Introduction to Statistics Theory (Fall 2018)

Midterm 1

Name: ________________

Results you may use directly:

- If $Y \sim$ Poisson(λ), then $\mathbb{E}Y = \text{var}(Y) = \lambda$.
- If $Y \sim Gamma(\alpha, \beta)$ with $f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$, then $\mathbb{E}(Y) = \alpha/\beta$, $\text{var}(Y) = \alpha/\beta^2$ and $cY \sim Gamma(\alpha, \beta/c)$.
- If $Y \sim \mathcal{X}_\nu^2$, then $Y \sim Gamma(\nu/2, 1/2)$.

1. If $Y_1, Y_2, \ldots, Y_n$ are iid from a Poisson distribution with parameter $0 < \lambda < \infty$, suppose we have a statistic $\bar{Y} = \frac{1}{n}\sum_{i=1}^n Y_i$.

(a) Find the mean and variance of $\bar{Y}$. (10 pt)

(b) When n is large, can we use central limit theorem to approximate the distribution of $\bar{Y}$? Why or why not? (5 pt)

(c) When n is large, construct an approximate two-sided 95% confidence interval for λ. ($\cancel{5}$ pt) 10.

(a) $\mathbb{E}\bar{Y} = \mathbb{E}Y_i = \lambda$

$\text{Var}\,\bar{Y} = \frac{\text{Var}\,Y_i}{n} = \frac{\lambda}{n}$

(b) Since $Y_1 \ldots Y_n$ are iid and $\lambda < \infty$,

$$\sqrt{n}\,\frac{\bar{Y} - \lambda}{\sqrt{\lambda}} \xrightarrow{D} N(0,1)$$

(c) At large n

$$\bar{Y} \overset{approx}{\sim} N\left(\lambda, \left(\sqrt{\frac{\lambda}{n}}\right)^2\right)$$

as we don't know $\sqrt{\frac{\lambda}{n}}$, we use $\sqrt{\frac{\bar{Y}}{n}}$ as an estimate

therefore an approximate 95% confidence interval is

$$\left(\bar{Y} - Z_{2.5\%}\sqrt{\frac{\bar{Y}}{n}},\ \bar{Y} + Z_{2.5\%}\sqrt{\frac{\bar{Y}}{n}}\right)$$

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2. If $Y_1, Y_2, \ldots, Y_n$ are iid from a certain distribution with mean μ and variance σ^2, assuming an even sample size $n = 2k$, consider an estimator

$$\hat{\sigma}^2 = \frac{1}{k}\sum_{i=1}^{k}(y_{2i} - y_{2i-1})^2$$

(a) Find $\mathbb{E}\hat{\sigma}^2$. (10 pt)

(b) Find an unbiased estimator for σ^2 by multiplying a constant to $\hat{\sigma}^2$ (i.e. in the form of $c\hat{\sigma}^2$, find c). (5 pt)

(a) $\mathbb{E}(Y_{2i} - Y_{2i-1})^2$

$= \mathbb{E}Y_{2i}^2 - 2\mathbb{E}Y_{2i}Y_{2i-1} + \mathbb{E}Y_{2i-1}^2$

Since $\mathbb{E}Y_{2i}^2 = \mathbb{E}Y_{2i-1}^2 = \sigma^2 + \mu^2$

due to iid, $\mathbb{E}Y_{2i}Y_{2i-1} = \mathbb{E}Y_{2i}\,\mathbb{E}Y_{2i-1} = \mu^2$.

$\Rightarrow \mathbb{E}(Y_{2i} - Y_{2i-1})^2 = 2\sigma^2$

$\Rightarrow \mathbb{E}\hat{\sigma}^2 = \frac{1}{k}\sum_{i=1}^{k} 2\sigma^2 = 2\sigma^2.$

(b). $\mathbb{E}\frac{1}{2}\hat{\sigma}^2 = \sigma^2.$

$\Rightarrow c = \frac{1}{2}.$

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3. If $Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2)$, assume that μ is known and we are estimating σ^2. Consider two estimators

$$W_1 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu)^2$$

$$W_2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

(a) Find the distribution for W_1. You may state the related theorem without proving it. (10 pt)

(b) Find the distribution for W_2. You may state the related theorem without proving it. (10 pt)

(b) Find the relative efficiency $R.E.(W_1, W_2)$. (5 pt)

(a). since $\frac{Y_i - \mu}{\sigma} \sim N(0,1)$

$\sum_{i=1}^{n} \frac{(Y_i - \mu)^2}{\sigma^2} \sim \chi^2_n$ equivalent to $\mathrm{Gamma}(\frac{n}{2}, \frac{1}{2})$

$\Rightarrow W_1 = \frac{\sigma^2}{n} \cdot \sum_{i=1}^{n} \frac{(Y_i - \mu)^2}{\sigma^2} \sim \mathrm{Gamma}(\frac{n}{2}, \frac{n}{2\sigma^2})$

(b). W_2 is the sample variance for an iid normal population

$\Rightarrow \frac{(n-1)W_2}{\sigma^2} \sim \chi^2_{n-1}$. equivalent to $\mathrm{Gamma}(\frac{n-1}{2}, \frac{1}{2})$

$\Rightarrow W_2 \sim \mathrm{Gamma}(\frac{n-1}{2}, \frac{n-1}{2\sigma^2})$

(c) $\mathrm{Var}(W_1) = \frac{\frac{n}{2}}{(\frac{n}{2\sigma^2})^2} = \frac{2}{n}\sigma^4.$

$\mathrm{Var}(W_2) = \frac{\frac{n-1}{2}}{(\frac{n-1}{2\sigma^2})^2} = \frac{2}{n-1}\sigma^4$

$\Rightarrow R.E.(W_1, W_2) = \frac{\mathrm{Var}(W_2)}{\mathrm{Var}(W_1)} = \frac{n}{n-1}$

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