(k)} + \underbrace{2u(k)}_{\eta(k)}$$

$$\hat{y}^\circ(k+1) = -\hat{a}_1(k)\, y(k) - \hat{a}_2(k)\, y(k-1) + \hat{b}_1(k) u(k-1) + 2u(k)$$

$$= \underbrace{[\hat{a}_1(k) \quad \hat{a}_2(k) \quad \hat{b}_1(k)]}_{\hat{\theta}^T} \underbrace{\begin{bmatrix} -y(k) \\ -y(k-1) \\ u(k-1) \end{bmatrix}}_{\phi(k)} + \underbrace{2u(k)}_{\eta(k)}$$

$$\varepsilon^\circ(k+1) = y(k+1) - \hat{y}^\circ(k+1) = \theta^T\phi(k) + \eta(k) - \hat{\theta}^T(k)\phi(k) - \eta(k)$$

$$= \theta^T\phi(k) - \hat{\theta}^T(k)\phi(k)$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k+1)\phi(k)\,\varepsilon^\circ(k+1)$$

with forgetting factor: $F(k+1) = \frac{1}{0.99}\left[F(k) - \frac{F(k)\phi(k)\phi^T(k)F(k)}{0.99 + \phi^T(k)F(k)\phi(k)}\right]$

$$\Rightarrow \hat{\theta}(k+1) = \hat{\theta}(k) + F(k+1)\phi(k)\,\varepsilon^\circ(k+1)$$

$$= \hat{\theta}(k) + \frac{1}{0.999}\left[F(k) - \frac{F(k)\phi^T(k)F(k)}{0.999 + \phi^T(k)F(k)\phi(k)}\right]\phi(k)\,\varepsilon^\circ(k+1)$$

$$= \hat{\theta}(k) + \frac{F(k)\phi(k)\,\varepsilon^\circ(k+1)}{0.999 + \phi^T(k)F(k)\phi(k)}$$

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(b) [4 points] Translate the parameter adaptation algorithm in (a) to a feedback block diagram and apply hyperstability theory for stability analysis.

a posteori parameter update, with forgeting factor

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \frac{1}{\lambda} F \phi(k) \varepsilon(k+1)$$

parameter estimation error

$$\tilde{\theta}(k) = \hat{\theta}(k) - \theta$$

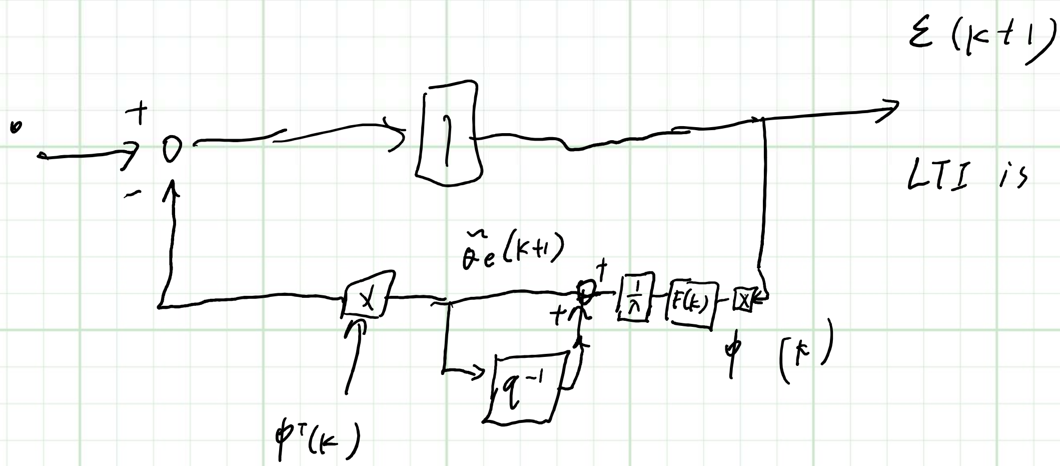
$$\tilde{\theta}(k+1) = \hat{\theta}(k+1) - \theta = \hat{\theta}(k) - \theta + \hat{\theta}(k) + \frac{1}{\lambda} F \phi(k) \varepsilon(k+1)$$

