Manual for the code used in, *A Coincidence Problem Related to the Λ-CDM Cosmological Model*, by Harri Ehtamo

Harri Ehtamo
Aalto University
harri.ehtamo@aalto.fi

Lauri Jokinen
Aalto University
lauri.jokinen@iki.fi

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Contents

1 Introduction

This manual describes the code that generates figures for, *A Coincidence Problem Related to the Λ-CDM Cosmological Model*, by Harri Ehtamo. The code is available at Github [1]. We use Matlab R2021b.

2 System of differential equations

Define, $a = a(t)$, and define T by $a(T) = 1$. The ΛR-model is defined with the following equations,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_T^2}{a^3}\Omega^{B+D} + \frac{H_T^2}{a^3}\alpha\Omega_t^{\Lambda\mathrm{R}} + H_T^2\Omega^{\Lambda}, \tag{1}$$

$$\Omega_t^{\Lambda\mathrm{R}} = \frac{H_T\sqrt{\Omega^{\Lambda}}}{a(t)}\int_0^t a(s)\, e^{-s\, H_T\sqrt{\Omega^{\Lambda}}}\ \mathrm{d}s, \tag{2}$$

with initial condition, $a(0) = a_0 > 0$. Here, $\Omega^{B+D} = \Omega^B + \Omega^D$, with $\Omega^B \geq 0$, $\Omega^D \geq 0$ constants; and $H_T > 0$, $\Omega^{\Lambda} > 0$, $\alpha \geq 0$ constants. (Pure) Friedmann model corresponds to the case $\alpha = 0$. Equation (1) is equivalent to,

$$\dot{a} = H_T\sqrt{\frac{1}{a}\Omega^{B+D} + \frac{\alpha}{a}\Omega_t^{\Lambda\mathrm{R}} + a^2\Omega^{\Lambda}}.$$

Then, define the integral in Equation (2) as, $b = b(t)$, which results in another ODE,

$$b = \int_0^t a(s)\, e^{-s\, H_T\sqrt{\Omega^{\Lambda}}}\ \mathrm{d}s \quad \Longleftrightarrow \quad \begin{cases} \dot{b} = a\, e^{-t\, H_T\sqrt{\Omega^{\Lambda}}}, \\ b(0) = 0, \end{cases} \tag{3}$$

and,

$$\Omega_t^{\Lambda\mathrm{R}} = \frac{H_T\sqrt{\Omega^{\Lambda}}}{a(t)}b(t).$$

Now, we have an ODE system which can be solved with, e.g., Matlab's ODE solver,

$$\begin{aligned} \dot{a} &= H_T\sqrt{\frac{1}{a}\Omega^{B+D} + \frac{1}{a}\alpha\left(\frac{H_T\sqrt{\Omega^{\Lambda}}}{a}b\right) + a^2\Omega^{\Lambda}}; \\ \dot{b} &= a\, e^{-t\, H_T\sqrt{\Omega^{\Lambda}}}; \\ a(0) &= a_0,\ b(0) = 0,\ t \geq 0. \end{aligned} \tag{4}$$

In the figures, we shift the time axis, s.t., the present time is at zero. This is done by replacing the time variable, t, by $t' = t - T$, where $a(T) = 1$. But for the purpose of this manual, it is sufficient to use t instead of t'.

Above, $a(t)$, $t \geq 0$, is called the scale factor of the model. The parameters Ω^B, Ω^D and Ω^{Λ} are the density parameters for baryonic matter, cold dark matter, and cosmological constant. The time dependent density parameter $\Omega_t^{\Lambda\mathrm{R}}$ arises as a consequence of a radiation field and it is called Λ-radiation; the constant α is an extra parameter related to the amount of Λ-radiation. The constant H_T is called the Hubble constant, and the Hubble parameter entering, e.g., in Figure 12, is defined by $H_t = \dot{a}/a$. Note that, since $a(T) = 1$, then $H_T = \dot{a}$, if the flatness equation in 3.2 holds.

