BU CAS Computer Science 132 (Spring 2013)
Geometric Algorithms

Quiz

Vector Spaces and Linear Transformations

Thursday, April 18, 2013

You have 50 minutes to complete this quiz. Your solutions must show all work.

You have the option to submit this quiz to be graded; **if you do not submit it, it will not count towards your grade.** If you submit this quiz to be graded, it will count as one homework assignment (the homework component of your overall course grade will remain 50%).

NAME	
BU ID	

Problem #1		*10*
Problem #2		*10*
Problem #3		*10*
Problem #4		*10*
Problem #5		*10*
Total		*50*

Problem 1. (10 pts.)

Determine if the following equation can be solved. If it can be solved, find the solution $x, y \in \mathbb{R}$. If it cannot be solved, explain why it cannot be solved by describing a relevant matrix property.

$$\begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We know that the span of the columns of $\begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix}$ is $\mathsf{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ (because $-3 \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$). Since $\begin{bmatrix} 1 \\ -2 \end{bmatrix} \notin \mathsf{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ (because there is no s such that $s \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$), this equation is not solvable.

Alternatively, we can say the matrix is not invertible. However, in general, it is not always sufficient to say that the matrix is not invertible; for example, even though the matrix is not invertible in the following equation, it is solvable:

$$\begin{bmatrix} -2 & 6 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}.$$

Problem 2. (10 pts.)

Consider the following linear transformations $f, g, h \in \mathbb{R}^2 \to \mathbb{R}^2$ (i.e., the domain is always $\mathbb{R}^2$ and the codomain is always $\mathbb{R}^2$):

$$f(v) = \begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix} \cdot v \qquad g(v) = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix} \cdot v \qquad h(v) = \begin{bmatrix} 1 & -2 \\ 7 & 5 \end{bmatrix} \cdot v$$

(a) Define $\mathsf{im}(f)$ as a span of one or more vectors and find its basis and dimension.

We have that:
$$\mathsf{im}(f) = \mathsf{span}\left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\}.$$
Since the space is the span of a single vector, one possible smallest basis is:
$$\left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\}.$$
Its dimension is the size of any basis, so:
$$\mathsf{dim}(\mathsf{im}(f)) = \left| \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\} \right| = 1.$$

(b) Either prove that g is surjective or provide a counterexample.

The linear transformation g is not surjective. Since the two columns of the matrix used to define g are linearly dependent, any w not on the line spanned by the two columns could not be an output of g; for example:

$$g(v) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$4 \cdot x + 1 \cdot y = 1$$
$$8 \cdot x + 2 \cdot y = 2 \text{ according to the first row equation}$$
$$8 \cdot x + 2 \cdot y = 0 \text{ according to the second row equation}$$

(c) Either prove that h is injective or provide a counterexample.

The matrix is invertible, so h is bijective, which means it is both surjective and injective. To show that it is injective, let the matrix be M. Then we have for any $v, v' \in \mathbb{R}^2$:

$$\begin{aligned} f(v) &= f(v') \\ M \cdot v &= M \cdot v' \\ M^{-1} \cdot M \cdot v &= M^{-1} \cdot M \cdot v' \\ v &= v' \end{aligned}$$

Problem 3. (10 pts.)

You are given the following data points:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find a polynomial of the form $f(x) = ax + b$ that is the least squares best fit for this data.

We first set up the matrix equation for fitting a curve to some points:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

The above equation (call it $M \cdot v = w$ has no solution. Thus, we find an orthonormal basis of the span of the columns of the matrix, project the vector on the right-hand side onto that span to obtain w^*, and create a new, solvable equation $M \cdot v^* = w^*$. The solution v^* to that equation is the least-squares approximate solution with error $||w - w^*||$.

The two columns vectors are already orthogonal (their dot product is 0), so it is sufficient to normalize them in order to create an orthonormal basis:

$$\mathsf{span}\left\{ \begin{bmatrix} \frac{-1}{\sqrt{(2)}} \\ 0 \\ \frac{1}{\sqrt{(2)}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}$$

We perform the projection to obtain the new equation:

$$w^* = \left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{\sqrt{(2)}} \\ 0 \\ \frac{1}{\sqrt{(2)}} \end{bmatrix} \right) \cdot \begin{bmatrix} \frac{-1}{\sqrt{(2)}} \\ 0 \\ \frac{1}{\sqrt{(2)}} \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right) \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = 0 + \begin{bmatrix} \frac{6}{3} \\ \frac{6}{3} \\ \frac{6}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

We can now build the new solvable equation and find its solution:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Thus, the polynomial that is the least-squares best fit for this data is $f(x) = 0 \cdot x + 2$ or just $f(x) = 2$.

Problem 4. (10 pts.)

Use the Gram-Schmidt process to find an orthonormal basis of the following vector space (you must use the Gram-Schmidt process to receive full credit):

$$
V = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}
$$

What is the dimension of the space?

First, we remove one vector from the spanning set of vectors because it is a linear combination of the other two:

$$
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 4 \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 2 \end{bmatrix}
$$

The set is now a basis of the form $\{v_1, v_2\}$. We need to make it an orthonormal basis $\{e_1, e_2\}$ using the Gram-Schmidt process for two steps. We first find e_1:

$$
e_1 = \frac{1}{1} \cdot u_1 = u_1 = v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Next, we find e_2:

$$
v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}
$$

$$
u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}
$$

$$
e_2 = \frac{1}{||u_2||} \cdot u_2 = \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}
$$

Problem 5. (10 pts.)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as:
$$f(v) = ||v||.$$
Show that f is not a linear transformation (i.e., find a counterexample).

It is sufficient to find a counterexample that contradicts one of the two properties of a linear transformation. For example:

$$f\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = 1$$

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1$$

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$= 0$$

$$0 \neq 2$$

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \neq f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$