Eigenvalue-eigenfunction (Laguerre)

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Example: Laguerre equation

$$x\frac{d^2}{dx^2}\phi(x) + (1-x)\frac{d}{dx}\phi(x) = \lambda\phi(x)$$

Some remarks:

* not self-adjoint

$P_0(x) = x \qquad P_1(x) = 1-x \neq \frac{dP_0}{dx}$

We need to multiply everything with the weight function

$$\rho(x) = \frac{1}{P_0(x)}\cdot\exp\left[\int^x \frac{P_1(x')}{P_0(x')}dx'\right]$$

$$\rho(x) = \frac{1}{x}\exp\left(\int^x \frac{1-x'}{x'}dx'\right)$$

$$= \frac{1}{x}e^{\int^x(\frac{1}{x'}-1)dx'}$$

$$= \frac{1}{x}e^{(\ln x - x + c)}$$

$$= \frac{1}{x}e^{\ln x}\cdot e^{-x}\cdot e^{c}$$

irrelevant constant ; $C=0$

$$= \frac{1}{x}\cdot x\cdot e^{-x}$$

$$\boxed{\rho(x) = e^{-x}}$$

Note: even if the ODE is not self-adjoint, we can still apply the series solution.

$$x\frac{d^2}{dx^2}\phi(x) + (1-x)\frac{d}{dx}\phi(x) = \lambda\phi(x)$$

Let: $\phi(x) = \sum_{j=0}^{\infty} a_j x^{j+s}$ (expansion about $x=0$)

$x=0$ is a regular singular point

$$\frac{d\phi}{dx} = \sum_{j=0}^{\infty} a_j(j+s)x^{j+s-1}$$

$$\frac{d^2\phi}{dx^2} = \sum_{j=0}^{\infty} a_j(j+s)(j+s-1)x^{j+s-2}$$

substitute to Laguerre

$$x\frac{d^2}{dx^2}\phi(x) + (1-x)\frac{d}{dx}\phi(x) = \lambda\phi(x)$$

$$\sum_{j=0}^{\infty} a_j(j+s)(j+s-1)x^{j+s-1} + (1-x)\sum_{j=0}^{\infty}a_j(j+s)x^{j+s-1} = \lambda\sum_{j=0}^{\infty}a_j x^{j+s}$$

$$\sum_{j=0}^{\infty}\left[a_j(j+s)(j+s-1) + a_j(j+s)\right]x^{j+s-1} = \sum_{j=0}^{\infty}\left[\lambda a_j + a_j(j+s)\right]x^{j+s}$$

Let us take the indicial equation

→ lowest power of x → when $j=0$ → $s-1$

$$a_0[(0+s)(0+s-1) + (0+s)] = 0$$

$$s^2 - s + s = 0$$

$$s^2 = 0$$

$$s = 0$$

⇒ only one series solution is guaranteed.

$$x\sum_{j=2}^{\infty}a_j j(j-1)x^{j-2} + (1-x)\sum_{j=1}^{\infty}a_j j x^{j-1} = \lambda\sum_{j=0}^{\infty}a_j x^j$$

$$\sum_{j=2}^{\infty}a_j j(j-1)x^{j-1} + \sum_{j=1}^{\infty}a_j j x^{j-1} - \sum_{j=1}^{\infty}a_j j x^j = \lambda\sum_{j=0}^{\infty}a_j x^j$$

We then obtain

Laguerre domain $0\le x<\infty$

$$a_{j+1} = \frac{j+\lambda}{(j+1)^2}a_j$$

Consider the case when j is very large

$$a_{j+1} \sim \frac{j}{j^2}a_j$$

$$a_{j+1} \sim \frac{1}{j}a_j$$

Observe

$$a_{j+1+1} \sim \frac{1}{j+1}a_{j+1} = \frac{1}{j+1}\frac{1}{j}a_j$$

$$a_{j+2} \sim \frac{1}{(j+1)j}a_j$$

$$a_{j+3} \sim \frac{1}{(j+2)(j+1)j}a_j$$

If we choose a very large index N, we get

$$a_{j+N} \sim \frac{1}{(j+N-1)(j+N-2)\dots j}a_j$$

$$a_{j+N} > \frac{a_j}{(j+N-1)!} > \frac{a_j}{(j+N)!}$$

Our series solution behaves as an exponential function

$$\phi(x) \sim \sum_{N\gg j} a_j\frac{x^N}{N!} \sim a_j e^x$$

We want our solution to be in a Hilbert space

$$\langle f|g\rangle = \int_0^\infty dx\, f^*(x)e^{-x}g(x)$$

well-defined.

To make normalization possible, we truncate the series

$$a_{n+1}^{(\lambda)} = \frac{n+\lambda}{(n+1)^2}a_n^{(\lambda)} \rightarrow a_{n+1}^{(\lambda)} = 0$$

this gives us

$$n+\lambda = 0$$

$$\lambda = -n$$

Eigenvalues are non-positive integers

$\lambda=0, \quad \phi_0(x) = a_0^{(0)}$

$\lambda=-1 \quad \phi_1(x) = a_0^{(1)} + a_1^{(1)}x$

recall: $a_{j+1}^{(\lambda)} = \frac{j+\lambda}{(j+1)^2}a_j^{(\lambda)}$

$$= a_0^{(1)} + \frac{0+(-1)}{(0+1)^2}a_0^{(1)}x$$

$$= a_0^{(1)}(1-x)$$

$\lambda=-2 \quad \phi_2(x) = a_0^{(2)} + a_1^{(2)}x + a_2^{(2)}x^2$

$$= a_0^{(2)} + \frac{0+(-2)}{(0+1)^2}a_0^{(2)}x + \frac{[1+(-2)]}{(1+1)^2}\cdot\frac{[0+(-2)]}{(0+1)^2}a_0^{(2)}x^2$$

