

*The Waisman Laboratory  
for Brain Imaging and Behavior*



University of Wisconsin  
**SCHOOL OF MEDICINE  
AND PUBLIC HEALTH**

# BMI/STAT-768

## Statistical Methods for Medical Image

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<https://github.com/laplacebeltrami/BMI768>

# Geometric Data Analysis

Geometric Data Analysis (GDA) is a field that focuses on understanding and interpreting the geometric structures inherent in complex datasets. It leverages mathematical and statistical techniques to explore properties such as shape, curvature, and topology, enabling the identification of patterns and relationships within the data. GDA encompasses methods like manifold learning, principal component analysis, and dimensionality reduction, which help in visualizing high-dimensional data in more interpretable forms.

Information  
geometry

Data embedding

Covariance  
matrix

Data manifolds

**GDA**

quantifies  
the shape  
of data

Symmetric positive  
definite matrices

Statistics on  
manifolds

PCA

Spectral clustering

Functional-PCA

Spectral geometry

# Unit Objectives

- 1) Understand curve and surface parametrization
- 2) Understand Riemannian metric tensors
- 3) Know how to compute Jacobian determinant

Use the *exponential map* example in understanding these concepts

## References

Widely used  
basic textbook  
on differential  
geometry

# *Differential Geometry of Curves and Surfaces*

**Manfredo P. do Carmo**

*Instituto de Matematica Pura e Aplicada (IMPA)  
Rio de Janeiro, Brazil*



# Differential Geometry of a Curve

Point  $p$  on the curve at  $u_0$

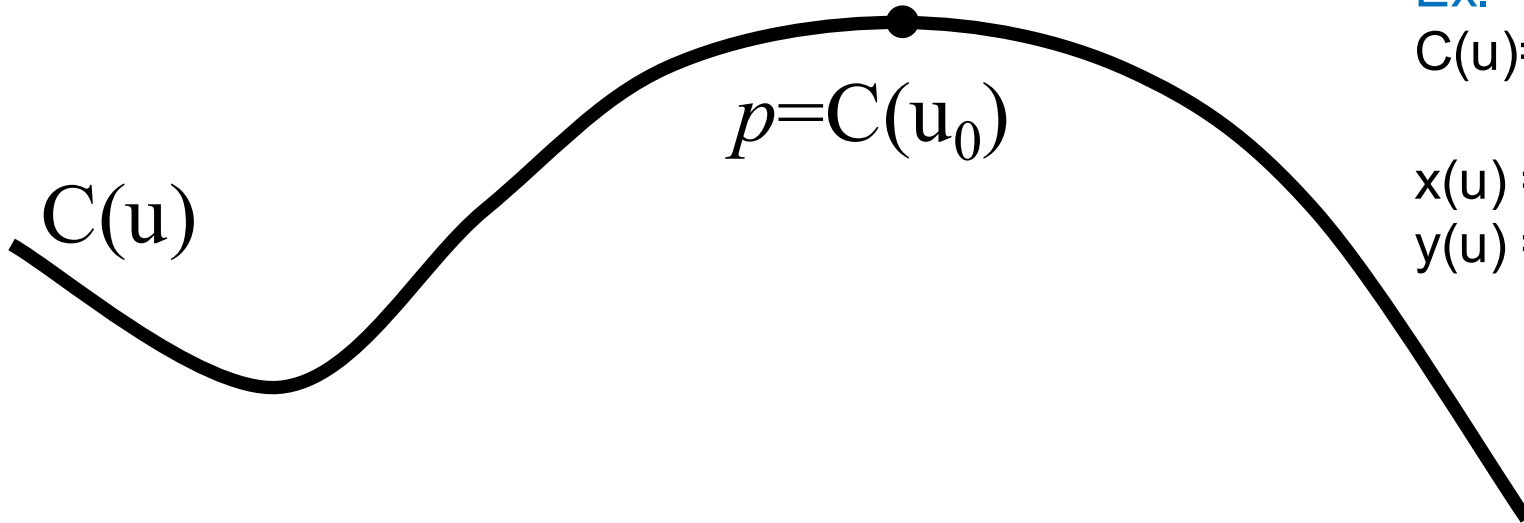
Smooth curves can be parameterized.

Ex.

