Counting : distinguishable & indistinguishable cases

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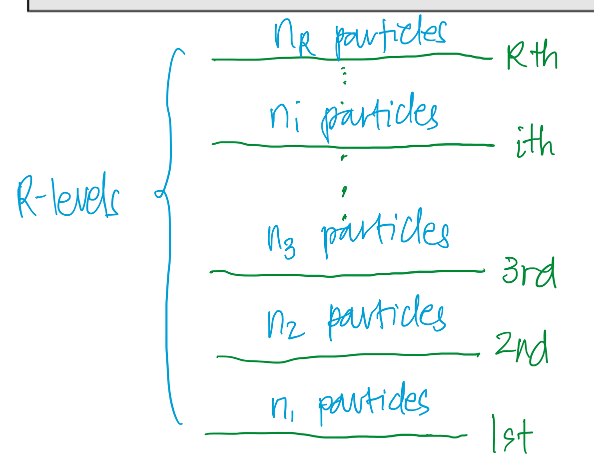
> ►*A system contains a number N of (non-interacting) particles, each of which can be in any of the quantum states of the system. The structure of the set of quantum states is such that there exist R energy levels with corresponding energies E_i and degeneracies g_i (i.e. the ith energy level contains g_i quantum states). Find the numbers of distinct ways in which the particles can be distributed among the quantum states of the system such that the ith energy level contains n_i particles, for $i = 1, 2, \ldots, R$, in the cases where the particles are*
>
> (i) *distinguishable with no restriction on the number in each state;*
> (ii) *indistinguishable with no restriction on the number in each state;*
> (iii) *indistinguishable with a maximum of one particle in each state;*
> (iv) *distinguishable with a maximum of one particle in each state.*



$n_1 + n_2 + n_3 + \ldots n_R = N$

⇒ ignore degeneracy for now

Total number of ways to arrange N distinguishable particles such that

n_1 is in the 1st level
n_2 is in the 2nd level
⋮
n_R is in the Rth level

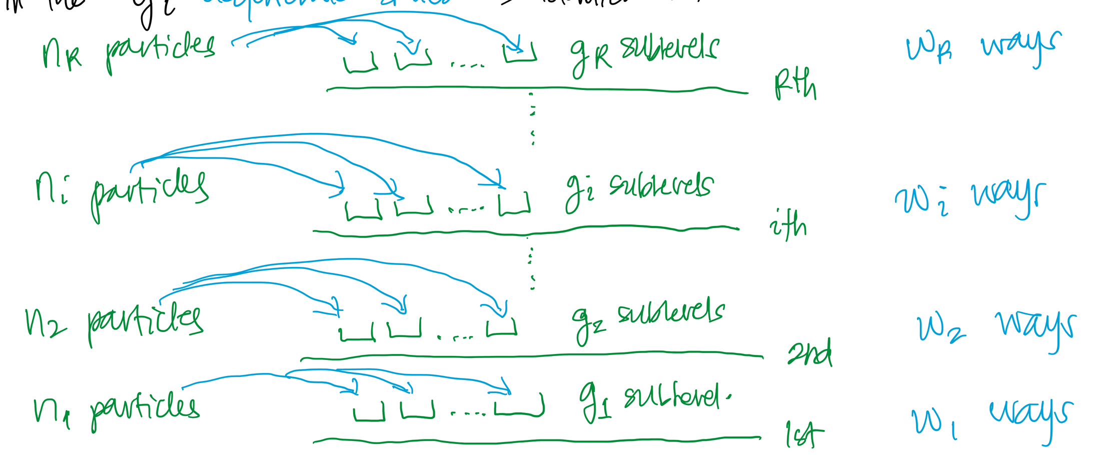
is $\dfrac{N!}{n_1!\, n_2! \ldots n_R!}$

Total number of ways to arrange N indistinguishable particles such that

n_1 is in the 1st level
n_2 is in the 2nd level
⋮
n_R is in the Rth level

is 1

Let w_i be the number of ways to distribute n_i particles in the g_i degenerate states → identical boxes



For distinguishable particles:

$$W\{n_1, n_2, \ldots n_R\} = \frac{N!}{n_1!\, n_2! \ldots n_R!}\, \omega_1 \times \omega_2 \times \omega_3 \cdots \times \omega_R$$

$$= \frac{N!}{n_1!\, n_2! \ldots n_R!} \prod_{i=1}^{R} \omega_i$$

For indistinguishable particles

$$W\{n_1, n_2, \ldots n_R\} = (1) \times \omega_1 \times \omega_2 \times \omega_3 \cdots \times \omega_R$$

$$= \prod_{i=1}^{R} \omega_i$$

How do we calculate ω_i?

