Eigenvectors and eigenvalues

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Let $\hat{A}$ be an $N \times N$ matrix.
Any non-zero vector $|x\rangle$ that satisfy

$$\hat{A}|x\rangle = \lambda |x\rangle$$

for some value λ is called a eigenvector of $\hat{A}$
The number λ is called eigenvalue.

We can rewrite the eigenvalue-eigenvector relation

$$\hat{A}|x\rangle = \lambda|x\rangle \longrightarrow (\hat{A}|x\rangle - \lambda|x\rangle) = 0$$

$$(\hat{A} - \lambda\hat{\mathbb{1}})|x\rangle = 0$$

non-zero vector

→ We do not want $(\hat{A} - \lambda\hat{\mathbb{1}})^{-1}$ to exist. no inverse
→ We force $(\hat{A} - \lambda\hat{\mathbb{1}})$ to be a "singular matrix".
→ Its inverse will not exist when

$\det(\hat{A} - \lambda\mathbb{1}) = 0$ ← characteristic equation/determinant
← secular equation/determinant
⊙ equation to get the eigenvalues

→ The sum of the eigenvalues & $\mathrm{Tr}(\hat{A})$ has the ff relation:

$$\sum_{n=1}^{N} \lambda_n = \mathrm{Tr}(\hat{A})$$

Q5. A triangular $N \times N$ looks like

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ 0 & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{NN} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix}$$

Show that the eigenvalue equation for every triangular matrix has the form

$$(\lambda - A_{11})(\lambda - A_{22})\cdots(\lambda - A_{NN}) = 0.$$

Extraction of eigenvector given the eigenvalue

Consider the matrix

$$\hat{A} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}$$

From Q5, the eigenvalues are $\{2, 3, 4\}$.
Suppose we want to get the eigenvector for the eigenvalue 3

$$\hat{A}|x\rangle = 3|x\rangle$$

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 3\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$2\alpha + 3\beta + \gamma = 3\alpha$$
$$3\beta + 4\gamma = 3\beta \rightarrow \gamma = 0$$
$$4\gamma = 3\gamma \rightarrow \gamma = 0$$

$$2\alpha + 3\beta + 0 = 3\alpha$$
$$3\beta = \alpha$$

It follows that

$$|x\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 3\beta \\ \beta \\ 0 \end{pmatrix} = \beta\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

can be fixed by the normalization condition

$$\| |x\rangle \| = 1$$

$$\beta^* \begin{pmatrix} 3 & 1 & 0 \end{pmatrix} \beta \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = |\beta|^2 (9 + 1 + 0) = 1$$
$$= |\beta|^2\, 10$$

$$|\beta|^2 = \frac{1}{10}$$

$$|\beta| = \frac{1}{\sqrt{10}}$$

$$|x_{\lambda=3}\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

Example:

Find the eigenvalues & eigenvectors of

$$\hat{A} = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$$

Solution: $\hat{A}|x\rangle = \lambda|x\rangle \rightarrow (\hat{A} - \lambda\hat{\mathbb{1}})|x\rangle = 0$

$|x\rangle$ is NOT a zero vector
$(\hat{A} - \lambda\hat{\mathbb{1}})^{-1}$ cannot exist

$$\det(\hat{A} - \lambda\hat{\mathbb{1}}) = 0$$

$$\det\left[\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0$$

$$\begin{vmatrix} 4-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$$-(4-\lambda)(2+\lambda) - (1)(-5) = 0$$
$$-(8 + 2\lambda - \lambda^2) + 5 = 0$$
$$\lambda^2 - 2\lambda - 3 = 0$$
$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = -1 \qquad \lambda = 3$$

$\lambda = -1$ eigenvalue

$$\hat{A}|x\rangle = (-1)|x\rangle$$

$$\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (-1)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\left.\begin{array}{l} 4\alpha - 5\beta = -\alpha \rightarrow 5\alpha - 5\beta = 0 \\ \alpha - 2\beta = -\beta \rightarrow 2\alpha - 2\beta = 0 \end{array}\right\} \alpha - \beta = 0 \text{ or } \alpha = \beta$$

The eigenvector associated with $\lambda = -1$ eigenvalue takes the form:

$$|x\rangle = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The constant α is fixed by normalizing $|x\rangle$

$$\| |x\rangle \| = 1$$

$$\langle x|x\rangle = \alpha^* \begin{pmatrix} 1 & 1 \end{pmatrix} \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$$

$$|\alpha|^2 (1+1) = 1$$

$$|\alpha|^2 = \frac{1}{2}$$

$$|\alpha| = \frac{1}{\sqrt{2}}$$

$$|x_{\lambda=-1}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda = 3$ eigenvalue

$$\hat{A}|x\rangle = 3|x\rangle$$

$$\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 3\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\left.\begin{array}{l} 4\alpha - 5\beta = 3\alpha \rightarrow \alpha - 5\beta = 0 \\ \alpha - 2\beta = 3\beta \rightarrow \alpha - 5\beta = 0 \end{array}\right\} \alpha = 5\beta$$

The eigenvector associated with $\lambda = 3$ eigenvalue takes the form:

$$|x\rangle = \begin{pmatrix} 5\beta \\ \beta \end{pmatrix} = \beta\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

The constant β is fixed by normalizing $|x\rangle$

$$\| |x\rangle \| = 1$$

$$\langle x|x\rangle = \beta^* \begin{pmatrix} 5 & 1 \end{pmatrix} \beta\begin{pmatrix} 5 \\ 1 \end{pmatrix} = 1$$

$$|\beta|^2 (25+1) = 1$$

$$|\beta|^2 = \frac{1}{26}$$

$$|\beta| = \frac{1}{\sqrt{26}}$$

$$|x_{\lambda=3}\rangle = \frac{1}{\sqrt{26}}\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Q6. Given the matrix

$$\hat{A} = \begin{pmatrix} 1 & k & 3 \\ -k & 2 & -k \\ 1 & k & 3 \end{pmatrix}$$

a) Find all the real possible values of k such that all the eigenvalues of $\hat{A}$ are real.
b) With k being real, what are the corresponding eigenvectors?