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + \frac{1}{\lambda} F \phi(k) \varepsilon(k+1)$$

posteriori prediction error

$$\varepsilon(k+1) = y(k+1) - \hat{y}(k+1) = \theta^T \phi(k) - \hat{\theta}^T \phi(k)$$

$$\varepsilon(k+1) = -\tilde{\theta}^T(k+1) \phi(k)$$



LTI is SPR

$$\tilde{\theta}(k+1) - \tilde{\theta}(k) = \frac{1}{\lambda} F \phi(k) \varepsilon(k+1)$$

$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1) \phi(k) \varepsilon(k+1) = \sum_{k=0}^{k_1} \left(\tilde{\theta}^T(k+1) F^{-1} \tilde{\theta}(k+1) - \tilde{\theta}^T(k+1) F^{-1} \tilde{\theta}(k) \right)$$

$$\lambda \sum_{k=0}^{k_1} \tilde{\theta}^T(k+1) \phi(k) \varepsilon(k+1) \pm \frac{1}{2} \tilde{\theta}(k) F^{-1} \tilde{\theta}(k) - \tilde{\theta}^T(k+1) F^{-1} \tilde{\theta}(k))$$

$$= \frac{1}{2} \lambda \tilde{\theta}^T(k_1+1) F^{-1} \tilde{\theta}(k_1+1) - \frac{1}{2} \tilde{\theta}^T(0) F^{-1} \tilde{\theta}(0) + \frac{1}{2} \lambda \left(F^{-\frac{1}{2}} \tilde{\theta}(k+1) - F^{-\frac{1}{2}} \tilde{\theta}(k) \right)^T \left(F^{-\frac{1}{2}} \tilde{\theta}(k+1) - F^{-\frac{1}{2}} \tilde{\theta}(k) \right)$$

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$$\sum_{k=0}^{k_1} \tilde{\theta}^T(k+1)\phi(k)\varepsilon(k+1) \geq -\frac{1}{2}\lambda\tilde{\theta}^T(0)F^{-1}\tilde{\theta}(0)$$

passive

The system is asymptotically stable

(c) [4 points] Obtain the equations of PAA with a parallel predictor and a constant adaptation gain to estimate the plant parameters.

Ther first parameter of B: $b_0 = 2$

$$y(k+1) = \phi^T(k)\theta + y(k)$$

$$\theta = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \end{bmatrix} \quad \phi(k) = \begin{bmatrix} -\hat{y}(k) \\ -\hat{y}(k-1) \\ u(k-1 \end{bmatrix}$$

posterior prediction: $\hat{y}(k+1) = \phi^T(k)\hat{\theta}(k+1) + \eta(k)$

Posteriory ---- error: $\varepsilon(k+1) = y(k+1) - \hat{y}(k+1)$

estimated parameter: $\hat{\theta}(k+1) = \hat{\theta}(k) + F\phi(k)\varepsilon(k+1)$

(d) [4 points] Provide conditions for hyperstability for the PAA in (c).

$$\tilde{\theta}(k+1) = \hat{\theta}(k+1) - \theta$$
$$= (\hat{\theta}(k) - \theta) + F(k)\phi(k)\varepsilon(k+1)$$
$$= \tilde{\theta}(k) + F(k)\phi(k)\varepsilon(k+1)$$

$$A(q^{-1})y(k+1) = B(q^{-1})u(k)$$

$$\hat{A}(q^{-1}, k+1)\hat{y}(k+1) = \hat{B}(q^{-1}, k+1)u(k)$$

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add and subtract terms:

