Special Matrices

Monday, September 30, 2024 10:39

Normal matrix: A matrix that commutes with its Hermitian conjugate

$$\hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A} \quad \leftarrow [\hat{A},\hat{A}^\dagger]=0$$

Example

$$\hat{A} = \begin{pmatrix} 1 & 1+i & -1 \\ -1 & 1 & 1+i \\ 1+i & -1 & 1 \end{pmatrix}$$

Its Hermitian conjugate is

$$\hat{A}^\dagger = \begin{pmatrix} 1 & -1 & 1-i \\ 1-i & 1 & -1 \\ -1 & 1-i & 1 \end{pmatrix}$$

Notice that:

$$\hat{A}\hat{A}^\dagger = \begin{pmatrix} 4 & -1+2i & -1-2i \\ -1-2i & 4 & -1+2i \\ -1+2i & -1-2i & 4 \end{pmatrix}$$

$$\hat{A}^\dagger\hat{A} = \begin{pmatrix} 4 & -1+2i & -1-2i \\ -1-2i & 4 & -1+2i \\ -1+2i & -1-2i & 4 \end{pmatrix}$$

Eigenvalues & eigenvectors of normal matrices

Let $\hat{A}$ be a normal matrix & consider the eigenvector-eigenvalue equation:

$$\hat{A}|x\rangle = \lambda|x\rangle$$

$$(\hat{A}-\lambda\hat{\mathbb{1}})|x\rangle = 0$$

Now, let us take the Hermitian conjugate.

$$\left[(\hat{A}-\lambda\hat{\mathbb{1}})|x\rangle = 0\right]^\dagger$$

$$\langle x|(\hat{A}-\lambda\hat{\mathbb{1}})^\dagger = 0$$

$$\langle x|(\hat{A}^\dagger-\lambda^*\hat{\mathbb{1}}) = 0$$

The inner-product with the given relation

$$\langle x|(\hat{A}^\dagger-\lambda^*\hat{\mathbb{1}})(\hat{A}-\lambda\hat{\mathbb{1}})|x\rangle = 0$$

$$\langle x|(\hat{A}^\dagger\hat{A}-\lambda\hat{A}^\dagger-\lambda^*\hat{A}+\lambda^*\lambda\hat{\mathbb{1}})|x\rangle = 0$$

But $\hat{A}$ is normal : $\hat{A}^\dagger\hat{A} = \hat{A}\hat{A}^\dagger$

$$\langle x|(\hat{A}\hat{A}^\dagger-\lambda\hat{A}^\dagger-\lambda^*\hat{A}+\lambda^*\lambda\hat{\mathbb{1}})|x\rangle = 0$$

$$\langle x|\left[(\hat{A}-\lambda\hat{\mathbb{1}})\hat{A}^\dagger-(\hat{A}-\lambda\hat{\mathbb{1}})\lambda^*\right]|x\rangle = 0$$

$$\langle x|\left[(\hat{A}-\lambda\hat{\mathbb{1}})(\hat{A}^\dagger-\lambda^*\hat{\mathbb{1}})\right]|x\rangle = 0$$

split the inner-product,

$$(\hat{A}^\dagger-\lambda^*\hat{\mathbb{1}})|x\rangle = 0 \quad \text{or}$$

$$\hat{A}^\dagger|x\rangle = \lambda^*|x\rangle$$

> For normal matrices $\hat{A}$:
> - The eigenvalues of $\hat{A}^\dagger$ are complex conjugates of the eigenvalues of $\hat{A}$
> - The two matrices $\hat{A}^\dagger$ and $\hat{A}$ share the same set of eigenvectors $\{|x\rangle\}$

Let us consider the two solutions to eigenvalue-eigenvector relation (still for normal matrices)

$$\hat{A}|x_i\rangle = \lambda_i|x_i\rangle$$
$$\hat{A}|x_j\rangle = \lambda_j|x_j\rangle \qquad (\lambda_i \neq \lambda_j)$$

Consider the second equation

$$\langle x_i|\left(\hat{A}|x_j\rangle = \lambda_j|x_j\rangle\right)$$

$$\langle x_i|\hat{A}|x_j\rangle = \lambda_j\langle x_i|x_j\rangle$$

this is $(\hat{A}^\dagger|x_i\rangle)^\dagger$ and since $\hat{A}$ is normal

& $\hat{A}|x_i\rangle = \lambda_i|x_i\rangle$

then $\hat{A}^\dagger|x_i\rangle = \lambda_i^*|x_i\rangle$ or

$$\left[\hat{A}^\dagger|x_i\rangle = \lambda_i^*|x_i\rangle\right]^\dagger$$

$$\langle x_i|\hat{A} = \lambda_i\langle x_i|$$

$$\lambda_i\langle x_i|x_j\rangle = \lambda_j\langle x_i|x_j\rangle$$

$$(\lambda_i-\lambda_j)\langle x_i|x_j\rangle = 0$$

since $\lambda_i \neq \lambda_j$

$$\langle x_i|x_j\rangle = 0$$

> - The distinct eigenvectors of a normal matrix forms an orthogonal set.

