Matrices: Basic properties

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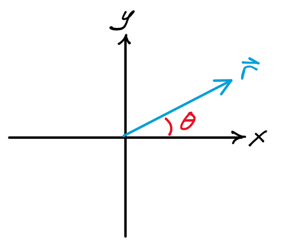
Physical/geometrical motivation

Consider a two-dimensional position vector



$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$

The components of $\vec{r}$ are

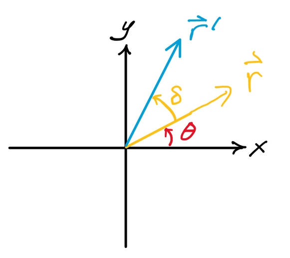
$x = r\cos\theta$

$y = r\sin\theta$

we can write

$|\vec{r}\rangle = \begin{pmatrix} x \\ y \end{pmatrix}$

Let us rotate $\vec{r}$ by an angle δ from its original orientation (no change in length)



$\vec{r}\,' = r\cos(\theta+\delta)\,\hat{i} + r\sin(\theta+\delta)\,\hat{j}$

Rewrite $\vec{r}\,'$ as a column matrix

$|\vec{r}\,'\rangle = \begin{pmatrix} r\cos(\theta+\delta) \\ r\sin(\theta+\delta) \end{pmatrix}$

Goal: retrieve $\vec{r}$ from $\vec{r}\,'$ so we can have a mathematical representation of the rotation.

$$|\vec{r}\,'\rangle = \begin{pmatrix} r\cos(\theta+\delta) \\ r\sin(\theta+\delta) \end{pmatrix}$$

$$= \begin{pmatrix} r\cos\theta\cos\delta - r\sin\theta\sin\delta \\ r\sin\theta\cos\delta + r\cos\theta\sin\delta \end{pmatrix}$$

We recognize $\left.\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right\}$ the original components of $\vec{r}$

$$= \begin{pmatrix} x\cos\delta - y\sin\delta \\ y\cos\delta + x\sin\delta \end{pmatrix}$$

$$= \begin{pmatrix} x\cos\delta - y\sin\delta \\ x\sin\delta + y\cos\delta \end{pmatrix}$$

We notice that entry in our column matrix is a result of an inner product. Consider the 1st element

$$x\cos\delta - y\sin\delta = x\cos\delta + y(-\sin\delta)$$

$$= \begin{pmatrix} \cos\delta & -\sin\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and the 2nd element

$$x\sin\delta + y\cos\delta = \begin{pmatrix} \sin\delta & \cos\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

combining the two operations in one framework, we have

$$|\vec{r}\,'\rangle = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$|\vec{r}\,'\rangle = \hat{R}\,|\vec{r}\rangle$; where R is a 2x2 matrix (rotation matrix)

Remark: a matrix transform a vector

Matrix - array of numbers with M rows & N columns

$$\hat{A} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{MN} \end{pmatrix}$$

- transforms an N-dimensional vector to an M-dimensional vector

The element of $\hat{A}$ in the ith row, jth column is denoted by A_{ij}

Basic Matrix Algebra

$\hat{A}, \hat{B} \rightarrow$ matrices

$x \rightarrow$ vector (column matrix)

$\lambda \rightarrow$ scalar

when a matrix acts on a vector, it gives a transformed vector

$$\hat{A}x = y \quad \text{or} \quad \sum_{j=1}^{N} A_{ij}x_j = y_i$$

From the algebra of vectors, we deduce the algebra of matrices

$$* \sum_{j=1}^{N}(\hat{A}+\hat{B})_{ij}\,x_j = \sum_{j=1}^{N} A_{ij}x_j + \sum_{j=1}^{N} B_{ij}x_j$$

$$\longrightarrow (\hat{A}+\hat{B})_{ij} = A_{ij} + B_{ij}$$

adding two matrices gives a new matrix with elements equal to the algebraic sum of the original two matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & 1+3 \end{pmatrix}$$

