Vector Spaces

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8.1 Vector spaces

A set of objects (vectors) **a**, **b**, **c**, … is said to form a *linear vector space* V if:

(i) the set is closed under commutative and associative addition, so that

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}, \tag{8.2}$$

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}); \tag{8.3}$$

(ii) the set is closed under multiplication by a scalar (any complex number) to form a new vector $\lambda\mathbf{a}$, the operation being both distributive and associative so that

$$\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}, \tag{8.4}$$

$$(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}, \tag{8.5}$$

$$\lambda(\mu\mathbf{a})=(\lambda\mu)\mathbf{a}, \tag{8.6}$$

where λ and μ are arbitrary scalars;

(iii) there exists a *null vector* $\mathbf{0}$ such that $\mathbf{a}+\mathbf{0}=\mathbf{a}$ for all $\mathbf{a}$;

(iv) multiplication by unity leaves any vector unchanged, i.e. $1\times\mathbf{a}=\mathbf{a}$;

(v) all vectors have a corresponding *negative vector* $-\mathbf{a}$ such that $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$. It follows from (8.5) with $\lambda=1$ and $\mu=-1$ that $-\mathbf{a}$ is the same vector as $(-1)\times\mathbf{a}$.

When λ, μ → are real → real vector space
→ are complex → complex vector space

If V is an N-dimensional vector space then any set of N linearly independent vectors

$$\{\vec{e}_i\}_{i=1}^{N} = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_3\}$$

forms a basis for V.

Thus, for any vector $\vec{v}\in V$, we can write

$$\vec{v}=\sum_{n=1}^{N} v_n \vec{e}_n$$

Note: The basis vector is not unique.
Choosing a different basis will NOT change $\vec{v}$, it will change the components

$$\{e_i'\}_{i=1}^{N}, \quad \vec{v}=\sum_{n=1}^{N} v_n' \vec{e}_n'$$

We may denote a vector using an array of numbers.
Consider an orthogonal basis set $\{\hat{e}_i\}_{i=1}^{N}$, we can write

$$\hat{e}_1=\begin{pmatrix}1\\0\\0\\ \vdots\\0\\0\end{pmatrix}, \quad \hat{e}_2=\begin{pmatrix}0\\1\\0\\0\\ \vdots\\0\end{pmatrix}, \quad \hat{e}_i=\begin{pmatrix}0\\ \vdots\\0\\1\\0\\ \vdots\\0\end{pmatrix} \leftarrow i^{th} \text{ entry}, \quad \hat{e}_N=\begin{pmatrix}0\\0\\ \vdots\\0\\1\end{pmatrix}$$

then a $\vec{v}\in V$

$$\vec{v}=\sum_{n=1}^{N} v_n\hat{e}_n = v_1\hat{e}_1+v_2\hat{e}_2+\dots+v_N\hat{e}_N=\begin{pmatrix}v_1\\v_2\\ \vdots\\v_n\\ \vdots\\v_N\end{pmatrix}$$

The inner product (generalization of scalar product)

The usual scalar product (of real vector space) can be represented by

$$\vec{a}\cdot\vec{b}=a^{T}b=(a_1, a_2 \dots a_N)\begin{pmatrix}b_1\\b_2\\ \vdots\\b_N\end{pmatrix}$$

$$= a_1b_1+a_2b_2+\dots+a_Nb_N$$

where T is for transpose.

For complex vector space, the scalar product is

$$\vec{a}\cdot\vec{b}=a^{\dagger}b=(a_1^*, a_2^* \dots a_N)\begin{pmatrix}b_1\\b_2\\ \vdots\\b_N\end{pmatrix} \qquad \|a\|^2=a^{\dagger}a\geq 0$$

Here, $\dagger$ (dagger) is transpose conjugate.

$$=a_1^*b_1+a_2^*b_2+a_3^*b_3\dots+a_N^*b_N$$

In bra-ket notation,

$$|a\rangle=\begin{pmatrix}a_1\\a_2\\ \vdots\\a_N\end{pmatrix}$$

$$\left[|a\rangle=\begin{pmatrix}a_1\\a_2\\ \vdots\\a_N\end{pmatrix}\right]^{\dagger}$$

$$\langle a| = (a_1^*\ a_2^* \dots a_N^*)$$

$\langle a|b\rangle$ can be complex

$\langle a|a\rangle =$

Gram-Schmidt Orthogonalization

Example: (ordinary vectors)

Form a set of orthonormal vectors from

$$\vec{c}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \vec{c}_2=\begin{pmatrix}1\\1\\2\end{pmatrix}, \quad \vec{c}_3=\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$$\|\vec{c}_1\|=\sqrt{\vec{c}_1\cdot\vec{c}_1}$$

$$\hat{A}=\frac{\vec{A}}{\sqrt{\vec{A}\cdot\vec{A}}}$$

$$\hat{A}\cdot\hat{A}=\frac{\vec{A}}{\sqrt{\vec{A}\cdot\vec{A}}}\cdot\frac{\vec{A}}{\sqrt{\vec{A}\cdot\vec{A}}}=\frac{\vec{A}\cdot\vec{A}}{\vec{A}\cdot\vec{A}}=1$$

Note that the above vectors are not orthogonal.

