BU CAS Mathematics 142 (Spring 2013)
Introduction to Linear Algebra

Practice Final Exam

Vector and Matrix Algebra, Vector Spaces, and Linear Transformations

You have 120 minutes to complete this exam. Your solutions must show all work.

The expected total for the exam is 120 points. However, the total number of points available is higher. **Choose any subset of the questions to answer. Any points you earn above 120 will count as extra credit.**

You are permitted to use one side of an 8.5 in. × 11 in. sheet filled with any information you choose. Please submit it with your exam. It will be returned to you with the graded exam.

NAME	
BU ID	

Problem #1		*10*
Problem #2		*10*
Problem #3		*10*
Problem #4		*10*
Problem #5		*10*
Problem #6		*10*
Problem #7		*10*
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Problem #10		*20*
Problem #11		*25*
Problem #12		*15*
Total		*120*
Extra		*30*

Problem 1. (10 pts.)

Solve the following equation for $x, y \in \mathbb{R}$.

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The matrix is invertible, so there is a unique solution $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$.

Problem 2. (10 pts.)

Determine whether each of the following sets of matrices is closed under matrix multiplication (you may assume these are all subsets of $\mathbb{R}^{n \times n}$). If it is closed, justify your answer; if it is not, provide a counterexample (e.g., in $\mathbb{R}^{2 \times 2}$).

(a) $\{M \mid \underline{M \text{ is upper triangular}} \text{ and } \underline{M \text{ is invertible}}\}$

Both the set of upper triangular matrices and the set of invertible matrices are closed under matrix multiplication. Thus, two matrices A and B are both upper triangular and invertible, then $A \cdot B$ must be triangular, and it must be invertible. Thus, the set is closed under matrix multiplication.

(b) $\{M \mid \underline{M \text{ is upper triangular}} \text{ or } \underline{M \text{ is diagonal}}\}$

For any two matrices A and B in the set, we have the following possibilities:

- if both matrices are diagonal, $A \cdot B$ is diagonal;
- if both matrices are upper triangular, $A \cdot B$ is upper triangular;
- if one matrix is diagonal and the other is upper triangular, then $A \cdot B$ is upper triangular because all diagonal matrices are by definition upper triangular;

In other words, this set is equivalent to the set of upper triangular matrices because all diagonal matrices are upper triangular. We know the set of upper triangular matrices is closed under multiplication.

(c) $\{M \mid \underline{M \text{ has no zero entries}}\}$

We know that a matrix with orthogonal columns and no zero entries will yield a matrix with zero entries when multiplied by its own transpose, so this set cannot be closed. Here is one such counterexample:

$$\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}^\top \cdot \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$$

Problem 3. (10 pts.)

Assume that $a, b \in \mathbb{R}$, $a \neq 0$, $b \neq 0$, and that the following equation is true:

$$
\begin{bmatrix} 26 \\ 14 \\ 39 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 1 & 6 & 5 \end{bmatrix} \begin{bmatrix} a & b^2 & 2 \\ -9a & 2b^2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1/a \\ 4/b^2 \\ 1 \end{bmatrix}
$$

Compute the following to obtain an explicit vector in $\mathbb{R}^2$:

$$
\begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 1 & 6 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 26 \\ 14 \\ 39 \end{bmatrix} = ?
$$

We expand the left-hand side of the second equation using the first equation, then cancel terms, then multiply:

$$
\begin{aligned}
\begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 1 & 6 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 26 \\ 14 \\ 39 \end{bmatrix} &= \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 1 & 6 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 1 & 6 & 5 \end{bmatrix} \begin{bmatrix} a & b^2 & 2 \\ -9a & 2b^2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1/a \\ 4/b^2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 1 & 6 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 1 & 6 & 5 \end{bmatrix} \begin{bmatrix} a & b^2 & 2 \\ -9a & 2b^2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1/a \\ 4/b^2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} a & b^2 & 2 \\ -9a & 2b^2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1/a \\ 4/b^2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1+4+2 \\ -9+2+3 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 7 \\ -4 \\ 4 \end{bmatrix}
\end{aligned}
$$

Problem 4. (10 pts.)

