DATA 3: Surface Mesh (2-Simplices)

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Abstract. This document provides a detailed explanation of the MATLAB codes and surface mesh data available at https://github.com/laplcebeltrami/hk. The dataset consists of paired inner (white matter) and outer (gray matter) cortical surface meshes from both autistic subjects and healthy controls. Each cortical surface is represented as a high-resolution triangular mesh with consistent vertex correspondence across subjects and between inner and outer surfaces, enabling pointwise geometric and statistical analysis. The accompanying MATLAB scripts in the folder demonstrate how to load, visualize, and manipulate these surface meshes, as well as how to compute vertexwise cortical thickness as the Euclidean distance between corresponding inner and outer surface vertices.

Triangular Mesh Representation (2-Simplices)

Anatomical surfaces often represented as triangulated surface meshes (Chung 2012). Each surface mesh is a collection of *2-simplices*, where a 2-simplex corresponds to a filled triangle. A surface mesh is therefore not a single simplex but a large assembly of many 2-simplices that together approximate a smooth two-dimensional surface embedded in three-dimensional space.

In MATLAB, each surface mesh is stored as a structure with two fields:

$$\texttt{vertices} \in \mathbb{R}^{N\times 3}, \qquad \texttt{faces} \in \mathbb{N}^{M\times 3}.$$

The matrix `vertices` contains the three-dimensional coordinates of the mesh vertices, with each row specifying an (x, y, z) location in physical space. The matrix `faces` encodes the mesh connectivity: each row consists of three integer indices referring to rows of `vertices`. These three indices define a single triangular face, i.e., a 2-simplex. The meshes used here consist of $N = 40{,}962$ vertices and $M = 81{,}920$ triangular faces, providing a high-resolution discretization of the cortical surface. Collectively, the vertices and faces form a two-dimensional simplicial complex embedded in $\mathbb{R}^3$.

Inner and Outer Cortical Surfaces

We use cortical brain surface meshes originally published in Chung, Robbins, Dalton, Davidson, Alexander & Evans (2005) and Chung, Robbins & Evans

(2005). For each subject, two surfaces are provided: an inner surface corresponding to the white matter boundary and an outer surface corresponding to the pial (gray matter) boundary. Importantly, the inner and outer surfaces share the same mesh topology, meaning that their `faces` arrays are identical. Only the vertex coordinates differ between the two surfaces. This one-to-one correspondence between vertices on the inner and outer surfaces is essential for pointwise geometric computations, such as cortical thickness estimation.

Loading the Surface Data

The following MATLAB commands load the inner and outer surfaces for autistic and control subjects, as well as a spherical template mesh that defines the common triangulation:

```
load autism_inner.mat
load autism_outer.mat
load control_inner.mat
load control_outer.mat

load sphere.mat
```

In addition to the cortical surface meshes, we provide a spherical template mesh stored in `sphere.mat`. This template defines a common triangulation that is shared across all subjects and across both inner and outer cortical surfaces. `sphere.faces` encodes a fixed mesh connectivity, while subject-specific surfaces are obtained by replacing only the vertex coordinates. This shared connectivity guarantees consistent vertex correspondence across subjects and between inner and outer surfaces. The spherical template mesh also serves as a natural reference domain for spectral representations of surface data. it can be used to construct spherical harmonic (SPAHRM) representations of cortical geometry and scalar fields defined on the surface, enabling multiscale analysis, smoothing, and statistical modeling in a common coordinate system (Chung et al. 2007, 2008).

Cortical thickness

The goal of cortical thickness analysis is to quantify the distance between the inner surface and the outer surface of the cortex. Because these two surfaces share identical mesh connectivity and vertex correspondence, a simple and commonly used approximation computes cortical thickness at each vertex as the Euclidean distance between corresponding vertices on the two surfaces (Chung 2012). However, this Euclidean approximation does not fully account for the geometry of the cortical folding. In regions of high curvature or deep sulci, the straight-line distance between corresponding vertices may underestimate or overestimate the true anatomical thickness. In addition, the Euclidean method implicitly assumes that thickness is measured along a straight path, which may not align with the local cortical normal direction. More sophisticated definitions of cortical thickness address these issues by modeling the space between the inner and outer surfaces as a continuous volume. One class of methods solves a Laplace equation between the two surfaces and defines thickness along streamlines of the resulting potential field. Another class of approaches uses heat diffusion or geodesic distance computations to trace paths that better follow the intrinsic geometry of the cortical ribbon. These methods provide more anatomically faithful thickness estimates but are computationally more demanding and require solving partial differential equations on complex surface domains.

