

The Waisman Laboratory
for Brain Imaging and Behavior



University of Wisconsin
**SCHOOL OF MEDICINE
AND PUBLIC HEALTH**

BMI / STAT - 768
Statistical Methods
for
Medical Image
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University of Wisconsin-Madison

<https://github.com/laplcebeltrami/BMI768>

Geometric Data Analysis

Geometric Data Analysis (GDA) is a field that focuses on understanding and interpreting the geometric structures inherent in complex datasets. It leverages mathematical and statistical techniques to explore properties such as shape, curvature, and topology, enabling the identification of patterns and relationships within the data. GDA encompasses methods like manifold learning, principal component analysis, and dimensionality reduction, which help in visualizing high-dimensional data in more interpretable forms.

Data embedding

Data manifolds

Statistics on
manifolds

Spectral clustering

Spectral geometry

Information
geometry

Covariance
matrix

Symmetric positive
definite matrices

PCA

Functional-PCA

GDA

quantifies
the shape
of data

Unit Objectives

- 1) Understand curve and surface parametrization
- 2) Understand Riemannian metric tensors
- 3) Know how to compute Jacobian determinant

Use the *exponential map* example in understanding these concepts

References

Widely used
basic textbook
on differential
geometry

Differential Geometry of Curves and Surfaces

Manfredo P. do Carmo

*Instituto de Matematica Pura e Aplicada (IMPA)
Rio de Janeiro, Brazil*



manfredo.carmo@impa.br

Differential Geometry of a Curve

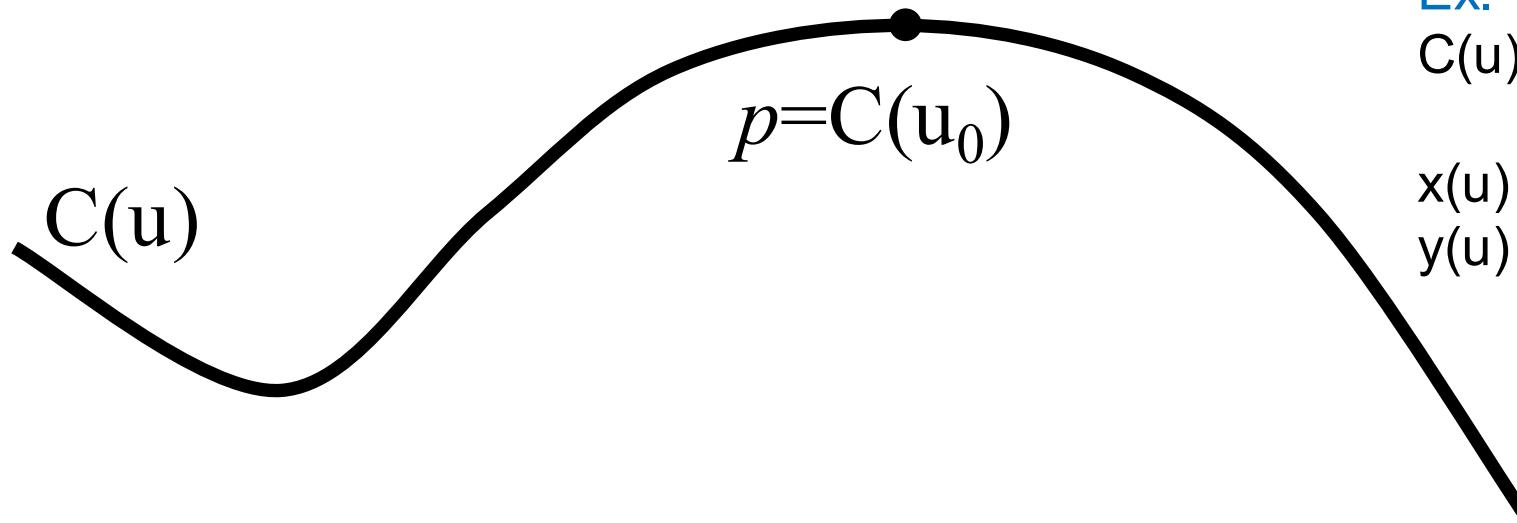
Point p on the curve at u_0

Smooth curves can
be parameterized.

Ex.

$$C(u) = (x(u), y(u))$$

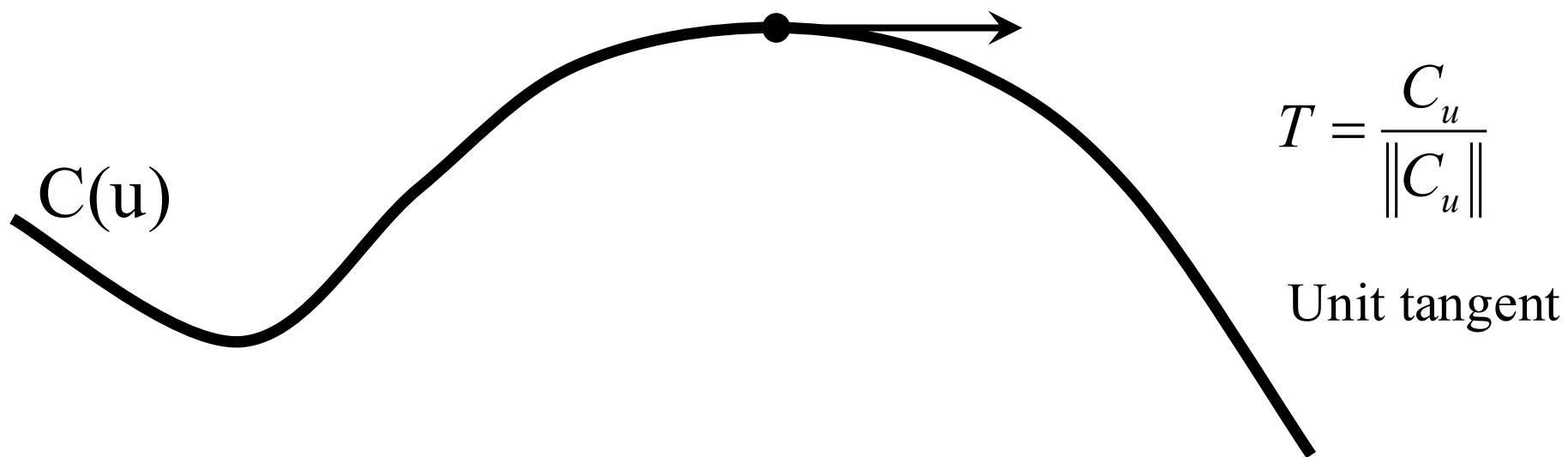
$$\begin{aligned}x(u) &= \cos(u) \\y(u) &= \sin(u)\end{aligned}$$



Tangent and normal vectors

Tangent T to the curve at u_0

$$C_u = \frac{\partial C(u)}{\partial u}$$



$$T = \frac{C_u}{\|C_u\|}$$

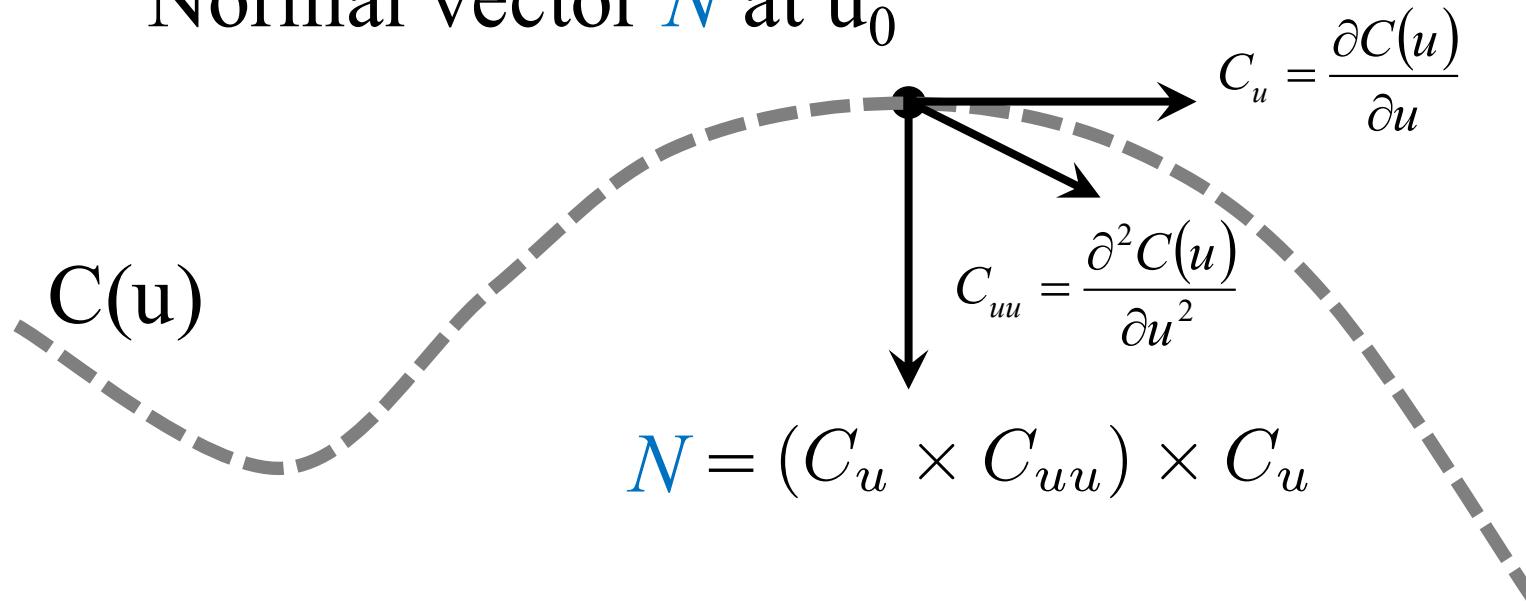
Ex.