Consider the following linear transformations $f, g, h \in \mathbb{R}^2 \to \mathbb{R}^2$ (i.e., the domain is always $\mathbb{R}^2$ and the codomain is always $\mathbb{R}^2$):

$$f(v) = \begin{bmatrix} -2 & 1 \\ 8 & -4 \end{bmatrix} \cdot v \qquad g(v) = \begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix} \cdot v \qquad h(v) = \begin{bmatrix} 1 & -2 \\ 7 & 5 \end{bmatrix} \cdot v$$

(a) Define $\mathsf{im}(f)$ as a span of one or more vectors and find its basis and dimension.

We have that:
$$\mathsf{im}(f) = \mathsf{span}\left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\}.$$

Since the space is the span of a single vector, one possible smallest basis is:
$$\left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\}.$$

Its dimension is the size of any basis, so:
$$\dim(\mathsf{im}(f)) = \left| \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\} \right| = 1.$$

(b) Either prove that g is surjective or provide a counterexample.

The linear transformation g is not surjective. Since the two columns of the matrix used to define g are linearly dependent, any w not on the line spanned by the two columns could not be an output of g; for example:

$$\begin{aligned} g(v) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 4 \cdot x + 1 \cdot y &= 1 \\ 8 \cdot x + 2 \cdot y &= 2 \text{ according to the first row equation} \\ 8 \cdot x + 2 \cdot y &= 0 \text{ according to the second row equation} \end{aligned}$$

(c) Either prove that h is injective or provide a counterexample.

The matrix is invertible, so h is bijective, which means it is both surjective and injective. To show that it is injective, let the matrix be M. Then we have for any $v, v' \in \mathbb{R}^2$:

$$\begin{aligned} f(v) &= f(v') \\ M \cdot v &= M \cdot v' \\ M^{-1} \cdot M \cdot v &= M^{-1} \cdot M \cdot v' \\ v &= v' \end{aligned}$$

Problem 5. (10 pts.)

You are given the following data points:

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find a polynomial of the form $f(x) = ax + b$ that is the least squares best fit for this data.

We first set up the matrix equation for fitting a curve to some points:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

The above equation (call it $M \cdot v = w$ has no solution. Thus, we find an orthonormal basis of the span of the columns of the matrix, project the vector on the right-hand side onto that span to obtain w^*, and create a new, solvable equation $M \cdot v^* = w^*$. The solution v^* to that equation is the least-squares approximate solution with error $||w - w^*||$.

The two columns vectors are already orthogonal (their dot product is 0), so it is sufficient to normalize them in order to create an orthonormal basis:

$$\mathsf{span}\left\{ \begin{bmatrix} \frac{-1}{\sqrt{(2)}} \\ 0 \\ \frac{1}{\sqrt{(2)}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right\}$$

We perform the projection to obtain the new equation:

$$w^* = \left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{-1}{\sqrt{(2)}} \\ 0 \\ \frac{1}{\sqrt{(2)}} \end{bmatrix} \right) \cdot \begin{bmatrix} \frac{-1}{\sqrt{(2)}} \\ 0 \\ \frac{1}{\sqrt{(2)}} \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right) \cdot \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = 0 + \begin{bmatrix} \frac{6}{3} \\ \frac{6}{3} \\ \frac{6}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

We can now build the new solvable equation and find its solution:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Thus, the polynomial that is the least-squares best fit for this data is $f(x) = 0 \cdot x + 2$ or just $f(x) = 2$.

Problem 6. (10 pts.)

Use the Gram-Schmidt process to find an orthonormal basis of the following vector space (you must use the Gram-Schmidt process to receive full credit):

$$V = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

What is the dimension of the space?