$$A(q^{-1})y(k+1) \pm A(q^{-1})\hat{y}(k+1) = B(q^{-1})u(k)$$

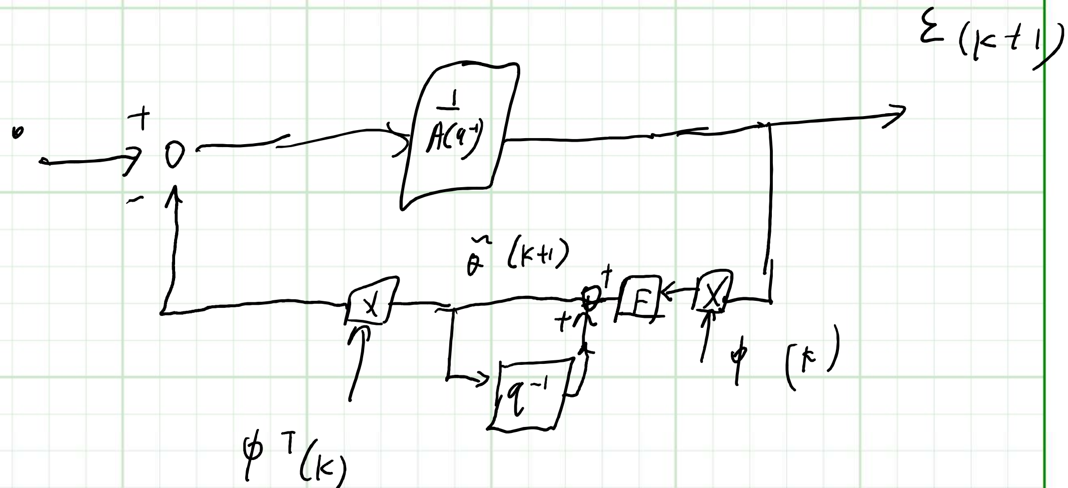
$$A(q^{-1})\varepsilon(k+1) = B(q^{-1})u(k) - A(q^{-1})\hat{y}(k+1)$$

$$= B(q^{-1})u(k) - A(q^{-1})\hat{y}(k+1) + \hat{A}(q^{-1},k+1)\hat{y}(k+1) - \hat{B}(q^{-1},k+1)u(k)$$

$$= [\hat{A}(q^{-1},k+1) - A(q^{-1})]\hat{y}(k+1) - [\hat{B}(q^{-1},k+1) - B(q^{-1})]u(k)$$

$$= \tilde{a}_1(k+1)\hat{y}(k) + \tilde{a}_2(k+1)\hat{y}(k-1) - \tilde{b}_1(k+1)u(k-1)$$

$$= -\tilde{\theta}(k+1)^T\phi(k)$$



For $F(k) = F$, the nonlinear block is passive shown in d)

For LTI block, $\frac{1}{A(z^{-1})}$ is SPR if $\sum_{i=1}^{n}|a_i| < 1$

However, $\frac{1}{A(z^{-1})}$ is not assured by $\frac{1}{A(z^{-1})}$ being stable alone

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(e) [8 points] Someone proposed to look at the system from a different perspective:

$$y(k+1) \longrightarrow \boxed{\frac{1+a_1 z^{-1}+a_2 z^{-2}}{2+b_1 z^{-1}}} \longrightarrow u(k)$$

i.e., to estimate $u(k)$ instead of $y(k+1)$ and use the estimation error of $u(k)$ for the parameter adaptation algorithm.

i. Obtain the RLS PAA with $\lambda = 0.999$.

ii. Obtain PAA with a parallel predictor and a constant adaptation gain.

i)

$$u(k) = \frac{1}{2}Y(k+1) + \frac{1}{2}a_1 Y(k) + \frac{1}{2}a_2 Y(k-1) - \frac{1}{2}b_1 u(k-1)$$

$$= \underbrace{\left[\frac{1}{2}a_1 \;\; \frac{1}{2}a_2 \;\; \frac{1}{2}b_1\right]}_{\theta^T} \underbrace{\begin{bmatrix} Y(k) \\ Y(k-1) \\ -u(k-1) \end{bmatrix}}_{\phi(k)} + \underbrace{\frac{1}{2}Y(k+1)}_{n(k)}$$

$$\hat{u}^\circ(k) = \frac{1}{2}\hat{a}_1 Y(k) + \frac{1}{2}\hat{a}_2 Y(k-1) - \frac{\hat{b}_1}{2} u(k-1) + \frac{1}{2}Y(k+1)$$

$$= \underbrace{\left[\frac{1}{2}\hat{a}_1(k) \;\; \frac{1}{2}\hat{a}_2(k) \;\; \frac{1}{2}\hat{b}_1(k)\right]}_{\hat{\theta}^T} \underbrace{\begin{bmatrix} Y(k) \\ Y(k-1) \\ -u(k-1) \end{bmatrix}}_{\phi(k)} + \underbrace{\frac{1}{2}Y(k+1)}_{\eta(k)}$$

$$\varepsilon^\circ(k+1) = Y(k+1) - \hat{Y}^\circ(k+1) = \theta^T\phi(k) + n(k) - \hat{\theta}^T(k)\phi(k) - \eta(k)$$

$$= \theta^T\phi(k) - \hat{\theta}^T(k)\phi(k)$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) + F(k+1)\phi(k)\,\varepsilon^\circ(k+1)$$

gain: $$F(k) = \frac{1}{0.999}\left[F(k+1) - \frac{F(k+1)\phi(k)\phi^T(k)F(k+1)}{0.999+\phi^T(k)F(k+1)\phi(k)}\right]$$

estimated parameter

$$\hat{\theta}(k) = \theta(k+1) + \frac{F(k+1)\phi(k)\varepsilon^\circ(k)}{0.999+\phi^T(k)F(k+1)\phi(k)}$$

