Linear differential operator

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Let $\mathcal{L}$ be a linear differential operator and $f(x)$ is a known function.
We seek to solve the equation

$\mathcal{L}\,y(x) = f(x)$; $y(x)$ is the unknown function subject to some boundary conditions.

In words, $\mathcal{L}$ acts on an unknown function $y(x)$ to give a known function $f(x)$

Some comments:

linear → let $g_1(x)$ and $g_2(x)$ be some nontrivial functions, and α_1 and α_2 be some arbitrary constants.

$\mathcal{L}$ is a linear operator when

$$\boxed{\mathcal{L}[\alpha_1 g_1(x)+\alpha_2 g_2(x)] = \mathcal{L}[\alpha_1 g_1(x)] + \mathcal{L}[\alpha_2 g_2(x)] = \alpha_1 \mathcal{L} g_1(x) + \alpha_2 \mathcal{L} g_2(x)}$$

Examples of nonlinear operator

* $\mathcal{L}[\alpha_1 g_1(x)+\alpha_2 g_2(x)] = \alpha_1^* \mathcal{L} g_1(x) + \alpha_2^* \mathcal{L} g_2(x)$ (anti-linear)
* $\mathcal{L}[\alpha_1 g_1(x)+\alpha_2 g_2(x)] = \beta_1 \mathcal{L} g_1(x) + \beta_2 \mathcal{L} g_2(x) + c(x)$

Differential operator → $\mathcal{L}$ performs differentiation (of some finite order) on the unknown function $y(x)$.

Some examples of differential operators

a) $\mathcal{L}\,y(x) = \frac{d^2}{dx^2}y(x)$

b) $\mathcal{L}\,y(x) = (2x+1)\frac{d^3}{dx^3}y(x) + x^2\frac{d^2}{dx^2}y(x)$

c) $\mathcal{L}\,y(x) = [y(x)+2x]\frac{d}{dx}y(x)$

Let's explore example (a).
Is (a) a linear differential operator?

let $y(x) = \alpha_1 g_1(x) + \alpha_2 g_2(x)$

$\mathcal{L} = \frac{d^2}{dx^2}$

$$\mathcal{L}\,y(x) = \frac{d^2}{dx^2}[\alpha_1 g_1(x)+\alpha_2 g_2(x)]$$
$$= \alpha_1\frac{d^2}{dx^2}g_1(x) + \alpha_2\frac{d^2}{dx^2}g_2(x)$$
$$= \alpha_1\mathcal{L}g_1(x) + \alpha_2\mathcal{L}g_2(x)$$

Q1. Demonstrate that (b) is a linear differential operator.

(c) is NOT a linear differential operator

To show this, let $y(x) = \alpha_1 g_1(x) + \alpha_2 g_2(x)$

Note: (c) $\mathcal{L}[\ ] = \{[\ ]+2x\}\frac{d}{dx}[\ ]$

$$\mathcal{L}\,y(x) = \mathcal{L}[\alpha_1 g_1(x)+\alpha_2 g_2(x)]$$
$$= \{[\alpha_1 g_1(x)+\alpha_2 g_2(x)]+2x\}\frac{d}{dx}[\alpha_1 g_1(x)+\alpha_2 g_2(x)]$$
$$= [\alpha_1 g_1(x)+\alpha_2 g_2(x)+2x]\left(\alpha_1\frac{d}{dx}g_1(x)+\alpha_2\frac{d}{dx}g_2(x)\right)$$
$$= \alpha_1^2 g_1(x)\frac{dg_1}{dx} + \alpha_1\alpha_2 g_1(x)\frac{dg_2}{dx} + \alpha_2\alpha_1 g_2(x)\frac{dg_1}{dx} + \alpha_2^2 g_2(x)\frac{dg_2}{dx} + 2\alpha_1 x\frac{dg_1}{dx} + 2\alpha_2 x\frac{dg_2}{dx}$$

we can try to reduce this to $\alpha_1\mathcal{L}g_1(x) + \alpha_2\mathcal{L}g_2(x)$

$$\alpha_1^2 g_1(x)\frac{dg_1}{dx} + \alpha_1\alpha_2 g_1(x)\frac{dg_2}{dx} + \alpha_2\alpha_1 g_2(x)\frac{dg_1}{dx} + \alpha_2^2 g_2(x)\frac{dg_2}{dx} + 2\alpha_1 x\frac{dg_1}{dx} + 2\alpha_2 x\frac{dg_2}{dx}$$
$$= \alpha_1\left\{[\alpha_1 g_1(x)+2x]\frac{dg_1}{dx}\right\} + \alpha_2\left\{[\alpha_2 g_2(x)+2x]\frac{dg_2}{dx}\right\} + \alpha_1\alpha_2\left[g_1(x)\frac{dg_2}{dx} + g_2(x)\frac{dg_1}{dx}\right]$$

Compare with $\mathcal{L}y = (y+2x)\frac{dy}{dx}$

$$\mathcal{L}[\alpha_1 g_1(x)+\alpha_2 g_2(x)] \neq \alpha_1\mathcal{L}g_1(x) + \alpha_2\mathcal{L}g_2(x)$$

Going back to the original problem

$$\mathcal{L}\,y(x) = f(x)$$

The given function $f(x)$ and the unknown function $y(x)$ are assumed to be elements of some Hilbert space.
Let $\{\phi_n(x)\}_{n=0}^{\infty}$ be a linearly independent set that spans the Hilbert space.

