Laplace Transform

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Abstract. The Laplace transform is a fundamental analytical tool for characterizing linear dynamical systems and smoothing operators through their spectral response. We present the Laplace transform as a principled framework for defining and interpreting low–pass filtering and exponential smoothing in dynamic data. By formulating smoothing operators as solutions to linear differential equations, the Laplace transform establishes a direct correspondence between time–domain dynamics and frequency–domain attenuation, yielding physically and statistically interpretable parameters such as decay rates and characteristic time scales. This provides the mathematical foundation for Laplace Transform based smoothing and related filtering techniques for dynamic data.

1 Laplace Transform in One Dimension

Let $f(t)$ be a real–valued function defined for $t \geq 0$ that is piecewise continuous and of at most exponential growth. The one–dimensional Laplace transform of f is defined as (Courant and Hilbert 1953, Hassani 1991, Widder 1945)

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt, \qquad s \in \mathbb{R}.$$

The Laplace transform measures the behavior of a time–domain signal $f(t)$ by comparing it against exponentially decaying test functions e^{-st}. For large values of s, the exponential term decays rapidly, so the transform emphasizes the early–time behavior of $f(t)$. For smaller values of s, the decay is slower, allowing contributions from later times to influence the integral. In this sense, s controls how strongly the distant past is discounted, and $F(s)$ encodes how the signal accumulates under exponential forgetting. Unlike the Fourier transform (Courant and Hilbert 1953, Hassani 1991), which decomposes a signal into oscillatory components, the Laplace transform emphasizes exponential growth and decay, thereby revealing stability properties and transient behavior. This makes it particularly well suited for analyzing the persistence and stability of dynamical systems.

Here are a few examples of the Laplace transform computed and displayed in MATLAB (Fig. 1). For input signal

$$f(t) = e^{-2t}, \qquad t \geq 0,$$

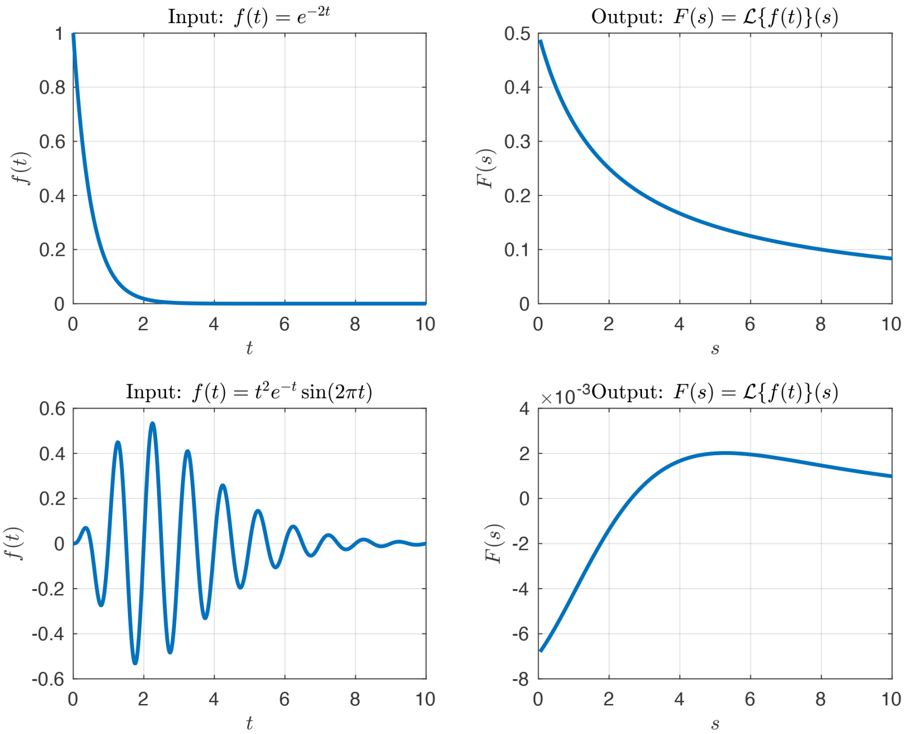

Fig. 1: Laplace transform. Time domain input $f(t)$ (left) and Laplace domain output $F(s) = \mathcal{L}\{f(t)\}(s)$ (right). **Top** shows a simple exponentially decaying signal $f(t) = e^{-2t}$, whose Laplace transform computed by `laplace()` is $F(s) = 1/(s+2)$. **Bottom** shows a more complex signal $f(t) = t^2 e^{-t}\sin(2\pi t)$ and its transform.