$$x(u) = \cos(u)$$

$$y(u) = \sin(u)$$

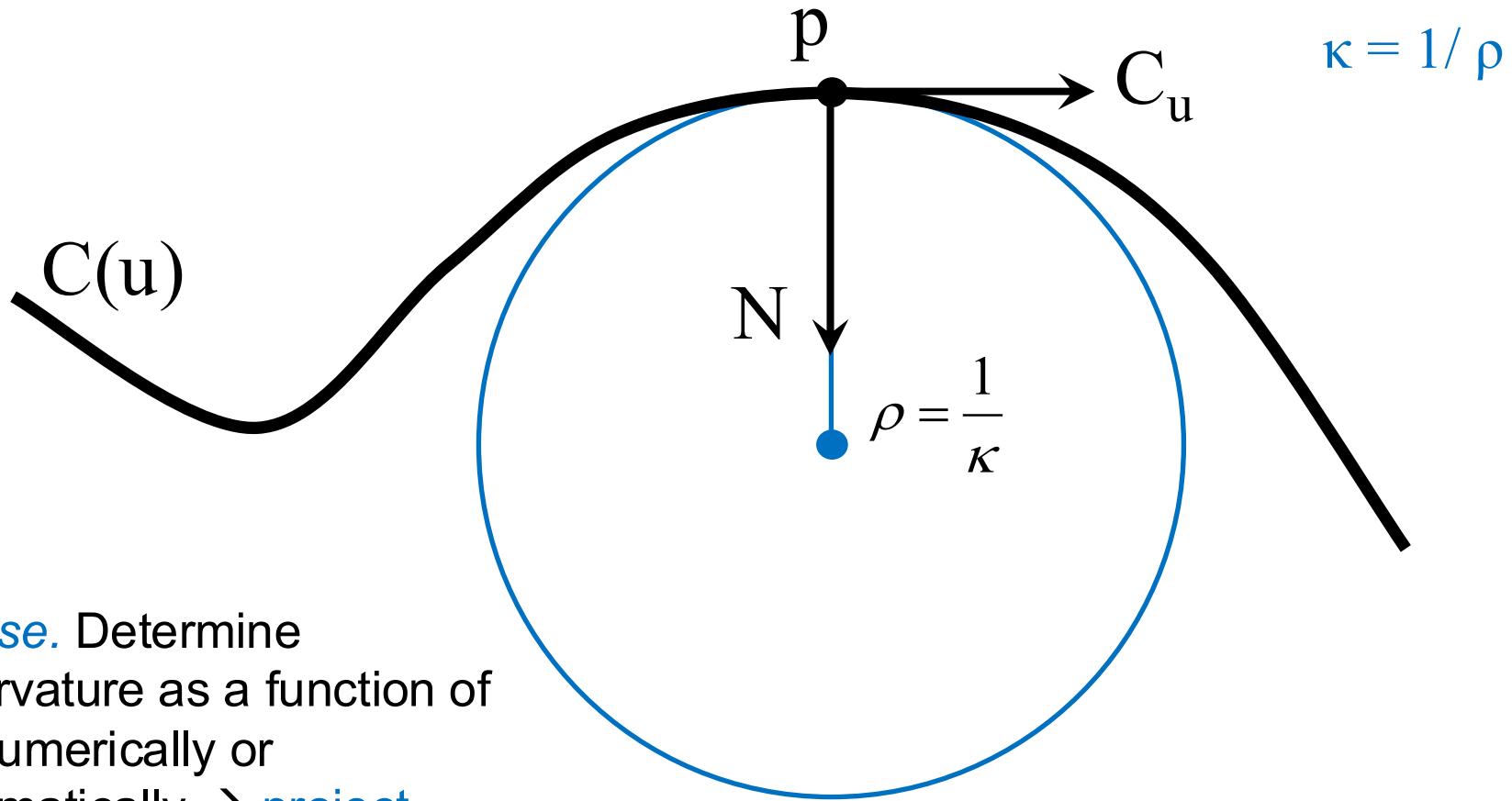
Normal vector of a curve

Normal vector N at u_0



1D curve has only one curvature

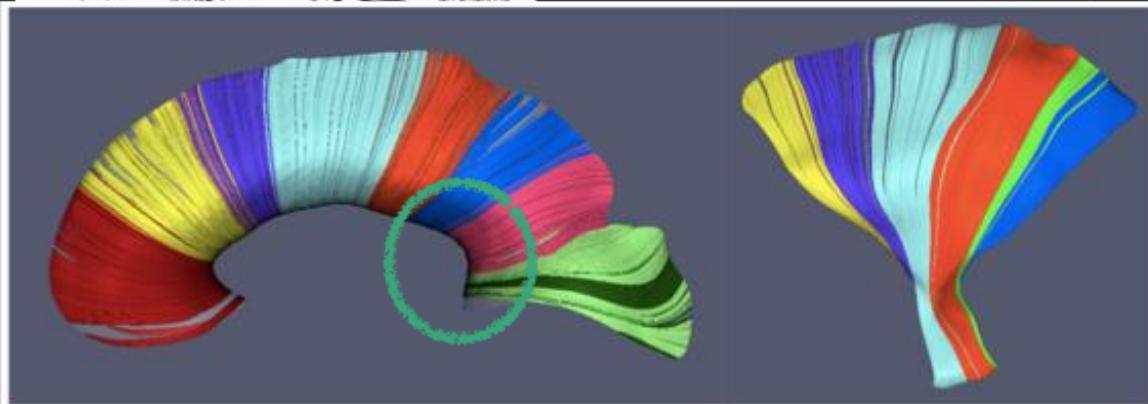
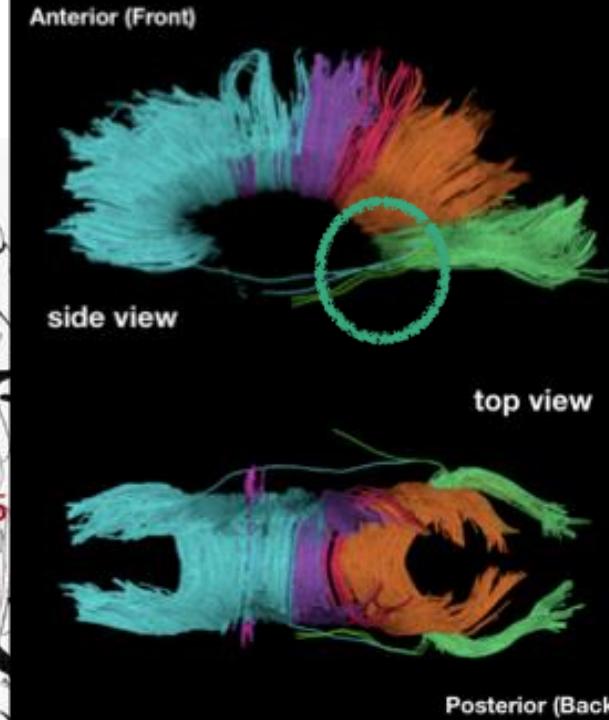
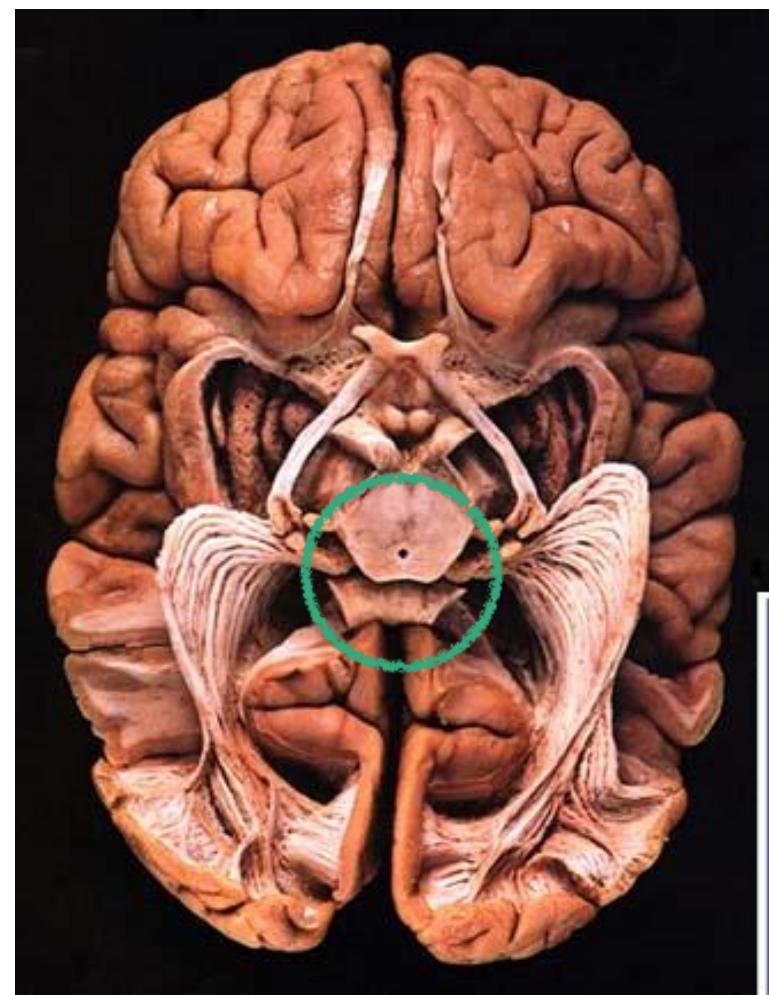
Curvature κ at u_0 and the radius ρ osculating circle



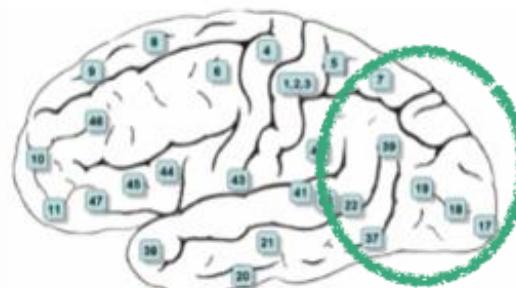
Exercise. Determine
the curvature as a function of
 $C(u)$ numerically or
mathematically → project

White matter fibers

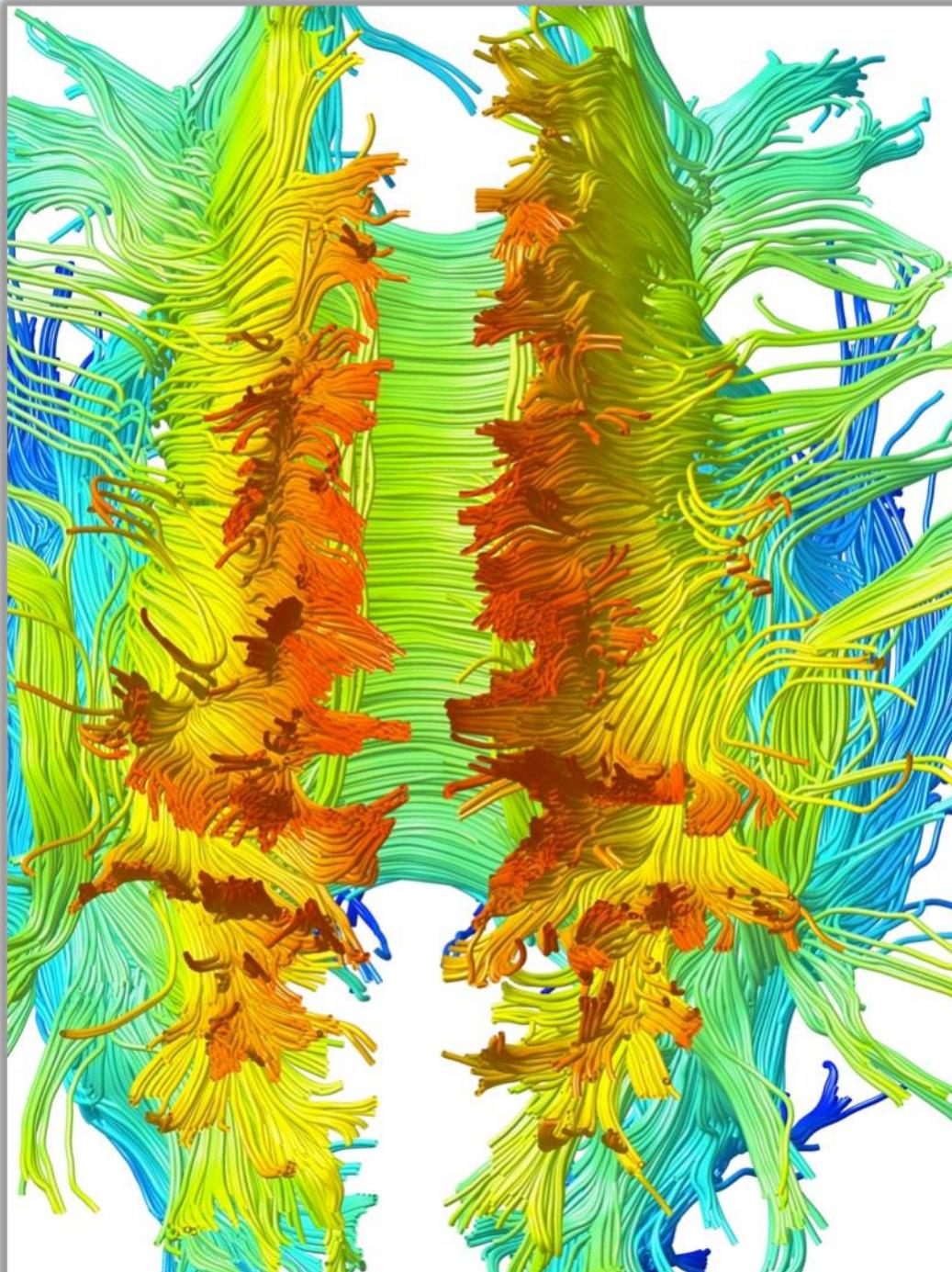
James Gee
Univ. Penn



Fibers passing through
the splenium of the
corpus callosum



Brodmann Area 2	Brodmann Area 8
Brodmann Area 4	Brodmann Area 9
Brodmann Area 5	Brodmann Area 10
Brodmann Area 6	Brodmann Area 18
Brodmann Area 7	Brodmann Area 19



Cosine series representation

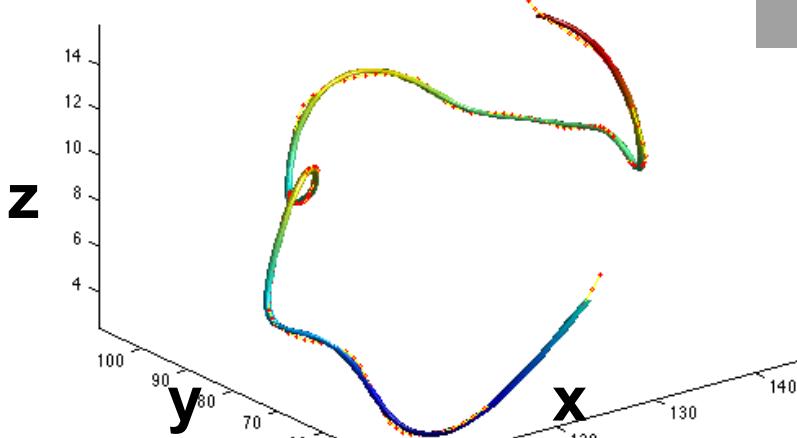
Cover art

Chung, M.K. 2012

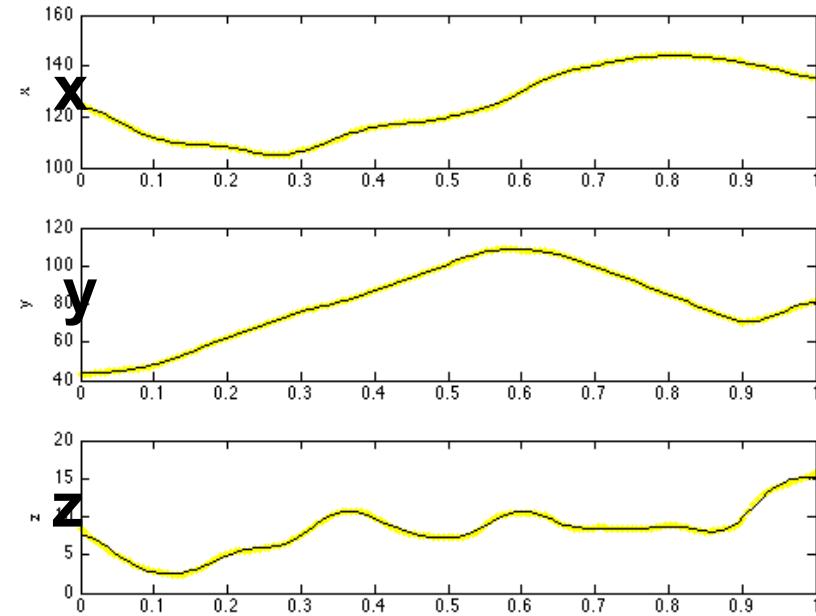
*Computational Neuroanatomy:
The Methods, World Scientific
Press*