Then,

$f(x) = \sum_{n=0}^{\infty} a_n\phi_n(x)$; a_ns are known

$y(x) = \sum_{n=0}^{\infty} c_n\phi_n(x)$; c_ns are not known

Suppose that each $\phi_n(x)$ satisfies the eigenfunction-eigenvalue relation

$$\mathcal{L}\phi_n(x) = \lambda_n\phi_n(x)$$

where λ_n (eigenvalue) is known constant for each $\phi_n(x)$ (eigenfunction).
Then, by using the linear independence of $\{\phi_n(x)\}_{n=0}^{\infty}$, the unknown function $y(x)$ is solved.

$$\mathcal{L}\,y(x) = f(x)$$
$$\mathcal{L}\left[\sum_{n=0}^{\infty} c_n\phi_n(x)\right] = \sum_{n=0}^{\infty} a_n\phi_n(x)$$
$$\sum_{n=0}^{\infty} c_n\mathcal{L}\phi_n(x) = \sum_{n=0}^{\infty} a_n\phi_n(x)$$
$$\sum_{n=0}^{\infty} c_n\lambda_n\phi_n(x) = \sum_{n=0}^{\infty} a_n\phi_n(x)$$
$$\sum_{n=0}^{\infty} (c_n\lambda_n - a_n)\phi_n(x) = 0$$

Note: $\{\phi_n\}$ is a linearly independent set

$$c_n\lambda_n - a_n = 0$$
$$\Rightarrow \boxed{c_n = \frac{a_n}{\lambda_n}}$$

Consider the integral $\int_a^b dx\, f^*(x)\,\mathcal{L}\,g(x)$. ← $\langle f|\mathcal{L}g\rangle$

Since $\mathcal{L}$ is a linear differential operator, then we can perform integration by parts until no more derivative is acting on $g(x)$

$$\int_a^b dx\, f^*(x)\,\mathcal{L}\,g(x) = \int_a^b dx\,[\mathcal{L}^{\dagger} f(x)]^* g(x) + \text{boundary terms}$$

* A new linear diff operator is now acting on $f(x)$. We call this the adjoint of $\mathcal{L}$.

Example: Let $\mathcal{L} = \alpha\frac{d}{dx}$, for some arbitrary (complex) constant α

$\int u\,dv = uv - \int v\,du$

$$\int_a^b dx\, f^*(x)\,\mathcal{L}\,g(x) = \int_a^b dx\, f^*(x)\,\alpha\frac{dg}{dx}$$
$$= \int_a^b dx\,\frac{dg}{dx}f^*(x)\,\alpha$$
$$= \underbrace{\alpha f^*(x)g(x)\Big|_a^b}_{\text{boundary terms}} - \int_a^b d[\alpha f^*(x)]\,g(x)$$
$$= \text{boundary terms} + \int_a^b dx\left(-\alpha\frac{df^*}{dx}\right)g(x)$$
$$\left(-\alpha^*\frac{df}{dx}\right)^* = \left(\mathcal{L}^{\dagger}f(x)\right)^*$$

Thus, the adjoint of $\mathcal{L} = \alpha\frac{d}{dx}$ is $\mathcal{L}^{\dagger} = -\alpha^*\frac{d}{dx}$

→ If $\mathcal{L} = \mathcal{L}^{\dagger}$, then the linear differential operator $\mathcal{L}$ is said to be self-adjoint

→ If $\mathcal{L} = \mathcal{L}^{\dagger}$ AND the boundary terms are zero, then $\mathcal{L}$ is said to be HERMETIAN.

Q2. Let $\mathcal{L} = \alpha(x)\frac{d}{dx}$ where $\alpha(x)$ is some complex function of real variable x. What is $\mathcal{L}^{\dagger}$?

$$\int_a^b dx\, f^*(x)\,\mathcal{L}\,g(x) = \int_a^b dx\,[\mathcal{L}^{\dagger}f(x)]^* g(x) + \text{boundary terms} \quad \text{(IBP)}$$
$$= \int_a^b dx\,[\mathcal{L}f(x)]^* g(x) + \text{boundary terms} \quad \text{(self-adjoint)}$$
$$= \int_a^b dx\,[\mathcal{L}f(x)]^* g(x) + 0 \quad \text{(Hermitian)}$$