(i) * distinguishable
* no restriction on number of particles per sublevel.

→ how many sublevels can the 1st particle occupy? → g_i
→ how many sublevels can the 2nd particle occupy? → g_i
⋮
→ how many sublevels can the n_i^{th} particle occupy? → g_i

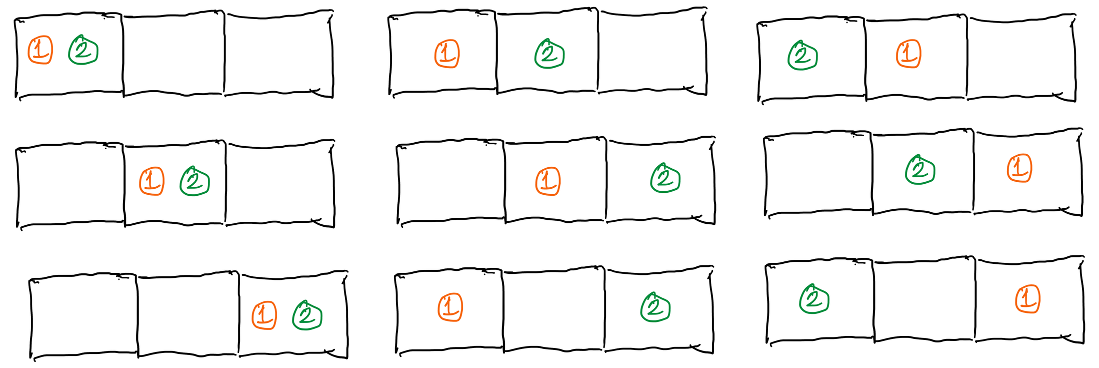
Thus, $\omega_i = g_i^{\,n_i}$

Example: $n_i = 2$ (two distinguishable particles ① ②)
$g_i = 3$ (three degenerate sublevels of the ith level)

→ no restriction on the number of particles per sublevel

$$\omega_i = g_i^{\,n_i} = 3^2 = 9$$

The 9 possible arrangements are



For distinguishable particles with no restriction on the number of particles in each degenerate sublevels:

$$W\{n_1, n_2, n_3, \ldots n_R\} = \frac{N!}{n_1!\, n_2! \ldots n_R!} \prod_{i=1}^{R} g_i^{\,n_i}$$

$$= N! \prod_{i=1}^{R} \frac{g_i^{\,n_i}}{n_i!} \quad \text{(Maxwell-Boltzmann statistics)}$$

(ii) * indistinguishable particles
* no restriction on number of particles in each sublevel

→ n_i : particles
→ $g_i - 1$: sublevel partitions

particles & partitions $= n_i + g_i - 1$

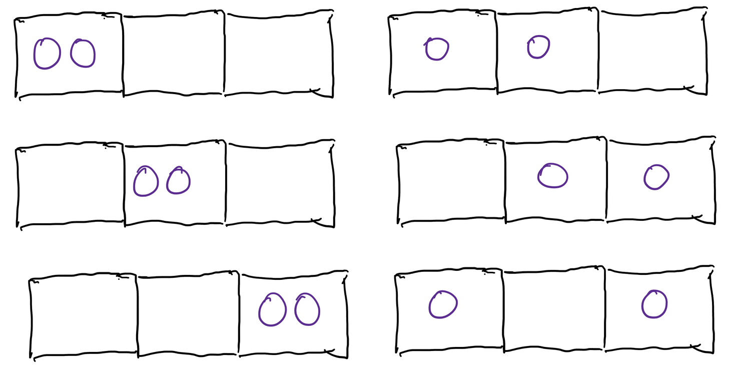
$$\omega_i = \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!}$$