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ii)

$$\phi(k) = \begin{bmatrix} -\hat{u}(k-1) \\ y(k) \\ y(k-1) \end{bmatrix} \quad \theta = \begin{bmatrix} \frac{b_1}{2} \\ \frac{a_1}{2} \\ \frac{a_2}{2} \end{bmatrix}$$

$$u(k) = \phi^T(k)\,\theta + \eta^{\circ}(k+1)$$

priori prediction : $\hat{u}^{\circ}(k) = \phi^T(k)\,\hat{\theta}(k+1) + \eta^{\circ}(k)$

priori prediction error : $\varepsilon^{\circ}(k) = u(k) - \hat{u}^{\circ}(k)$

estimated parameter : $\hat{\theta}(k) = \hat{\theta}(k+1) + F(k)\,\phi(k)\,\varepsilon^{\circ}(k)$

constant : $F(k) = F$

$$\hat{\theta}(k) = \hat{\theta}(k+1) + F\phi(k)\,\varepsilon^{\circ}(k)$$

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(f) [16 points] Choose numerical values of a_1 and a_2 such that $|a_1| + |a_2| < 1$. Choose b_1 such that $|b_1| < 2$. Modify the sp_predict.m and p_predict.m codes from homework 3 to test the PAAs that you derived in (a), (c). List all values that you use, including actual values of the plant parameters, the initial condition of your parameter prediction vector, and the initial adaptation gain. Submit your entire coding package. Plot the parameter convergence of your PAAs and verify whether they converge to the true values or not.

$$u(k) \longrightarrow \boxed{\frac{2 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}} \longrightarrow y(k+1)$$

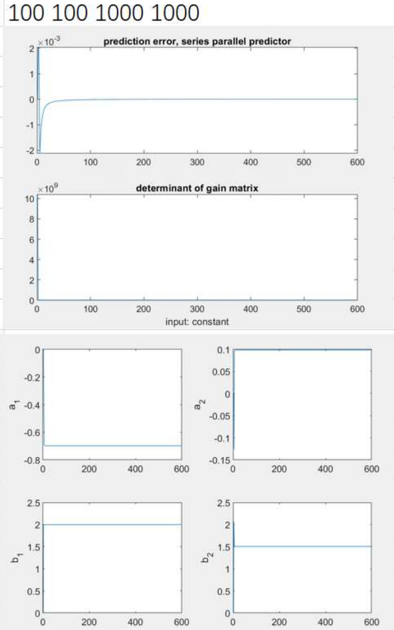
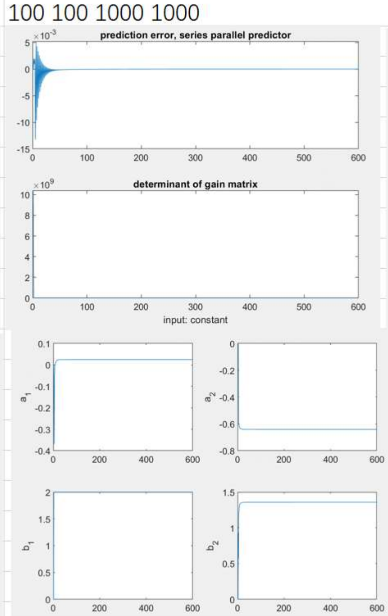
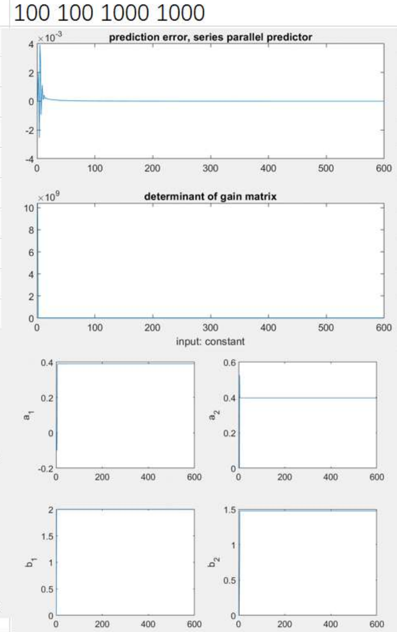
$$u(k) \rightarrow \boxed{\frac{2z + b_1}{z^2 + a_1 z + a_2}} \rightarrow y(k)$$

