Eigenvalue-eigenfunctions: Legendre

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Objective: Obtain the eigenvalues and eigenfunctions of our linear differential operator

$$\mathcal{L}\,\phi_n(x) = \lambda_n \phi_n(x)$$

* homogeneous linear differential equation
* λ_n : eigenvalues → to be determined
* $\phi_n(x)$: eigenfunctions → to be determined

Generally, we can write

$$\frac{d^2}{dx^2}\phi_n(x) + p(x)\frac{d}{dx}\phi_n(x) + Q(x)\,\phi_n(x) = 0$$

Mode of solution → power series (Frobenius Method)
– infinite series in powers of $(x-x_0)$
x_0 → expansion point

ordinary (expansion) point → when $P(x)$ and $Q(x)$ are well defined at x_0. no singularity

Example: Legendre Equation

$$\mathcal{L}\,\phi(x) = \lambda\,\phi(x) \quad ; \quad -1 \le x \le 1$$

$$-(1-x^2)\frac{d^2}{dx^2}\phi(x) + 2x\frac{d}{dx}\phi(x) = \lambda\,\phi(x)$$

→ This ODE always appear in physics
e.g. angular part of ∇^2 in spherical form $(x=\cos\theta)$
→ The differential operator is already self-adjoint

$$\mathcal{L} = -(1-x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx}$$

$P_0(x) = -(1-x^2)$, $P_1(x) = 2x$, $P_1(x) = \frac{dP_0(x)}{dx}$

We solve this using the Frobenius Method (series solution)

1) Assume a general series form

$$\phi(x) = x^c \sum_{j=0}^{\infty} a_j x^j \qquad x_0 = 0$$

$$\phi(x) = \sum_{j=0}^{\infty} a_j x^{s+j}$$

s & all the a_j are to be determined $(x-x_0)$

2) Get the needed derivatives

$$\frac{d\phi}{dx} = \sum_{j=0}^{\infty} a_j (s+j)\,x^{s+j-1}$$

$$\frac{d^2\phi}{dx^2} = \sum_{j=0}^{\infty} a_j (s+j)(s+j-1)\,x^{s+j-2}$$

at $j=0$: $(s+j)(s+j-1) = 0$ → $s=0$, $s=1$

or $s(s-1)=0$ (indicial equation)

The indicial equation tells us that there are two solutions to our 2nd order D.E.

solution for $s=0$: $\phi(x) = \sum_{j=0}^{\infty} a_j x^j$

solution for $s=1$: $\phi(x) = \sum_{j=0}^{\infty} a_j x^{j+1}$

3) Obtain the recurrence relation for a_j for each solution for $s=0$

$$-(1-x^2)\sum_{j=2}^{\infty} a_j\, j(j-1)\,x^{j-2} + 2x\sum_{j=1}^{\infty} a_j\, j\, x^{j-1} = \lambda \sum_{j=0}^{\infty} a_j x^j$$

$$\underbrace{-\sum_{j=2}^{\infty} a_j j(j-1) x^{j-2}} + \sum_{j=2}^{\infty} a_j j(j-1) x^j + \sum_{j=1}^{\infty} a_j 2j\, x^j = \lambda \sum_{j=0}^{\infty} a_j x^j$$

offset in terms of powers of x
make this conform with the other terms by shifting the index.

$$-\sum_{j+2=2}^{\infty} a_{j+2}(j+2)(j+2-1)\,x^{(j+2)-2}$$

$$= -\sum_{j=0}^{\infty} a_{j+2}(j+2)(j+1)\,x^j$$

Collecting same powers of x

$$-a_{j+2}(j+2)(j+1) + a_j\, j(j-1) + a_j 2j = a_j\lambda$$

$$a_j\, j(j-1) + a_j 2j - a_j\lambda = (j+2)(j+1)\,a_{j+2}$$

$$a_{j+2} = \frac{j(j-1)+2j-\lambda}{(j+2)(j+1)}\,a_j$$

↳ recurrence relation
you can obtain a_{j+2} by knowing the previous a_j value

Notice from the recurrence relation

$$a_{j+2} = \frac{j(j-1)+2j-\lambda}{(j+2)(j+1)}\,a_j \xrightarrow{j=0} a_2 = \frac{-\lambda}{2\cdot 1}a_0$$

$$\xrightarrow{j=1} a_3 = \frac{2-\lambda}{3\cdot 2}a_1$$

$j=4$: $a_6 = \square\, a_4$; $\frac{a_0}{a_0} = a_0$
$j=5$: $a_7 = \square\, a_5$; $a_7 = \square\, a_1$

