# Introduction to Probability (Spring 2019)

## Exam 3

Name: ______________

ATTENTION!!

- Show clearly how you derive the result. Only having the right answer without support reasoning $\Rightarrow$ 1 pt only.

1. For two random variables

$$f(x, y) = 1/2, \quad 0 \le x \le 2, 0 \le y \le 1$$

Find

(a) $\mathbb{E}(X)$. (5 pt)

(b) $\mathbb{E}(Y)$. (5 pt)

(c) $\mathrm{Cov}(X, Y)$. (5 pt)

(a)

$$\mathbb{E}X = \int_0^1\int_0^2 x \cdot \frac{1}{2} \cdot dx \cdot dy$$

$$= 1$$

(b)

$$\mathbb{E}Y = \int_0^1\int_0^2 \cdot y \frac{1}{2} \cdot dx \cdot dy$$

$$= \frac{1}{2}$$

(c)

$$\mathbb{E}XY = \int_0^1\int_0^2 x \cdot y \cdot \frac{1}{2} \cdot dx \cdot dy$$

$$= \frac{1}{2}$$

$$\mathrm{Cov}(X, Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$$

$$= 0$$

2. For two random variables

$$f(x, y) = 2e^{-x}, \quad 0 \leq y \leq x \leq 2y$$

Find

(a) $f(x)$. (5 pt) Note that $x/2 \leq y \leq x$.

(b) $f(y)$. (5 pt) Note that $y \leq x \leq 2y$.

(c) $f(x \mid y)$. (5 pt)

(d) $f(y \mid x)$. (5 pt)

(a) $f(x) = \int_{x/2}^{x} f(x,y) \cdot dy$

$= \int_{x/2}^{x} 2 \cdot e^{-x} \cdot dy$

$= x \cdot e^{-x}$, $\quad x \geq 0$

(b). $f(y) = \int_{y}^{2y} f(x,y) \cdot dx$

$= \int_{y}^{2y} 2 e^{-x} \cdot dx$

$= 2 \cdot (e^{-y} - e^{-2y})$, $\quad y \geq 0$

(c) $f(x|y) = \dfrac{f(x,y)}{f(y)} = \dfrac{e^{-x}}{e^{-y} - e^{-2y}}$, $\quad y \leq x \leq 2y$

(d) $f(y|x) = \dfrac{f(x,y)}{f(x)} = \dfrac{2}{x}$, $\quad \dfrac{x}{2} \leq y \leq x$

3. For two random variables $Y_1$ and $Y_2$, assume that $Y_1$ follows a Gamma distribution with $\alpha = 2$ and $\beta = 1$,

$$f(y_1) = y_1 \exp(-y_1), \quad y_1 > 0$$

and $Y_2$ given $Y_1 = y_1$ follows a uniform distribution $(0, y_1)$

$$f(y_2 \mid y_1) = 1/y_1, \quad 0 < y_2 < y_1.$$

Find

(a) $f(y_1, y_2)$. (5 pt) Hint: the definition $f(y_1 \mid y_2) = f(y_1, y_2)/f(y_2)$.

(b) $\mathbb{E}(Y_2 \mid Y_1 = y_1)$. (5 pt)

(c) Use result from (a) and to find $f(y_2)$. (5 pt)

(a) $f(y_1, y_2) = f(y_2 | y_1) \cdot f(y_1)$

$= \exp(-y_1), \quad 0 < y_2 < y_1$

(b) $\mathbb{E}(Y_2 | Y_1 = y_1) = \int_0^{y_1} y_2 \, f(y_2 | y_1) \cdot dy_2$

$= \int_0^{y_1} y_2 \cdot \frac{1}{y_1} \cdot dy_2$

$= \frac{y_1^2}{2 \cdot y_1}$

$= \frac{y_1}{2}$

(c). $f(y_2) = \int_{y_2}^{\infty} f(y_1, y_2) \cdot dy_1$

$= -e^{-y_1} \Big|_{y_2}^{\infty}$

$= e^{-y_2}, \quad y_2 > 0$