Matlab visualization

White matter fiber tract data



parameterization



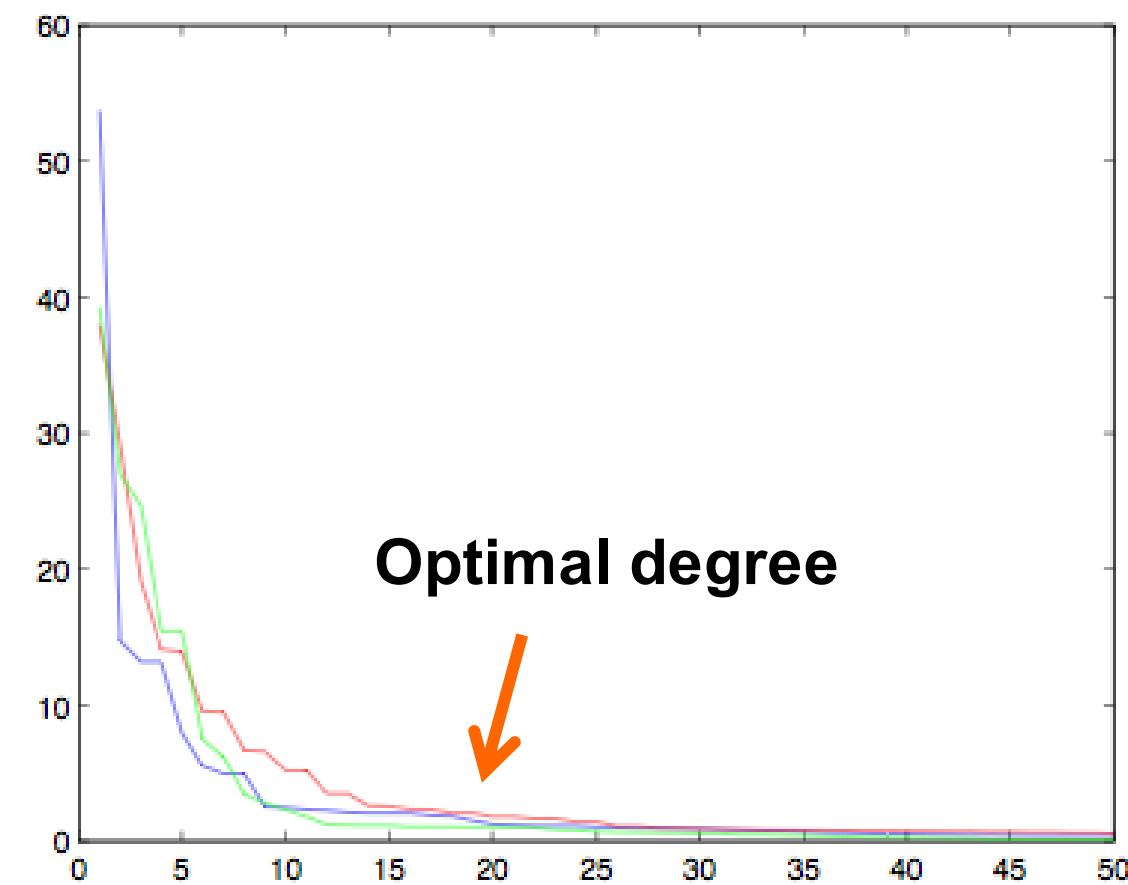
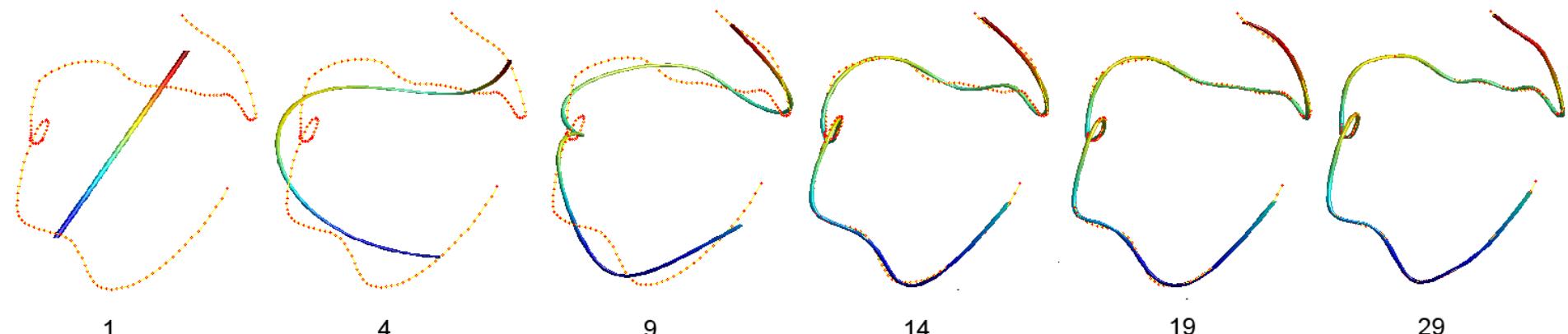
88.1799	56.6336	5.7367
-12.4775	-11.2552	-2.0791
2.4336	-15.4428	-0.4021
4.3956	2.2733	-0.9354
-0.0106	-0.0674	0.6999
2.1773	-2.4194	-0.1176
0.5808	0.8390	1.2942
0.0615	-0.1893	0.1188
-0.2629	0.7524	0.1089
0.7909	-0.7276	-0.1901
0.5458	0.6236	0.6939
0.4295	-0.4337	0.2185
0.2150	0.4157	0.0254
0.1584	-0.1973	0.0762
-0.1557	0.2466	-0.1086
0.0632	-0.0978	-0.0208
0.0389	-0.0143	-0.0284
-0.0014	-0.1193	0.1970
0.0004	0.0129	-0.0198
0.1342	0.0002	0.0260

Any tract can be compactly parameterized with only 60 coefficients.
Why?

basis expansion

→ $(x, y, z)' = \sum_{l=0}^{19} \beta_l \cos(l\pi t)$

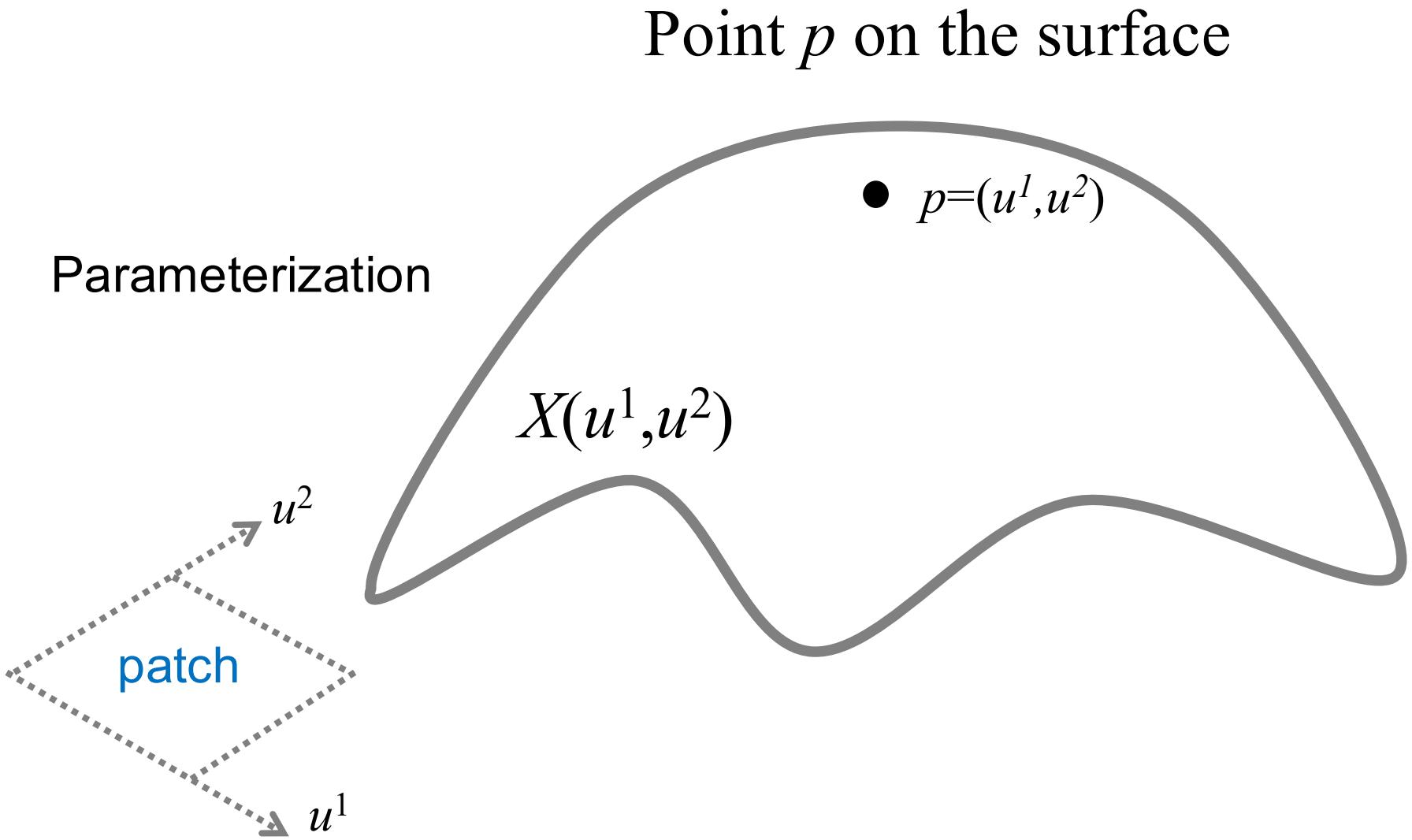
Cosine series representation



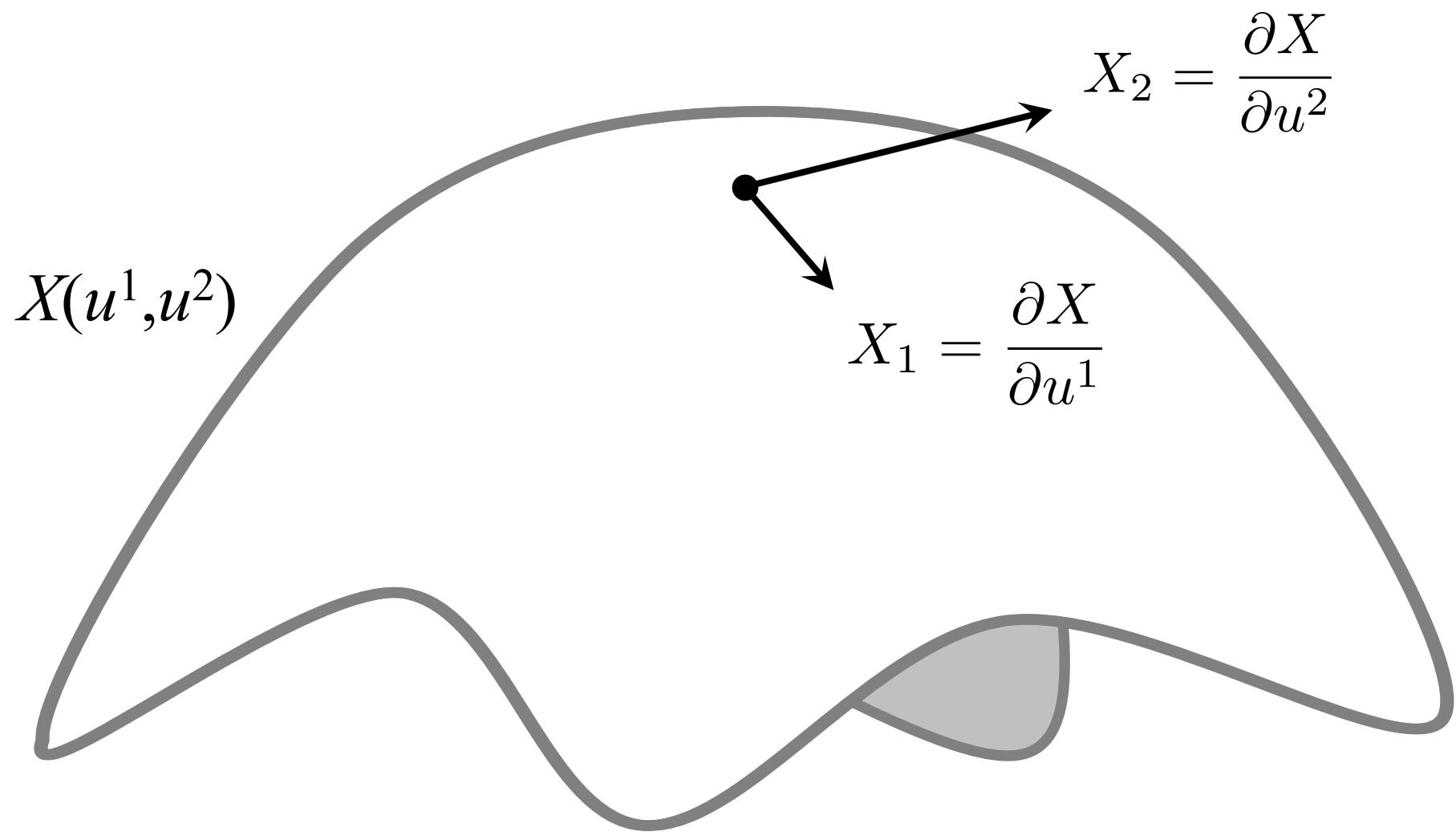
The optimal degree chosen using the forward model selection.