Solutions can be separated to odd and even solutions

even solutions : $a_0 \ne 0$, $a_1 = 0$ $(s=0)$
odd solution : $a_0 = 0$, $a_1 \ne 0$ $(s=1)$

The general solution to the Legendre equation is

$$\phi(x) = a_0\,\phi_{\text{even}}(x) + a_1\,\phi_{\text{odd}}(x)$$

⇒ $\phi_{\text{even}}(x)$ and $\phi_{\text{odd}}(x)$ are linearly independent
⇒ Both $\phi_{\text{even}}(x)$ and $\phi_{\text{odd}}(x)$ are infinite series about $x=0$ with the same recurrence relation

The infinite series are convergent only when $|x|<1$ ✓ $\boxed{-1 \le x \le 1}$

Consider the j terms in the series

$$a_{j+2} = \frac{j(j-1)+2j-\lambda}{(j+2)(j+1)}\,a_j$$

$$= \frac{j^2+j-\lambda}{j^2+3j+2}\,a_j$$

large j: $a_{j+2} \sim \frac{j^2}{j^2}a_j$ or $a_{j+2} \sim a_j$

This means that

$$a_0\,\phi_{\text{even}}(x) = a_0 + a_2 x^2 + a_4 x^4 + \dots\dots \underbrace{a_j x^j + a_j x^{j+2} + a_j x^{j+4} + \dots}$$

large j behavior : $a_{j+2} \sim a_j$

when we set $x = \pm 1$

$$a_0\,\phi_{\text{even}}(1) = a_0 + a_2 + a_4 + \dots\dots \underbrace{a_j + a_j + a_j + \dots}$$

these numbers are NOT decreasing. This term will not converge !!!

④ Fix the eigenvalue λ by imposing the series to be finite (polynomial)

Terminate the infinite series ! $\phi_\ell(x) = \sum_{j=0}^{\infty} a_j x^j \to \sum_{j=0}^{\ell} a_j x^j$

ℓ ⇒ max integer (j)

$$a_{\ell+2} = \frac{\ell(\ell-1)+2\ell-\lambda}{(\ell+2)(\ell+1)}\,a_\ell$$

a_ℓ is now the coefficient of the highest power of x

at $j=\ell$ ⇒ the series will stop

$$\ell(\ell-1)+2\ell-\lambda = 0$$

$$\ell^2-\ell+2\ell-\lambda = 0$$

$$\ell^2+\ell-\lambda = 0$$

$$\boxed{\lambda = \ell(\ell+1)} \quad \text{for } \ell = 0, 1, 2, \dots$$

We obtained the eigenvalue in our eigenvalue-eigenfunction problem
⇒ Impose that our solution is finite at $x=\pm 1$

$|x|<1$

Let us list down some eigenvalues and their corresponding un-normalized eigenfunctions.

Note: for each eigenvalue we have one recurrence relation

$$\phi_\ell(x) = \sum_j a_j^{(\lambda)} x^j \quad \leftarrow$$

Note: $\lambda = \ell(\ell+1)$

$\ell = 0 \to \lambda = 0$ $\qquad \phi_0(x) = a_0^{(0)}$

$\ell = 1 \to \lambda = 1(1+1) = 2$ $\qquad \phi_1(x) = a_1^{(2)}x$ ✓

$\ell = 2 \to \lambda = 2(2+1) = 6$ $\qquad \phi_2(x) = a_0^{(6)} + a_2^{(6)}x^2$

⋮

use the recurrence formula

Recall:

$$a_{j+2}^{(\lambda)} = \frac{j(j-1)+2j-\lambda}{(j+2)(j+1)}\,a_j^{(\lambda)}$$

$$\phi_2(x) = a_0^{(6)} + \frac{0(0-1)+2\cdot 0-6}{(0+2)(0+1)}\,a_0^{(6)}x^2$$

$$= a_0^{(6)}(1-3x^2) \;✓$$

$\ell = 3 \to \lambda = 3(3+1) = 12$ $\qquad \phi_3(x) = a_1^{(12)}x + a_3^{(12)}x^3$

$$\phi_3(x) = a_1^{(12)}x + \frac{1(1-1)+2\cdot 1-12}{(1+2)(1+1)}\,a_1^{(12)}x^3$$

$$= a_1^{(12)}\left(x - \frac{5}{3}x^3\right)$$

Notice that all $\phi_\ell(x)$ will have a common factor $\underbrace{a_0^{(\lambda)} \text{ or } a_1^{(\lambda)}}$

the values can be fixed by normalization

Legendre ODE is already self-adjoint, so $p(x)=1$ and

$$\langle f|g\rangle = \int_{-1}^{1} dx\, f^*(x)\,g(x)$$

Q4. List down the $\ell=4$ & $\ell=5$ eigenvalues and obtain the corresponding eigenfunctions. ~~Normalize $\phi_0(x)$ to $\phi_3(x)$~~