$$C(u) = (x(u), y(u))$$

$$x(u) = \cos(u)$$

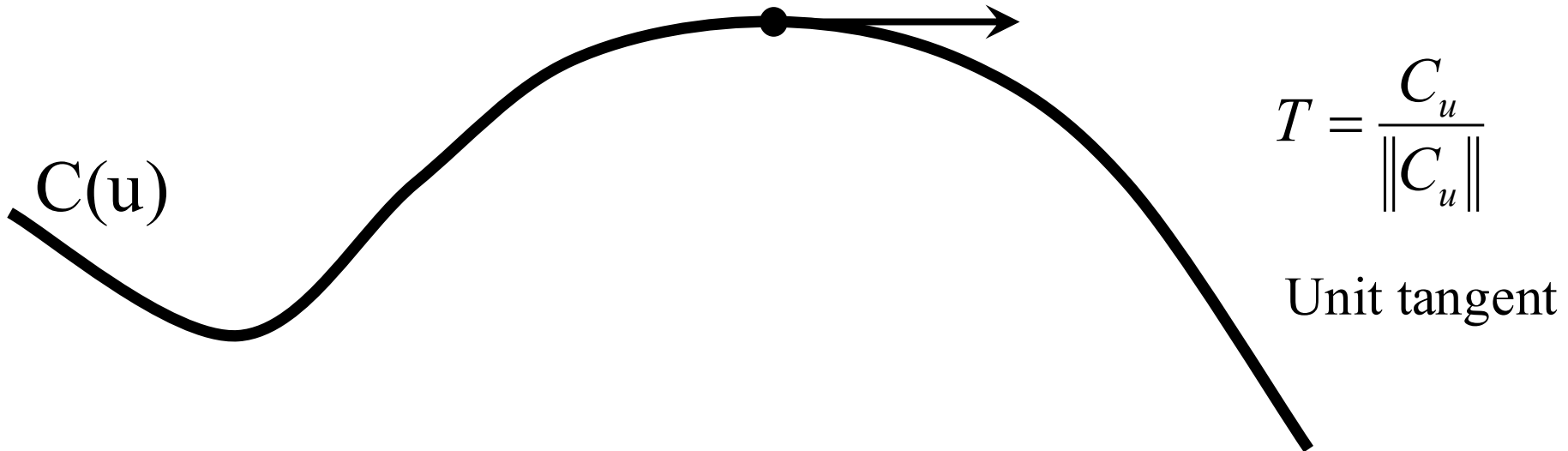
$$y(u) = \sin(u)$$



# Tangent and normal vectors

Tangent  $T$  to the curve at  $u_0$

$$C_u = \frac{\partial C(u)}{\partial u}$$



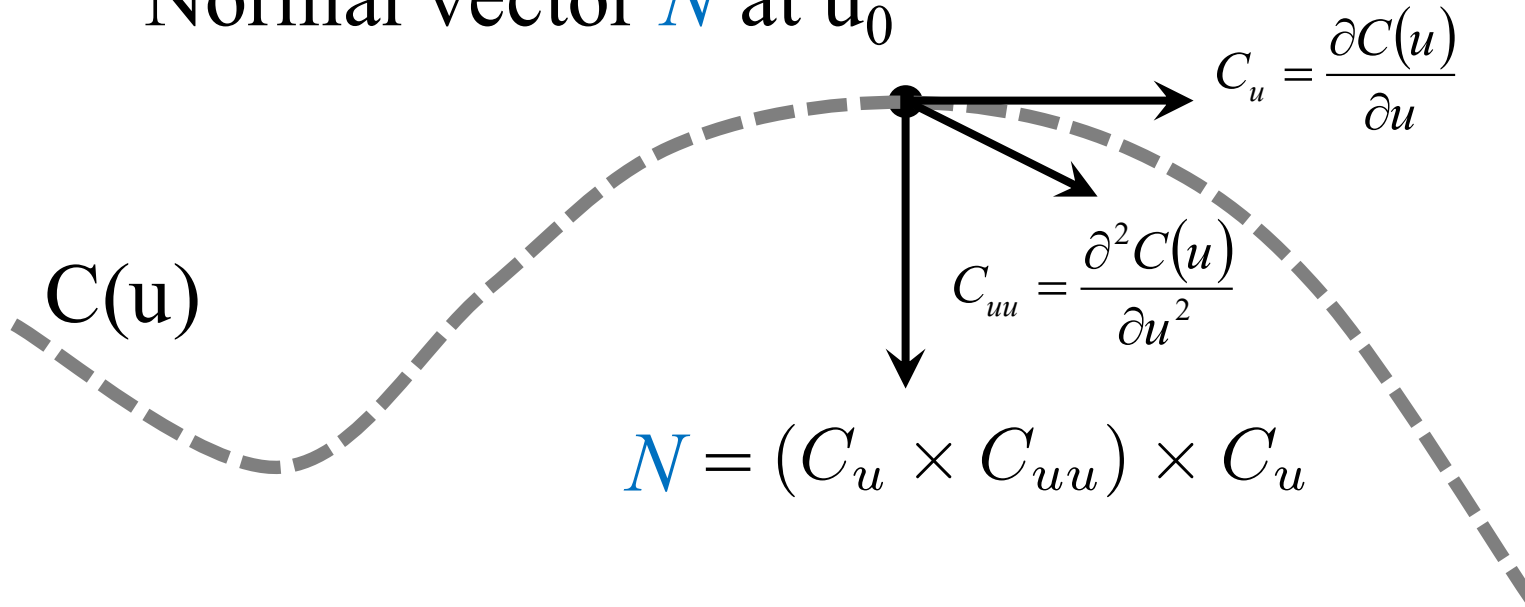
*Ex.*

$$x(u) = \cos(u)$$

$$y(u) = \sin(u)$$

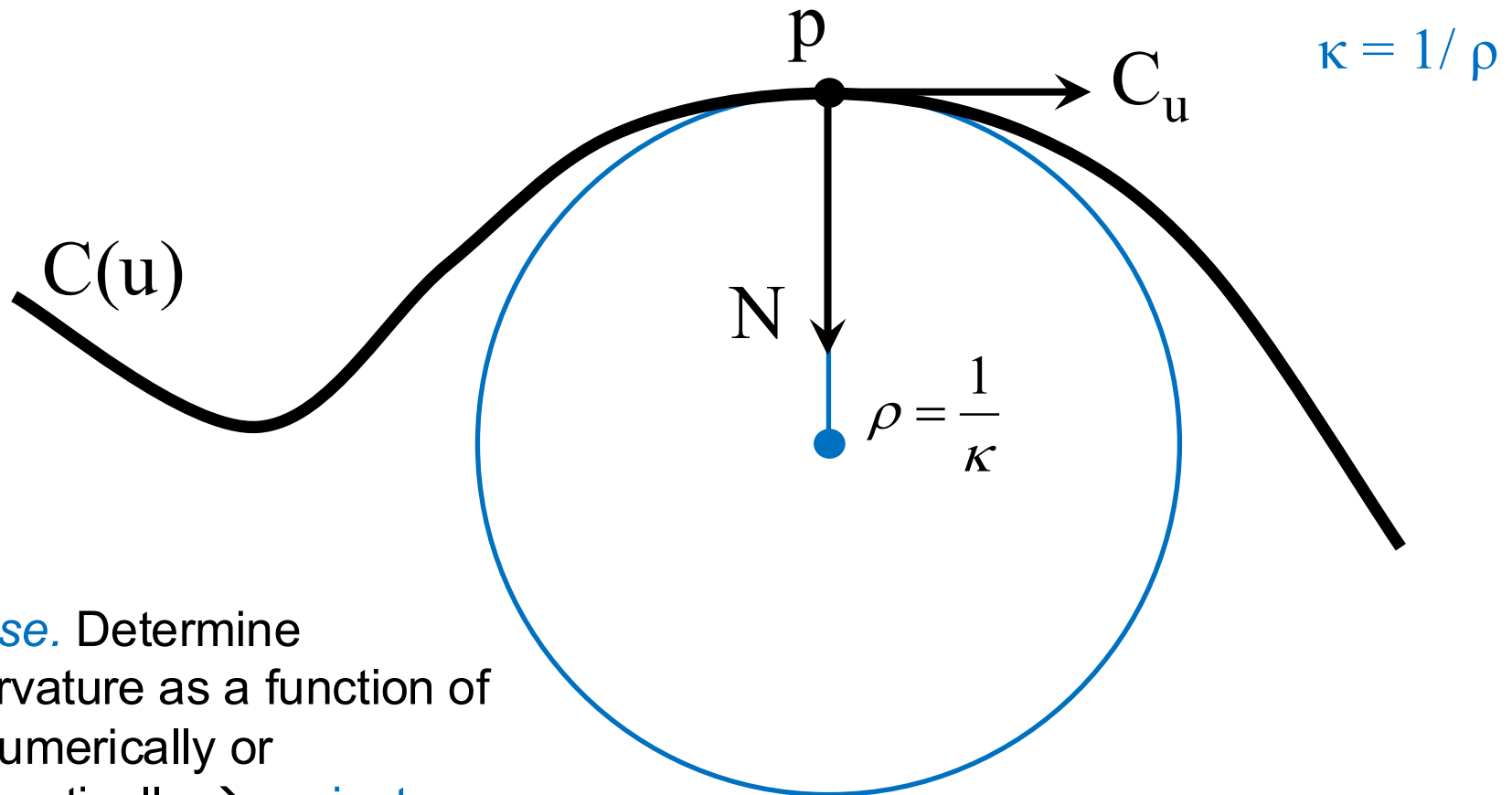
# Normal vector of a curve

Normal vector  $N$  at  $u_0$



# 1D curve has only one **curvature**

Curvature  $\kappa$  at  $u_0$  and the radius  $\rho$  osculating circle



**Exercise.** Determine the curvature as a function of  $C(u)$  numerically or mathematically → **project**

# White matter fibers

James Gee  
Univ. Penn

Anterior (Front)

side view

top view

Posterior (Back)

Fibers passing through  
the splenium of the  
corpus callosum





*Cosine series representation*

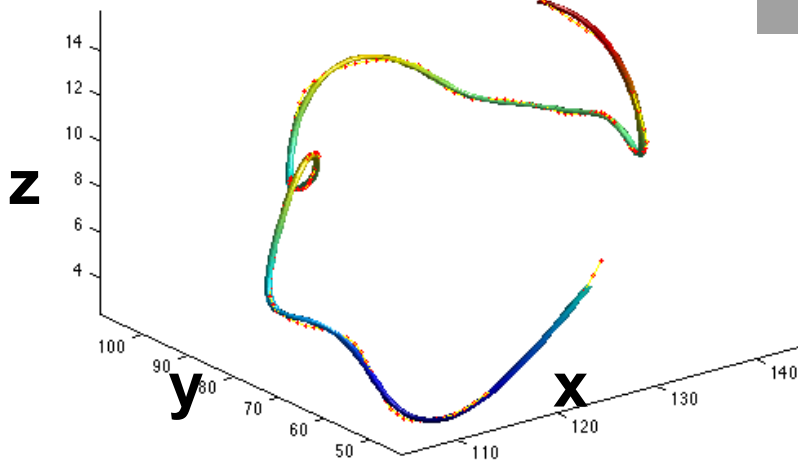
*Cover art*

*Chung, M.K. 2012*

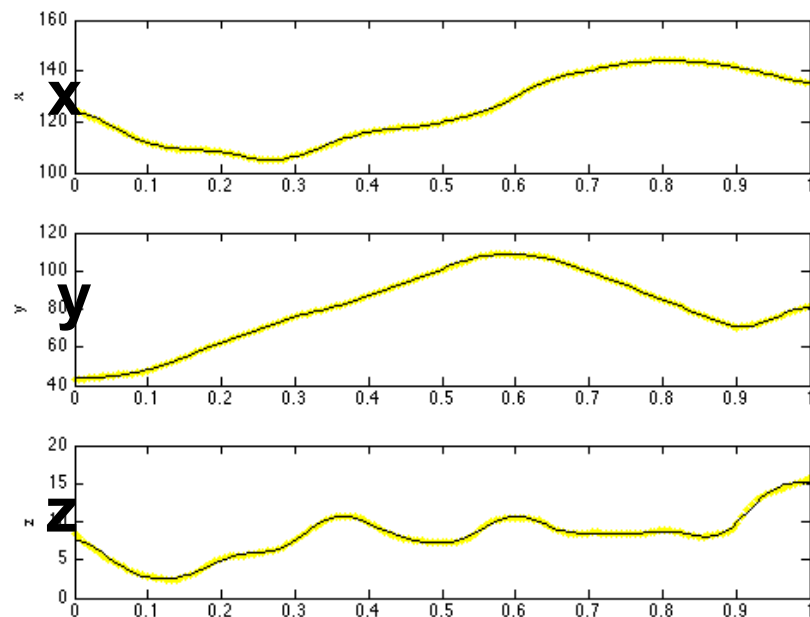
*Computational Neuroanatomy:  
The Methods, World Scientific  
Press*

*Matlab visualization*

# White matter fiber tract data



parameterization



88.1799	56.6336	5.7367
-12.4775	-11.2552	-2.0791
2.4336	-15.4428	-0.4021
4.3956	2.2733	-0.9354
-0.0106	-0.0674	0.6999
2.1773	-2.4194	-0.1176
0.5808	0.8390	1.2942
0.0615	-0.1893	0.1188
-0.2629	0.7524	0.1089
0.7909	-0.7276	-0.1901
0.5458	0.6236	0.6939
0.4295	-0.4337	0.2185
0.2150	0.4157	0.0254
0.1584	-0.1973	0.0762
-0.1557	0.2466	-0.1086
0.0632	-0.0978	-0.0208
0.0389	-0.0143	-0.0284
-0.0014	-0.1193	0.1970
0.0004	0.0129	-0.0198
0.1342	0.0002	0.0260

Any tract can be compactly parameterized with only 60 coefficients.

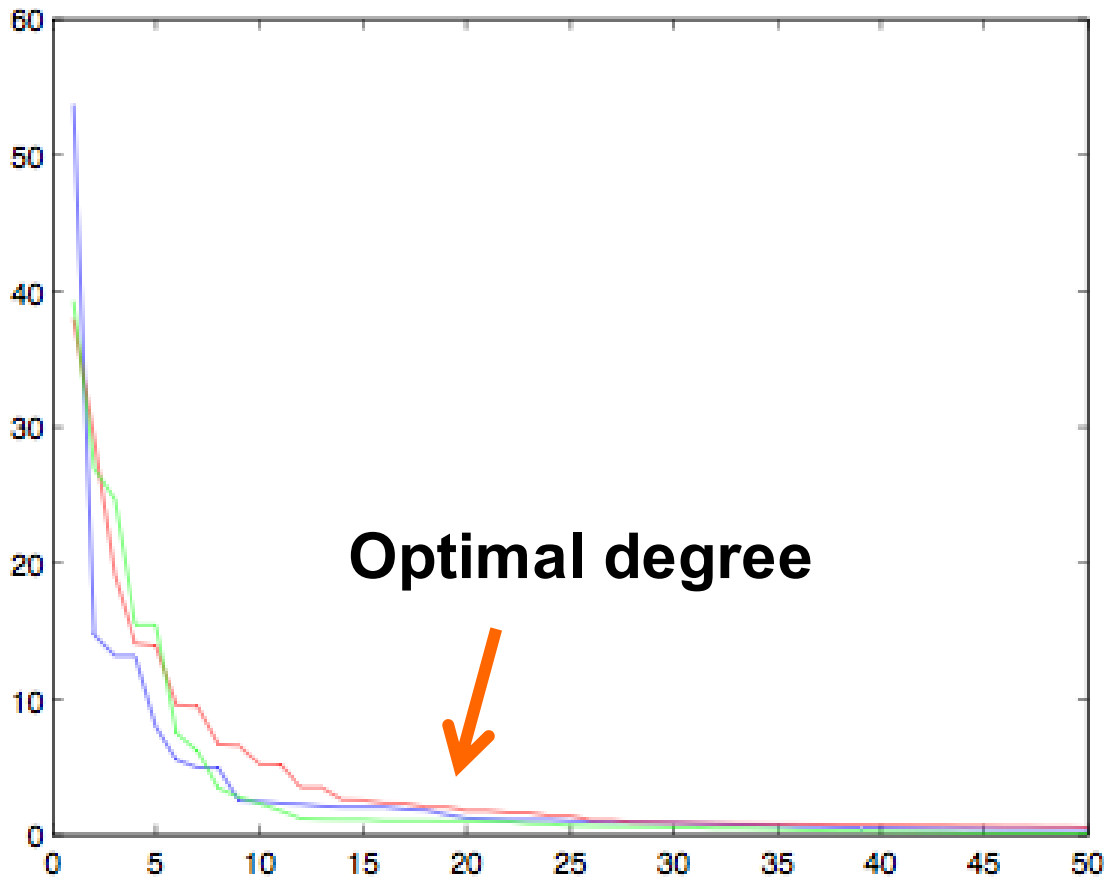
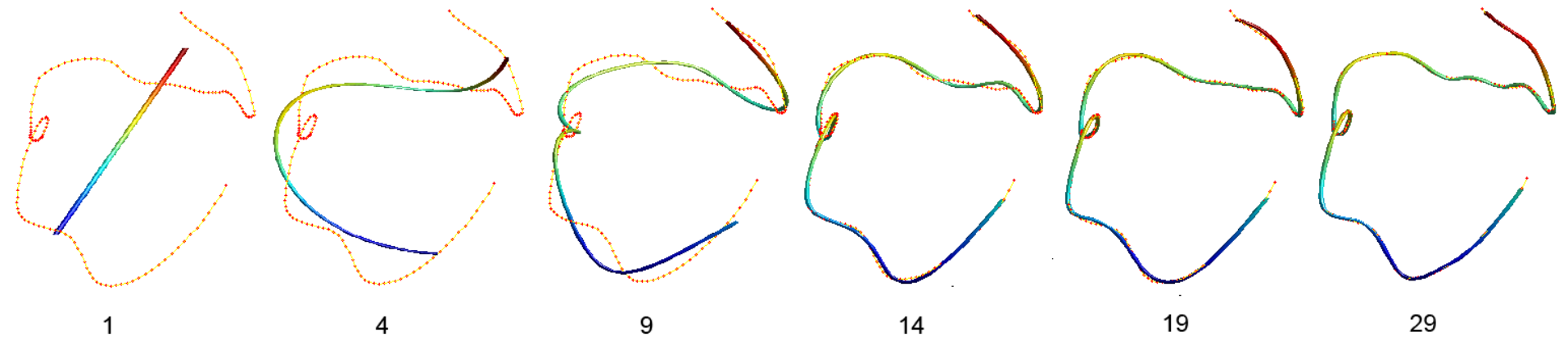
**Why?**

basis expansion



$$(x, y, z)' = \sum_{l=0}^{19} \beta_l \cos(l\pi t)$$

# Cosine series representation



The optimal degree chosen using the forward model selection.