for a) → sp RLS, forgetting factor : $\lambda = 0.99$

plant parameters ; a_1 a_2 b_1 b_2

I, C

I. A, G F

	A	B	C	D
1	(in matlab notation)			
2	a1	-0.7	0.1	0.4
3	a2(abs(a1)+abs(a2)<1	0.1	-0.7	0.4
4	b1=2	2	2	2
5	b2<2(b1 in question)	1.5	1.5	1.5
6				
7	F: 1/10	100 100 1000 1000	100 100 1000 1000	100 100 1000 1000
8–27				
28				
29				
30	observation	has one overshoot and converges quickly	oscillation occoured and converge slower	small amount of occilation and converge quickly
31				
32		Vary b2 did not see significant change		
33		for sine input, it will converge and has zero error		

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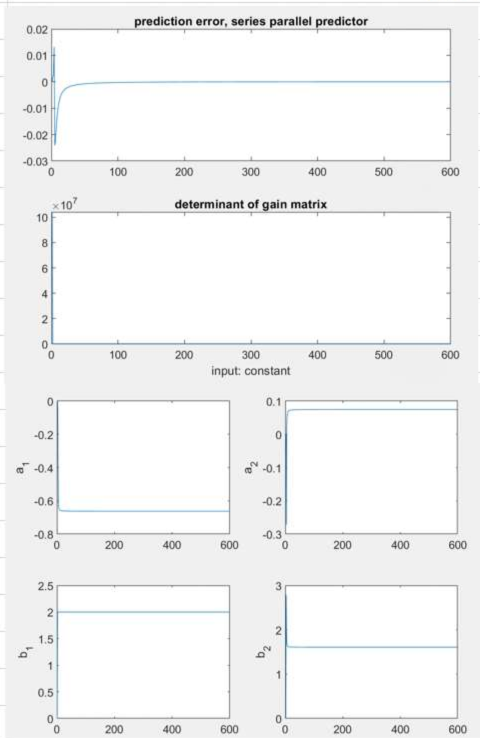
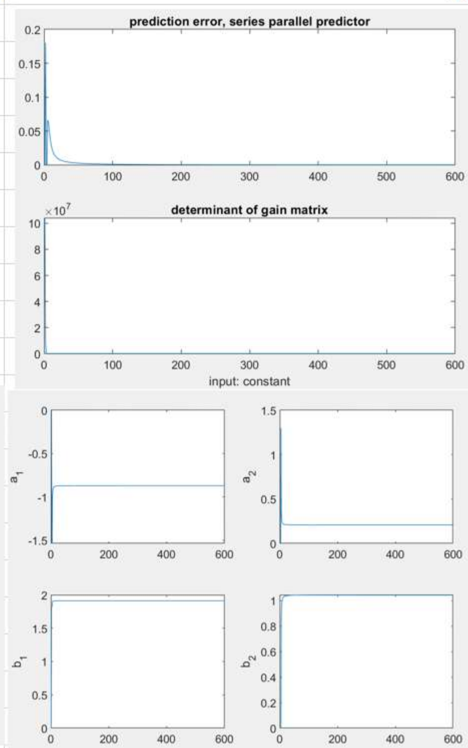
Initial adaptation gain		
f1	10	1000
f2	10	1000
f3	1000	10
f4	1000	10
		
	It has larger overshoot, still converge	It has largest overshoot, still converge
Initial condition:	It only has minimum effect on the plot. Doesn't affect overall behavior.	

