Degeneracy

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Q11. Prove that if $\hat{U} = \exp(i\hat{H})$ is unitary, then $\hat{H}$ is Hermitian.

Example in [Riley 2006] p. 282

Exercise:
calculate the determinant by using the 2nd row

The eigenvalues are $\lambda = 4$ (multiplicity of 1)
$\lambda = -2$ (multiplicity of 2)

The eigenvector for the eigenvalue $\lambda = +4$ is

$$|x_{\lambda=4}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

We do the eigenvectors for $\lambda = -2$

$$\begin{pmatrix}1 & 0 & 3\\0 & -2 & 0\\3 & 0 & 1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = -2\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$$

$$\alpha + 3\gamma = -2\alpha \rightarrow \gamma = -\alpha$$
$$-2\beta = -2\beta \quad \text{(useless)}$$
$$3\alpha + \gamma = -2\gamma \rightarrow \gamma = -\alpha$$

Note: there will be two independent constants left in our eigenvector: α and β
Under the degenerate $\lambda = -2$ are two independent eigenvectors waiting for us to unravel.

For eigenvalue -2, we have $\begin{pmatrix}\alpha\\\beta\\-\alpha\end{pmatrix}$

Use the fact that α & β are independent constants:

$$\begin{pmatrix}\alpha\\\beta\\-\alpha\end{pmatrix} = \alpha\begin{pmatrix}1\\0\\-1\end{pmatrix} + \beta\begin{pmatrix}0\\1\\0\end{pmatrix}$$

We have identified two eigenvectors.

Exercise: Prove that the three eigenvectors are linearly independent.
$\Rightarrow$ show that $\det(\hat{M}) \neq 0$

Example: Consider a 3-D vector given by

$$|v\rangle = \begin{pmatrix}i\\1\\1-i\end{pmatrix}$$

Express $|v\rangle$ in terms of the normalized eigenvectors of $\hat{H} = \begin{pmatrix}1 & 0 & 3\\0 & -2 & 0\\3 & 0 & 1\end{pmatrix}$

Solution:
We are looking for the coefficients $\{v_n\}_{n=1}^{N}$ such that

$$|v\rangle = v_1|\hat{e}_1\rangle + v_2|\hat{e}_2\rangle + v_3|\hat{e}_3\rangle$$

From the discussion, the normalized eigenvectors of $\hat{H}$ are

$$|\hat{e}_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, \quad |\hat{e}_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad |\hat{e}_3\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

since $\hat{H}$ is Hermitian, the eigenvectors are orthogonal.

$$|v\rangle = v_1|\hat{e}_1\rangle + v_2|\hat{e}_2\rangle + v_3|\hat{e}_3\rangle$$
$$\langle\hat{e}_1|\,[\,|v\rangle = v_1|\hat{e}_1\rangle + v_2|\hat{e}_2\rangle + v_3|\hat{e}_3\rangle\,]$$
$$\langle\hat{e}_1|v\rangle = v_1\underbrace{\langle\hat{e}_1|\hat{e}_1\rangle}_{=1 \text{ normalized}} + v_2\underbrace{\langle\hat{e}_1|\hat{e}_2\rangle}_{=0 \text{ orthogonal}} + v_3\underbrace{\langle\hat{e}_1|\hat{e}_3\rangle}_{=0 \text{ orthogonal}}$$

$$v_1 = \langle\hat{e}_1|v\rangle$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}i\\1\\1-i\end{pmatrix}$$
$$= \frac{1}{\sqrt{2}}[i + 0 + (1-i)]$$
$$= \frac{1}{\sqrt{2}}$$

Exercise: Solve for v_2 and v_3

Thus,

$$|v\rangle = \frac{1}{\sqrt{2}}|\hat{e}_1\rangle + \frac{(-1+2i)}{\sqrt{2}}|\hat{e}_2\rangle + |\hat{e}_3\rangle$$

Remark: the eigenvectors of an $N\times N$ Hermitian matrix form a "complete" basis set.

Defective Matrix

Consider the matrix given by

$$\hat{A} = \begin{pmatrix}1 & 2 & 3 & 4\\0 & 1 & 2 & 3\\0 & 0 & 1 & 2\\0 & 0 & 0 & 2\end{pmatrix}$$

Exercise: Solve for the eigenvectors of $\hat{A}$ $\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}$

eigenvalue $\lambda = 2$
eigenvalue $\lambda = 1$

Remark: The two 4-D eigenvectors

$$\delta\begin{pmatrix}24\\7\\2\\1\end{pmatrix} \text{ and } \alpha\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

cannot span a 4-dimensional vector space.

- The eigenvectors of $\hat{A}$ (4×4) do not form a "complete" basis set for a 4-D vector space.

Q12. Consider a vector

$$|v\rangle = \begin{pmatrix}1+i\\1-i\\2\\1\end{pmatrix}$$

and the matrix

$$\hat{M} = \begin{pmatrix}1 & i\sqrt{8} & 0 & 0\\-i\sqrt{8} & 1 & i\sqrt{8} & 0\\0 & -i\sqrt{8} & 1 & 0\\0 & 0 & 0 & 3\end{pmatrix}$$

We want to express $|v\rangle$ as a superposition of the normalized eigenvectors of $\hat{M}$.

a) Verify that $\hat{M}$ is Hermitian.
b) obtain the normalized eigenvectors of $\hat{M}$.
c) What are the components of $|v\rangle$ with eigenvectors of $\hat{M}$ as the basis?