Q7. Verify that the eigenvectors are orthogonal for the normal matrix

$$\hat{A} = \begin{pmatrix} 1 & 1+i & -1 \\ -1 & 1 & 1+i \\ 1+i & -1 & 1 \end{pmatrix}$$

Q8. Verify that $\hat{A}^\dagger$ in Q7 has the same set of eigenvectors but its eigenvalues are complex conjugates of that in $\hat{A}$.

Eigenvalues of an inverse of a matrix

Given a matrix $\hat{A}$ with an inverse $\hat{A}^{-1}$, we have

$$\hat{A}|x\rangle = \lambda|x\rangle$$

$$\hat{A}^{-1}\left[\hat{A}|x\rangle = \lambda|x\rangle\right]$$

$$\underbrace{\hat{A}^{-1}\hat{A}}_{\hat{\mathbb{1}}}|x\rangle = \lambda\hat{A}^{-1}|x\rangle$$

Thus,

$$\hat{A}^{-1}|x\rangle = \frac{1}{\lambda}|x\rangle$$

> - $\hat{A}$ & $\hat{A}^{-1}$ share the same set of eigenvectors $\{|x\rangle\}$
> - The eigenvalues of $\hat{A}^{-1}$ is equal to the reciprocal of the eigenvalues of $\hat{A}$
>
> note: $\Rightarrow$ If at least one of the eigenvalues of $\hat{A}$ is zero, then $\hat{A}^{-1}$ cannot exist.

Hermitian and Anti-Hermitian matrices

Hermitian matrix : $\hat{H} = \hat{H}^\dagger$

Anti-Hermitian matrix : $\hat{\mathbb{H}} = -\hat{\mathbb{H}}^\dagger$

Both are normal matrices → their eigenvectors are orthogonal

$$\hat{H}\hat{H}^\dagger = \hat{H}\hat{H} = \hat{H}^\dagger H$$
$$\hat{H}\hat{H}^\dagger = \hat{H}^\dagger\hat{H} \leftarrow \text{normal}$$

Consider : $\hat{H}|x\rangle = \lambda|x\rangle$

since $\hat{H}$ is also a normal matrix, then $\hat{H}^\dagger|x\rangle = \lambda^*|x\rangle$

but $\hat{H}$ is Hermitian ($\hat{H}^\dagger = \hat{H}$), so $\hat{H}|x\rangle = \lambda^*|x\rangle$

Notice the LHS of the starting equation is the same as

Thus,

$$\lambda|x\rangle = \lambda^*|x\rangle \quad \text{or}$$

$$(\lambda-\lambda^*)|x\rangle = 0$$

not a zero vector

$$\therefore \quad \lambda = \lambda^* \quad \text{(what does this mean?)}$$

> - The eigenvalues of a Hermitian matrix are real-valued numbers.

Consider : $\hat{\mathbb{H}}|x\rangle = \lambda|x\rangle$

Since $\hat{H}$ is also a normal matrix : $\hat{\mathbb{H}}^\dagger|x\rangle = \lambda^*|x\rangle$

Note that $\mathbb{H}$ is anti-Hermitian ($\hat{\mathbb{H}} = -\hat{\mathbb{H}}^\dagger$) : $-\hat{\mathbb{H}}|x\rangle = \lambda^*|x\rangle$

$$\hat{\mathbb{H}}|x\rangle = -\lambda^*|x\rangle$$

Compare with the starting line

$$\lambda|x\rangle = -\lambda^*|x\rangle$$

$$\lambda = -\lambda^* \quad \text{(when will this happen?)}$$

> - The eigenvalues of an anti-Hermitian matrix are pure imaginary numbers.

Q9. Revisit our old friends $\{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$

a) Check if they are Hermitian.

b) Obtain their eigenvalues & eigenvectors

Unitary matrices

$$\hat{U}^\dagger = \hat{U}^{-1}$$

Unitary matrices are also normal matrices

$$\hat{U}^\dagger\hat{U} = \hat{U}^{-1}\hat{U} = \hat{\mathbb{1}}$$
$$\hat{U}\hat{U}^\dagger = \hat{U}U^{-1} = \hat{\mathbb{1}}$$

It follows that

$$\hat{U}^\dagger\hat{U} = \mathbb{1}$$

$$\det\left[\hat{U}^\dagger\hat{U} = \mathbb{1}\right]$$

$$\det(\hat{U}^\dagger\hat{U}) = \det\mathbb{1}$$

$$\det(\hat{U}^\dagger)\det(\hat{U}) = \det(\mathbb{1})$$

$$\det(U^\dagger)\cdot\det(\hat{U}) = 1$$

Recall

$$\det(\hat{U}^\dagger) = \det(\hat{U})^*$$

$$\det(\hat{U})^*\det(\hat{U}) = 1$$

$$|\det(U)|^2 = 1$$

> The modulus of the determinant of a unitary matrix is 1

Note: A complex number z with $|z| = 1$ can always be represented as

$$z = \exp(i\delta)$$

where δ is real-valued.