Mid 1f

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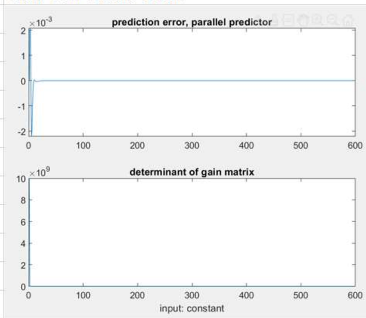
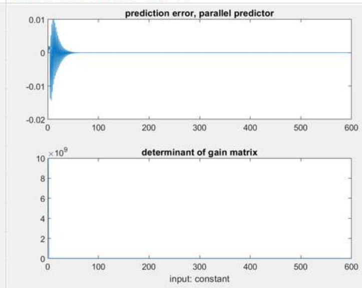
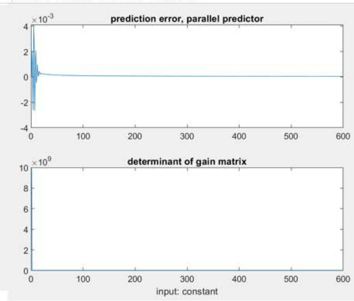
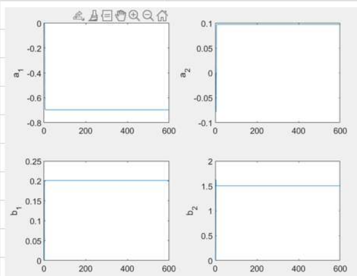
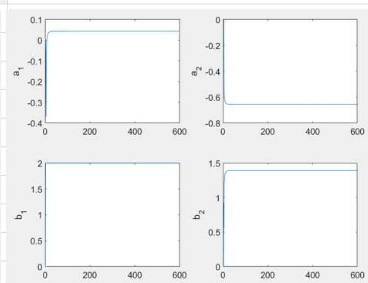
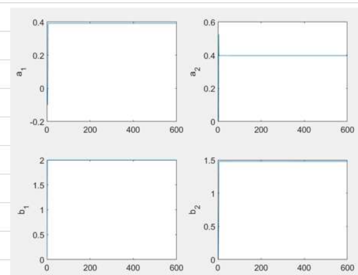
f): for c) → P constant adoptation gain

	A	B	C	D
1	(in matlab notation)			
2	a1	-0.7	0.1	0.4
3	a2(abs(a1)+abs(a2)<1	0.1	-0.7	0.4
4	b1=2	2	2	2
5	b2<2(b1 in question)	1.5	1.5	1.5
6				
7	F: 1/10	100 100 1000 1000	100 100 1000 1000	100 100 1000 1000
8–18				
19–28				
29				
30	observation	has one overshoot and converges quickly	oscillation occoured and converge slower	small amount of occilation and converge quickly

Mid1f

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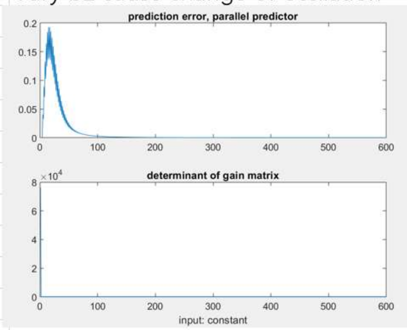
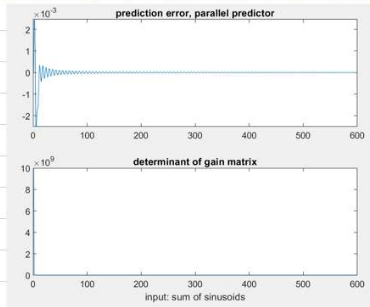
Vary b2 cause change of ocsilation

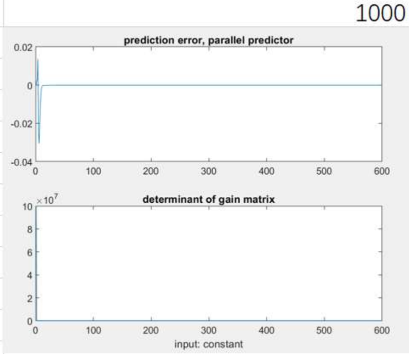


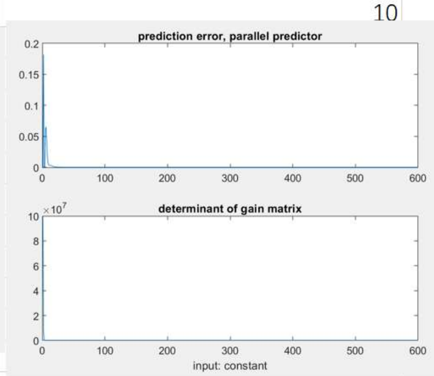
For sine input, it has some residule error