Exercise. Compute Curvature and display on top of 3D curve.

Differential Geometry of a Surface



Tangent vectors



Example: quadratic polynomial surface

$$x = u^1$$

$$y = u^2$$

$$z = \beta_0 + \beta_1 u^1 + \beta_2 u^2 + \beta_3 u^1 u^2 + \beta_4 (u^1)^2 + \beta_5 (u^2)^2$$

$$X(u^1, u^2) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u^1 \\ u^2 \\ \beta_0 + \beta_1 u^1 + \dots + \beta_5 (u^2)^2 \end{pmatrix}$$

$$\frac{\partial X}{\partial u^1} = \begin{pmatrix} 1 \\ 0 \\ \beta_1 + \beta_3 u^2 + 2\beta_4 u^1 \end{pmatrix}$$

At origin $(u^1, u^2) = (0, 0)$,

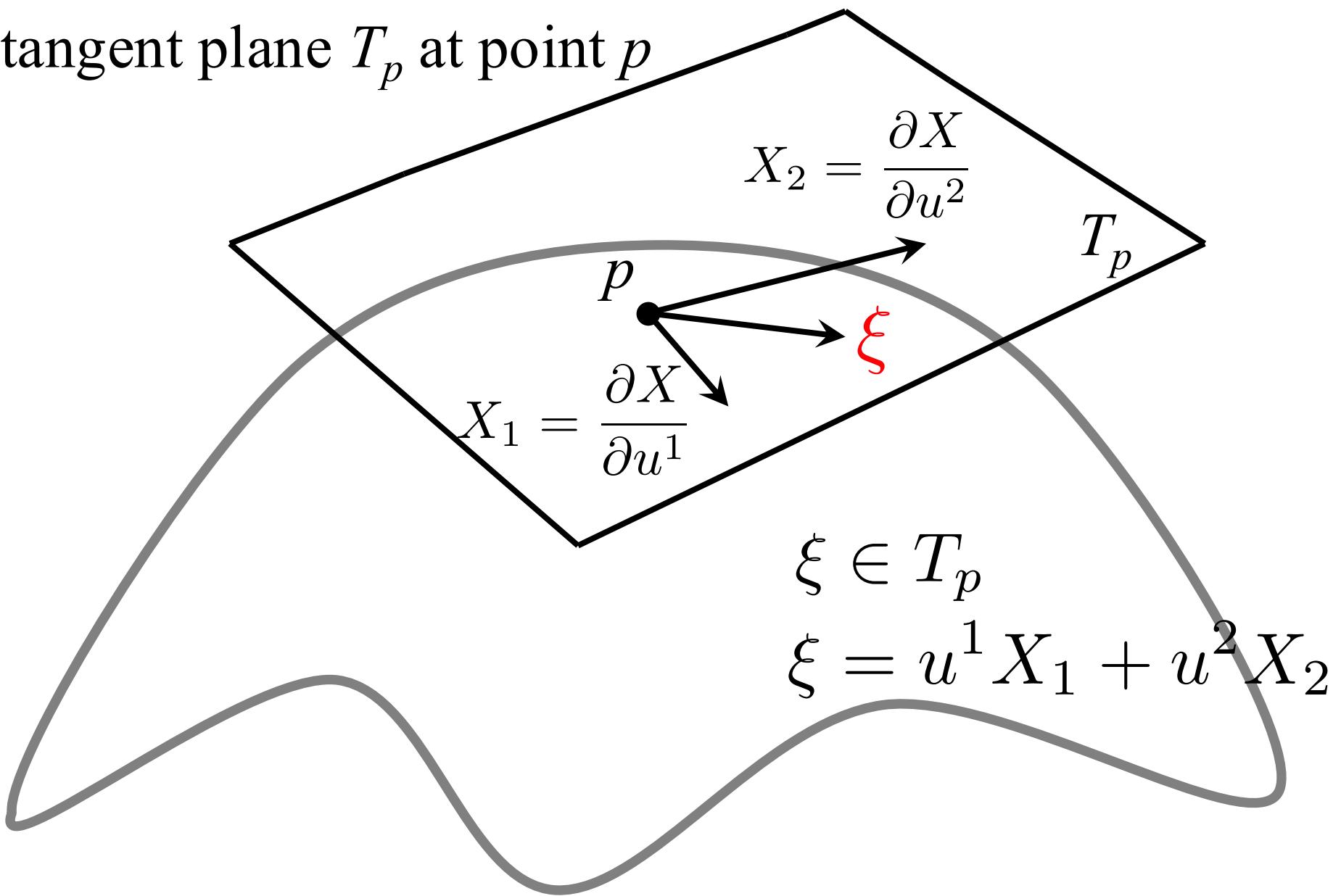
$$\frac{\partial X}{\partial u^2} = \begin{pmatrix} 0 \\ 1 \\ \beta_2 + \beta_3 u^1 + 2\beta_5 u^2 \end{pmatrix}$$

$$\frac{\partial X}{\partial u^1} = \begin{pmatrix} 1 \\ 0 \\ \beta_1 \end{pmatrix}$$

$$\frac{\partial X}{\partial u^2} = \begin{pmatrix} 0 \\ 1 \\ \beta_2 \end{pmatrix}$$

Tangent plane

tangent plane T_p at point p



First Fundamental Form

Differential form

$$d\xi = du^1 X_1 + du^2 X_2$$

$$d\xi^2 =^d \langle d\xi, d\xi \rangle = \sum_{i,j} \langle X_i, X_j \rangle du^i du^j$$

*Riemannian
metric*

Metric tensor

$$g_{ij} = \langle X_i, X_j \rangle \longrightarrow g = (g_{ij})$$

Example: quadratic surface

$$x = u^1$$

$$y = u^2$$

$$z = \beta_0 + \beta_1 u^1 + \beta_2 u^2 + \beta_3 u^1 u^2 + \beta_4 (u^1)^2 + \beta_5 (u^2)^2$$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ \beta_1 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ \beta_2 \end{pmatrix}$$

$$g_{11} = \langle X_1, X_1 \rangle = \|X_1\|^2 = 1 + \beta_1^2$$

$$g_{12} = \langle X_1, X_2 \rangle = \beta_1 \beta_2$$

$$g_{22} = \langle X_2, X_2 \rangle = \|X_2\|^2 = 1 + \beta_2^2$$

$$\rightarrow g = \begin{pmatrix} 1 + \beta_1^2 & \beta_1 \beta_2 \\ \beta_1 \beta_2 & 1 + \beta_2^2 \end{pmatrix}$$

positive
definite
symmetric

Riemannian metric tensor

Riemannian manifold (M, g) is a smooth manifold M with inner product g on the tangent space $T_p(M)$. At each point p , *the metric varies smoothly in such a way that if X_1 and X_2 are differentiable tangent vectors, g is also a smooth function.* The family of such inner products is called a Riemannian metric tensor.

If the metric doesn't vary, it's just the boring Euclidean space.

Riemannian metric tensor

Question. If g_1, g_2, \dots, g_n are metrics, under what condition $c_1 g_1 + c_2 g_2 + \dots + c_n g_n$ is metric as well?

Area element

parameter space $N \rightarrow$ manifold M

$$\text{Area of manifold } M = \int_M d\mu(p) = \int_N \sqrt{\det g} du^1 du^2$$



Local surface = Jacobian
area element determinant

$$J = \sqrt{\det g}$$

measures the amount of area with respect to the unit **patch area**

Exercise: Show surface area is invariant under parameterization

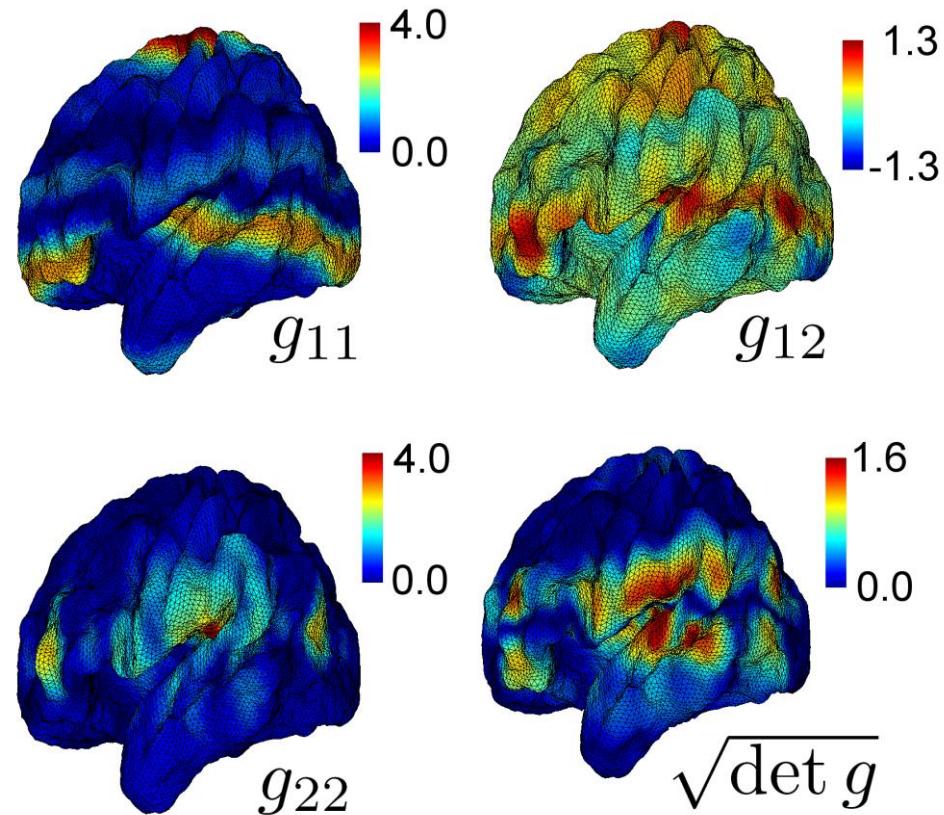
Tensor-based morphometry

Surface parameterization:

$$u = (u^1, u^2) \square X(u)$$

Riemannian metric tensor:

$$g_{ij} = \left\langle \frac{\square X}{\square u^i}, \frac{\square X}{\square u^j} \right\rangle$$



Chung et al. 2003 *NeuroImage* 18:198-213

Chung et al. 2008. *IEEE Transactions on Medical Imaging* 27:1143-1151

Jacobian matrix and its determinant

$$\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\mathbf{x} = \{x_1, \dots, x_n\} \in \mathbb{R}^n$$

Jacobian matrix

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \dots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Not symmetric

Jacobian determinant

$$\det \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

Can we equate?