the corresponding Laplace transform is

$$F(s) = \int_0^\infty e^{-st} e^{-2t}\, dt = \frac{1}{s+2}.$$

It can be computed using MATLAB's symbolic language:

```
syms t s
f = exp(-2*t);
F = laplace(f, t, s);

tvals = 1:0.01:10;
fvals = subs(f, t, tvals);
plot(tvals, fvals)
```

A more complex example, shown in Fig. 1 (bottom row), is $f(t) = t^2 e^{-t} \sin(2\pi t)$, whose Laplace transform is given symbolically in Matlab as

$$F(s) = \frac{4\pi(2s+2)^2}{\big((s+1)^2 + 4\pi^2\big)^3} - \frac{4\pi}{\big((s+1)^2 + 4\pi^2\big)^2}.$$

A more complex example is $f(t) = e^{-2t}, (t \geq 0)$ with the corresponding Laplace transform

$$F(s) = \int_0^\infty e^{-st} e^{-2t}\, dt = \frac{1}{s+2}$$

is given in Fig. 1-bottom.

In general, when data are too complex to admit a simple analytic representation, we approximate the underlying signal by projecting it onto a structured family of basis functions $\{\phi_k\}$ whose Laplace transforms are analytically known. Suppose that $f(t)$ is represented as

$$f(t) \;\approx\; \sum_{k=1}^{K} c_k\, \phi_k(t).$$

Then we have

$$\mathcal{L}\{f\}(s) \;\approx\; \sum_{k=1}^{K} c_k\, \mathcal{L}\{\phi_k\}(s).$$

The coefficients c_k serve as compact, task–relevant features, enabling downstream modeling, prediction, or inference to be carried out directly in the coefficient space rather than on the raw signal.

Exponentials $\phi_k(t) = e^{-\alpha_k t}$ are the most natural basis (Widder 1945), as their Laplace transforms are obtained in closed form as

$$\mathcal{L}\{e^{-\alpha_k t}\}(s) = \frac{1}{s + \alpha_k}.$$

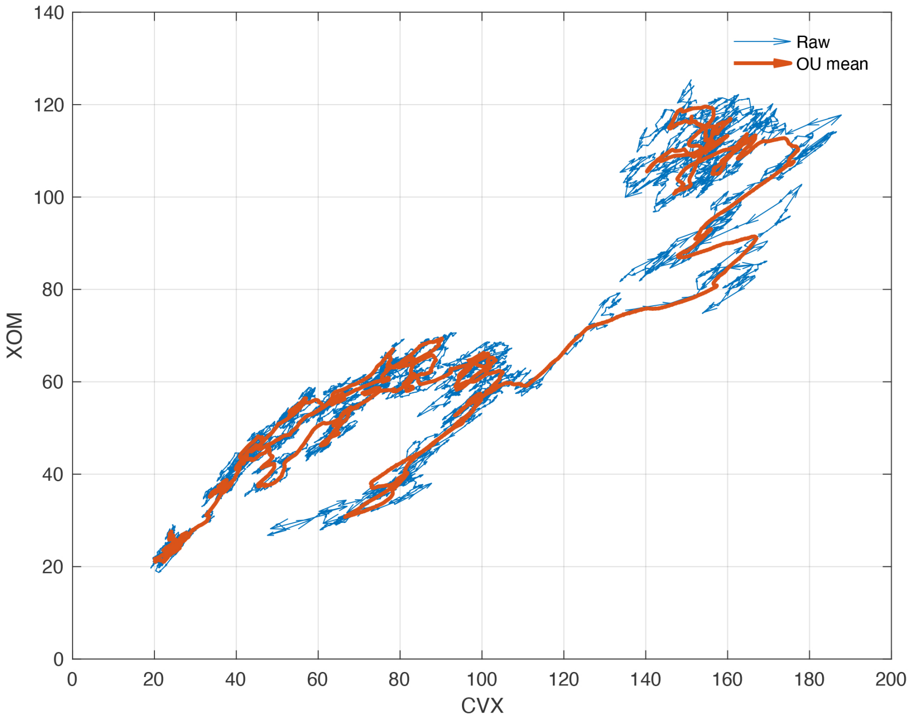

Fig. 2: Time–varying vector field of stock prices and its Ornstein–Uhlenbeck (OU) mean. Daily closing prices of Chevron (CVX) and Exxon Mobil (XOM) are embedded as trajectories in the (CVX, XOM) price plane. Arrows represent one–day price increments $(x_{t+1} - x_t,\ \ y_{t+1} - y_t)$, yielding a time–dependent velocity field. Thin arrows show the raw day–to–day price changes, which are dominated by high–frequency fluctuations. Thick arrows show the Ornstein–Uhlenbeck mean (with $\lambda = 20$, trading–day), revealing persistent co–movement and the dominant drift structure shared by the two stocks.

Bibliography

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