2.1 Solving the ODE system

Let's study the system with respect to the parameter $a_0 > 0$. Ideally, we would set $a_0 = 0$, however, the equation for $\dot{a}$ is singular at that point. However, Matlab's stiff ODE solver `ode23s` is able to solve the ODE with very small a_0. We shall use $a_0 = 10^{-16}$ which provides enough accuracy for our purposes.

For the sake of completeness, we should study this approximation carefully, since the system *may* behave very differently if, say, $a_0 = b_0 > 0$ are small numbers. So, as an attempt to justify the above approximation, we simulate the system with randomly sampled initial conditions. The system's state variables are (a, b, t), initial conditions are (a_0, b_0, t_0), and the ideal initial conditions are $(0, 0, 0)$. In Figure 1, we have sampled random initial conditions under different constraints. We see that the curves, including the black curve, seem to converge as we approach the ideal initial conditions. This indicates that an ideal curve exists and the above approximation, $(10^{-16}, 0, 0)$, should indeed be close to the ideal curve.

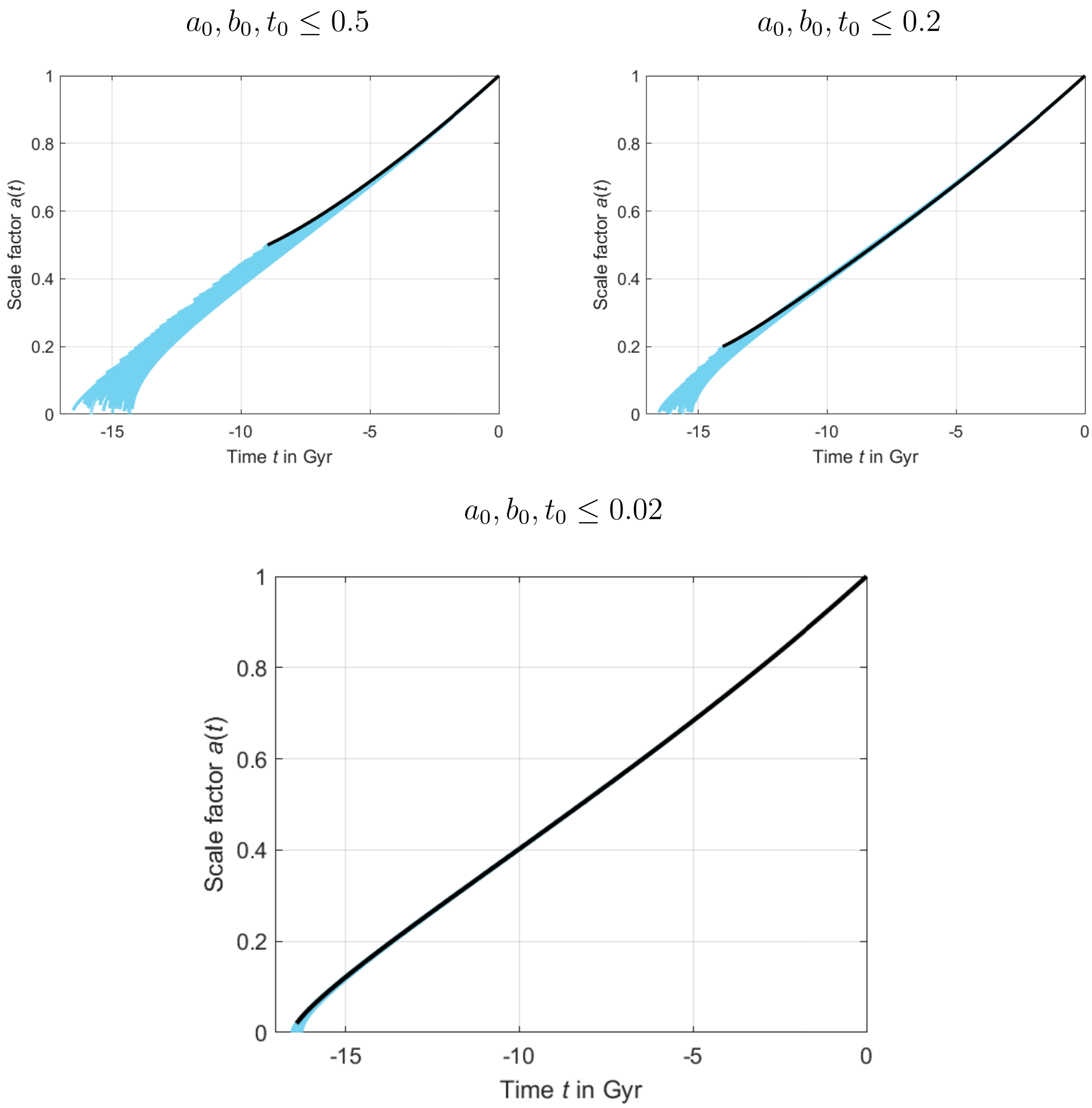

Figure 1: The ODE system solved with random positive initial conditions, namely, (a_0, b_0, t_0). The curve in black color denotes the largest a_0 allowed by the constraint, and $b_0 = t_0 = 0$. The Figures are produced with parameters, $\Omega^{B+D} = 0.049$, $\alpha = 1$, $\Omega^{\Lambda} = 0.6881$, and, $H_T = 0.0688$.

The solver `ode23s` is versatile and provides various options. We use these options to increase the precision of the solver, see code for details. The function `ode23s` also provides framework for finding user-defined *events* during integration. The solver then attempts to find these events with required precision. As an example, define two such events.

Suppose, we want to solve T from the equation $a(T) = 1$. This equation is trivial to define as an event. In Figure 3, we want to find the maximum point of $\Omega_t^{\Lambda\mathrm{R}}$. This is found by solving the equation,