*Exercise.* Compute Curvature and display on top of 3D curve.

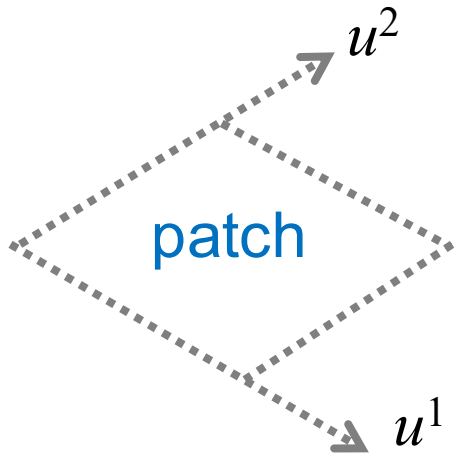
# Differential Geometry of a Surface

Point  $p$  on the surface

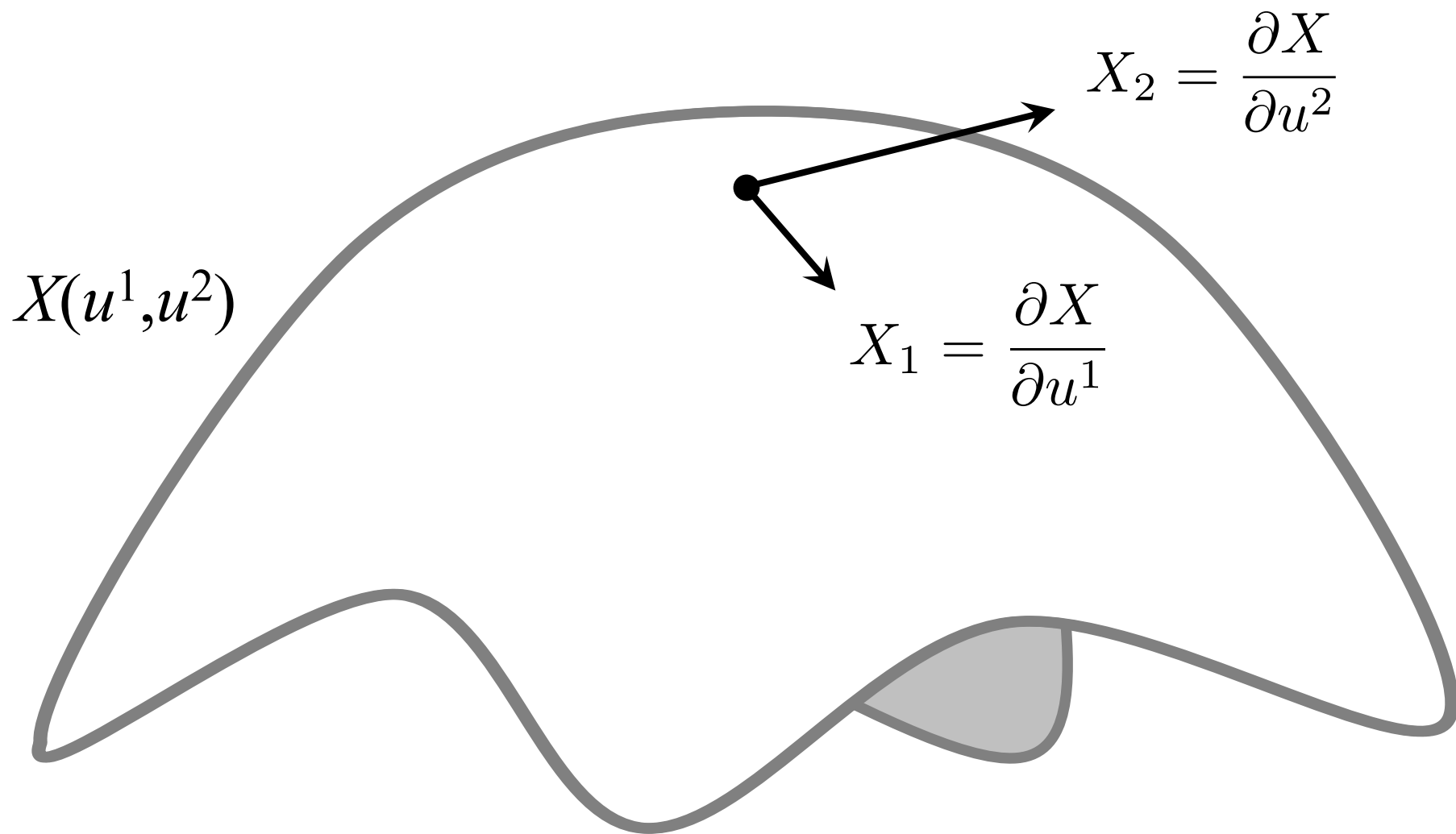
•  $p=(u^1,u^2)$

Parameterization

$X(u^1,u^2)$



# Tangent vectors



# Example: quadratic polynomial surface

$$x = u^1$$

$$y = u^2$$

$$z = \beta_0 + \beta_1 u^1 + \beta_2 u^2 + \beta_3 u^1 u^2 + \beta_4 (u^1)^2 + \beta_5 (u^2)^2$$

$$X(u^1, u^2) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u^1 \\ u^2 \\ \beta_0 + \beta_1 u^1 + \dots + \beta_5 (u^2)^2 \end{pmatrix}$$

$$\frac{\partial X}{\partial u^1} = \begin{pmatrix} 1 \\ 0 \\ \beta_1 + \beta_3 u^2 + 2\beta_4 u^1 \end{pmatrix}$$

At origin  $(u^1, u^2) = (0, 0)$ ,

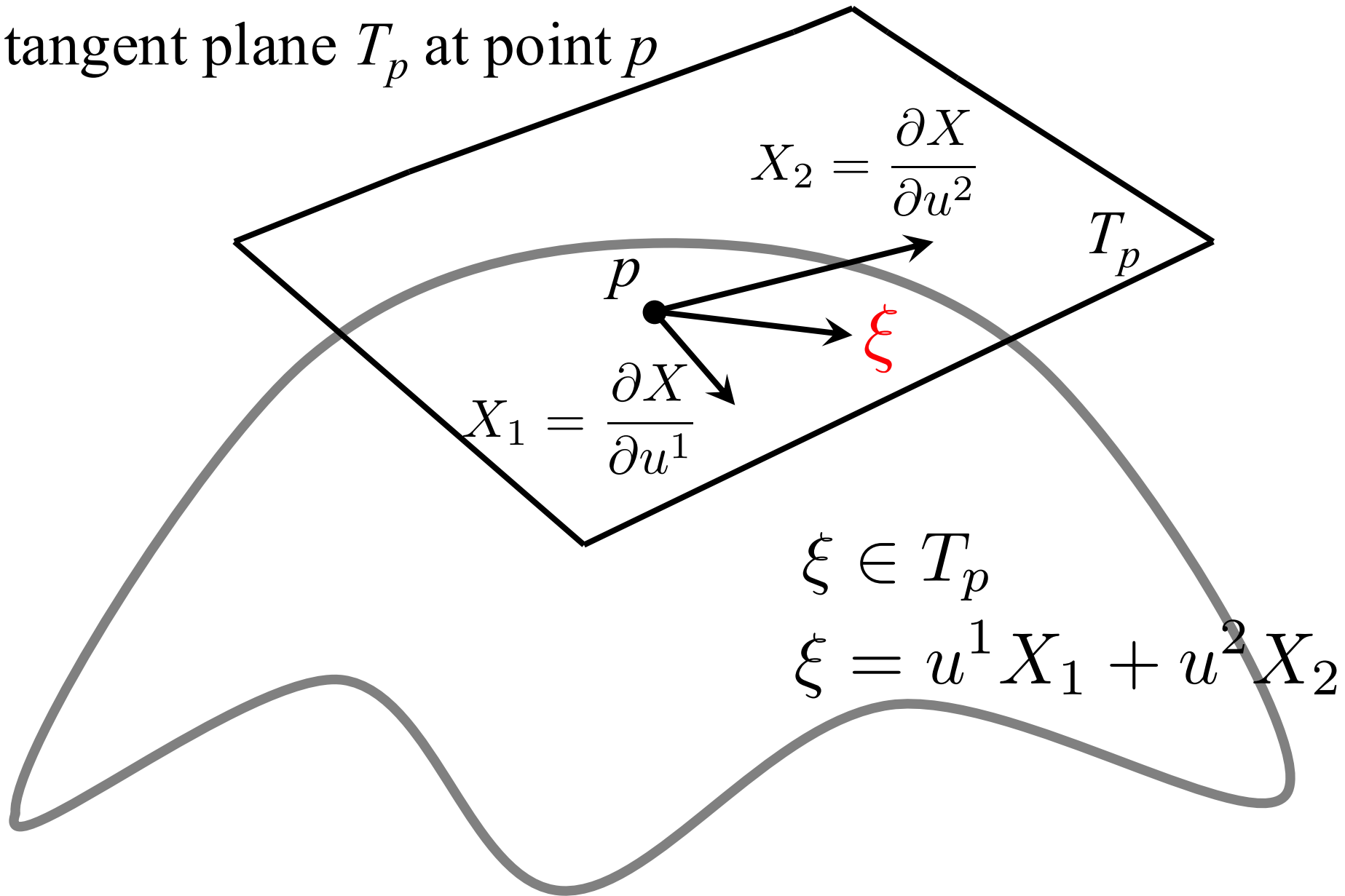
$$\frac{\partial X}{\partial u^2} = \begin{pmatrix} 0 \\ 1 \\ \beta_2 + \beta_3 u^1 + 2\beta_5 u^2 \end{pmatrix}$$

$$\frac{\partial X}{\partial u^1} = \begin{pmatrix} 1 \\ 0 \\ \beta_1 \end{pmatrix}$$

$$\frac{\partial X}{\partial u^2} = \begin{pmatrix} 0 \\ 1 \\ \beta_2 \end{pmatrix}$$

# Tangent plane

tangent plane  $T_p$  at point  $p$



# First Fundamental Form

Differential form

$$d\xi = du^1 X_1 + du^2 X_2$$

$$d\xi^2 \stackrel{\text{Riemannian metric}}{=} \langle d\xi, d\xi \rangle = \sum_{i,j} \langle X_i, X_j \rangle du^i du^j$$

Metric tensor

$$g_{ij} = \langle X_i, X_j \rangle \longrightarrow g = (g_{ij})$$

## Example: quadratic surface

$$x = u^1$$

$$y = u^2$$

$$z = \beta_0 + \beta_1 u^1 + \beta_2 u^2 + \beta_3 u^1 u^2 + \beta_4 (u^1)^2 + \beta_5 (u^2)^2$$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ \beta_1 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ \beta_2 \end{pmatrix}$$

$$g_{11} = \langle X_1, X_1 \rangle = \|X_1\|^2 = 1 + \beta_1^2$$

$$g_{12} = \langle X_1, X_2 \rangle = \beta_1 \beta_2$$

$$g_{22} = \langle X_2, X_2 \rangle = \|X_2\|^2 = 1 + \beta_2^2$$

$$\rightarrow g = \begin{pmatrix} 1 + \beta_1^2 & \beta_1 \beta_2 \\ \beta_1 \beta_2 & 1 + \beta_2^2 \end{pmatrix}$$

positive  
definite  
symmetric

# Riemannian metric tensor

Riemannian manifold  $(M, g)$  is a smooth manifold  $M$  with inner product  $g$  on the the tangent space  $T_p(M)$ . *At each point  $p$ , the metric varies smoothly in such a way that if  $X_1$  and  $X_2$  are differentiable tangent vector,  $g$  is also a smooth function. The family of such inner products is called a Riemannian metric tensor.*

If the metric doesn't vary, it's just the boring Euclidean space.