Thus, for a unitary matrix $\hat{U}$, we have

$$\det(\hat{U}) = \exp(i\delta)$$

Consider the relation :

$$\hat{U}|x\rangle = \lambda|x\rangle$$

$$\hat{U}^{-1}\left[\hat{U}|x\rangle = \lambda|x\rangle\right]$$

$$\mathbb{1}|x\rangle = \lambda\hat{U}^{-1}|x\rangle$$

$\hat{U}$ is unitary, ($\hat{U}^{-1} = \hat{U}^\dagger$)

$$|x\rangle = \lambda\underbrace{\hat{U}^\dagger|x\rangle}$$

$$|x\rangle = \lambda\lambda^*|x\rangle$$

or

$$(1-\lambda\lambda^*)|x\rangle = 0$$

since $|x\rangle$ is not a zero vector, then

$$\lambda\lambda^* = 1$$
$$|\lambda|^2 = 1$$

> - The eigenvalues of a unitary matrix are complex numbers with modulus equal to 1.

Thus, $\lambda = \exp(i\alpha)$ where α is a real number

In quantum mechanics, most transformations are represented mathematically as unitary operators.

→ unitary operators preserve the norm of a quantum state

Recall that $\| |x\rangle \|^2 = \langle x|x\rangle$

Let $|y\rangle = \hat{U}|x\rangle$

$$\left[|y\rangle = \hat{U}|x\rangle\right]^\dagger$$

$$\langle y| = \langle x|\hat{U}^\dagger$$

Get the norm of $|y\rangle$

$$\langle y|y\rangle = (\langle x|\hat{U}^\dagger)(\hat{U}|x\rangle)$$

$$= \langle x|\hat{U}^\dagger\hat{U}|x\rangle \quad \text{but } \hat{U} \text{ is unitary}$$

$$= \langle x|\hat{\mathbb{1}}|x\rangle$$

$$\langle y|y\rangle = \langle x|x\rangle$$

> - The norm (square) of $|x\rangle$ is preserved under unitary transformation.

Q10. Using the matrix form of $\hat{U} = \exp(-i\theta_y\hat{\sigma}_y)$

1) Show that $\hat{U}$ is unitary.

2) Let $|v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$ and

consider another vector $|v_2\rangle = \hat{U}|v_1\rangle$

Demonstrate explicitly that

a) $|v_1\rangle$ is not equal to $|v_2\rangle$

b) the norm of $|v_1\rangle$ is preserved under unitary transformation.

Commuting matrices

Let $\hat{A}$ and $\hat{B}$ be two commuting matrices:

$$[\hat{A},\hat{B}] = 0 \quad \text{or} \quad \hat{A}\hat{B} = \hat{B}\hat{A}$$

We start with the eigenvalue-eigenvector equation for $\hat{A}$:

$$\hat{A}|x\rangle = \lambda|x\rangle$$

$$\hat{B}\left[\hat{A}|x\rangle = \lambda|x\rangle\right]$$

$$\hat{B}\hat{A}|x\rangle = \lambda\hat{B}|x\rangle$$

$\hat{A}$ & $\hat{B}$ commutes

$$\hat{A}\hat{B}|x\rangle = \lambda\hat{B}|x\rangle$$

$$\hat{A}|y\rangle = \lambda|y\rangle$$

compare this with

$$\hat{A}|x\rangle = \lambda|x\rangle$$

→ is $|y\rangle$ a new eigenvector of $\hat{A}$ with some eigenvalue λ? No.

→ The eigenvector of $\hat{A}$ for the eigenvalue λ is unique up to some scaling factor.

→ $|y\rangle$ is not a new eigenvector, it is only proportional to $|x\rangle$

$$|y\rangle = \mu|x\rangle$$

recall that $|y\rangle = \hat{B}|x\rangle$. It follows that

$$\hat{B}|x\rangle = \mu|x\rangle$$

This is an eigenvalue-eigenvector relation for $\hat{B}$

> - Two commuting operators share the same set of eigenvectors

- In quantum mechanics, commuting operators correspond to compatible observables.
  - Ex. In a spherically symmetric system, the Hamiltonian (energy operator) is compatible with the angular momentum operator.
- Non-compatible observables correspond to non-commuting operators. Measuring one will mess up the outcome of the other.
  - Ex $[x, \vec{p}_x] = i\hbar\mathbb{1} \longrightarrow \Delta x\cdot\Delta p_x \geq \frac{\hbar}{2}$