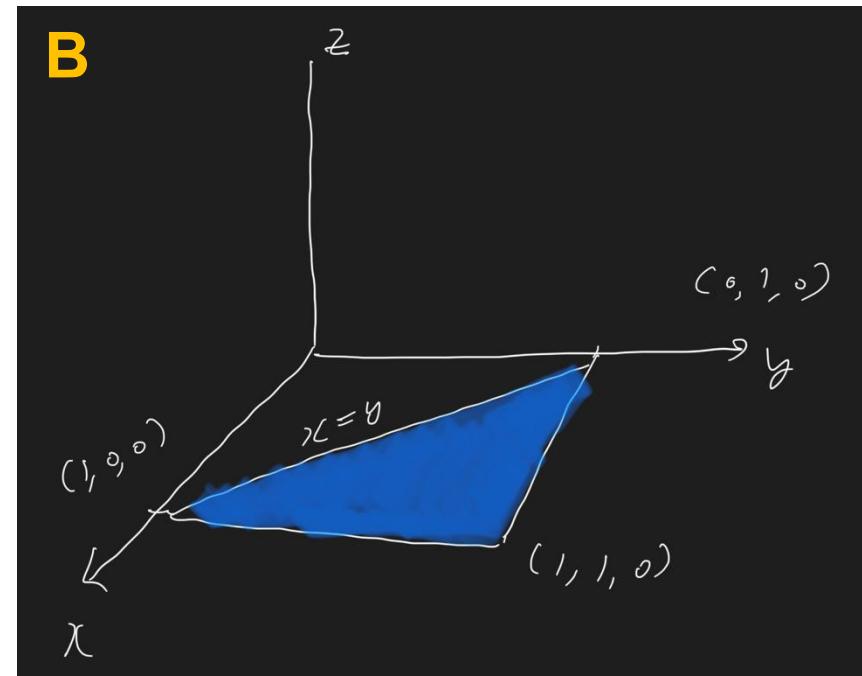
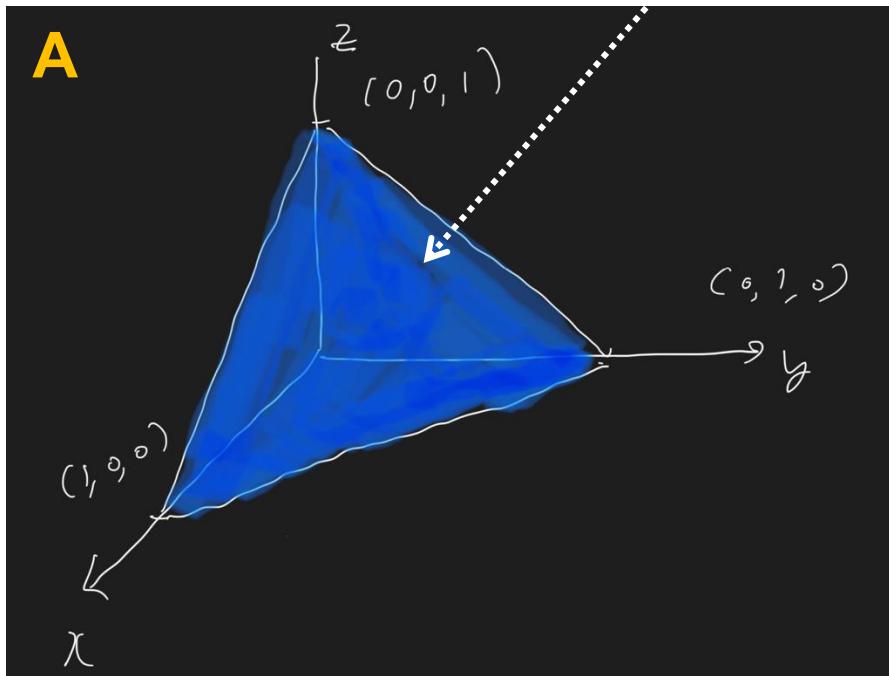
$$\det \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \sqrt{\det g}$$

Relation between Jacobian matrix and metric tensors

The 3×2 Jacobian matrix J of mapping from parameter space \mathcal{N} to cortical surface \mathcal{M} is given by $J = (\partial_\theta \hat{\nu}, \partial_\varphi \hat{\nu})$. The Riemannian metric tensors are $g = (g_{ij}) = J^t J$. The component is given by $g_{ij} = \partial_i \hat{\nu} \cdot \partial_j \hat{\nu}$ with the vector inner product \cdot . The Riemannian metric tensors measure the amount of deviation of a cortical surface from a flat Euclidean plane. If the cortical surface is flat, we obtain $g_{ij} = \delta_{ij}$, the identity matrix. The Riemannian metric tensors enable us to compute the local *area element* $\sqrt{\det g}$. The area element measures the amount of the transformed area in \mathcal{M} of the unit area in the parameterized space \mathcal{N} via the mapping ν . Fig. 8 shows the estimation of the metric tensors for a subject. Using the area element, the total surface area of \mathcal{M} can be written as

$$\mu(\mathcal{M}) = \int_0^{2\pi} \int_0^\pi \sqrt{\det g(\theta, \varphi)} d\theta d\varphi.$$

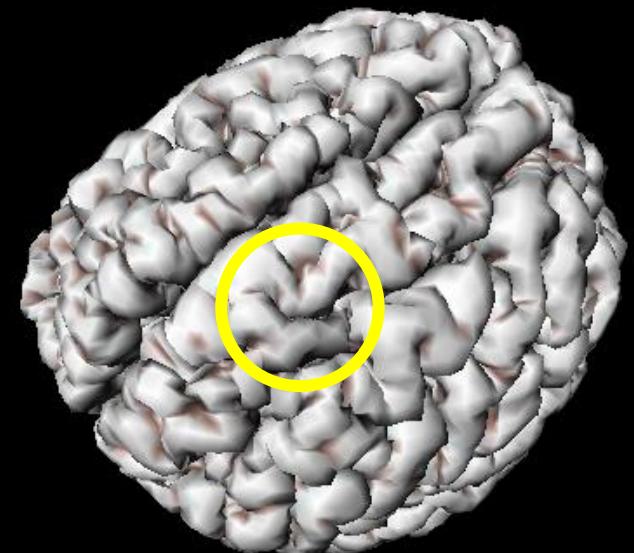
Exercise: probability distribution on 2-simplex

$$f(x, y, z)$$


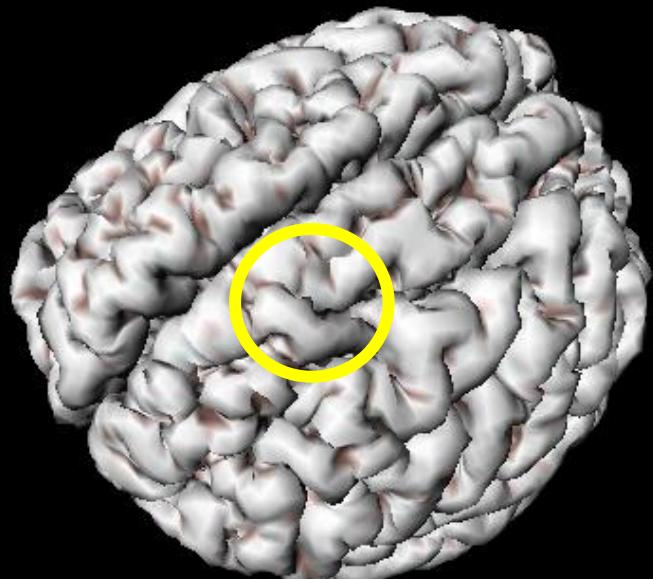
Dirichlet distribution on a triangle domain

Exercise. See if you can find a distribution on triangle B.

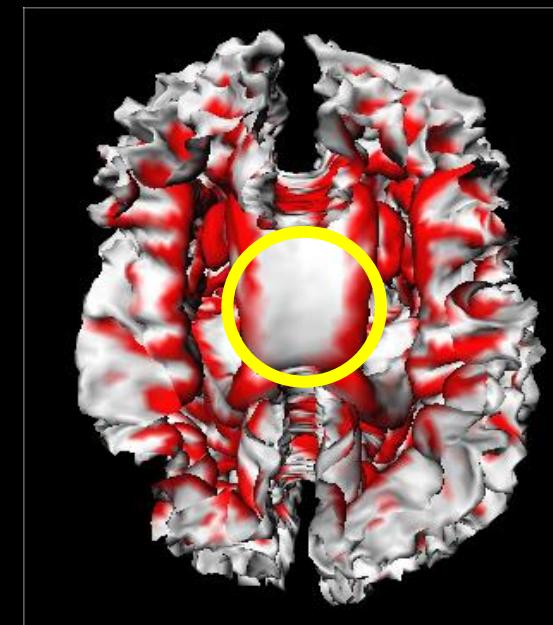
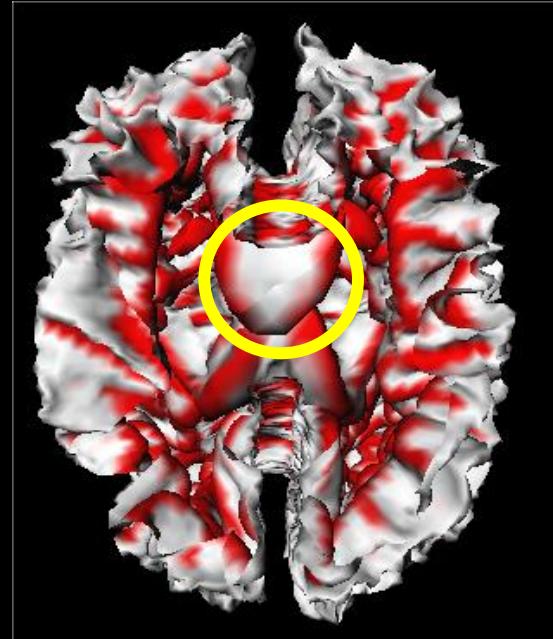
How to quantify brain surface growth and atrophy?



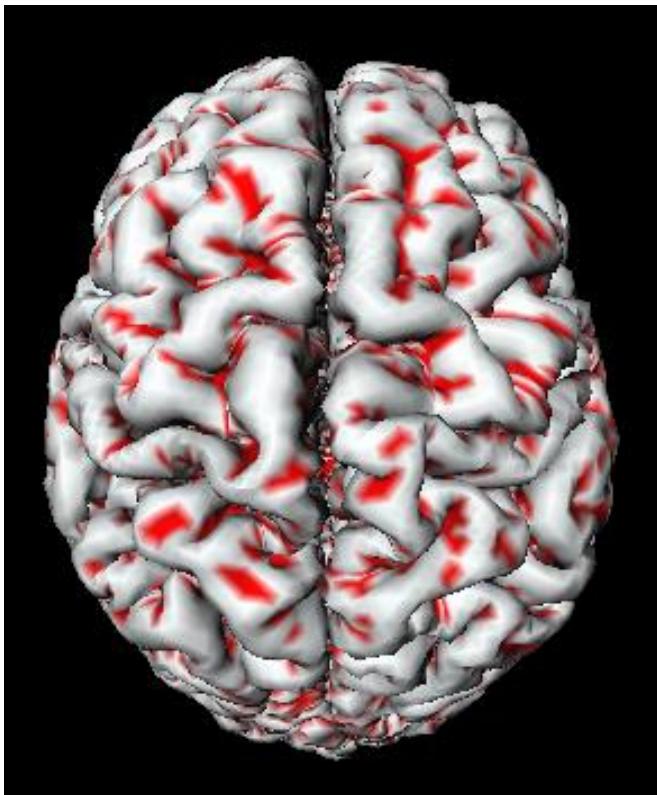
14 year old



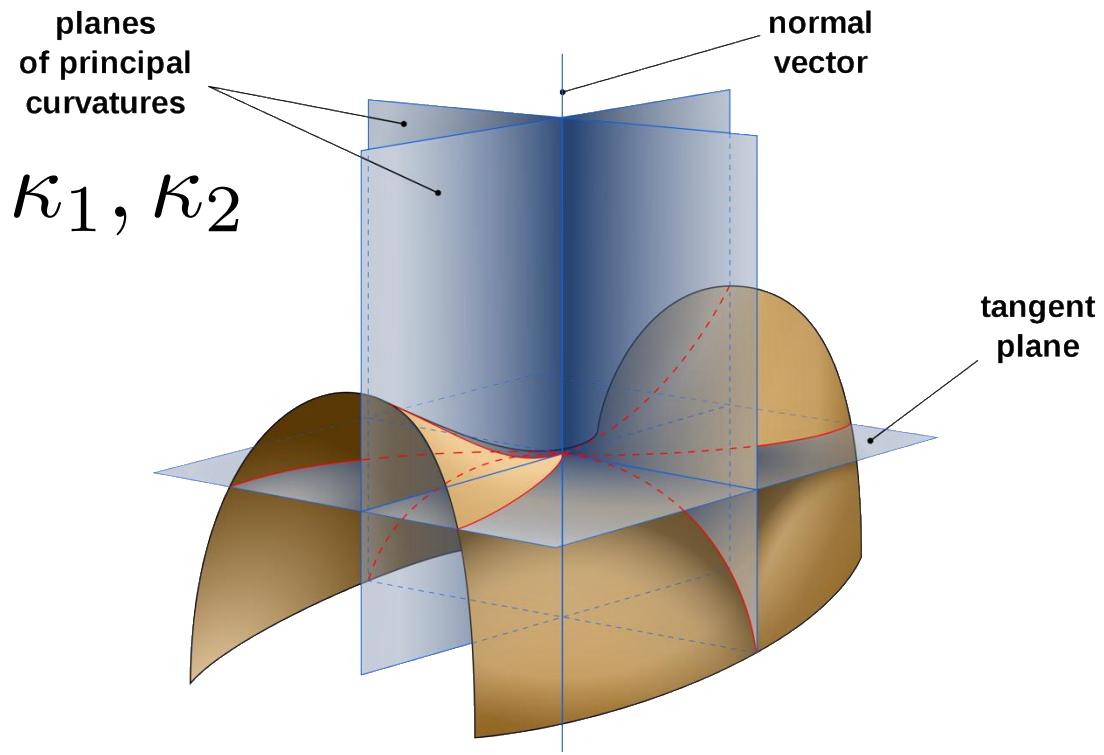
19 year old



Mean and Gaussian curvatures



mean curvature
Gaussian curvature



[Chung et al. 2003 CVPR Tensor-based surface modeling and analysis](#)



Matlab
demo

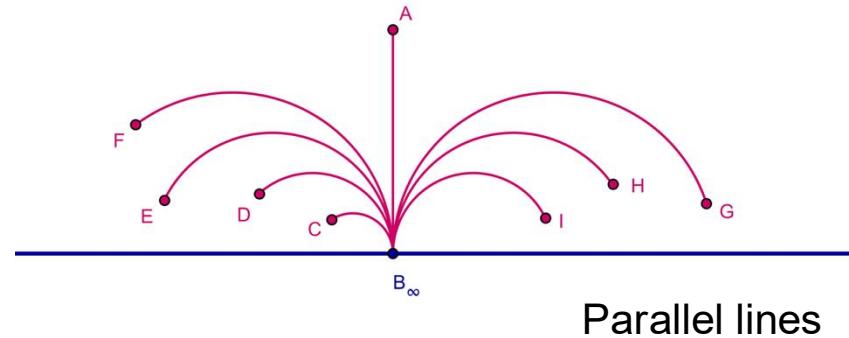
Hyperbolic space

Non-Euclidean space with constant negative Gaussian curvature.
Ex. Saddle surface

Poincare half-plane

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

Metric:
$$\frac{dx_1^2 + \dots + dx_n^2}{x_n^2}$$



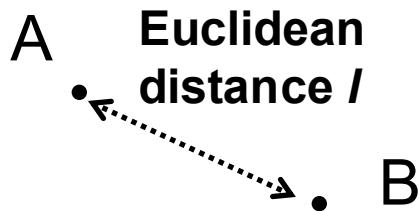
Poincare disk

$$\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 < 1\}$$

$$r^2 = \|x\|_2^2 = x_1^2 + \dots + x_n^2$$

You can have L2-norm as well. Why?