# Riemannian metric tensor

*Question.* If  $g_1, g_2, \dots, g_n$  are metrics, under what condition  $c_1 g_1 + c_2 g_2 + \dots + c_n g_n$  is metric as well?

# Area element

parameter space  $N \rightarrow$  manifold  $M$

$$\text{Area of manifold } M = \int_M d\mu(p) = \int_N \sqrt{\det g} \, du^1 du^2$$



Local surface area element = Jacobian determinant

$$J = \sqrt{\det g}$$

measures the amount of area with respect to the unit **patch area**

**Exercise:** Show surface area is invariant under parameterization

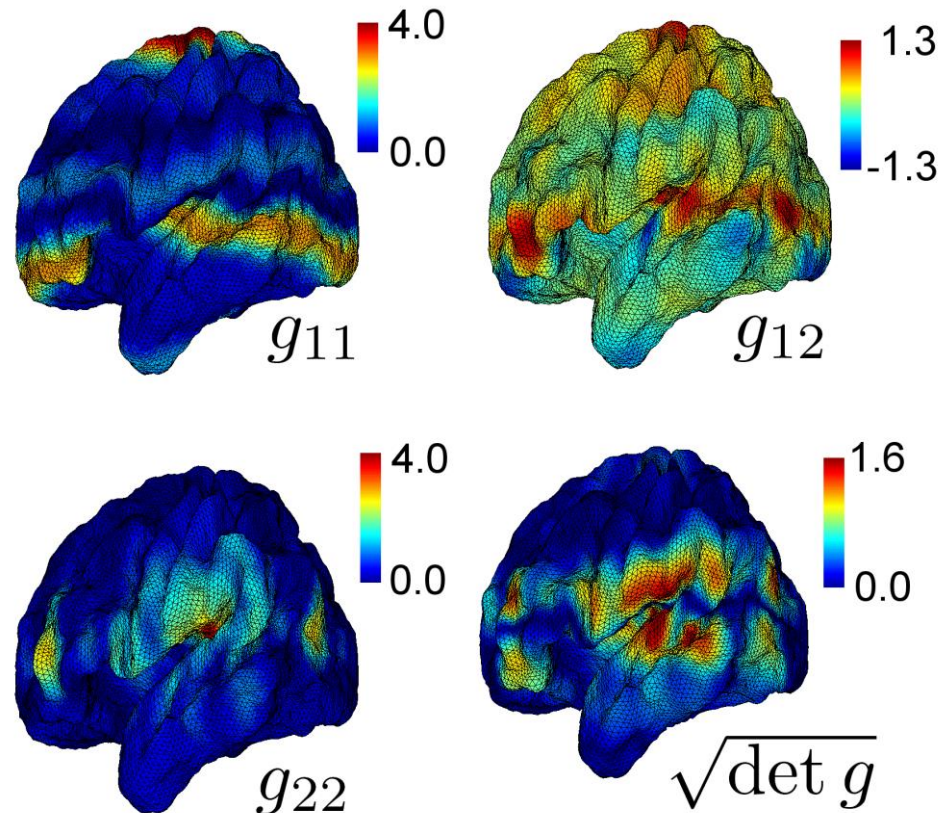
# Tensor-based morphometry

Surface parameterization:

$$u = (u^1, u^2) \mapsto X(u)$$

Riemannian metric tensor:

$$g_{ij} = \left\langle \frac{\partial X}{\partial u^i}, \frac{\partial X}{\partial u^j} \right\rangle$$



*Chung et al. 2003 Neurolmage 18:198-213*

*Chung et al. 2008. IEEE Transactions on Medical Imaging 27:1143-1151*

# Jacobian matrix and its determinant

$$\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\mathbf{x} = \{x_1, \dots, x_n\} \in \mathbb{R}^n$$

Jacobian matrix

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \left[ \frac{\partial \mathbf{f}}{\partial x_1} \quad \dots \quad \frac{\partial \mathbf{f}}{\partial x_n} \right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Not symmetric

Jacobian determinant  $\det \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$

Can we equate?

$$\det \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \sqrt{\det g}$$

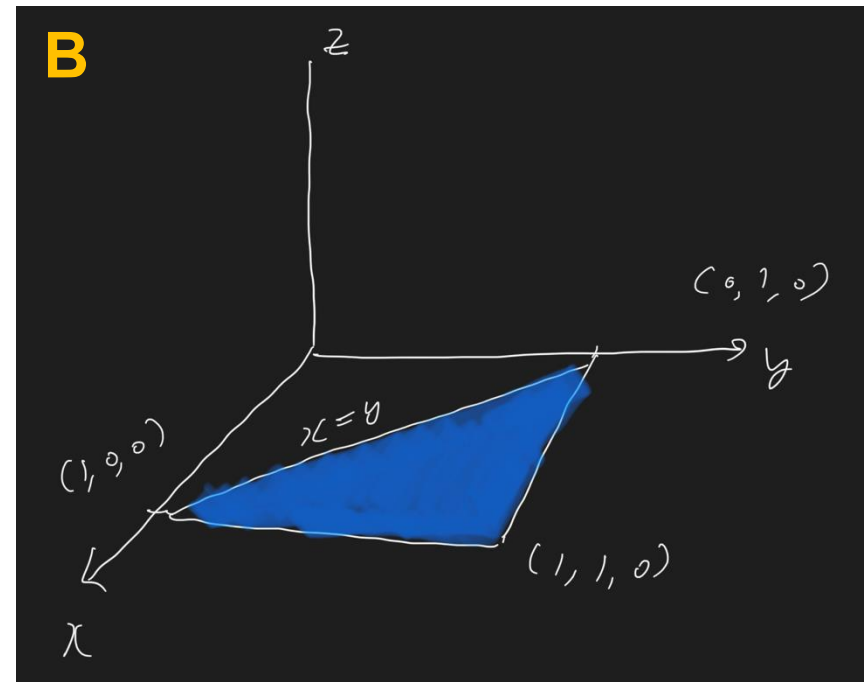
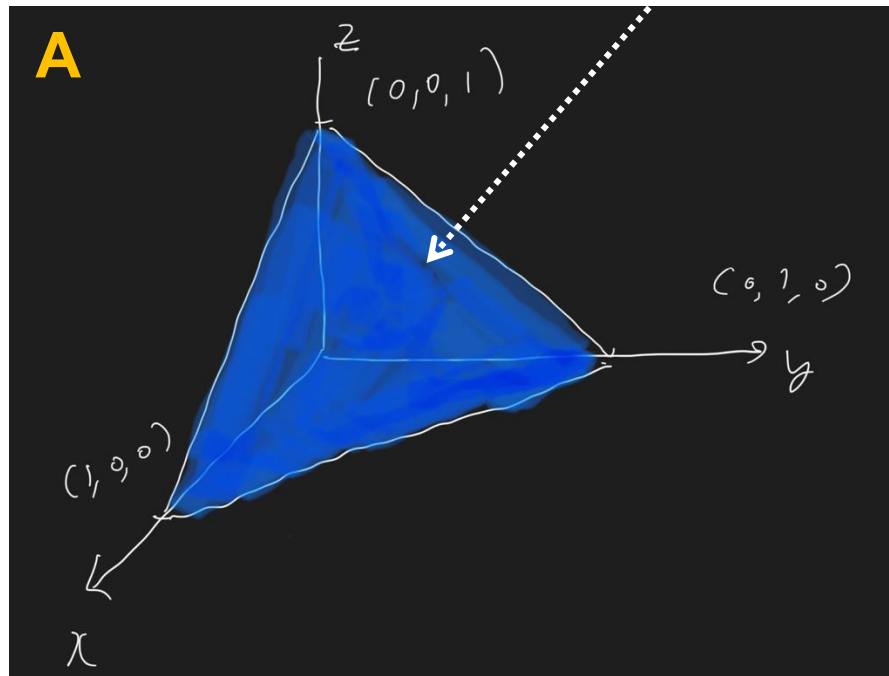
# Relation between Jacobian matrix and metric tensors

The  $3 \times 2$  Jacobian matrix  $J$  of mapping from parameter space  $\mathcal{N}$  to cortical surface  $\mathcal{M}$  is given by  $J = (\partial_\theta \hat{\nu}, \partial_\varphi \hat{\nu})$ . The Riemannian metric tensors are  $g = (g_{ij}) = J^t J$ . The component is given by  $g_{ij} = \partial_i \hat{\nu} \cdot \partial_j \hat{\nu}$  with the vector inner product  $\cdot$ . The Riemannian metric tensors measure the amount of deviation of a cortical surface from a flat Euclidean plane. If the cortical surface is flat, we obtain  $g_{ij} = \delta_{ij}$ , the identity matrix. The Riemannian metric tensors enable us to compute the local *area element*  $\sqrt{\det g}$ . The area element measures the amount of the transformed area in  $\mathcal{M}$  of the unit area in the parameterized space  $\mathcal{N}$  via the mapping  $\nu$ . Fig. 8 shows the estimation of the metric tensors for a subject. Using the area element, the total surface area of  $\mathcal{M}$  can be written as

$$\mu(\mathcal{M}) = \int_0^{2\pi} \int_0^\pi \sqrt{\det g}(\theta, \varphi) d\theta d\varphi.$$

# Exercise: probability distribution on 2-simplex

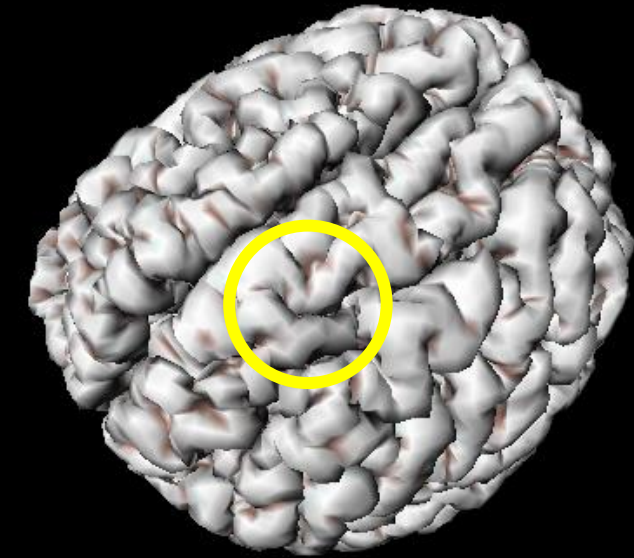
$f(x, y, z)$



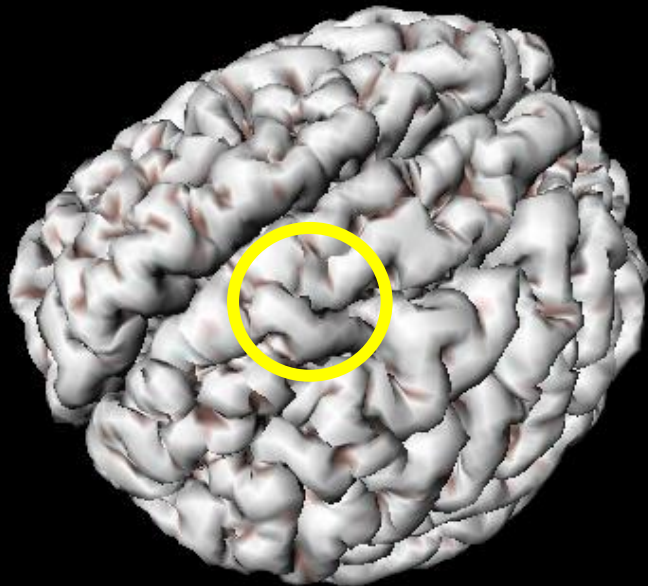
Dirichlet distribution on a triangle domain