$$* \sum_{j=1}^{N}(\lambda\hat{A})_{ij}\,x_j = \lambda\sum_{j=1}^{N} A_{ij}x_j$$

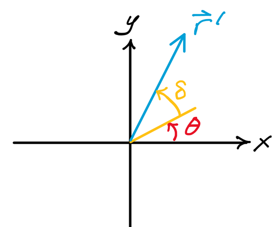
$$\longrightarrow (\lambda\hat{A})_{ij} = \lambda A_{ij} \qquad 2\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & \cdot \\ \cdot & \cdot \end{pmatrix}$$

$$* \sum_{j=1}^{N}(\underbrace{\hat{A}}_{M\times L}\underbrace{\hat{B}}_{L\times N})_{ij}\,x_j = \sum_{k=1}^{L}\hat{A}_{ik}(\hat{B}x)_k = \sum_{j=1}^{N}\sum_{k=1}^{L} A_{ik}B_{kj}x_j$$

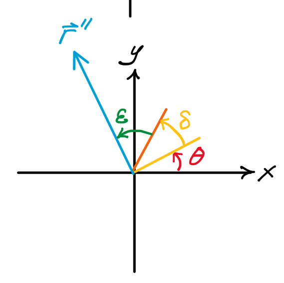
$$\longrightarrow (\hat{A}\hat{B})_{ij} = \sum_{k=1}^{L} A_{ik}B_{kj}$$

* The product of $\hat{A}\,(M\times L)$ & $\hat{B}\,(L\times N)$ gives new $M\times N$ matrix $\hat{A}\hat{B}$ with elements $(\hat{A}\hat{B})_{ij} = \sum_{k=1}^{L} A_{ij}B_{kj}$

Visualizing matrix Multiplication



Suppose we further rotate $\vec{r}\,'$. We want to see the combined effect on the original $\vec{r}$.



$$|\vec{r}\,''\rangle = \begin{pmatrix} r\cos(\theta+\delta+\varepsilon) \\ r\sin(\theta+\delta+\varepsilon) \end{pmatrix}$$

$$= \begin{pmatrix} r[\cos(\theta+\delta)\cos\varepsilon - \sin(\theta+\delta)\sin\varepsilon] \\ r[\sin(\theta+\delta)\cos\varepsilon + \cos(\theta+\delta)\sin\varepsilon] \end{pmatrix}$$

$$= \begin{pmatrix} r\cos(\theta+\delta)\cos\varepsilon - r\sin(\theta+\delta)\sin\varepsilon \\ r\cos(\theta+\delta)\sin\varepsilon + r\sin(\theta+\delta)\cos\varepsilon \end{pmatrix}$$

we recognize that $x' = r\cos(\theta+\delta)$, $y' = r\sin(\theta+\delta)$

$$= \begin{pmatrix} x'\cos\varepsilon - y'\sin\varepsilon \\ x'\sin\varepsilon - y'\cos\varepsilon \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

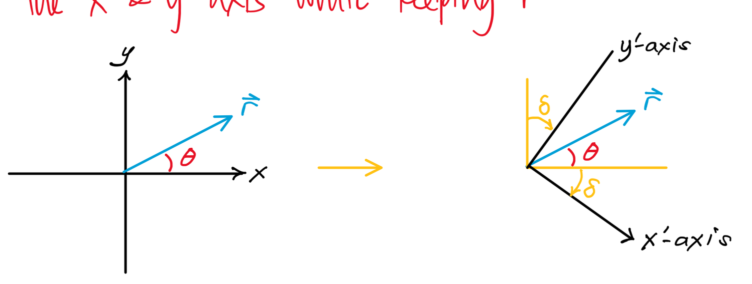
but $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix}\begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$|\vec{r}\,''\rangle = \hat{R}_\varepsilon\hat{R}_\delta\,|\vec{r}\rangle$$

Remark: matrix multiplication can be interpreted as composition of functions $(f\circ g)(x)$

Q3 Suppose that instead of rotating $\vec{r}$, we rotate the x & y axis while keeping $\vec{r}$



Express the transformation of coordinate system in terms of matrices.