$$\frac{d}{dt}\Omega_t^{\Lambda\mathrm{R}} = 0 \iff \frac{d}{dt}\left(\frac{b}{a}\right) = 0 \iff a\,\dot{b} - b\,\dot{a} = 0,$$

which we define as an event.

3 Defining parameters

The ODE system has parameters H_T, Ω^B, Ω^D, Ω^Λ and α. Here, we provide different parameter definitions which are then later used in various Figures. To numerically solve equations in the code, we use Matlab's algorithm `fsolve`, and for unconstrained optimization we use `fminunc`. We increase the precision of these algorithms by using certain options, which are visible in the code.

3.1 Unit conversion for H_T

The ODE system's only parameter whose unit depends on time is H_T, and in the figures, we use time unit of Gyrs, i.e., 10^9 years. Since the value of H_T is given in unit $\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$, we convert it to Gyrs^{-1}. With a year of 365.25 days, and using IAU 2012 definitions [2], we have an exactly defined conversion factor for the unit conversion (shown below). As an example, below we convert $H_T = 67.26\ \mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$.

```
convertion_factor = 0.001 * 3.15576 / (1.495978707 * 6.48 / pi);
%%%               = 0.001022712165045695
H_T = 67.26 * convertion_factor;
%%% = 0.06878762022097345 (in unit 1/Gyrs)
```

This value for H_T is used in Figures 1-10. In Figures 11-12 the used values are written in the Figure.

3.2 Flatness equation

This section describes the function `flatness_solve_Omega_L` in the code. The flatness equation is,

$$\Omega^{B+D} + \Omega^\Lambda + \alpha\Omega_T^{\Lambda\mathrm{R}} = 1,$$

that for given Ω^{B+D}, α only depends on the parameter Ω^Λ. This equation for Ω^Λ is solved iteratively by using a suitable iteration method. Note, that at every iteration step, the equations (4) are solved with that Ω^Λ to get $\Omega_T^{\Lambda\mathrm{R}}$.

E.g., for parameter values $\alpha = 1$, $\Omega^B = 0.049$ and $\Omega^D = 0$, the flatness equation implies, $\Omega^\Lambda \approx 0.6881$.