Metric: $4 \frac{r^2}{(1 - r^2)^2}$



$$dist(A, B) = \ln\left(\frac{1+l}{1-l}\right) = 2\operatorname{artanh}(l)$$

Inverse hyperbolic function

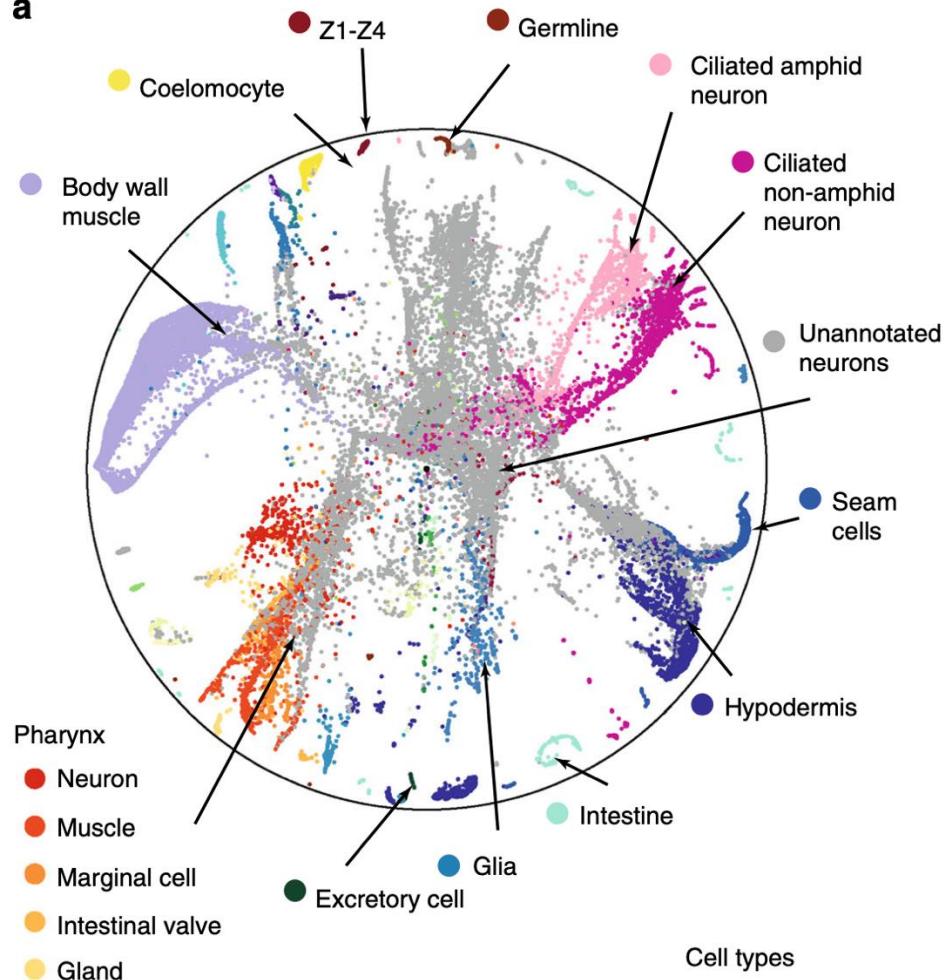
Exercise: Compute the distance in the Poincare disk

Embedding graphs to Poincare disk

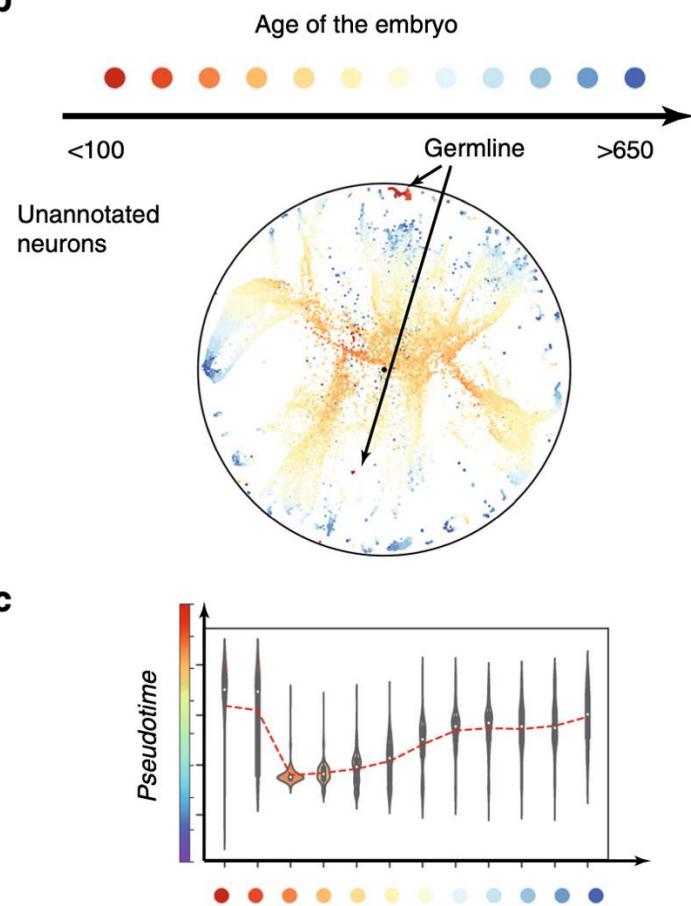
NATURE COMMUNICATIONS | <https://doi.org/10.1038/s41467-020-16822-4>

ARTICLE

a



b

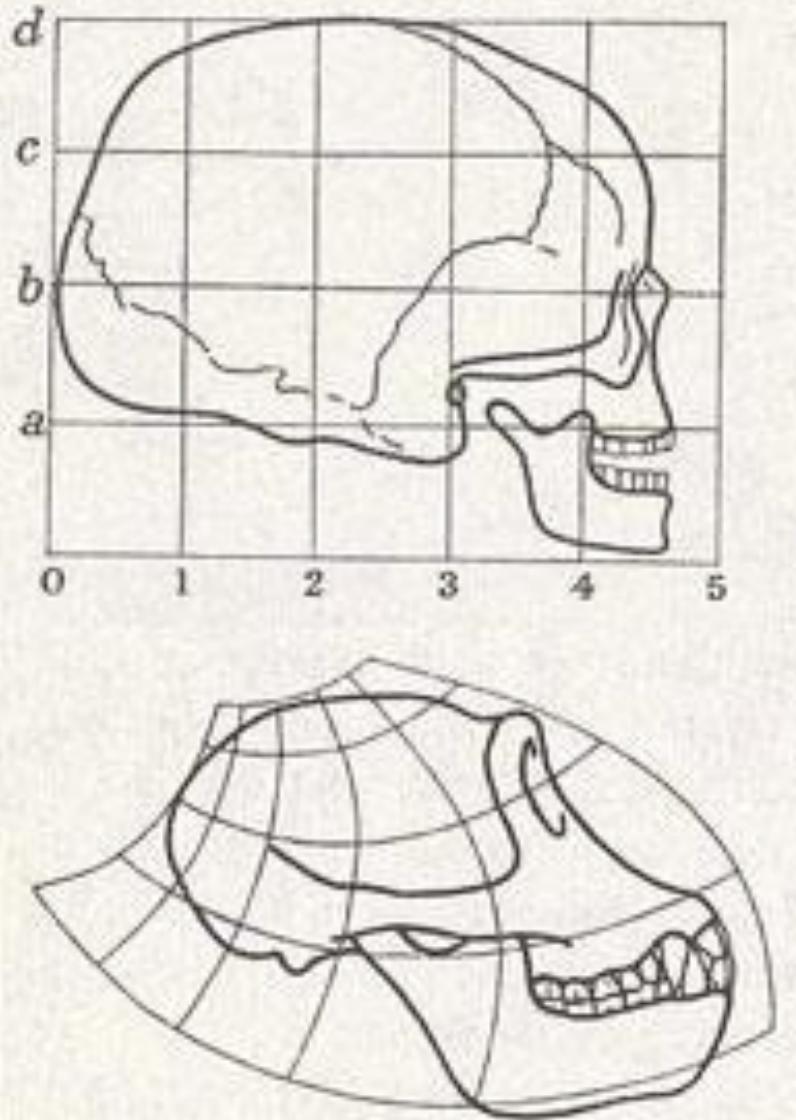


c



Deformable shape model

D'Arcy Thompson 1860-1948



figuratively speaking, the 'p

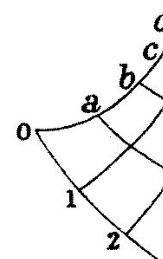


Fig. 178. Co-ordinates on the Cartesian

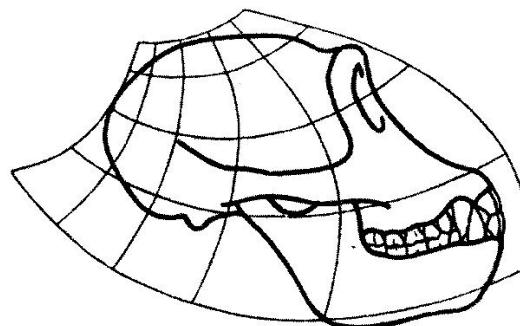
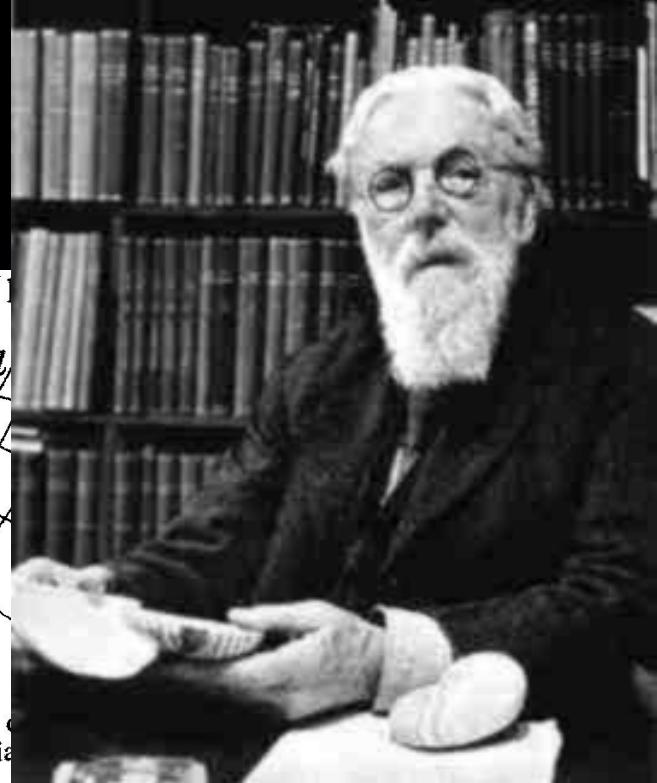


Fig. 179. Skull of chimpanzee.