**Exercise.** See if you can find a distribution on triangle B.

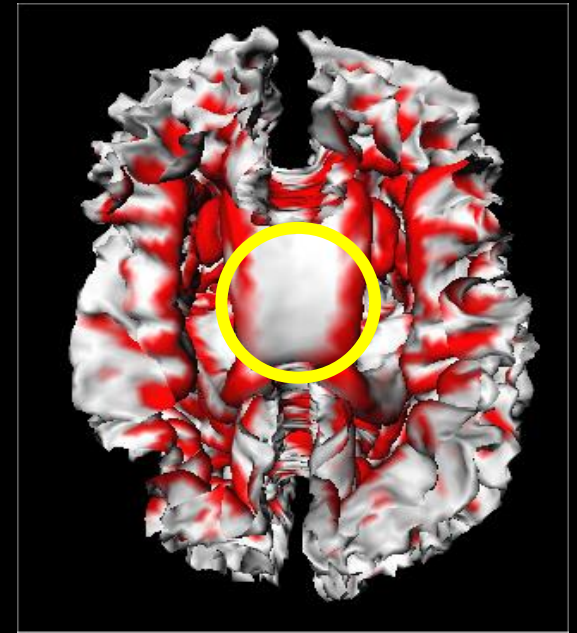
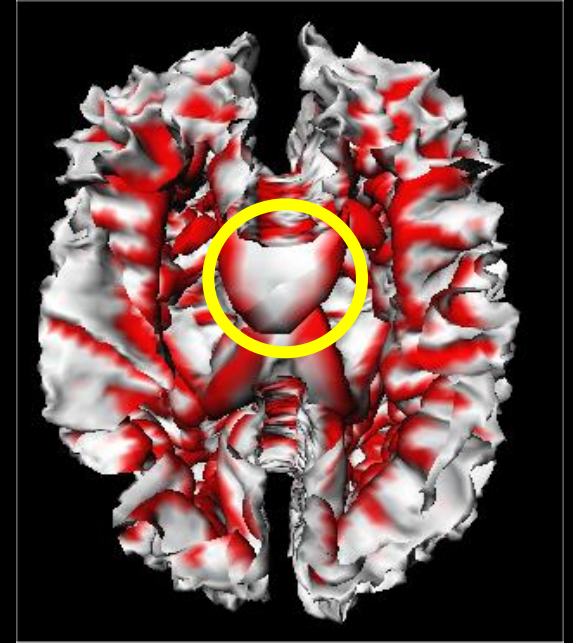
# How to quantify brain surface growth and atrophy?



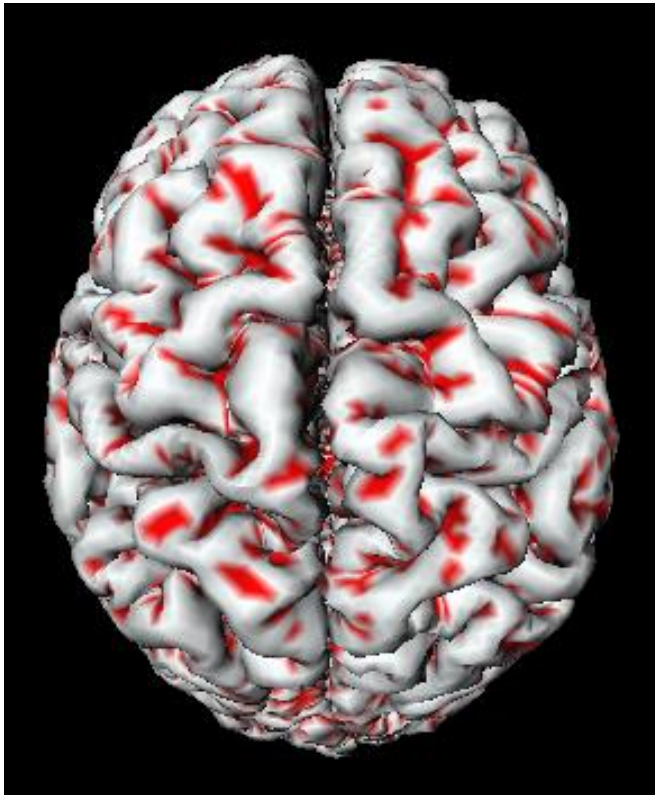
14 year old



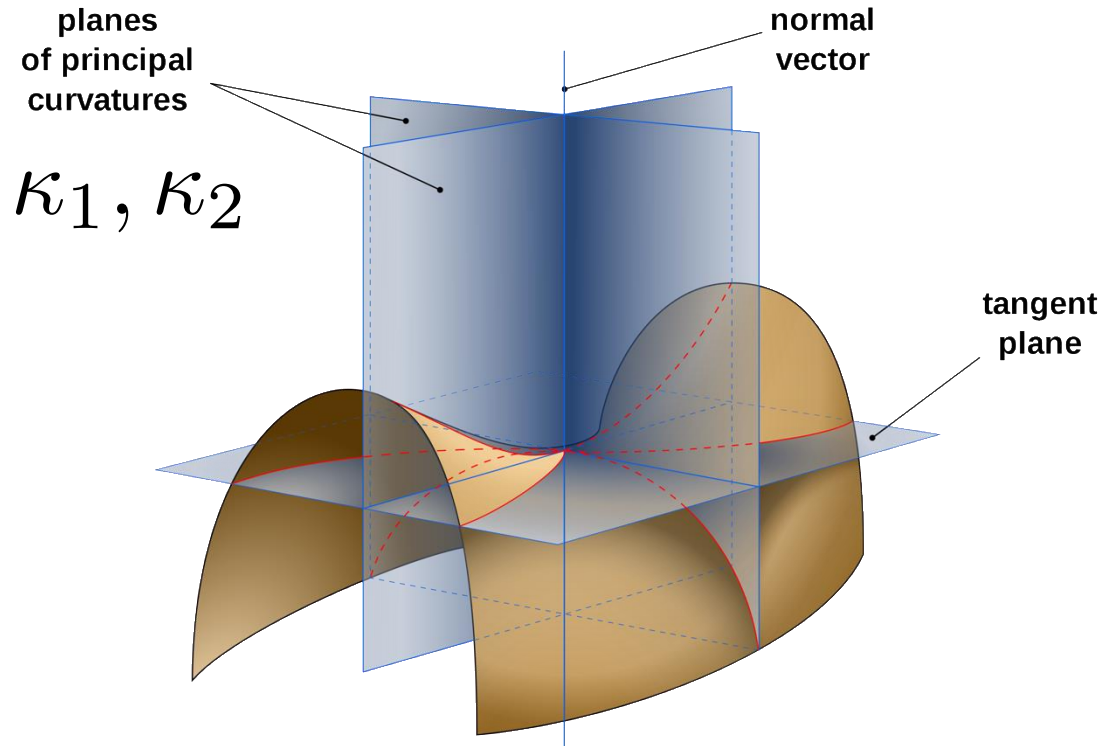
19 year old



# Mean and Gaussian curvatures



mean curvature  
Gaussian curvature



[Chung et al. 2003 CVPR Tensor-based surface modeling and analysis](#)



**Matlab  
demo**

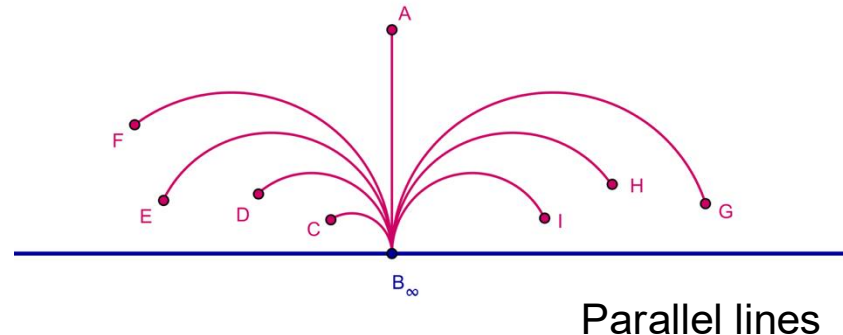
# Hyperbolic space

Non-Euclidean space with constant negative Gaussian curvature.  
Ex. Saddle surface