How is this different from our "active" transformation of the given vector?

Commutativity & the Commutator

In most cases, matrix multiplication is NOT commutative

$$\hat{A}\hat{B} \neq \hat{B}\hat{A}$$

* suppose $M \neq N$, then an $M\times L$ matrix $\hat{A}$ and an $L\times N$ matrix $\hat{B}$ can give an $M\times N$ matrix $\hat{C} = \hat{A}\hat{B}$. since $M \neq N$, then $\hat{B}\hat{A}$ is NOT defined

* Consider square matrices $(M=N)$

The Pauli matrices (important in the description of spin in Q.M & isospin in nuclear particle physics)

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Notice that $\hat{\sigma}_x\hat{\sigma}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0+i & 0+0 \\ 0+0 & -i+0 \end{pmatrix}$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\hat{\sigma}_y\hat{\sigma}_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0-i & 0+0 \\ 0+0 & i+0 \end{pmatrix}$$

$$= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\hat{\sigma}_x\hat{\sigma}_y \neq \hat{\sigma}_y\hat{\sigma}_x$$

Since $\hat{A}\hat{B}$ is NOT necessarily equal to $\hat{B}\hat{A}$, then one can obtain a new matrix by defining the operation

$$[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \quad \text{(the commutator of } \hat{A} \text{ \& } \hat{B})$$

Note that

$$[\hat{B},\hat{A}] = \hat{B}\hat{A} - \hat{A}\hat{B}$$

$$= -(\hat{A}\hat{B} - \hat{B}\hat{A})$$

$$[\hat{B},\hat{A}] = -[\hat{A},\hat{B}]$$

Going back to the Pauli matrices,

$$\hat{\sigma}_x\hat{\sigma}_y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{\sigma}_y\hat{\sigma}_x = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\sigma}_x\hat{\sigma}_y = i\hat{\sigma}_z \qquad \hat{\sigma}_y\hat{\sigma}_x = -i\hat{\sigma}_z$$

Thus, $[\hat{\sigma}_x,\hat{\sigma}_y] = \hat{\sigma}_x\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_x$

$$= i\hat{\sigma}_z - (-i\hat{\sigma}_z)$$

or $[\hat{\sigma}_x,\hat{\sigma}_y] = 2i\hat{\sigma}_z$

Q4. By considering all possible combinations, show that:

$$[\hat{\sigma}_\ell,\hat{\sigma}_m] = 2i\epsilon_{\ell mn}\,\hat{\sigma}_n$$

$$\epsilon_{\ell mn} = \begin{cases} 1 & \text{for even permutation of } \ell mn \\ -1 & \text{for odd permutation of } \ell mn \\ 0 & \text{for repeated permutation of } \ell mn \end{cases}$$

Not commutative but associative

$$(\hat{A}\hat{B})\hat{C} = \hat{A}(\hat{B}\hat{C})$$

Consider the LHS

$$[(\hat{A}\hat{B})\hat{C}]_{ij} = \sum_k (AB)_{ik}C_{kj}$$

$$= \sum_k\sum_\ell A_{i\ell}B_{\ell k}C_{kj}$$

$$= \sum_\ell A_{i\ell}\Big(\sum_k B_{\ell k}C_{kj}\Big)$$

$$= \sum_\ell A_{i\ell}(\hat{B}\hat{C})_{\ell j}$$

$$= [\hat{A}(\hat{B}\hat{C})]_{ij}$$

Q5 Consider 3 square matrices of the same dimension $\hat{A}$, $\hat{B}$, and $\hat{C}$. By direct expansion and some arrangement, prove the Jacobi identity

$$[\hat{A},[\hat{B},\hat{C}]] = [\hat{B},[\hat{A},\hat{C}]] - [\hat{C},[\hat{A},\hat{B}]].$$

Q6. Show that

$$(\hat{A}+\hat{B})(\hat{A}-\hat{B}) = \hat{A}^2 - \hat{B}^2$$

if and only if $[\hat{A},\hat{B}] = 0$.