Eigenvalue-eigenfunction II

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Classification of 2nd order homogeneous linear O.D.E.

$$\frac{d^2}{dx^2}\phi(x) + P(x)\frac{d}{dx}\phi(x) + Q(x)\phi(x) = 0$$

1) ordinary point of an ODE

$x = x_0$ is an ordinary point when $P(x)$ and $Q(x)$ are single-valued

→ we are guaranteed with two series solution

2) singular point of an ODE

$x = x_0$ when $P(x)$ or $Q(x)$, or both diverge

* regular singular point

$(x-x_0)P(x)$, $(x-x_0)^2 Q(x)$ — must be both single-valued at $x = x_0$

→ at least one series solution around $x = x_0$ is still possible

* irregular singular point/essential singular point

$(x-x_0)P(x)$, $(x-x_0)^2 Q(x)$ — either or both are divergent at $x = x_0$

Table 7.1 Singularities of Some Important ODEs.

Equation	Regular Singularity $x =$	Irregular Singularity $x =$
1. Hypergeometric (canonical form) $x(x-1)y'' + [(1+a+b)x + c]y' + aby = 0$	$0, 1, \infty$	$\cdots$
2. Legendre[a] $(1-x^2)y'' - 2xy' + l(l+1)y = 0$	$-1, 1, \infty$	$\cdots$
3. Chebyshev $(1-x^2)y'' - xy' + n^2y = 0$	$-1, 1, \infty$	$\cdots$
4. Confluent hypergeometric (canonical form) $xy'' + (c-x)y' - ay = 0$	0	∞
5. Bessel $x^2y'' + xy' + (x^2 - n^2)y = 0$	0	∞
6. Laguerre[a] $xy'' + (1-x)y' + ay = 0$	0	∞
7. Simple harmonic oscillator (canonical form) $y'' + \omega^2 y = 0$	$\cdots$	∞
8. Hermite $y'' - 2xy' + 2\alpha y = 0$	$\cdots$	∞

[a] The associated equations have the same singular points.

Consider the Legendre's equation

$$(1-x^2)\frac{d^2}{dx^2}\phi(x) - 2x\frac{d}{dx}\phi(x) + \ell(\ell+1)\phi(x) = 0$$

where ℓ is a fixed number

Standard form

$$\frac{d^2}{dx^2}\phi(x) - \underbrace{\frac{2x}{1-x^2}}_{P(x)}\frac{d}{dx}\phi(x) + \underbrace{\frac{\ell(\ell+1)}{1-x^2}}_{Q(x)}\phi(x) = 0$$

$$P(x) = -\frac{2x}{1-x^2} = -\frac{2x}{(1-x)(1+x)}$$

$$Q(x) = \frac{\ell(\ell+1)}{1-x^2} = \frac{\ell(\ell+1)}{(1-x)(1+x)}$$

not regular at $x = \pm 1$

Notice

$$(1+x)P(x) = -\frac{2x}{1-x}$$

$$(1+x)^2 Q(x) = \frac{(1+x)\ell(\ell+1)}{1-x}$$

regular (single-valued) at $x = -1$

and

$$(1-x)P(x) = -\frac{2x}{1+x}$$

$$(1-x)^2 Q(x) = \frac{(1-x)\ell(\ell+1)}{1+x}$$

regular (single-valued) at $x = +1$

Thus, $x = \pm 1$ are regular singular points of the Legendre equation

To verify if $|x| \to \infty$ is regular, we use substitution

$$w = \frac{1}{x}$$

$$\frac{d\phi}{dx} = \frac{d\phi}{dw}\frac{dw}{dx} = \frac{d\phi}{dw}\left(-\frac{1}{x^2}\right) = -w^2\frac{d\phi}{dw}$$ ✓

$$\frac{d^2\phi}{dx^2} = \frac{d}{dx}\left(\frac{d\phi}{dx}\right) = \frac{dw}{dx}\frac{d}{dw}\left(-w^2\frac{d\phi}{dw}\right)$$

$$= \left(-\frac{1}{x^2}\right)\left(-2w\frac{d\phi}{dw} - w^2\frac{d^2\phi}{dw^2}\right)$$

$$= -w^2\left(-2w\frac{d\phi}{dw} - w^2\frac{d^2\phi}{dw^2}\right)$$ ✓

$$(1-x^2)\frac{d^2}{dx^2}\phi(x) - 2x\frac{d}{dx}\phi(x) + \ell(\ell+1)\phi(x) = 0$$

$$\to w^2(w^2-1)\frac{d^2\phi}{dw^2} + 2w^3\frac{d\phi}{dw} + \ell(\ell+1)\phi = 0$$

$$\to \frac{d^2\phi}{dw^2} + \underbrace{\frac{2w}{w^2-1}}_{\text{regular at } w=0 \text{ (w/c is } |x|\to\infty)}\frac{d\phi}{dw} + \underbrace{\frac{\ell(\ell+1)}{w^2(w^2-1)}}_{\text{blows up at } w=0 \text{ (w/c is at } |x|\to\infty)}\phi = 0$$

this means that $|x| \to \infty$ is a singular point

$$w P(w) = \frac{2w^2}{w^2-1}$$

$$w^2 Q(w) = \frac{\ell(\ell+1)}{w^2-1}$$

both are regular at $w = 0$ ($|x| \to \infty$)

Thus $|x| \to \infty$ is a regular singular point

This is just an FYI.

We just expand at $x=0$ w/c is an ordinary point for Legendre equation

Q6. Verify that the Bessel's equation has a regular singularity at $x=0$ & an irregular singularity at $|x| \to \infty$

Example: Laguerre equation

$$x\frac{d^2}{dx^2}\phi(x) + (1-x)\frac{d}{dx}\phi(x) = \lambda\phi(x)$$

Some remarks:

* not self-adjoint

$P_0(x) = x \qquad P_1(x) = 1-x \neq \frac{dP_0}{dx}$

we need to multiply everything with the weight function

$$\rho(x) = \frac{1}{P_0(x)}\cdot\exp\left[\int\frac{P_1(x)}{P_0(x)}dx\right]$$

$$\rho(x) = \frac{1}{x}\exp\left(\int^x \frac{1-x'}{x'}dx'\right)$$

$$= \frac{1}{x}e^{\int^x(\frac{1}{x'}-1)dx'}$$

$$= \frac{1}{x}e^{(\ln x - x + c)}$$

$$= \frac{1}{x}e^{\ln x}\cdot e^{-x}e^{c}$$

irrelevant constant; $c=0$

$$= \frac{1}{x}\cdot x\cdot e^{-x}$$

$$\boxed{\rho(x) = e^{-x}}$$

Table 18.2 Laguerre Polynomials

$L_0(x) = 1$

$L_1(x) = -x + 1$

$2!\,L_2(x) = x^2 - 4x + 2$

$3!\,L_3(x) = -x^3 + 9x^2 - 18x + 6$

$4!\,L_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24$

$5!\,L_5(x) = -x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120$

$6!\,L_6(x) = x^6 - 36x^5 + 450x^4 - 2400x^3 + 5400x^2 - 4320x + 720$