## Poincare half-plane

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

Metric: 
$$\frac{dx_1^2 + \dots + dx_n^2}{x_n^2}$$



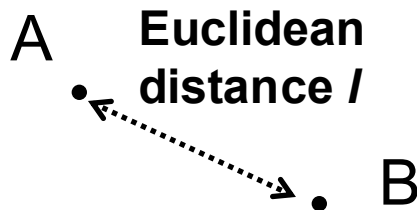
# Poincare disk

$$\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 < 1\}$$

$$r^2 = \|x\|_2^2 = x_1^2 + \dots + x_n^2$$

*You can have L2-norm as well. Why?*

Metric:  $4 \frac{r^2}{(1 - r^2)^2}$



$$\text{dist}(A, B) = \ln\left(\frac{1+l}{1-l}\right) = 2\text{artanh}(l)$$

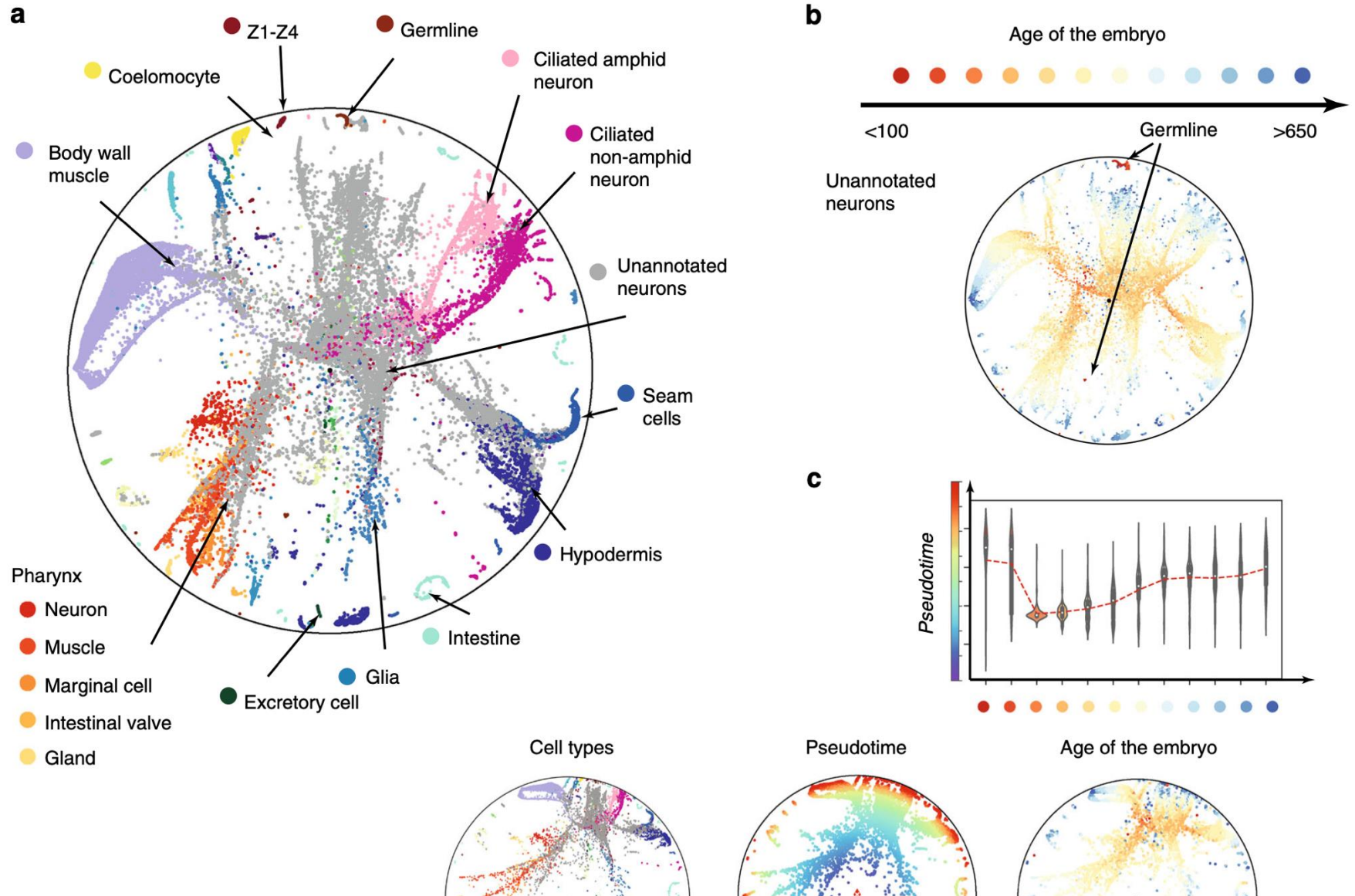
*Inverse hyperbolic function*

*Exercise: Compute the distance in the Poincare disk*

# Embedding graphs to Poincare disk

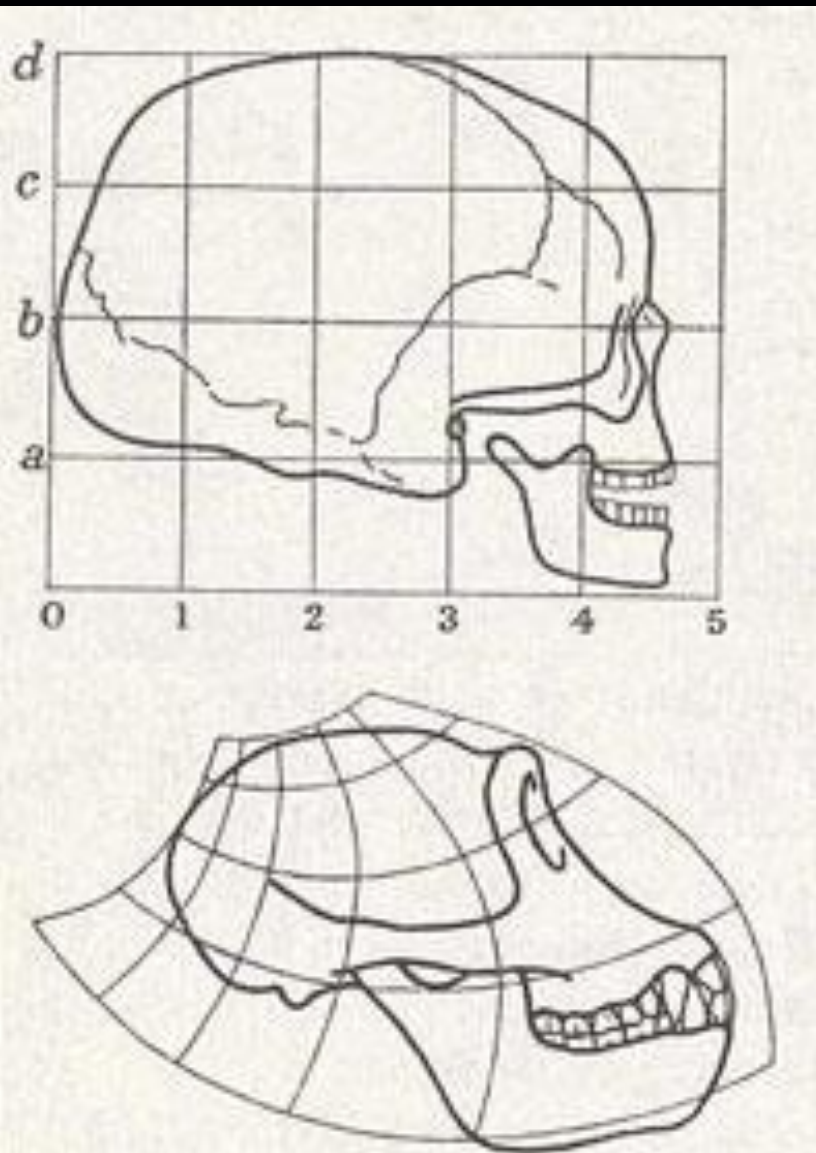
NATURE COMMUNICATIONS | <https://doi.org/10.1038/s41467-020-16822-4>

ARTICLE



# Deformable shape model

## D'Arcy Thompson 1860-1948



figuratively speaking, the

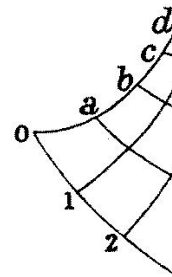


Fig. 178. Co-ordinates of the Cartesian

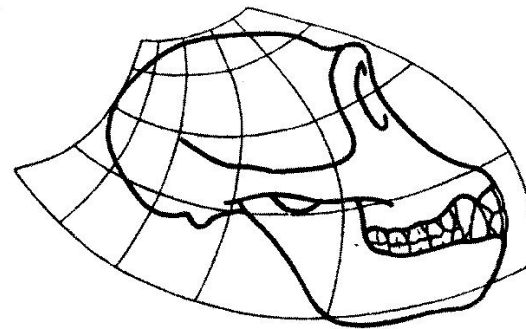
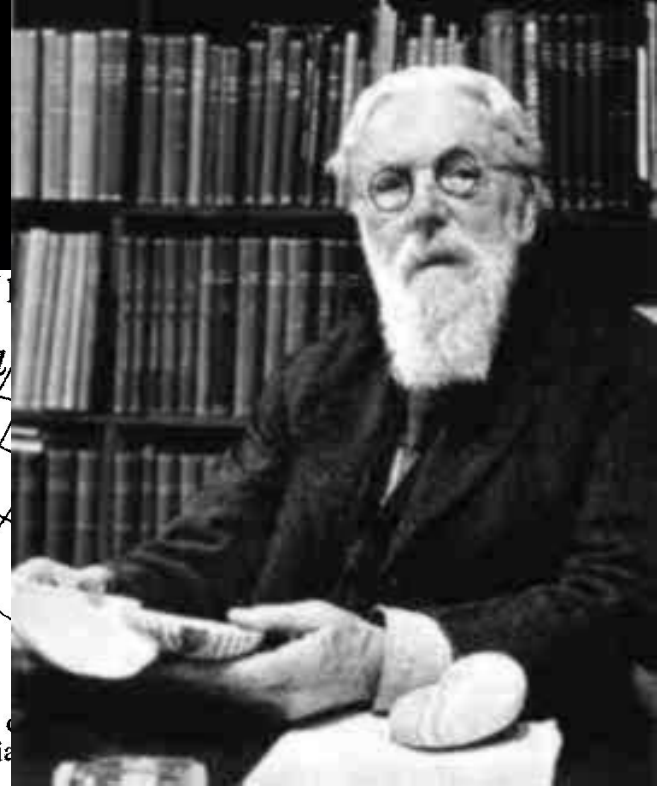


Fig. 179. Skull of chimpanzee.

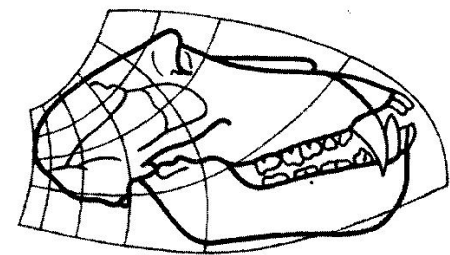


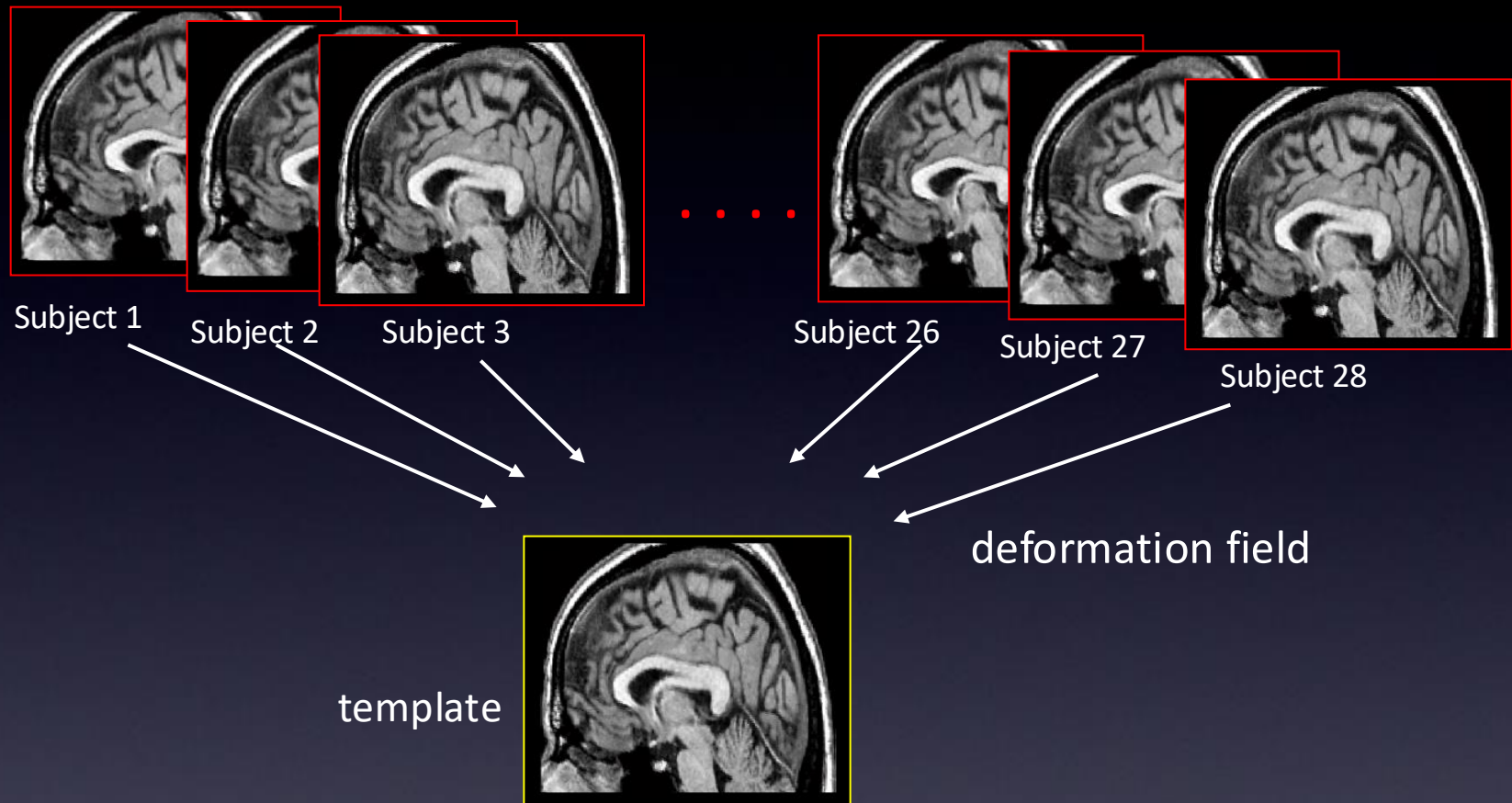
Fig. 180. Skull of baboon.