PROJECT 5: Manifold-on-Manifold

Goal. The goal of this project is to introduce students to geometric data analysis by treating complex datasets as objects living on nonlinear geometric spaces rather than in ordinary Euclidean space. Students will learn how geometric structure arises naturally in surface meshes, covariance matrices, and other simplicial complex–based data, and how respecting this structure leads to more meaningful analysis and interpretation.

Description. In this project, students will analyze data that are inherently geometric, meaning that the data elements either live on curved spaces or are constrained by topology. The primary dataset consists of cortical surface meshes represented as triangulated surfaces (2-simplicial complexes), with paired inner and outer cortical surfaces. However, the methods developed in this project are not limited to surface data, and other in-class datasets—such as multivariate time series, covariance matrices, or graph- and simplicial-complex–based data—may also be used to explore the same geometric principles.

Along mesh vertices, one can construct covariance or correlation matrices from surface-based measurements such as cortical thickness and curvature. The central objective is to investigate how statistical dependence between surface locations is structured by the underlying cortical geometry and why this dependence should be analyzed directly on the cortical surface rather than in Euclidean space. By constructing covariance and correlation matrices indexed by surface vertices, we respect the spatial organization and vertex correspondence encoded by the mesh, which enables covariance structure to be interpreted in anatomically meaningful and geometrically consistent terms.

We will examine how these covariance and correlation matrices change with cortical geometry and neighborhood structure, and we will assess how geometric distortions and folding patterns influence apparent statistical dependence. The key challenge is that the resulting data objects are themselves manifold-valued: covariance and correlation matrices are constrained to be symmetric positive definite, forming a curved manifold, while their indices correspond to locations on another curved manifold (the cortical surface). In other words, the project addresses a setting in which *manifold-valued data are organized over a manifold domain*, creating a “manifold-on-manifold” statistical problem.

Learning Outcomes. Students will develop a principled understanding of geometric data analysis and why many modern datasets cannot be analyzed appropriately using naïve Euclidean methods. Students will learn how covariance and correlation matrices constructed from surface-based measurements inherit geometric constraints from both the cortical surface domain and the symmetric positive definite matrix space, and how these constraints influence valid statistical operations. They will gain experience working with triangulated surface meshes, manifold-valued data, and geometry-aware representations, and will learn to critically compare Euclidean and geometry-respecting approaches across multiple data modalities, including cortical surfaces, multivariate time series, and simplicial complex–based datasets.

PROJECT 6: Topological Analysis of Simplex Data

Goal. The goal of this project is to show how topological data analysis reveals large-scale structure in geometric data that is invisible to standard pointwise or purely statistical methods. Using brain surface meshes as a motivating example, we will learn how multiscale topological features emerge from surface-based measurements and how topology provides a complementary perspective on shape, organization, and variability beyond traditional geometric analysis. The project can be applied to any in-class dataset that gives rise to a *nontrivial* simplicial complex.

Description. In this project, we analyze cortical surface data using persistent homology to study the global structure of surface-based measurements. The cortical surface is represented as a triangular mesh, with scalar quantities such as mean curvature defined at each vertex. Mean curvature is particularly suitable because it reflects cortical folding patterns and induces ridge–valley structures on the surface, leading to hierarchical, tree-like organization of regions across scales. Rather than analyzing these values independently at each location, students will examine how large-scale topological structure emerges as the surface is progressively built according to the ordering of the scalar values. To do this, we construct a filtration on the surface mesh by gradually adding vertices, edges, and triangles in order of increasing scalar value. At early stages of the filtration, only regions with low curvature values are present; as the filtration grows, regions merge, branches form, and loops may appear or disappear, reflecting the multiscale organization of cortical folds. Building on these representations, we will develop and evaluate statistical inference procedures for topological features, with the goal of identifying which aspects of cortical organization are stable, variable, or biologically meaningful.

Learning Outcomes. By completing this project, students will understand how persistent homology captures multiscale structure in geometric data, how filtrations are constructed on surface meshes from scalar measurements, and why topological features provide information beyond standard geometric or statistical summaries. We will develop intuition for when and why topological analysis is useful, and will be able to transfer the same conceptual framework to other structured datasets, including surface-based measurements, time series–derived representations, and data defined on graphs or simplicial complexes.

Bibliography

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