Introduction to Probability (Spring 2019)

Exam 2

Name: ________________

ATTENTION!!

- Show clearly how you derive the result. Only having the right answer without support reasoning $\Rightarrow$ 1 pt only.

1. If a discrete random variable X has the moment generating function

$$\mathbb{E}e^{tX} = \cosh(t) = \frac{e^{2t}+1}{2e^t}.$$

Find

(a) $\mathbb{E}X$. (5 pt)

(b) Var$X$. (5 pt)

(c) The probability mass function $p(x)$. (10 pt)

$$\cosh(t) = \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}$$

(a) $$\left.\frac{d\cosh(t)}{dt}\right|_{t=0} = \left.\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right|_{t=0} = 0$$ 3pt

$$\Rightarrow \mathbb{E}X = 0$$ 2pt

(b) $$\frac{d^2\cosh(t)}{dt^2} = \left.\frac{1}{2}e^{t} + \frac{1}{2}e^{-t}\right|_{t=0} = 1$$ 3pt

$$\Rightarrow \mathbb{E}X^2 = 1$$

$$\Rightarrow \text{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 1$$ 2pt

(c) $$p(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 1 \\ \frac{1}{2}, & \text{if } x = -1 \end{cases}$$

2. $X \sim \text{Poisson}(\lambda)$ with probability mass function

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

consider a random variable $Y = \frac{X-\lambda}{\lambda}$.

(a) Find $\mathbb{E}Y$. (5 pt)

(a) Find $\text{Var}(Y)$. (5 pt)

(a) Use Chebyshev's inequality to find an upper bound for $\text{P}(|Y| \geq \lambda)$. (10 pt)

(a) since $X \sim \text{Poisson}(\lambda)$,

$\mathbb{E}X = \lambda$ 2pt

using linear property of expectation

$\mathbb{E}Y = \frac{\mathbb{E}X - \lambda}{\lambda} = 0$ 3pt

(b) $\text{var } X = \lambda$ 2pt

$\text{Var } Y = \frac{\text{var } X}{\lambda^2} = \frac{1}{\lambda}$ 3pt

(c) $P(|Y| \geq \lambda)$

$= P(|Y - 0| \geq \lambda)$

$\leq \frac{\text{var } Y}{\lambda^2} = \frac{1}{\lambda^3}$

6pt 4pt

3. If $Y \sim \text{Binomial}(n, p)$, with probability mass function:

$$p(y) = \binom{n}{y} p^y (1-p)^{n-y}, \quad y = 0, \dots, n$$

find

(a) $\mathbb{E}Y(Y-1)(Y-2)$. (10 pt)

(b) $\mathbb{E}(Y+1)^{-1}$. (10 pt)

(a) $\mathbb{E}\, Y(Y-1)(Y-2)$

$$= \sum_{y=0}^{n} y(y-1)(y-2) \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}$$ 4pt

$$= \sum_{y=3}^{n} y(y-1)(y-2) \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}$$

$$\overset{y=x+3}{=} \sum_{x=0}^{n-3} \frac{n(n-1)(n-2)(n-3)!}{x!(n-3-x)!} p^{x+3} (1-p)^{n-3-x}$$ 4pt

$$= n(n-1)(n-2) p^3 \sum_{x=0}^{n-3} \left[\frac{(n-3)!}{x!(n-3-x)!} p^x (1-p)^{n-3-x} \right]$$

$$= n(n-1)(n-2)p^3$$ 2pt

(b) $\mathbb{E}(Y+1)^{-1}$

$$= \sum_{y=0}^{n} \frac{1}{y+1} \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}$$

$$\underset{y=x-1}{\overset{x=y+1}{=}} \sum_{x=1}^{n+1} \frac{1}{x} \frac{n!}{(x-1)!(n+1-x)!} p^{x-1} (1-p)^{n-x+1}$$ 4pt

$$= \sum_{x=1}^{n+1} \frac{1}{p(n+1)} \left[\frac{(n+1)!}{x!(n+1-x)!} p^x (1-p)^{n+1-x} \right]$$

$$= \sum_{x=0}^{n+1} \frac{1}{p(n+1)} \left[\frac{(n+1)!}{x!\,(n+1-x)!} p^{x} (1-p)^{n+1-x} \right]$$

$$- \frac{1}{p(n+1)} \frac{(n+1)!}{0!\,(n+1-0)!} p^{0} (1-p)^{n+1-0}$$ 4pt

$$= \frac{1}{p(n+1)} - \frac{1}{p(n+1)} \cdot (1-p)^{n+1}$$ 2pt