Initial adaptation gain

f1	10	1000
f2	10	1000
f3	1000	10
f4	1000	10





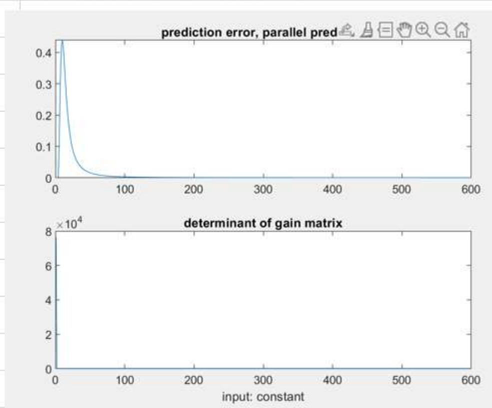
It has larger overshoot, still converge

It has larger overshoot, still converge

Initial condition:

It cause slower convergence and larger overshoot. Still converge. Doesn't affect overall behavior.

y1=5;y2=6;u1=7;u2=8;



Mid 2

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2. [10 points] Following notations introduced in class, prove that *parallel PAA with an adjustable compensator* is asymptotically hyperstable under a constant positive definite adaptation gain $F(k) = F \succ 0$.

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots a_n q^{-n}$$

$$\hat{C}(q^{-1}) = 1 + \hat{c}_1 q^{-1} + \cdots \hat{c}_n q^{-n}$$

$$v(k+1) = \hat{C}(q^{-1}, k+1)\,\varepsilon(k+1)$$

$$v^{\circ}(k+1) = \varepsilon^{\circ}(k+1) + \sum_{i=1}^{n} \hat{c}_i(k)\,\varepsilon(k+1-i)$$

$$\hat{\theta}_e(k+1) = \hat{\theta}_e(k) + \frac{F_e(k)\phi_e(k)}{1 + \phi_e^T(k) F_e(k) \phi_e(k)} v^{\circ}(k+1)$$

$$\hat{\theta}_e(k) = [\hat{\theta}^T(k) \quad \hat{c}_1(k) \quad \cdots \quad \hat{c}_n(k)]^T$$

$$\phi_e(k) = [\phi^T(k) \quad -\varepsilon(k) \quad \cdots \quad -\varepsilon(k+1-n)]^T$$

$$F_e^{-1}(k+1) = \lambda_1(k) F_e^{-1}(k) + \lambda_2(k)\phi_e(k)\phi_e^T(k)$$

a Posteriori:

$$\hat{\theta}_e(k+1) = \hat{\theta}_e(k) + F_e(k)\phi_e(k) v(k+1)$$

$$= \hat{\theta}_e(k) + F\phi_e(k) v(k+1) \quad \text{constant } F$$

$$\tilde{\theta}_e(k+1) = \hat{\theta}_e(k+1) - \theta_e$$

$$= \hat{\theta}_e(k) - \theta_e + F\phi_e(k) v(k+1)$$

$$= \tilde{\theta}_e(k) + F\phi_e(k) v(k+1)$$

$$A(q^{-1})\varepsilon(k+1) = -\tilde{\theta}^T(k+1)\phi(k)$$

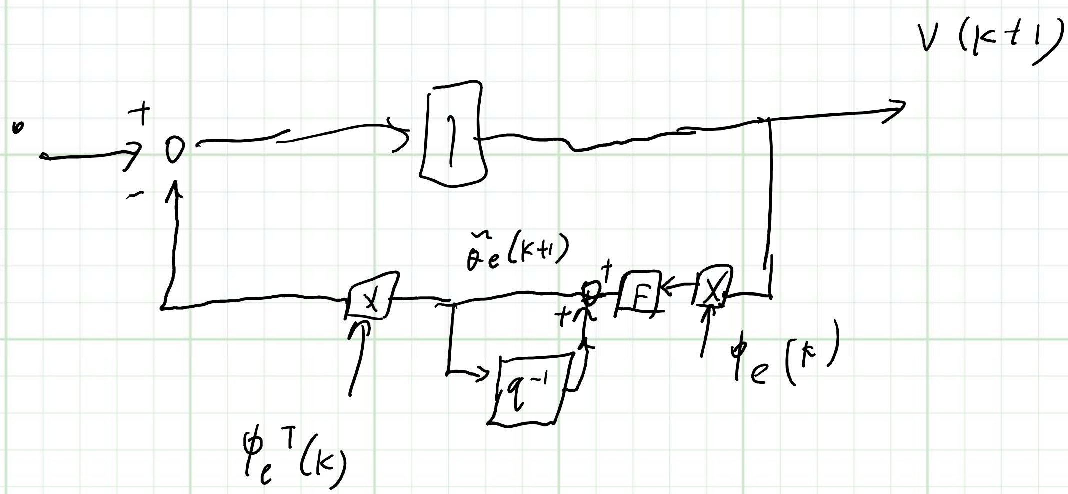
$$\varepsilon(k+1) = (1 - A(q^{-1}))\varepsilon(k+1) - \tilde{\theta}^T(k+1)\phi(k)$$