diagram  
I have sk  
is obviou  
differs or  
anthropo

**On Growth and Form**  
**D'Arcy Thompson**

In Fig. 180  
oon, and it  
order, and  
ion.<sup>1</sup> These  
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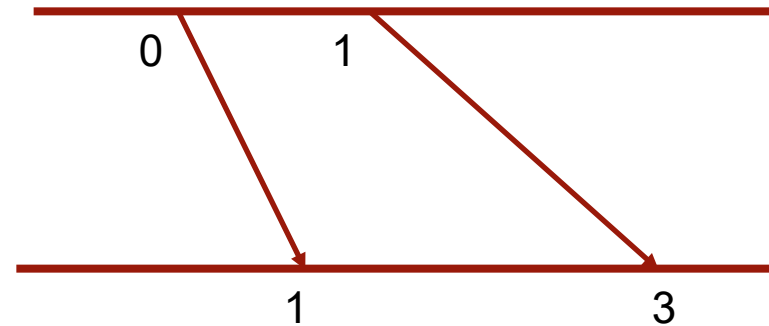
# Tensor-based morphometry framework



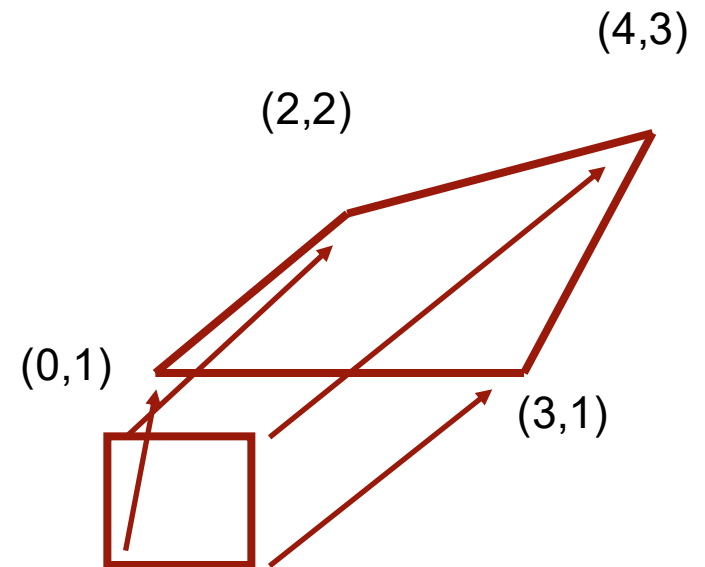
MRIs will be warped into a template and anatomical differences can be compared at a common reference frame.

# Jacobian determinant in Euclidean space

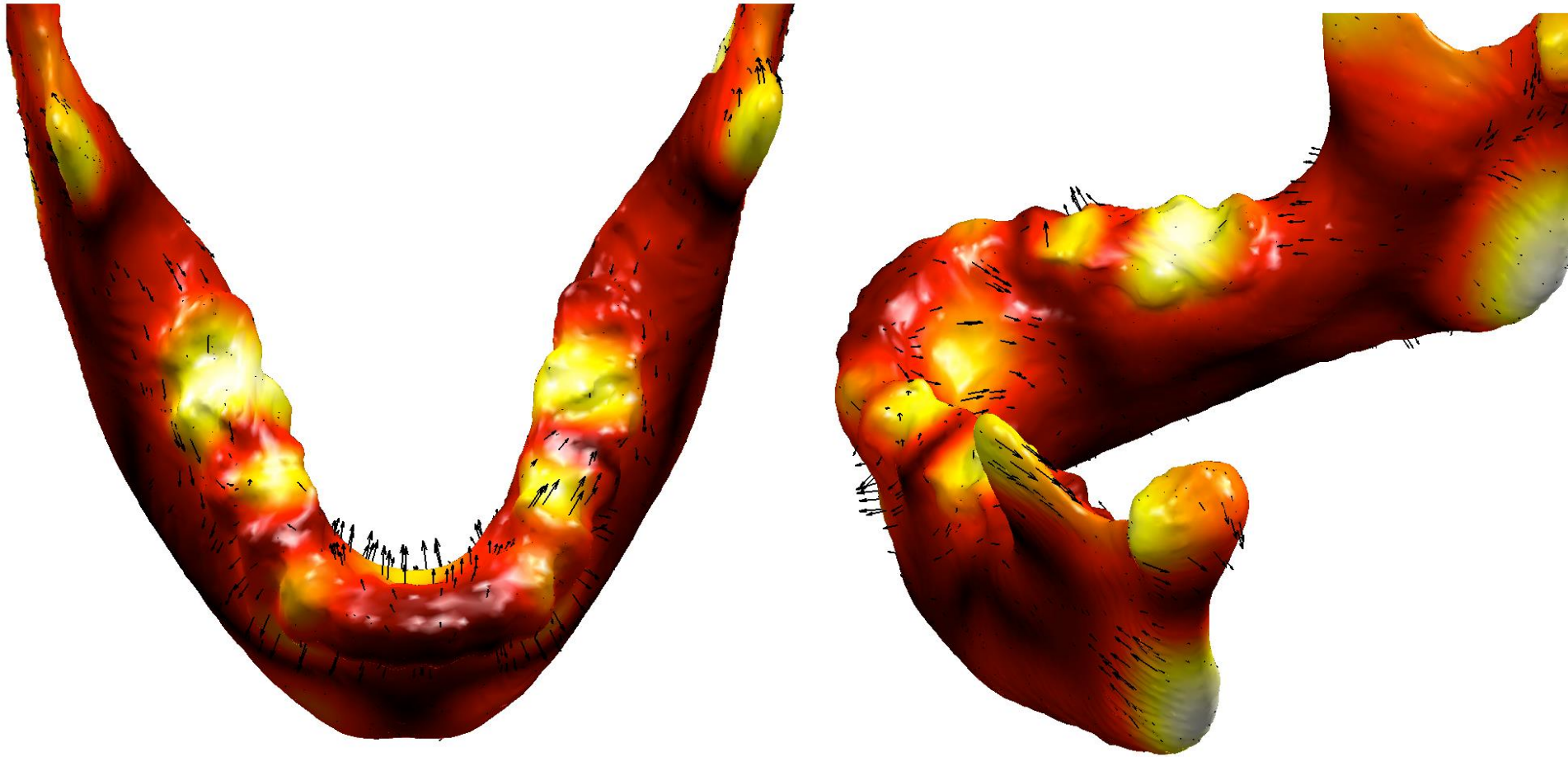
- 1D:  $x' = 2x + 1$   
 $J(x) = 2$



- 2D:  $x' = 2x + y + 1$   
 $y' = x + 2y$   
 $J(x, y) = 4 - 1 = 3$



# 3D Displacement vector field on surface template



# Computing Jacobian determinant via metric tensors

$$d_1, d_2, d_3 = d(x_1, x_2, x_3)$$

target position

Initial position

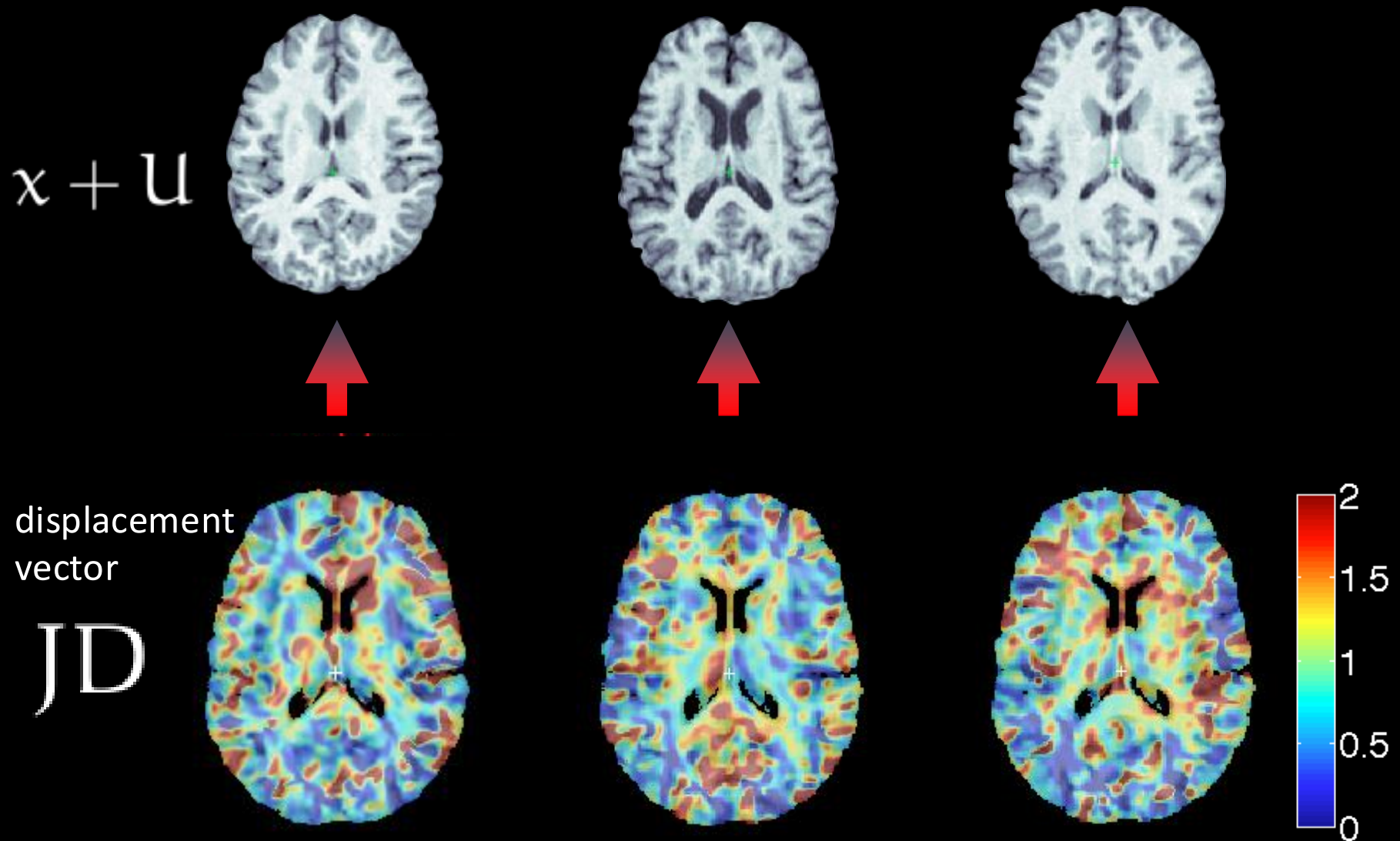
$$U(x_1, x_2, x_3) = d(x_1, x_2, x_3) - (x_1, x_2, x_3)$$

