HW6

1. a. $f_p(y) = p^y (1-p)^{1-y}$

$\Rightarrow \log f_p(y) = y \cdot \log p + (1-y)\log(1-p)$

$$\frac{\partial \log f_p(y)}{\partial p} = \frac{y}{p} - \frac{1-y}{1-p}$$

$$\frac{\partial^2 \log f_p(y)}{\partial p^2} = -\frac{y}{p^2} - \frac{1-y}{(1-p)^2}$$

$$I(p) = \mathbb{E}\left[-\frac{\partial^2 \log f_p(y)}{\partial p^2}\right] = \left(\frac{p}{p^2} + \frac{1-p}{(1-p)^2}\right)$$

$$= \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

because $\mathbb{E}Y = p$

$\mathbb{E}(1-Y) = 1-p$

b.

$V(Y_i) = p(1-p)$

$V(\bar{Y}) = \frac{p(1-p)}{n}$

c. $\mathrm{eff} = \frac{1/[I(p)\cdot n]}{V(\bar{Y})} = 1$

$\Rightarrow \hat{p}$ is efficient.

2. $\mathbb{E}Y = \int_0^1 y \cdot \theta \cdot y^{\theta-1} \cdot dy$

$= \theta \cdot \frac{y^{\theta+1}}{\theta+1} \Big|_0^1$

$= \frac{\theta}{\theta+1}.$

$\mathbb{E}Y^2 = \int_0^1 y^2 \cdot \theta \cdot y^{\theta-1} \cdot dy$

$= \theta \cdot \frac{y^{\theta+2}}{\theta+2} \Big|_0^1 = \frac{\theta}{\theta+2}$

$V(Y) = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \frac{\theta}{\theta+2} - \left(\frac{\theta}{\theta+1}\right)^2$

For $\bar{Y}$, $\mathbb{E}(\bar{Y}) = \mathbb{E}Y = \frac{\theta}{\theta+1}$

$V(\bar{Y}) = \frac{V(Y)}{n} = \left(\frac{\theta}{\theta+2} - \left(\frac{\theta}{\theta+1}\right)^2\right) / n$

$\longrightarrow 0$ as $n \to \infty$.

$\Rightarrow$ $\bar{Y}$ is a consistent estimator for $\frac{\theta}{\theta+1}$.