$$= a_0^{(2)}\left[1-2x+\left(-\tfrac{1}{4}\right)(-2)x^2\right]$$

$$= a_0^{(2)}\left(1-2x+\tfrac{1}{2}x^2\right)$$

Table 18.2 Laguerre Polynomials

$L_0(x) = 1$
$L_1(x) = -x+1$
$2!\,L_2(x) = x^2-4x+2$
$3!\,L_3(x) = -x^3+9x^2-18x+6$
$4!\,L_4(x) = x^4-16x^3+72x^2-96x+24$
$5!\,L_5(x) = -x^5+25x^4-200x^3+600x^2-600x+120$
$6!\,L_6(x) = x^6-36x^5+450x^4-2400x^3+5400x^2-4320x+720$

Laguerre equation & Laguerre polynomials

$$x^2\frac{d^2}{dx^2}L_n(x) + (1-x)\frac{d}{dx}L_n(x) = -nL_n(x) \; ; \quad 0\le x<\infty$$

$n = 0, 1, 2, 3\dots$

weight function: e^{-x}

$$\langle L_n|L_m\rangle = \int_0^\infty L_n(x)L_m(x)e^{-x}dx = \delta_{mn}$$

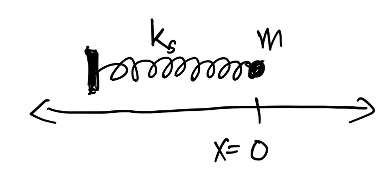
Q7. Consider the Hermite equation

$$\left(\frac{d^2}{dx^2} - 2x\frac{d}{dx}\right)\phi(x) = \lambda\phi(x) \; ; \quad -\infty<x<+\infty$$

a) Obtain the two linearly independent infinite series solution by specifying the corresponding recurrence relation.

b) Why do we need to truncate the infinite series into polynomials?

c) What are the 1st three lowest eigenvalues and their corresponding eigenfunctions?



Hooke's Law

$$F = -k_s x$$

Newton's 2nd law

$$F = ma$$

$$m\ddot{x} = -k_s x$$

$$\ddot{x} = -\frac{k_s}{m}x$$

$$\ddot{x} = -\omega^2 x \; ; \quad \omega^2 = \frac{k_s}{m}$$

angular frequency

$x(t) = A\cos(\omega t+\delta)$ ← position

$\dot{x}(t) = -A\omega\cdot\sin(\omega t+\delta)$ ← velocity

$$F = -\frac{dU}{dx}$$

$$-\frac{dU}{dx} = -kx$$

$U(x) = \frac{1}{2}kx^2$ (potential energy)

$$U(x) = \frac{1}{2}m\omega^2x^2$$

$$\text{kinetic energy} = \frac{p^2}{2m}$$

Total energy of oscillator with mass m

$$\frac{p^2}{2m} + U(x) = E$$

We go quantum: $p \rightarrow i\hbar\frac{d}{dx}$ (1D)

$$p^2 \rightarrow -\hbar^2\frac{d^2}{dx^2}$$

In QM, the particle is now described by a wavefunction $\psi(x)$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}m\omega^2x^2\psi(x) = E\psi(x)$$

$$\left[\frac{d^2}{dx^2} - \frac{m^2\omega^2}{\hbar^2}x^2\right]\psi(x) = -\frac{2mE}{\hbar^2}\psi(x)$$

Let us introduce a dimensionless variable ξ such that $x = \alpha\xi$

Here α contains all the constants that will give us the unit of length.

$$\left[\frac{1}{\alpha^2}\frac{d^2}{d\xi^2} - \frac{\alpha^2m^2\omega^2}{\hbar^2}\xi^2\right]\psi(\xi) = -\frac{2mE}{\hbar^2}\psi(\xi)$$

or

$$\left[\frac{d^2}{d\xi^2} - \frac{\alpha^4m^2\omega^2}{\hbar^2}\xi^2\right]\psi(\xi) = -\frac{2\alpha^2mE}{\hbar^2}\psi(\xi)$$

We are interested in solutions of $\psi(x)$ that are normalizable

$$\psi(x)\rightarrow 0 \quad \text{as} \quad x\rightarrow\pm\infty$$

We let $\psi(\xi) = g(\xi)H_n(\xi)$

← Hermite polynomials

$$\frac{d^2}{d\xi^2}H_n - 2\xi\frac{dH_n}{d\xi} + 2nH_n = 0 \quad \text{for } n=0,1,2\dots$$

Some Hermite polynomials

$$H_0(\xi) = 1$$

$$H_1(\xi) = 2\xi$$

$$H_2(\xi) = 4\xi^2-2$$

$$H_3(\xi) = 8\xi^3-12\xi$$

$\vdots$

Q8. Substitute $\psi(\xi) = g(\xi)H_n(\xi)$ to

$$\left[\frac{d^2}{d\xi^2} - \frac{\alpha^4m^2\omega^2}{\hbar^2}\xi^2\right]\psi(\xi) = -\frac{2\alpha^2mE}{\hbar^2}\psi(\xi)$$

and obtain the following

a) $g(\xi)$

b) the constant α in the substitution $x=\alpha\xi$

c) the eigenvalue energy $E_n = (n+\frac{1}{2})\hbar\omega$

All of these are obtained by mere comparison with the Hermite ODE. Enjoy! :)

Hermite ODE:

$$\frac{d^2}{d\xi^2}H_n - 2\xi\frac{dH_n}{d\xi} + 2nH_n = 0$$

for $n = 0, 1, 2\dots$