Displacement vector field

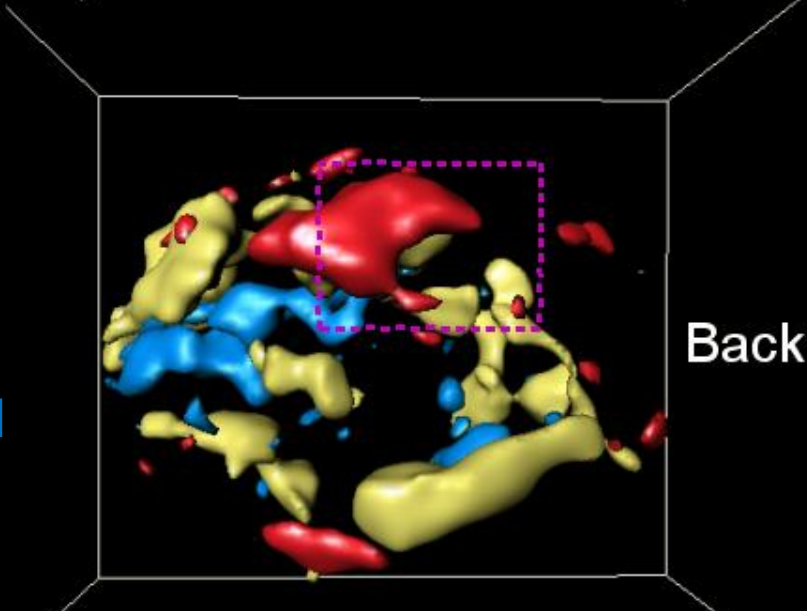
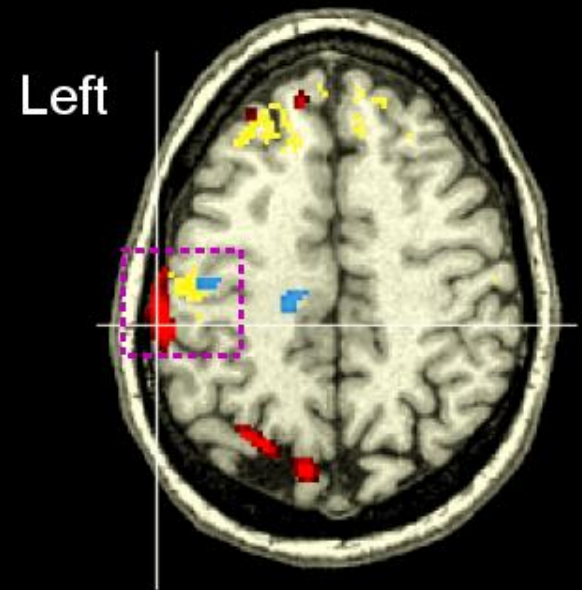
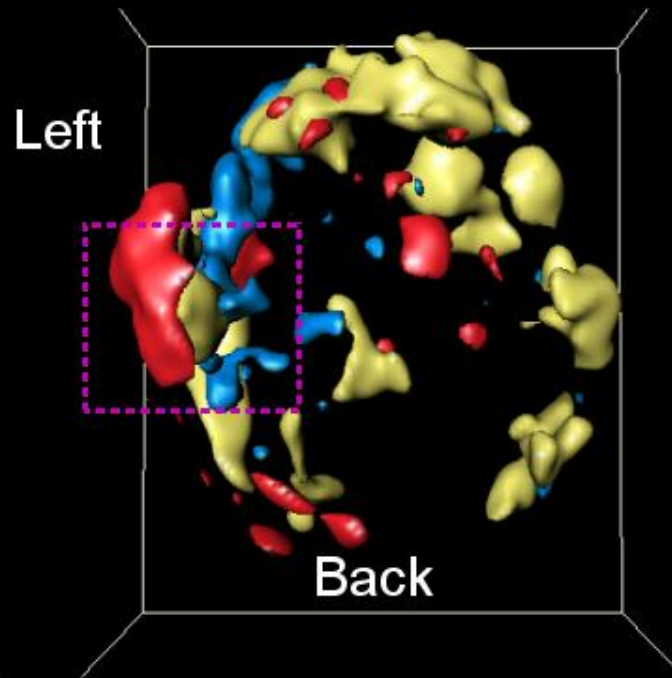
Jacobian determinant

$$J(x) = \det \frac{\partial d(x)}{\partial x} = \det \begin{bmatrix} \frac{\partial d}{\partial x_1} & \frac{\partial d}{\partial x_2} & \frac{\partial d}{\partial x_3} \\ \frac{\partial d}{\partial x_1} & \frac{\partial d}{\partial x_2} & \frac{\partial d}{\partial x_3} \\ \frac{\partial d}{\partial x_1} & \frac{\partial d}{\partial x_2} & \frac{\partial d}{\partial x_3} \end{bmatrix}$$

# Jacobian determinant in 3D



# Brain growth in children



Chung et al., 2001  
NeuroImage  
14:595-606

# 3D SPM

Back

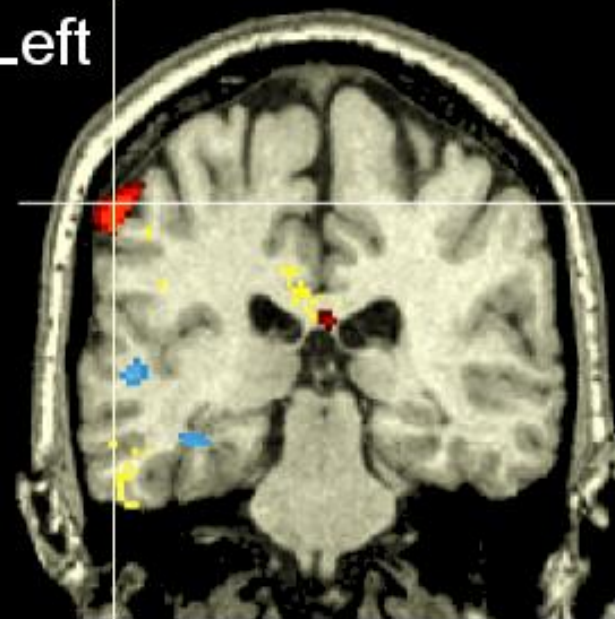
Right

Front

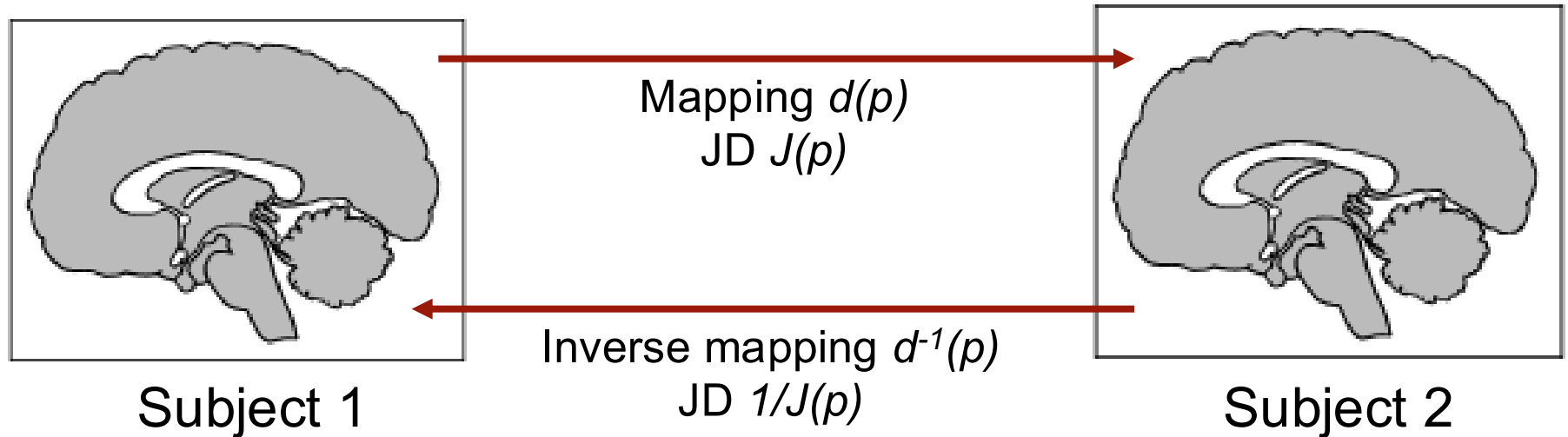
**Red:** Tissue growth  $p < 0.025$   
**Blue:** Tissue loss  $p < 0.025$   
**Yellow:** Structure displacement  $p < 0.05$



Left



# Statistical properties of Jacobian determinant



- $J(p) > 0$  for one-to-one mapping
- $J(p) > 1$  volume increase;  $J(p) < 1$  volume decrease
- Due to symmetry, the statistical distribution of  $J(p)$  and  $1/J(p)$  should be *identical*.

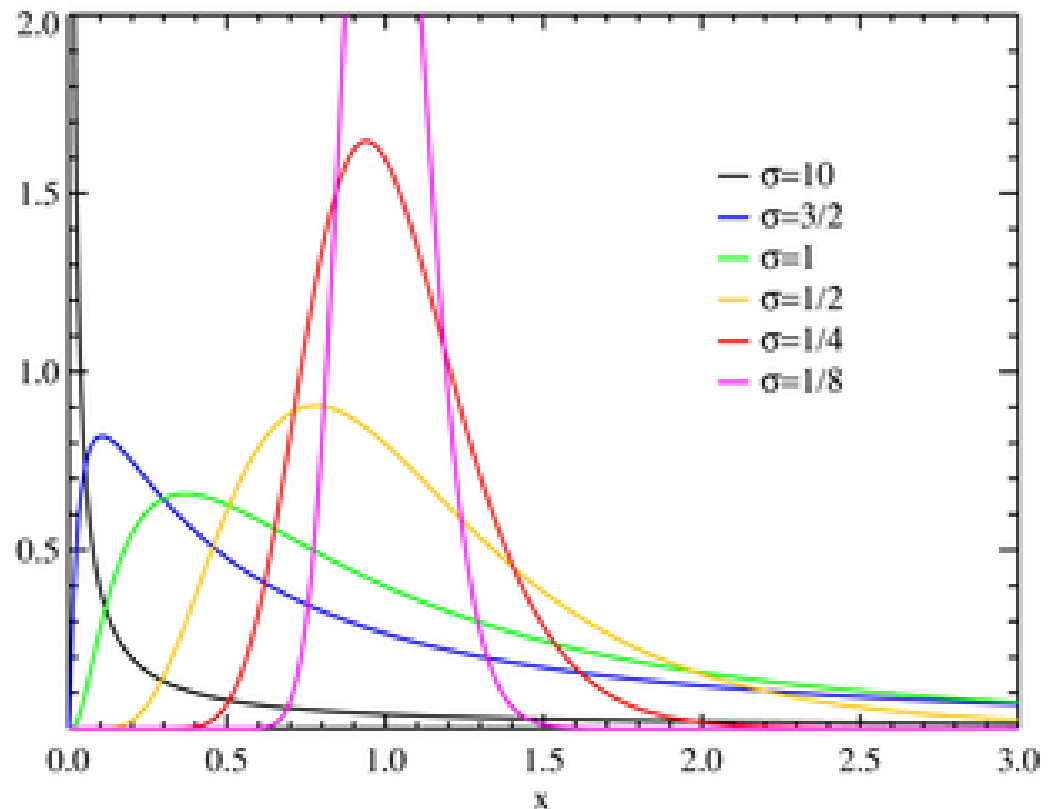
# Lognormality of JD

- Domain  $-\infty < \log J(p) < \infty$
- If  $J(p)=1$ ,  $\log J(p) = 0$
- Symmetry:  $\log[J^{-1}(p)] = -\log J(p)$
- These 3 properties show that JD can be modeled as *lognormal distribution*.

# Lognormal distribution

Random variable  $X$  is log-normally distributed if  $\log X$  is normally distributed.

*Question:* Some lognormal distribution looks normal so how do we check if data follows normal or lognormal?



# Unit Outcomes

- 1) Understand curve and surface parametrization
- 2) Understand Riemannian metric tensors
- 3) Compute Jacobian determinant

# Self assessment questions

1) Can you work out lecture materials using **exponential map**.

2) Given two surfaces, compute the **Jacobian determinant** of area change

2) Given displacement vector field, compute the Jacobian determinant.

**Research problem (Hard)**. Compute the surface normal vector of a given surface mesh.