3.3 Benchmark model's parameters

In the benchmark F-model, in the figures F-model b, the parameters are,

$$\Omega^B = 0.049, \quad \Omega^D = 0.268, \quad \Omega^\Lambda = 0.683, \quad \alpha = 0.$$

In other (pure) F-models, i.e., in models with $\alpha = 0$, also other Ω^B, Ω^D and Ω^Λ parameters are used. Note, that for $t = T$, $a(T) = 1$, $H_T = \dot{a}(T)$, the flatness condition $\Omega^{B+D} + \Omega^\Lambda = 1$ should hold for the parameters. This model implies $T^b \approx 13.800$ Gyrs.

3.4 Optimal Ω^B

This section describes the function `optimal_Omega_B`. Fix $\alpha = 1$ and $\Omega^D = 1$, which implies the following flatness equation, $\Omega^B + \Omega^\Lambda + \Omega_t^{\Lambda\mathrm{R}} = 1$. We then solve the following maximization problem,

$$\begin{aligned} &\max_{\Omega^B} \Omega_T^{\Lambda\mathrm{R}}, \\ \text{s.t.} \qquad &\Omega^B + \Omega^\Lambda + \Omega_T^{\Lambda\mathrm{R}} = 1. \end{aligned}$$

We use Matlab's unconstrained optimization algorithm, so the constraint needs to be satisfied at each iteration, i.e., we solve the value of Ω^Λ as in 3.2 at each iteration step. This solving method is quite slow due to the layered iterations, but it is sufficient and accurate for the purpose.

The solution to this problem is denoted as, $\Omega^B_{\mathrm{opt}} \approx 0.0458$, and, $\Omega^{\Lambda\mathrm{R}}_{T,\mathrm{opt}} \approx 0.2629$.

3.5 Age-optimal Ω^D and α version 1

E.g., in Figure 8, we have two curves with $\alpha\Omega_T^{\Lambda\mathrm{R}} \in \{0.06, 0.16\}$ and age of the Universe is matched to the benchmark model's age, namely, $T^b \approx 13.800$ Gyrs, see definition in 3.3. In addition, $\Omega^B = 0.049$ is fixed. In mathematical terms, we solve the parameters Ω^D and α from the following system of equations,

$$\begin{aligned} \alpha\Omega_T^{\Lambda\mathrm{R}} &= c, \\ T^b &= T, \\ \Omega^{B+D} + \Omega^\Lambda + \alpha\Omega_T^{\Lambda\mathrm{R}} &= 1, \end{aligned}$$

where parameter c is given, and $a(T) = 1$.

3.6 Age-optimal Ω^D and α version 2

This part describes, e.g., the black curve in Figure 9, and function `optimal_Omega_D_and_alpha` in the code. We aim to find a model with following property,

$$\frac{\Omega^D + \alpha\Omega_T^{\Lambda\mathrm{R}}}{\Omega^B} = \frac{\Omega^{\Lambda\mathrm{R}}_{T,\mathrm{opt}}}{\Omega^B_{\mathrm{opt}}} \approx 5.7380,$$

where optimal parameters are as in 3.4. As described above, we match the age of the Universe to the benchmark model's age, and fix $\Omega^B = 0.049$. In mathematical terms, we solve the parameters Ω^D and α from the following system,

$$\begin{aligned}\frac{\Omega^D + \alpha\Omega_T^{\Lambda\mathrm{R}}}{\Omega^B} &= \frac{\Omega_{T,\mathrm{opt}}^{\Lambda\mathrm{R}}}{\Omega_{\mathrm{opt}}^B}, \\ T_B &= T, \\ \Omega^{B+D} + \Omega^\Lambda + \alpha\Omega_T^{\Lambda\mathrm{R}} &= 1,\end{aligned}$$

where $a(T) = 1$. Note, that we can replace the third equation by combining the first and third equations. This yields an equivalent system,

$$\begin{aligned}\frac{\Omega^D + \alpha\Omega_T^{\Lambda\mathrm{R}}}{\Omega^B} &= \frac{\Omega_{T,\mathrm{opt}}^{\Lambda\mathrm{R}}}{\Omega_{\mathrm{opt}}^B}, \\ T_B &= T, \\ \Omega^\Lambda &= 1 - \frac{\Omega_{\mathrm{opt}}^{\Lambda\mathrm{R}}}{\Omega_{\mathrm{opt}}^B}\Omega^B - \Omega^B.\end{aligned}$$

With this system, we no longer need to solve the flatness equation iteratively, since the value of Ω^Λ is fixed. The solution to this system is denoted as, $\Omega_{\mathrm{opt}}^D \approx 0.2589$, and, $\alpha_{\mathrm{opt}} \approx 0.0832$.

