Introduction to Statistics Theory (Fall 2018)

Midterm 2

Name: ________________

Results you may use directly:

- If $Y \sim \mathrm{N}(\mu, \sigma)$, then it has density function

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}.$$

- Values from z-table: $z_{0.05} = 1.64$, $z_{0.025} = 1.96$.

1. Suppose that $\bar{X}$ is the sample mean of n_1 random sample points from a Normal distribution $\mathrm{N}(\mu, \sigma_1^2)$, and $\bar{Y}$ is the sample mean of n_2 random sample points from a Normal distribution $\mathrm{N}(\mu, \sigma_2^2)$. The two samples are independent. Assuming $\sigma_1^2 < \infty$ and $\sigma_2^2 < \infty$, consider the estimator

$$\hat{\mu} = w\bar{X} + (1-w)\bar{Y}$$

with $0 < w < 1$.

(a) Show that $\hat{\mu}$ is an unbiased estimator for μ (10pt).

(b) Find the variance for $\hat{\mu}$ (10pt).

(c) Show that $\hat{\mu}$ is a consistent estimator for μ, as $n_1 \to \infty$ and $n_2 \to \infty$ (5pt).

Since $E\bar{X} = \mu$ and $E\bar{Y} = \mu$.

(a) $E\hat{\mu} = w\mu + (1-w)\mu = \mu$

(b) $var(\bar{X}) = \frac{\sigma_1^2}{n_1}$

$var(\bar{Y}) = \frac{\sigma_2^2}{n_2}$

$var(\hat{\mu}) = w^2 \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}(1-w)^2$

(c) Since $var(\hat{\mu}) \to 0$ as $n_1 \to \infty$, $n_2 \to \infty$

$\hat{\mu}$ is a consistent estimator for μ.

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2. If $Y_1, \ldots, Y_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2)$,

(a) Show that $\sum_{i=1}^n Y_i^2$ and $\sum_{i=1}^n Y_i$ are sufficient statistics for σ^2 (5 pt).

(b) Show that $\sum_{i=1}^n Y_i^2$ and $\sum_{i=1}^n Y_i$ are *minimum* sufficient statistics for σ^2 (5 pt).

(c) Rewrite $s^2 = \frac{1}{n-1}\sum_{i=1}^n (Y_i - \bar{Y})^2$ in terms of $\sum_{i=1}^n Y_i^2$ and $\sum_{i=1}^n Y_i$ (5 pt).

(d) Explain why s^2 is the minimum variance unbiased estimator for σ^2. You can use the fact that $\mathbb{E}s^2 = \sigma^2$ without proving it (5 pt).

(a) the likelihood function is

$$L(Y_1 - Y_n) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\left\{-\frac{\sum_{i=1}^n (Y_i-\mu)^2}{2\sigma^2}\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\left(-\frac{\sum_{i=1}^n Y_i^2 - 2\sum_{i=1}^n Y_i\mu + n\mu^2}{2\sigma^2}\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\left(-\frac{\sum Y_i^2 - 2\sum Y_i + n\mu^2}{2\sigma^2}\right) \cdot 1$$

using factorization theorem.

$\sum Y_i^2$ and $\sum Y_i$ are sufficient for σ^2.

(b) $$\frac{L(Y_1 -- Y_n)}{L(X_1 -- X_n)} = \exp\left\{-\frac{(\sum Y_i^2 - \sum X_i^2) - 2(\sum Y_i - \sum X_i)\cdot\mu}{2\sigma^2}\right\}$$

and is free from σ^2 if and only if $\sum Y_i = \sum X_i$, $\sum Y_i^2 = \sum X_i^2$.

(c) $$S^2 = \frac{1}{n-1}\left(\sum_{i=1}^n Y_i^2 - 2\sum_{i=1}^n Y_i \bar{Y} + n\bar{Y}^2\right)$$

$$= \frac{1}{n-1}\left(\sum Y_i^2 - 2\frac{(\sum Y_i)(\sum Y_i)}{n} + \frac{(\sum Y_i)^2}{n}\right)$$

$$= \frac{1}{n-1}\left(\sum Y_i^2 - \frac{(\sum Y_i)^2}{n}\right).$$

(d) Since $\mathbb{E}S^2 = \sigma^2$ By Rao-Blackwell theorem

$\mathbb{E}(S^2 \mid \sum Y_i, \sum Y_i^2) = S^2$ S^2 is MVUE for σ^2

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3. Suppose that we flipped a coin for $n = 1000$ times, and observed heads for $Y = 480$ times. Denote the probability of observing head in a single coin flip with p. Consider the hypothesis test:

$$H_0 : p = 0.5 \qquad H_a : p < 0.5$$

(a) Use central limit theorem to find the approximate distribution for $\hat{p} = Y/n$ (10 pt).

(b) At $\alpha = 0.05$, find the rejection region (10 pt).

(c) Should we reject the null hypothesis? (5 pt)

(a). By CLT.

$$\hat{p} \overset{approx}{\sim} N\left(p, \frac{p(1-p)}{n}\right)$$

(b) under H_0

$$P\left(\hat{p} < p_0 - \sqrt{\frac{p_0(1-p_0)}{n}} \cdot Z_{0.05} \,\middle|\, p_0 = 0.5\right)$$

$$= 0.05.$$

Rejection region is

$$R = \{\hat{p} < 0.474\}$$

(c). since $\hat{p} = \frac{480}{1000} = 0.48 \notin R$

we do not reject H_0

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