$$\vec{c}_1\cdot\vec{c}_2=(1\ 1\ 1)\begin{pmatrix}1\\1\\2\end{pmatrix}=1+1+2=4\neq 0$$

$$\vec{c}_1\cdot\vec{c}_3=(1\ 1\ 1)\begin{pmatrix}1\\0\\2\end{pmatrix}=1+0+2=3\neq 0$$

$$\vec{c}_2\cdot\vec{c}_3=(1\ 1\ 2)\begin{pmatrix}1\\0\\2\end{pmatrix}=1+0+4=5\neq 0$$

SOLUTION:

① Normalize the 1st vector:

$$\hat{\mathbb{C}}_1=\frac{\vec{c}_1}{\sqrt{\vec{c}_1\cdot\vec{c}_1}}=\frac{1}{\sqrt{1^2+1^2+1^2}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\hat{\mathbb{C}}_1=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

② Construct a vector, using $\vec{c}_2$, that is orthogonal to $\hat{\mathbb{C}}_1$:

$$\vec{\mathbb{C}}_2=\vec{c}_2-(\hat{\mathbb{C}}_1\cdot\vec{c}_2)\hat{\mathbb{C}}_1$$

$$=\begin{pmatrix}1\\1\\2\end{pmatrix}-\frac{4}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$=\begin{pmatrix}1-\frac{4}{3}\\1-\frac{4}{3}\\2-\frac{4}{3}\end{pmatrix}=\begin{pmatrix}-\frac{1}{3}\\-\frac{1}{3}\\\frac{2}{3}\end{pmatrix}$$

$$\vec{\mathbb{C}}_2=-\frac{1}{3}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$\hat{\mathbb{C}}_1\cdot\vec{c}_2=\left(\frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\right)\begin{pmatrix}1\\1\\2\end{pmatrix}=\frac{1+1+2}{\sqrt{3}}=\frac{4}{\sqrt{3}}$$

Notice that by construction, $\vec{\mathbb{C}}_2$ is already orthogonal to $\hat{\mathbb{C}}_1$

$$\vec{\mathbb{C}}_2=\vec{c}_2-(\hat{\mathbb{C}}_1\cdot\vec{c}_2)\hat{\mathbb{C}}_1$$

$$\vec{\mathbb{C}}_2\cdot\hat{\mathbb{C}}_1=\vec{c}_2\cdot\hat{\mathbb{C}}_1-(\hat{\mathbb{C}}_1\cdot\vec{c}_2)\underbrace{(\hat{\mathbb{C}}_1\cdot\hat{\mathbb{C}}_1)}_{=1}$$

$$=\vec{c}_2\cdot\hat{\mathbb{C}}_1-\hat{\mathbb{C}}_1\cdot\vec{c}_2 \qquad =1 \ \hat{\mathbb{C}}_1 \text{ is already normalized}$$

$$=0$$

③ Normalize $\vec{\mathbb{C}}_2$:

$$\hat{\mathbb{C}}_2=\frac{\vec{\mathbb{C}}_2}{\sqrt{\vec{\mathbb{C}}_2\cdot\vec{\mathbb{C}}_2}}=\frac{-\frac{1}{3}\begin{pmatrix}1\\1\\-2\end{pmatrix}}{\sqrt{\frac{2}{3}}}=-\sqrt{\frac{3}{2}}\cdot\frac{1}{3}\begin{pmatrix}1\\1\\-2\end{pmatrix}=-\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$\vec{\mathbb{C}}_2\cdot\vec{\mathbb{C}}_2=-\frac{1}{3}(1\ 1\ {-2})\left(-\frac{1}{3}\right)\begin{pmatrix}1\\1\\-2\end{pmatrix}=\frac{1}{9}\left[1^2+1^2+(-2)^2\right]=\frac{6}{9}=\frac{2}{3}$$

④ Construct the third vector from $\vec{c}_3$ that is orthogonal to both $\hat{\mathbb{C}}_1$ & $\hat{\mathbb{C}}_2$

$$\vec{\mathbb{C}}_3=\vec{c}_3-\underbrace{(\vec{c}_3\cdot\hat{\mathbb{C}}_2)\hat{\mathbb{C}}_2}_{\text{ensures orthogonality of } \vec{c}_3 \text{ with } \hat{\mathbb{C}}_2}-\underbrace{(\vec{c}_3\cdot\hat{\mathbb{C}}_1)\hat{\mathbb{C}}_1}_{\text{ensures orthogonality of } \vec{c}_3 \text{ with } \hat{\mathbb{C}}_1}$$

$$=\begin{pmatrix}1\\0\\2\end{pmatrix}-\left(-\frac{1}{\sqrt{6}}\right)(1+0-4)\left(-\frac{1}{\sqrt{6}}\right)\begin{pmatrix}1\\1\\-2\end{pmatrix}-\frac{1}{\sqrt{3}}[1+0+2]\left(\frac{1}{\sqrt{3}}\right)\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$=\begin{pmatrix}1\\0\\2\end{pmatrix}+\frac{3}{6}\begin{pmatrix}1\\1\\-2\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\vec{\mathbb{C}}_3=\begin{pmatrix}1/2\\-1/2\\0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

⑤ Last step: normalize $\vec{\mathbb{C}}_3$

$$\hat{\mathbb{C}}_3=\frac{\vec{\mathbb{C}}_3}{\sqrt{\vec{\mathbb{C}}_3\cdot\vec{\mathbb{C}}_3}}=\frac{1}{2\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

Q2. Consider the following non-orthogonal basis set

$$\hat{e}_1=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \quad \hat{e}_2=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \quad \hat{e}_3=\begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \quad \hat{e}_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$$

orthogonalize them by keeping the form of $\vec{e}_1$.