Table 15.1 Legendre Polynomials

$P_0(x) = 1$
$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2-1)$ ✓
$P_3(x) = \frac{1}{2}(5x^3-3x)$
$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$
$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$
$P_6(x) = \frac{1}{16}(231x^6-315x^4+105x^2-5)$
$P_7(x) = \frac{1}{16}(429x^7-693x^5+315x^3-35x)$
$P_8(x) = \frac{1}{128}(6435x^8-12012x^6+6930x^4-1260x^2+35)$

$\phi_\ell(x) = a_\ell P_\ell(x)$

Legendre polynomials are usually listed in their unnormalized form.

The orthogonality condition takes the form

$$\langle P_n|P_m\rangle = \int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \frac{2}{2n+1}\,\delta_{nm}$$

Recall $\qquad \ell = 0, 1, 2, \dots$

$$\mathcal{L}\,\phi(x) = \ell(\ell+1)\,\phi(x) \quad ; \quad -1 \le x \le 1$$

$$-(1-x^2)\frac{d^2}{dx^2}\phi(x) + 2x\frac{d}{dx}\phi(x) = \ell(\ell+1)\,\phi(x)$$

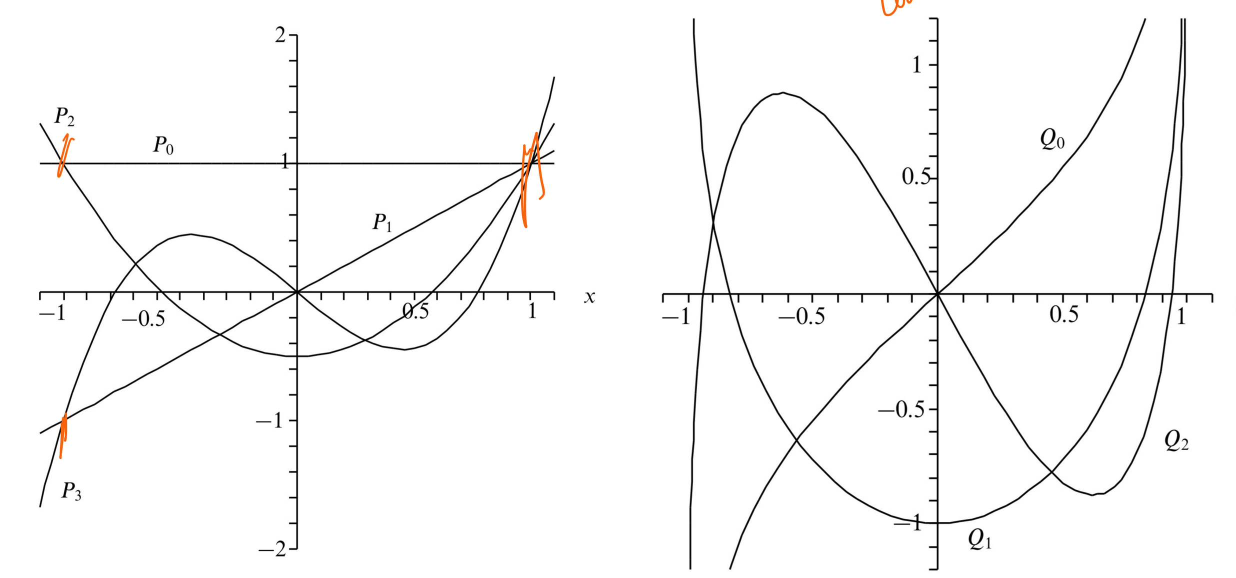
For a given ℓ, the general solution to the Legendre equation is

$$\phi(x) = C_1 P_\ell(x) + C_2 Q_\ell(x)$$

Legendre function of the 2nd kind

P_ℓ:
→ finite polynomial
→ finite-valued for all x in $-1 \le x \le 1$

Q_ℓ:
→ infinite series (continuation of the truncated P_ℓ)
→ finite-valued only for x in $-1 < x < 1$
→ not used in lower physics



Q5. Using the requirement that $P_\ell(x)$ and $Q_\ell(x)$ are linearly independent, obtain the 1st order ODE that can be used to solve for $Q_\ell(x)$ given $P_\ell(x)$.
Obtain a formula to calculate $Q_\ell(x)$ given $P_\ell(x)$

Hint: Extract the Wronskian using only the 2nd order ODE (see page 362 Arfken)

Reduction of order

$$W[f,g] = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf'$$

$W = 0 \to$ linearly Dependent

$W \ne 0 \to$ linearly independent