First, we remove one vector from the spanning set of vectors because it is a linear combination of the other two:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 4 \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

The set is now a basis of the form $\{v_1, v_2\}$. We need to make it an orthonormal basis $\{e_1, e_2\}$ using the Gram-Schmidt process for two steps. We first find e_1:

$$e_1 = \frac{1}{1} \cdot u_1 = u_1 = v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Next, we find e_2:

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$e_2 = \frac{1}{||u_2||} \cdot u_2 = \frac{1}{\sqrt{6}} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

Problem 7. (10 pts.)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as:
$$f(v) = ||v||.$$
Show that f is not a linear transformation (i.e., find a counterexample).

It is sufficient to find a counterexample that contradicts one of the two properties of a linear transformation. For example:

$$f\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = 1$$

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1$$

$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$
$$= 0$$
$$0 \neq 2$$
$$f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \neq f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Problem 8. (10 pts.)

A function $f(x) = ax^3 + bx^2 + cx + d$ has slope 4 at $x = 0$ and intersects the following points:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 29 \end{bmatrix}.$$

Set up an appropriate matrix equation to find the coefficients of f and solve it.

We know that the derivative of f is $f'(x) = 3ax^2 + 2bx + c$. Thus, if the slope of f is 4 at $x = 0$, this means:

$$f'(0) = 4.$$

We can set up the following matrix equation to solve for the coefficients a, b, c, and $d \in \mathbb{R}$:

$$\begin{bmatrix} (-1)^3 & (-1)^2 & (-1) & 1 \\ (0)^3 & (0)^2 & (0) & 1 \\ (2)^3 & (2)^2 & (2) & 1 \\ 3(0)^2 & 2(0) & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 29 \\ 4 \end{bmatrix}$$

Problem 9. (10 pts.)

Let $\left\{ \begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} \right\}$ be an orthonormal basis. If $a = \frac{\sqrt{3}}{2}$ and $c + d > \frac{\sqrt{3}-1}{2}$, find b, c, and d.

We know that the following are true:

$$\begin{bmatrix} a \\ c \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = 0$$

$$\left\| \begin{bmatrix} a \\ c \end{bmatrix} \right\| = 1$$

$$\left\| \begin{bmatrix} b \\ d \end{bmatrix} \right\| = 1$$

Thus, we have:

$$ab + cd = 0$$
$$a^2 + c^2 = 1$$
$$b^2 + d^2 = 1$$

This implies that:

$$c = 1 - \left(\frac{\sqrt{3}}{2}\right)^2$$
$$= \pm\frac{1}{2}$$

Since we must have $c + d > 0$, we let $c = \frac{1}{2}$, so:

$$\frac{\sqrt{3}}{2}b + \frac{1}{2}d = 0$$
$$b = \pm\frac{1}{2}$$
$$d = \pm\frac{\sqrt{3}}{2}$$

Since we want $c + d > 0$, we let $d = \frac{\sqrt{3}}{2}$ and $b = -\frac{\sqrt{3}}{2}$.

Problem 10. (20 pts.)

You are given the following data points:

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Find a polynomial of the form $f(x) = ax + b$ that is the least squares best fit for this data.
Hint: use the formula for computing a projection onto the image of a matrix.

We are looking for an approximate solution to the overdetermined system:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}.$$

We project $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ onto the image of the matrix above:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}^{\top} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}^{\top} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 2 \\ 5/2 \end{bmatrix}.$$

Now, if we replace $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ with its projection $\begin{bmatrix} 3/2 \\ 2 \\ 5/2 \end{bmatrix}$, the system is no longer overdetermined:

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3/2 \\ 2 \\ 5/2 \end{bmatrix}.$$

The approximate solution is then:

$$f(x) = \frac{1}{2}x + 2.$$

Problem 11. (25 pts.)

Consider the following vector subspaces of $\mathbb{R}^3$:

$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\} \qquad W = \text{span}\left\{ \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix} \right\}$$

You are in a spaceship positioned at $\begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix}$ and a signal transmitter is positioned at $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(a) Suppose the transmitter's signal beam only travels in two opposite directions along V. What is the shortest distance your spaceship must travel to intercept the signal beam?