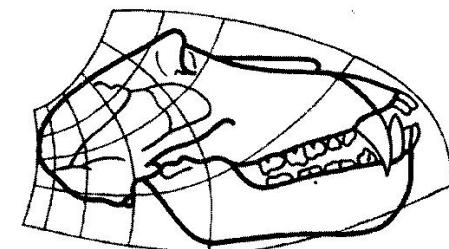


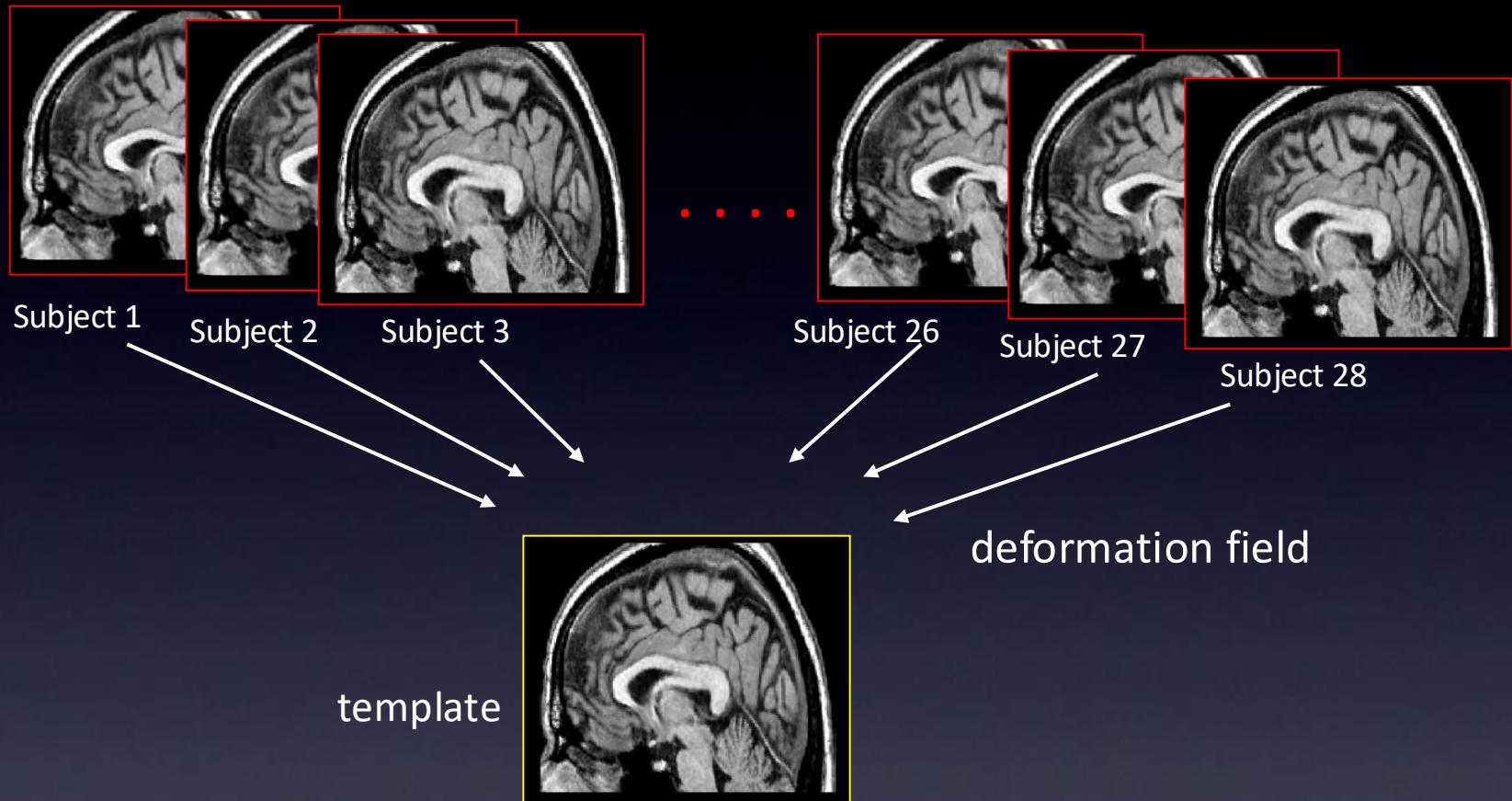
Fig. 180. Skull of baboon.

diagram
I have sh
is obviou
differs or
anthropo

On Growth and Form
D'Arcy Thompson

In Fig. 180
oon, and it
order, and
ion.¹ These
another by

Tensor-based morphometry framework

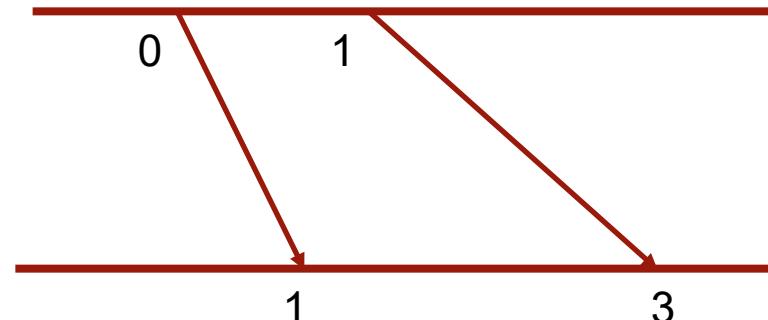


MRIs will be warped into a template and anatomical differences can be compared at a common reference frame.

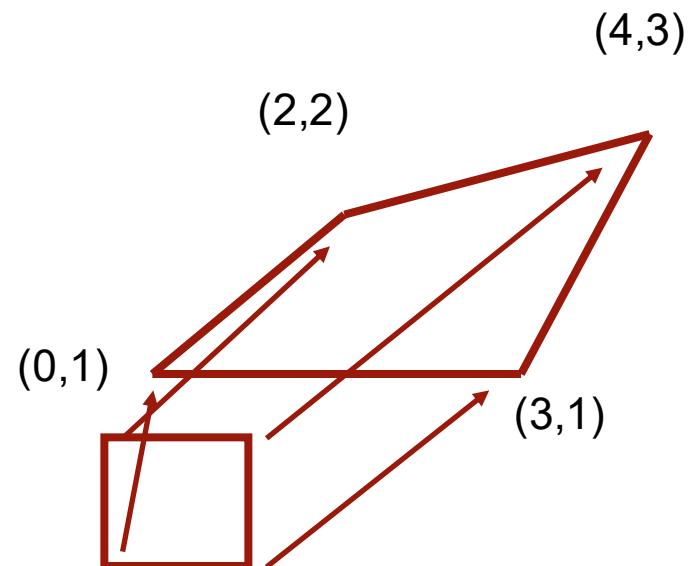
Jacobian determinant in Euclidean space

- 1D: $x' = 2x + 1$

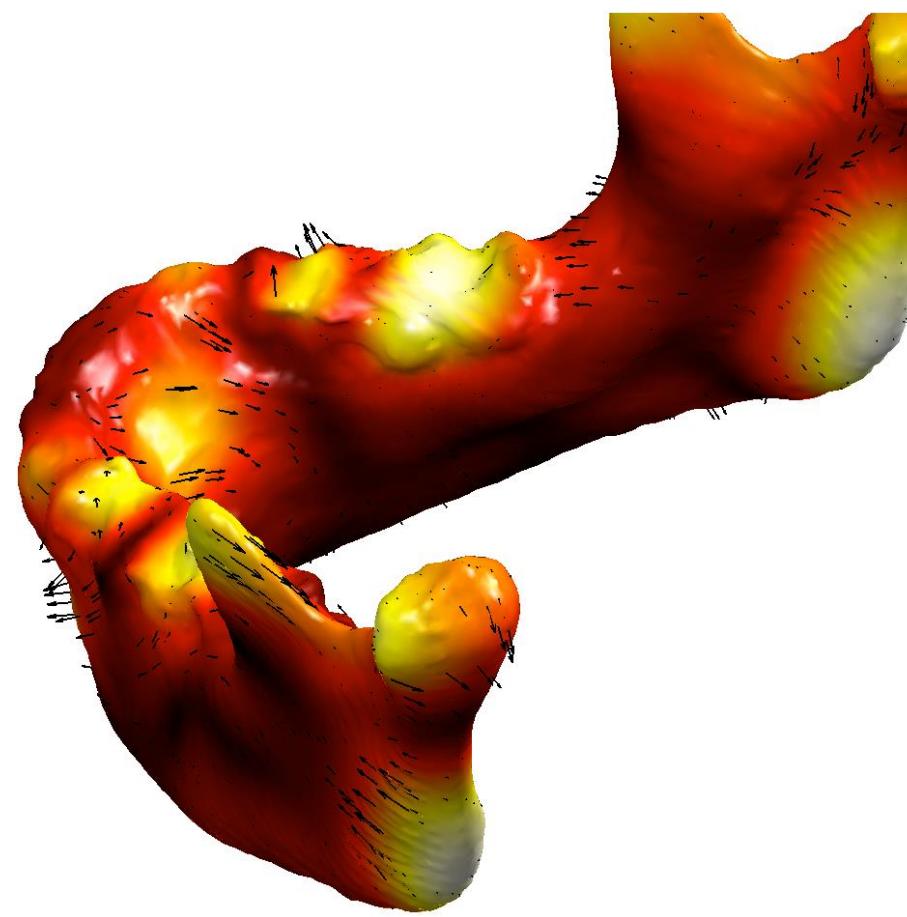
$$J(x) = 2$$



- 2D:
 $x' = 2x + y + 1$
 $y' = x + 2y$
 $J(x, y) = 4 - 1 = 3$



3D Displacement vector field on surface template



Computing Jacobian determinant via metric tensors

$$d_1, d_2, d_3 = d(x_1, x_2, x_3)$$

target position

Initial position

$$U(x_1, x_2, x_3) = d(x_1, x_2, x_3) - (x_1, x_2, x_3)$$

Displacement vector field

Jacobian determinant

$$J(x) = \det \frac{\boxed{d(x)}}{\boxed{x}} = \det \frac{\boxed{\boxed{d_j}}}{\boxed{\boxed{x_i}}}$$

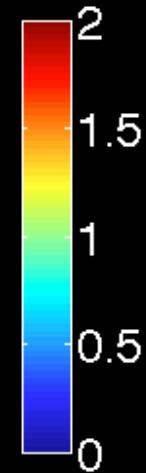
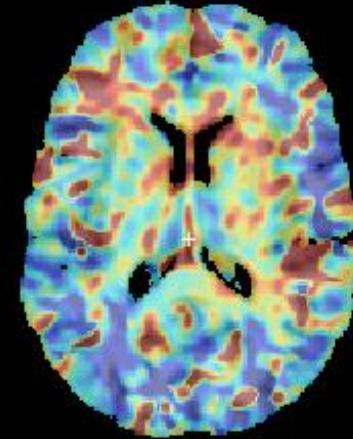
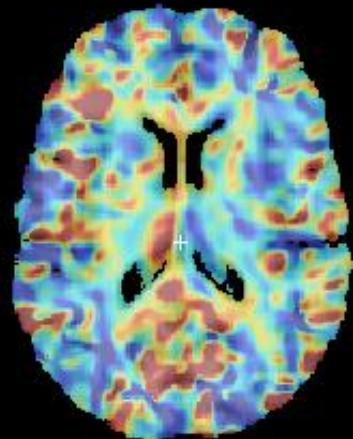
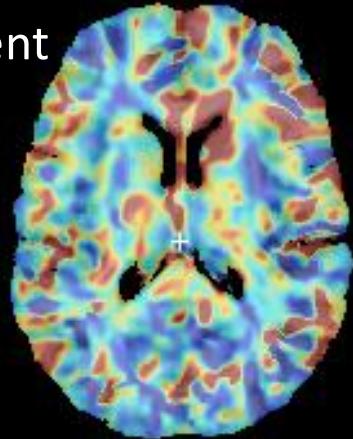
Jacobian determinant in 3D