4 Figures

Here, we describe the contents of each figure.

4.1 Figures 1 and 2

We use the values written in the Figures. Missing parameters for the ΛR-model are found in the example in 3.2.

4.2 Figure 3

In the red graph we have $\Omega^B = 0.049$, $\alpha = 0$, parameter Ω^Λ gets the value from the example in 3.2 and Ω^D is solved from the flatness equation.

The blue dotted graph has parameters as in the example in 3.2.

4.3 Figure 4

We have three cases: $\Omega^{B+D} \in \{0.2,\ 0.27,\ 0.3\}$. In all cases we have $\Omega^B = 0.049$ and Ω^Λ is as in the example in 3.2. The parameter α is solved from the flatness equation.

In the Friedmann curve, we have $\Omega^D = \Omega_{\mathrm{opt}}^{\Lambda\mathrm{R}}$, $\Omega^B = 0.049$, $\alpha = 0$ and Ω^Λ has the same value as above.

4.4 Figure 5

In each graph, we have $\Omega^D = 0$, $\alpha = 1$ and Ω^Λ is solved from the flatness equation. We let Ω^B take a few certain values including Ω_{opt}^B, see 3.4.

4.5 Figure 6 and 7

We have $\Omega^D = 0$ and $\alpha = 1$, and we let Ω^B vary to produce the curves. We solve the flatness equation at each curve point, and record the $\Omega_T^{\Lambda R}$-values. In the linear approximation curve we use,

$$\Omega^B + \frac{1 - \left(1 + \sqrt{\Omega^\Lambda}\right)\exp\left(-\sqrt{\Omega^\Lambda}\right)}{\sqrt{\Omega^\Lambda}} + \Omega^\Lambda = 1,$$

instead of the flatness equation. Models used in this graph are also used to form the Tables 1 and 2.

4.6 Figure 8

The benchmark model's parameters are described in 3.3. One curve is defined with,

$$\Omega^{B+D} = 0.2, \quad \alpha = 0.68, \quad \Omega^B = 0.049,$$

and another with,

$$\Omega^{B+D} = 0.27, \quad \alpha = 0.3, \quad \Omega^B = 0.049.$$

In both, Ω^Λ is given by the flatness equation.

In the two remaining graphs, the parameters are as in 3.5, where we fix $\alpha\Omega_T^{\Lambda R} \in \{0.06, 0.16\}$.

4.7 Figures 9 and 10

The benchmark model's parameters are described in 3.3. There are three curves with $\alpha\Omega_T^{\Lambda R} \in \{0.01, 0.02, 0.03\}$. These models are defined as in 3.5. For the black curves, see 3.6.

4.8 Figures 11 and 12

Define $H_t = \dot{a}/a$. Here, we have different values for H_T, all of which need to be converted to Gyrs, see 3.1. For the F-models we have $\alpha = 0$, and all other required parameters are visible.

We choose the ΛR-model's parameter α, s.t., its age of the Universe matches with the model "F-model 1", while $\Omega^{B+D} = 0.307$, $\Omega^B = 0.049$ and Ω^Λ is solved from the flatness equation.

References

[1] Lauri Jokinen. *Code for "A Coincidence Problem Related to the Λ-CDM Cosmological Model"*. 2023. URL: `https://github.com/lapamatoz/Lambda-R-model`.

[2] Wikipedia. *Parsec*. visited in 2023. URL: `https://en.wikipedia.org/wiki/Parsec`.