Determinants

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The determinant of $N \times N$ matrix $\hat{A}$ is another number that we associate with $\hat{A}$

$$\det(\hat{A}) = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1j} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2j} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{ij} & \cdots & A_{iN} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{Nj} & \cdots & A_{NN} \end{vmatrix}$$

How do we calculate $\det(\hat{A})$?

* let us define first minor of the element A_{ij}

– the determinant of the $(N-1)\times(N-1)$ matrix obtained by removing all the elements of the ith row and the jth column of $\hat{A}$

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1j} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2j} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{ij} & \cdots & A_{iN} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{Nj} & \cdots & A_{NN} \end{pmatrix} \rightarrow M_{ij} = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{vmatrix}$$

→ N-1 columns (ith row deleted)
→ N-1 rows (jth row deleted)

* cofactor of the element of A_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Example:

let $\hat{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

What is the minor of A_{32}?
What is the cofactor of A_{32}?

Solution:

$$M_{32} = \begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix}$$

$$C_{32} = (-1)^{3+2} M_{32} = (-1)\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix}$$

Given an $N \times N$ matrix $\hat{A}$,

$$\hat{A} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1j} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2j} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{ij} & \cdots & A_{iN} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{Nj} & \cdots & A_{NN} \end{pmatrix}$$

the determinant of $\hat{A}$ is calculated using the laplace expansion of minors

$$\det(\hat{A}) = \sum_{j=1}^{N} A_{ij} (-1)^{i+j} M_{ij}$$

→ only j is summed, i is fixed

alternatively,

$$\det(\hat{A}) = \sum_{i=1}^{N} A_{ij} (-1)^{i+j} M_{ij}$$

→ only i is summed, j is fixed

Exercise:

$$\hat{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

1) calculate the $\det(\hat{A})$ by expanding the first row.
2) calculate the $\det(\hat{A})$ by expanding the first column.

Properties of determinants

i) on the transpose

$$\det(\hat{A}^T) = \det(\hat{A})$$

ii) on the conjugate and Hermitian conjugate

$$\det(\hat{A}^*) = (\det(\hat{A}))^*$$
$$\det(\hat{A}^\dagger) = (\det(\hat{A}))^*$$

iii) interchanging two rows or two columns changes the sign of the determinant but not its magnitude.

iv) removing factors

→ If all the elements of a single row (or column) of $\hat{A}$ have a common factor λ, then

$$\det(\hat{A}) = \lambda \det(\hat{A}') \quad ; \text{ where } \hat{A}' \text{ is } \hat{A} \text{ with } \lambda \text{ factored out from one row (or one column)}$$

→ If every element of the $N \times N$ matrix $\hat{A}$ is multiplied by a constant factor λ, then

$$\det(\lambda\hat{A}) = (\lambda^N)\det(\hat{A})$$

v) Identical rows or columns.

If any two rows (columns) of $\hat{A}$ are identical or are multiples of one another, then it can be shown that

$$\det(\hat{A}) = 0$$

vi) Adding a constant multiple of one row (column) to another.

The determinant of a matrix is unchanged in value by adding to the elements of one row (column) any fixed multiple of the elements of another row (column).

vii) Determinant of a product

If $\hat{A}$ and $\hat{B}$ are square matrices of the same order then

$$\det(\hat{A}\hat{B}) = \det(\hat{A})\det(\hat{B}) = \det(\hat{B})\det(\hat{A})$$

Also, $\det(\hat{A}\hat{B}\ldots\hat{Z}) = \det(\hat{A})\det(\hat{B})\ldots\det(\hat{Z})$

Q3.

8.3 Using the properties of determinants, solve with a minimum of calculation the following equations for x:

$$\text{(a)} \begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0, \qquad \text{(b)} \begin{vmatrix} x+2 & x+4 & x-3 \\ x+3 & x & x+5 \\ x-2 & x-1 & x+1 \end{vmatrix} = 0.$$

Inverse of a Matrix

The inverse of $\hat{A}$ denoted by $\hat{A}^{-1}$ with the property

$$\hat{A}^{-1}\hat{A} = \hat{\mathbb{1}}_N \text{ and } \hat{A}\hat{A}^{-1} = \hat{\mathbb{1}}_N$$

where $\hat{\mathbb{1}}_N$ is $N \times N$ identity matrix.