Example: $n_i = 2$ (two indistinguishable particles ○ ○)
$g_i = 3$ (three degenerate sublevels of ith level)

→ no restriction on the number of particles per level

$$\omega_i = \frac{(n_i + g_i - 1)}{n_i!\,(g_i - 1)!} = \frac{(2+3-1)!}{2!\,(3-1)!} = \frac{4!}{2!\,2!} = \frac{\cancel{4}\cdot 3\cdot 2\cdot 1}{\cancel{2}\cdot\cancel{2}}$$

$$\omega_i = 6$$



$$W\{n_1, n_2, \ldots, n_R\} = (1) \prod_{i=1}^{R} \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!}$$

$$= \prod_{i=1}^{R} \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} \quad \text{(Bose-Einstein statistics)}$$

(iii) * indistinguishable particles
* maximum of one particle per sublevel

→ 1st particle ⟶ g_i options
→ 2nd particle ⟶ $g_i - 1$ options
⋮
→ n_i^{th} particle ⟶ $g_i - n_i + 1$ options

$$g_i(g_i - 1) \ldots [g_i - (n_i - 1)] = \frac{g_i(g_i - 1) \ldots [g_i - (n_i - 1)]\,(g_i - n_i)!}{(g_i - n_i)!}$$

$$= \frac{g_i!}{(g_i - n_i)!}$$

But the particles are indistinguishable
→ ordering does not matter
→ there are $n_i!$ possible ordering

we get,

$$\omega_i = \frac{g_i!}{n_i!\,(g_i - n_i)!}$$

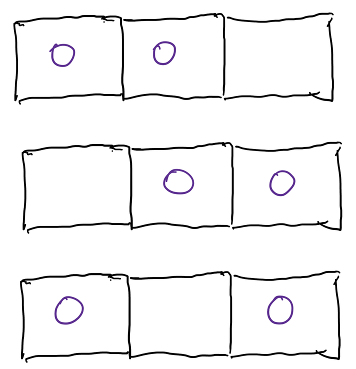
note: For distinguishable particles we do not divide by n_i!

Example: $n_i = 2$ (two indistinguishable particles)
$g_i = 3$ (three degenerate sublevels of the ith level)

→ maximum of one particle per sublevel

$$\omega_i = \frac{g_i!}{n_i!\,(g_i - n_i)!} = \frac{3!}{2!\,(3-2)!} = 3$$

the three possible arrangements are:



thus,

$$W\{n_1, n_2, \ldots, n_R\} = (1) \prod_{i=1}^{R} \frac{g_i!}{n_i!\,(g_i - n_i)!}$$

$$= \prod_{i=1}^{R} \frac{g_i!}{n_i!\,(g_i - n_i)!} \quad \text{(Fermi-Dirac statistics)}$$

(iv) * distinguishable particles but
* maximum of one particle per sublevel

→ 1st particle ⟶ g_i options
→ 2nd particle ⟶ $g_i - 1$ options
⋮
→ n_i^{th} particle ⟶ $g_i - n_i + 1$ options

$$g_i(g_i - 1) \ldots [g_i - (n_i - 1)] = \frac{g_i(g_i - 1) \ldots [g_i - (n_i - 1)]\,(g_i - n_i)!}{(g_i - n_i)!}$$

$$= \frac{g_i!}{(g_i - n_i)!}$$

Note: We do not need to divide by n_i!
→ particles are distinguishable
→ order DOES MATTER

$$\omega_i = \frac{g_i!}{(g_i - n_i)!}$$

Example: $n_i = 2$ (two distinguishable particles ① ②)
$g_i = 3$ (three degenerate sublevels of the ith level)

→ maximum of one particle per sublevel

$$\omega_i = \frac{g_i!}{(g_i - n_i)!} = \frac{3!}{(3-2)!} = 6$$

E1 : tabulate the 6 possible arrangements.

thus,

$$W\{n_1, n_2, \ldots n_R\} = \frac{N!}{n_1!\, n_2! \ldots n_R!} \prod_{i=1}^{R} \frac{g_i!}{(g_i - n_i)!}$$

$$= N! \prod_{i=1}^{R} \frac{g_i!}{n_i!\,(g_i - n_i)!} \quad \text{(no name statistics)}$$