cont. + Conditional Probability

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Let A be the union of n mutually exclusive events A_i.
It follows that

$$A \cap B = (A_1 \cup A_2 \cup A_3 \cup \dots A_n) \cap B$$
$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \dots \cup (A_n \cap B)$$

Note that, since the set $\{A_i\}$ are mutually exclusive, then the set $\{A_i \cap B\}_{i=1}^{n}$ are also mutually exclusive,

$$Pr(A \cap B) = \sum_{i=1}^{n} Pr(A_i \cap B)$$

If we let $\{A_i\}$ exhaust the sample space, then, $A \cap B = S \cap B = B$ and

$$Pr(B) = \sum_{i=1}^{n} Pr(A_i \cap B)$$

Conditional Probability

the probability that a particular event occurs given the occurence of another event.

$Pr(B|A)$ – the probability of event B given that event A has occured.

- We already know how to calculate the prob. of the union of events.
- When the conditional probability is known, then the probability of the intersection of events can be calculated:

$$Pr(A \cap B) = Pr(A) \cdot Pr(B|A)$$

equivalently, since $Pr(B \cap A) = Pr(A \cap B)$

$$Pr(B \cap A) = Pr(B) \cdot Pr(A|B).$$

It follows that:

$$Pr(B|A) = \frac{Pr(A \cap B)}{Pr(A)} \quad \& \quad Pr(A|B) = \frac{Pr(B \cap A)}{Pr(B)}$$

If events A and B are mutually exclusive, then

$$Pr(A|B) = Pr(B|A) = 0$$

Two events A and B are statistically independent if

$$Pr(A|B) = Pr(A) \quad \text{or} \quad Pr(B|A) = Pr(B).$$

Thus, for statistically independent events

$$Pr(A \cap B) = Pr(A) \cdot Pr(B)$$

Example:

Find the probability of drawing two aces at random from a pack of cards

i) when the first card is replaced at random into the pack before the second card is drawn.

ii) when the first card is put aside after being drawn

Solution:

sample space: drawing a card a card from a regular pack

$n_S = 52$

event A: drawing an ace : $n_A = 4$

$$Pr(A) = \frac{4}{52}$$

event B: 2nd draw of an ace : $n_B = 4$

i) We are looking for $Pr(A \cap B)$

$$Pr(A \cap B) = Pr(A) \cdot Pr(B|A)$$ ← prob. of drawing a 2nd ace given that the 1st ace is already drawn.

since the 1st drawn ace is replaced, then

$$Pr(B|A) = Pr(B)$$ ← events A & B here are statistically independent

$$Pr(A \cap B) = Pr(A) \cdot Pr(B)$$
$$= \frac{4}{52} \cdot \frac{4}{52}$$
$$= \frac{1}{13} \cdot \frac{1}{13}$$
$$= \frac{1}{169}$$ ← this is with replacement

ii) Here, the 1st drawn card was not replaced

$$Pr(B|A) \neq Pr(B)$$

Now, $n_S = 52 - \underbrace{1}_{\text{1st draw}} = 51$

$n_{B|A} = 4 - \underbrace{1}_{\text{1st card drawn was an ace}} = 3$

$$Pr(B|A) = \frac{n_{B|A}}{n_S} = \frac{3}{51}$$

$$Pr(A \cap B) = \frac{4}{52} \cdot \frac{3}{51}$$ ← this is without replacement
$$= \frac{1}{221}$$

Exercise: Find the probability of drawing an ace in the 1st draw & a jack in the 2nd draw.

i) with replacement of the 1st card drawn
ii) without replacement of the 1st card drawn

Some additional results

1) Consider a collection of statistically independent events $A_1, A_2, \dots A_n$

$$Pr(A_i \cap A_j) = Pr(A_i)\,Pr(A_j) \quad i \neq j$$
$$Pr(A_i \cap A_j \cap A_k) = Pr(A_i)\,Pr(A_j)\,Pr(A_k) \quad i \neq j \neq k$$
$$\vdots$$
$$Pr\left(\cap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} Pr(A_i)$$

2) Let an event A be the union of n mutually exclusive events $\{A_i\}_{i=1}^{n}$. Then

$$Pr(A \cap B) = Pr\left(\left(\cup_{i=1}^{n} A_i\right) \cap B\right)$$
$$= Pr\left(\cup_{i=1}^{n} [A_i \cap B]\right)$$
$$= \sum_{i=1}^{n} Pr(A_i \cap B)$$

Thus,
$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)} = \sum_{i=1}^{n} \frac{Pr(A_i \cap B)}{Pr(B)}$$

$$Pr(A|B) = \sum_{i=1}^{n} Pr(A_i|B)$$

3) Let $\{A_i\}_{i=1}^{n}$ be a set of mutually exclusive events that exhausts the sample space S.

From our previous result:
$$Pr(B) = \sum_{i=1}^{n} Pr(A_i \cap B)$$

Note that we can write
$$Pr(A_i \cap B) = Pr(A_i)\,Pr(B|A_i)$$

Thus
$$Pr(B) = \sum_{i=1}^{n} Pr(A_i)\,Pr(B|A_i) \; ; \quad \{A_i\}_{i=1}^{n} \text{ exhausts } S$$

Example: Fig 30.5 p 113 Riley, et al.

solution on the board

Q3 Read the discussion & example in p. 1131 (Riley, et al.)
Apply that formula to solve Exercise 30.5 in p 1211.
(DO NOT USE THE SOLUTION IN THE SOLUTION MANUAL)

Baye's theorem

Recall that:

$$Pr(B \cap A) = Pr(A \cap B)$$
$$Pr(B)\,Pr(A|B) = Pr(A)\,Pr(B|A)$$

$$Pr(A|B) = \frac{Pr(A)}{Pr(B)}\,Pr(B|A)$$

sometime $Pr(B)$ is not known

we use,
$$Pr(B) = Pr(A)\,Pr(B|A) + Pr(\bar{A})\,Pr(B|\bar{A})$$

$$Pr(A|B) = \frac{Pr(A)\,Pr(B|A)}{Pr(A)\,Pr(B|A) + Pr(\bar{A})\,Pr(B|\bar{A})}$$

Remark:

$$Pr(A|B) \neq Pr(B|A)$$