The closest interception point is the projection of the ship's position onto V. First, we find the unit vector that spans V:

$$u = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{-2}{\sqrt{5}} \end{bmatrix}.$$

Next, we project the spaceship's position onto $\text{span}u$ using the formula $(u \cdot v) \cdot u$:

$$p = \left(\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{-2}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix} \right) \cdot \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{-2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 16 \\ 0 \\ -32 \end{bmatrix}.$$

Finally, the distance is

$$\left\| \begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix} - \begin{bmatrix} 16 \\ 0 \\ -32 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 4 \\ 15 \\ 2 \end{bmatrix} \right\| = \sqrt{245} = 7\sqrt{5}$$

(b) Suppose the transmitter is also rotating around the axis collinear with W. What is the shortest distance your spaceship must travel to reach a position at which it can intercept the signal beam?

Because the transmitter is rotating, the beam sweeps around and covers a plane that is perpendicular to the axis of rotation. The closest interception point is the projection of the ship's position onto $W^\perp$. We have that $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ is orthogonal to W. We find another vector orthogonal to W and $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$:

$$
\begin{aligned}
-4x + y + 2z &= 0 \\
x - 2z &= 0 \\
z &= \frac{1}{2}x \\
y &= 3x
\end{aligned}
$$

Let $x = 2$. Thus, we want to project the spaceship's position onto the image of:

$$
M = \begin{bmatrix} 1 & 2 \\ 0 & 6 \\ -2 & 1 \end{bmatrix}.
$$

The projection is then:

$$
\begin{aligned}
p &= M \cdot (M^\top \cdot M)^{-1} \cdot M^\top \cdot \begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix} \\
&= M \cdot \frac{1}{205} \cdot \begin{bmatrix} 41 & 0 \\ 0 & 5 \end{bmatrix} \cdot M^\top \cdot \begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix} \\
&= \frac{1}{205} \cdot \begin{bmatrix} 4280 \\ 3000 \\ -6060 \end{bmatrix}
\end{aligned}
$$

As in part (a), the distance is $\left\| \begin{bmatrix} 20 \\ 15 \\ -30 \end{bmatrix} - \frac{1}{205} \cdot \begin{bmatrix} 4280 \\ 3000 \\ -6060 \end{bmatrix} \right\| \approx \begin{bmatrix} -0.9 \\ 0.4 \\ -0.4 \end{bmatrix}.$

Problem 12. (15 pts.)

Let vectors $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ be system state descriptions that represent population quantities in two locations (e.g., x is the number of people in the city and y is the number of people in the suburbs). Let f represent a change in the quantities over the course of one year if the economy is doing well, and let g represents a change in the quantities over the course of one year if the economy is not doing well:

$$f(v) = \begin{bmatrix} 60 & 120 \\ 20 & 40 \end{bmatrix} \cdot v \qquad g(v) = \begin{bmatrix} 0.06 & 0.12 \\ 0.02 & 0.04 \end{bmatrix} \cdot v$$

If the initial state is $v_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and after 100 years the state is $\begin{bmatrix} 300000 \\ 100000 \end{bmatrix}$, during how many years of this period was the economy doing well?

We first determine the eigenvalue of v_0 for both matrices:

$$\begin{bmatrix} 60 & 120 \\ 20 & 40 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 300 \\ 100 \end{bmatrix} = 100 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.06 & 0.12 \\ 0.02 & 0.04 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix} = 0.1 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Thus, if the economy after 100 years is in the state $\begin{bmatrix} 300000 \\ 100000 \end{bmatrix}$, the following equations represents this constraint (we do not care about the order because multiplication of scalars is commutative, and we have replaced all matrices in the term with scalars):

$$100^n \cdot 0.1^m \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 300000 \\ 100000 \end{bmatrix}$$
$$n + m = 100$$

The first equation can be simplified:

$$10^{2n} \cdot 10^{-m} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 10^5 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$10^{2n} \cdot 10^{-m} = 10^5$$
$$2n - m = 5$$

Given two equations with two unknowns, we can now solve for n and m:

$$2n - (100 - n) = 5$$
$$3n = 105$$
$$n = 35$$
$$m = 65$$

Thus, there were 35 years of a good economy and 65 years of a poor economy (in some order).