$$v(k+1) = \varepsilon(k+1) + (\hat{c}(q^{-1})-1)\,\varepsilon(k+1)$$
$$= -\tilde{\theta}^T(k+1)\,\phi(k) + (1-A(q^{-1}))\,\varepsilon(k+1) + (\hat{c}(q^{-1})-1)\,\varepsilon(k+1)$$

$$= -\tilde{\theta}^T(k+1)\phi(k) - (A(q^{-1}) - \hat{c}(q^{-1}))\,\varepsilon(k+1)$$

$$= -\tilde{\theta}^T(k+1)\phi(k) - \tilde{c}(q^{-1})\,\varepsilon(k+1)$$

$$= -\tilde{\theta}^T(k+1)\,\phi_e(k)$$



For LTI block: 1 is SPR

Non linear block:

$$\tilde{\theta}_e(k+1) = \tilde{\theta}_e(k) + F\phi_e(k)\,v(k+1)$$

$$\tilde{\theta}_e(k+1) - \tilde{\theta}_e(k) = F\phi_e(k)\,v(k+1)$$

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$$\sum_{k=0}^{k_1} \tilde{\theta}_e^T(k+1) F^{-1} \tilde{\theta}_e(k) v(k+1) = \sum_{k=0}^{k_1} \left(\tilde{\theta}_e^T(k+1) F^{-1} \tilde{\theta}_e(k+1) - \tilde{\theta}_e^T(k+1) F^{-1} \tilde{\theta}_e(k) \right)$$

$$= \sum_{k=0}^{k_1} \tilde{\theta}_e^T(k+1) F^{-1} \tilde{\theta}_e(k+1) \pm \frac{1}{2} \tilde{\theta}_e(k) F^{-1} \tilde{\theta}_e(k) - \tilde{\theta}_e^T(k+1) F^{-1} \tilde{\theta}_e(k)$$

$$= \sum_{k=0}^{k_1} \frac{1}{2} \left[\tilde{\theta}_e(k+1) F^{-1} \tilde{\theta}_e(k+1) - \tilde{\theta}_e^T(k) F^{-1} \hat{\theta}_e(k) \right]$$

$$+ \sum_{k=0}^{k_1} \frac{1}{2} \left[\tilde{\theta}_e^T(k+1) F^{-1} \tilde{\theta}_e(k+1) - 2 \hat{\theta}_e^T(k+1) F^{-1} \tilde{\theta}_e(k) + \tilde{\theta}_e^T(k) + \tilde{\theta}_e^T(k) F^{-1} \tilde{\theta}_e(k) \right]$$

$$= \left. \frac{1}{2} \tilde{\theta}_e^T(k_1+1) F^{-1} \tilde{\theta}_e(k_1+1) - \frac{1}{2} \tilde{\theta}_e^T(0) F^{-1} \tilde{\theta}_e(0) \right\} \geq -\frac{1}{2} \tilde{\theta}_e^T(0) F^{-1} \tilde{\theta}(0)$$

$$+ \left(F^{\frac{1}{2}} \tilde{\theta}_e(k+1) - F^{-\frac{1}{2}} \tilde{\theta}_e(k) \right)^T \left(F^{-\frac{1}{2}} \tilde{\theta}_e(k+1) - F^{-\frac{1}{2}} \tilde{\theta}_e(k) \right) \geq 0$$

passive

so it is asymptotically hyperstable

For $F(k) = F \succ 0$