The elements of $\hat{A}^{-1}$ is obtained by

$$(\hat{A}^{-1})_{ik} = \frac{(C^T)_{ik}}{\det(\hat{A})} = \frac{C_{ki}}{\det(\hat{A})}$$

where $\hat{C}$ is the cofactor matrix of the elements of $\hat{A}$
$\hat{C}^T$ is the transpose of the cofactor matrix

Example: Find the inverse of

$$\hat{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Solution:

⓪ Find the $\det(\hat{A})$

$$\det(\hat{A}) = A_{11}A_{22} - A_{12}A_{21}$$

if the $\det(\hat{A}) \neq 0$, the inverse exists.
if the $\det(\hat{A}) = 0$, there is no inverse

① construct the cofactor & obtain $\hat{C}^T$

$$C_{11} = (-1)^{1+1}A_{22} \qquad C_{12} = (-1)^{1+2}A_{21}$$
$$C_{21} = (-1)^{2+1}A_{12} \qquad C_{22} = (-1)^{2+2}A_{11}$$

$$\hat{C} = \begin{pmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{pmatrix}$$

$$\hat{C}^T = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Thus,

$$\hat{A}^{-1} = \frac{\hat{C}^T}{\det(\hat{A})} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

►*Find the inverse of the matrix*

$$A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & -2 & -2 \\ -3 & 3 & 2 \end{pmatrix}.$$

We first determine $|A|$:

$$|A| = 2[-2(2) - (-2)3] + 4[(-2)(-3) - (1)(2)] + 3[(1)(3) - (-2)(-3)] = 11. \tag{8.58}$$

This is non-zero and so an inverse matrix can be constructed. To do this we need the matrix of the cofactors, C, and hence C^T. We find

$$C = \begin{pmatrix} 2 & 4 & -3 \\ 1 & 13 & -18 \\ -2 & 7 & -8 \end{pmatrix} \quad \text{and} \quad C^T = \begin{pmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{pmatrix},$$

and hence

$$A^{-1} = \frac{C^T}{|A|} = \frac{1}{11}\begin{pmatrix} 2 & 1 & -2 \\ 4 & 13 & 7 \\ -3 & -18 & -8 \end{pmatrix}. \blacktriangleleft \tag{8.59}$$

Properties of the inverse

i) $(\hat{A}^{-1})^{-1} = \hat{A}$

ii) $(\hat{A}^T)^{-1} = (\hat{A}^{-1})^T$

iii) $(\hat{A}^\dagger)^{-1} = (\hat{A}^{-1})^\dagger$

iv) $(\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1}$

v) $(\hat{A}\hat{B}\ldots\hat{G})^{-1} = \hat{G}^{-1}\ldots\hat{B}^{-1}\hat{A}^{-1}$

Consider the relation

$$(\hat{A}^{-1})\hat{A} = \hat{\mathbb{1}}_N$$

$$\det[(\hat{A}^{-1})\hat{A}] = \det(\hat{\mathbb{1}}_N)$$

$$\det(\hat{A}^{-1})\det(\hat{A}) = 1$$

$$\text{or } \det(\hat{A}^{-1}) = \frac{1}{\det(\hat{A})}$$

Q4. The elements of a matrix $\hat{K}$ is given by

$$K_{ij} = a_i b_j \quad ; \text{ where } a_i \text{ \& } b_j \text{ are real numbers}$$

a) Construct a 2x2 matrix $\hat{S}$

$$\hat{S} = (\mathbb{1}_2 + i\hat{K})(\mathbb{1}_2 - i\hat{K})^{-1}.$$

what are the elements of $\hat{S}$ in terms of $a_1, a_2, b_1, \& b_2$?

b) What is $\hat{S}^{-1}$ in terms of $\hat{K}$?
Show that $\hat{S}^{-1}\hat{S} = \mathbb{1}_2$ explicitly.

Linear independence & inverse of a matrix

If $\vec{A}, \vec{B}$, and $\vec{C}$ are linearly independent vectors, then any vector $\vec{V}$ in 3D space can be written as

$$\vec{V} = \alpha\vec{A} + \beta\vec{B} + \gamma\vec{C}$$

How do you know if a given set of vectors forms a linearly independent set?

let

$$\vec{A} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{B} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{C} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Do they form a linearly independent set?

Apply the definition:

$$\alpha\vec{A} + \beta\vec{B} + \gamma\vec{C} = 0$$

$$\alpha\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\alpha + \beta + \gamma = 0$$
$$\beta = 0$$
$$\beta + \gamma = 0$$

or

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{A}, \vec{B}$, and $\vec{C}$ are linearly independent if and only if the only solution is $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

when this can happen?

When the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ has an inverse!

There is no need to solve for the actual inverse.

→ Just check if the inverse exist.
→ If $\det(M) = 0$, M^{-1} is NOT defined
→ For M^{-1} to exist, we must show that

$$\det(M) \neq 0$$