$x + u$

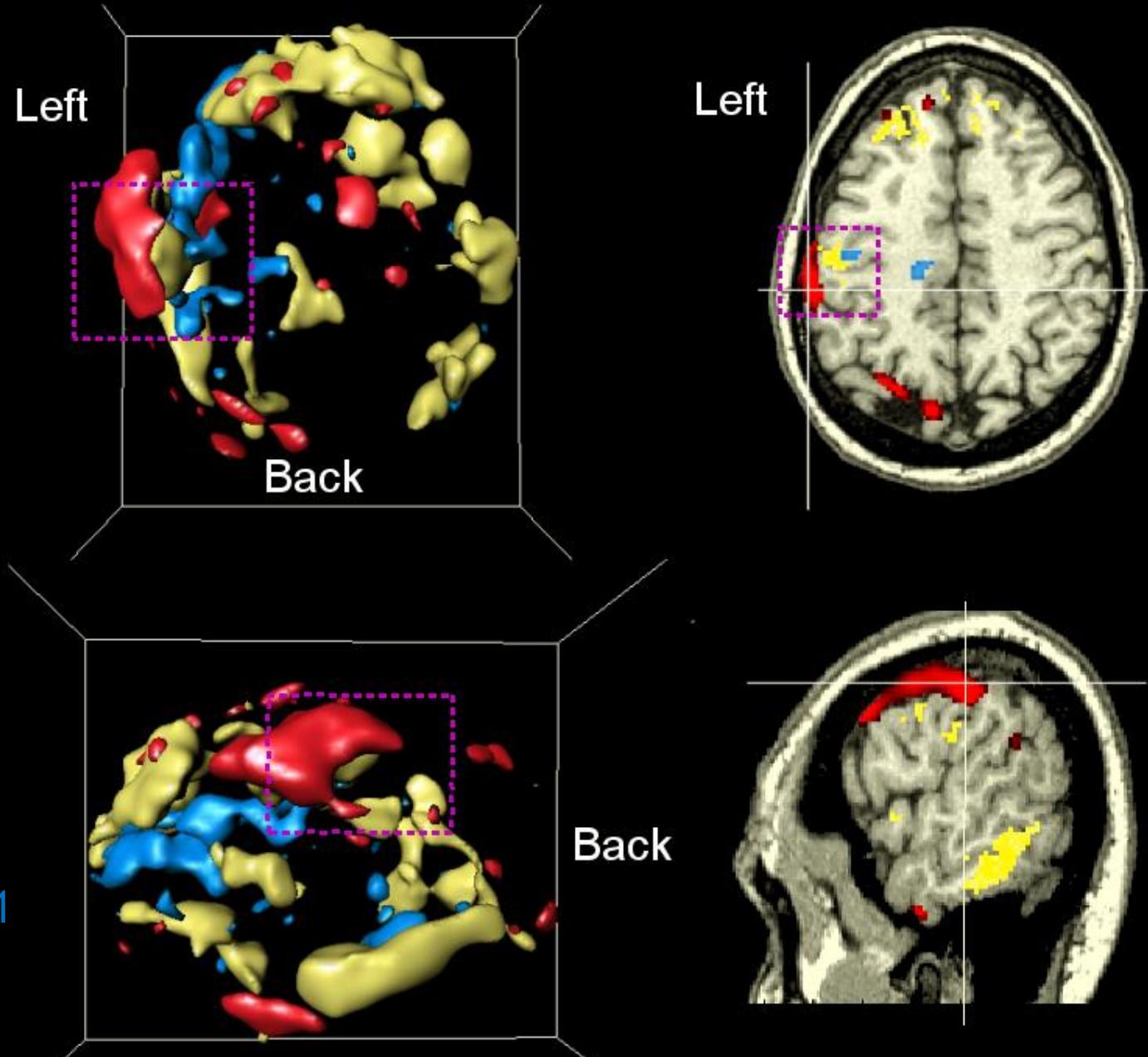


displacement
vector

JD

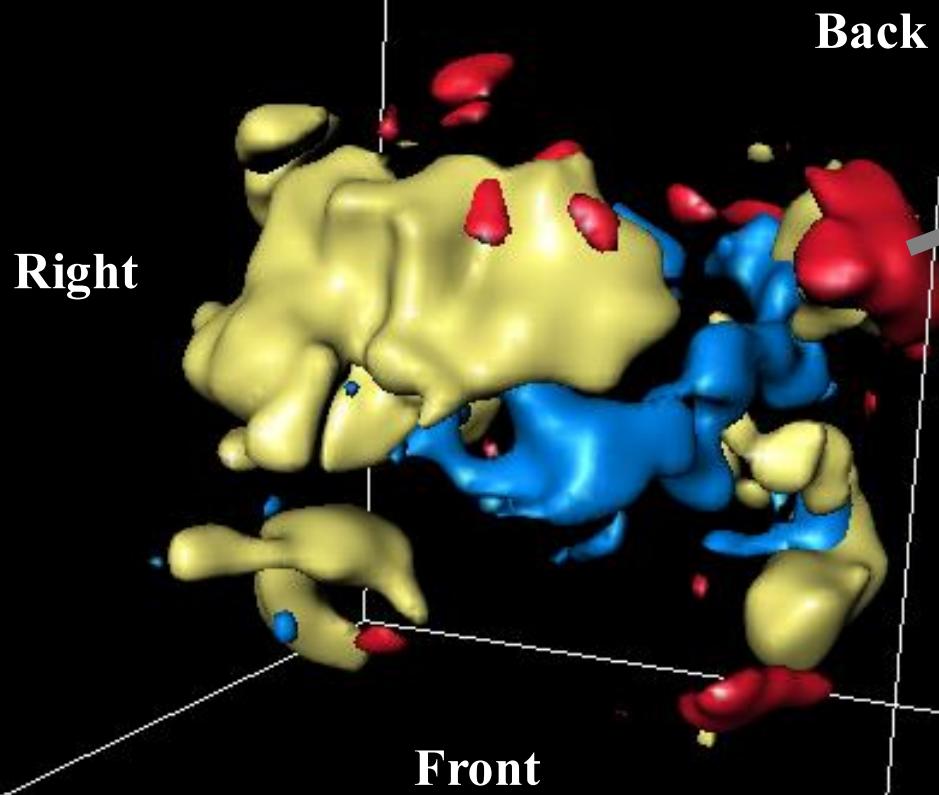


Brain growth in children



Chung et al., 2001
NeuroImage
14:595-606

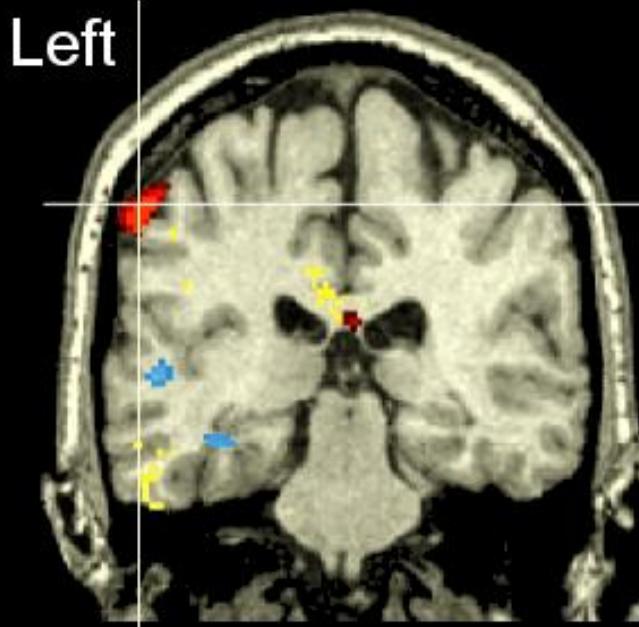
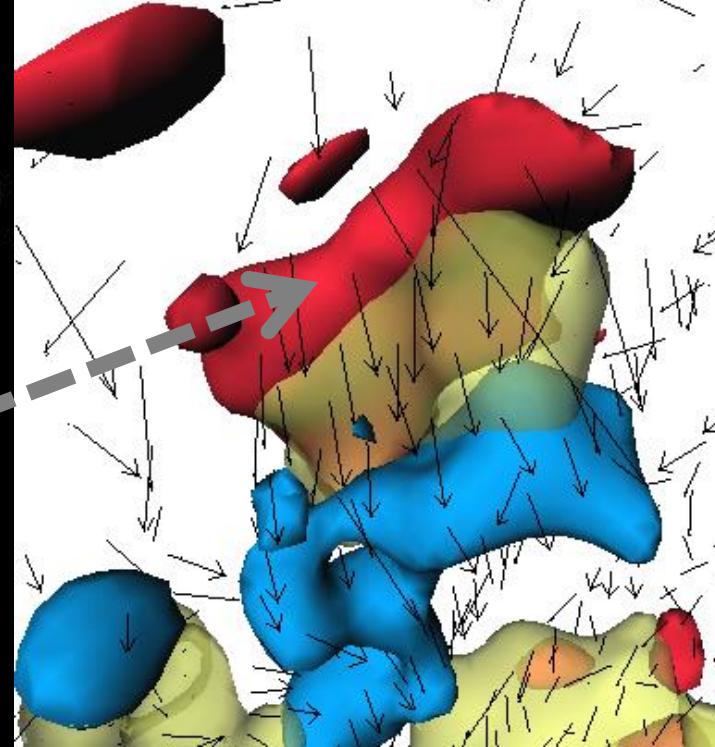
3D SPM



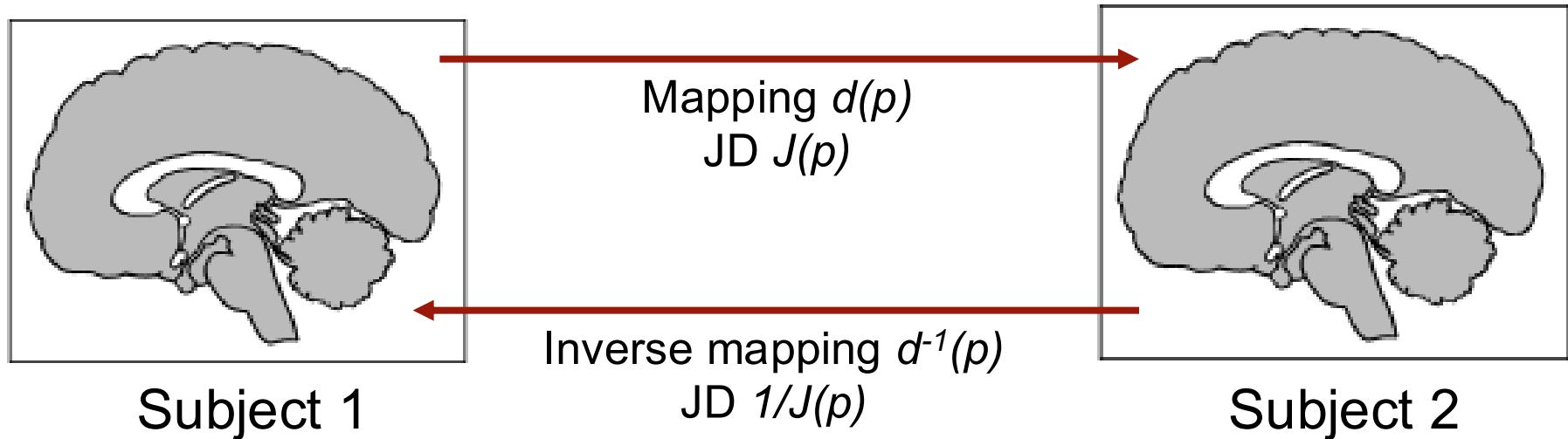
Red: Tissue growth $p < 0.025$

Blue: Tissue loss $p < 0.025$

Yellow: Structure displacement $p < 0.05$



Statistical properties of Jacobian determinant



- $J(p) > 0$ for one-to-one mapping
- $J(p) > 1$ volume increase; $J(p) < 1$ volume decrease
- Due to symmetry, the statistical distribution of $J(p)$ and $1/J(p)$ should be *identical*.

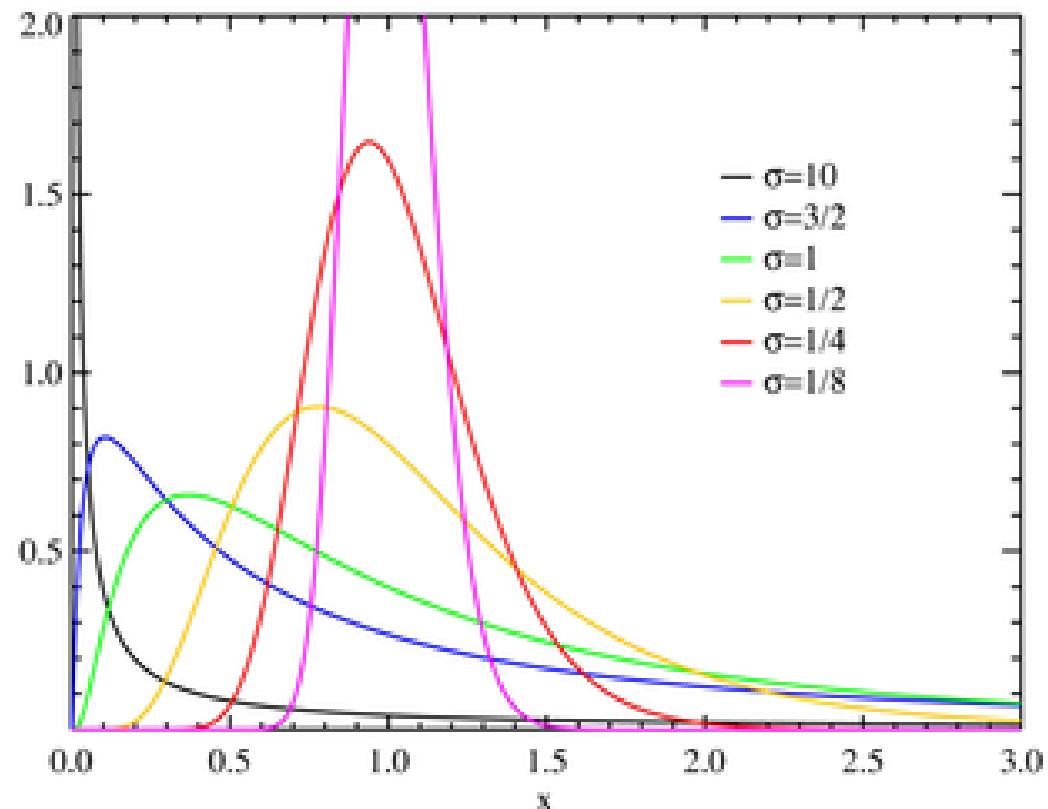
Lognormality of JD

- Domain $-\infty < \log J(p) < \infty$
- If $J(p)=1$, $\log J(p)=0$
- Symmetry: $\log[J^{-1}(p)] = -\log J(p)$
- These 3 properties show that JD can be modeled as *lognormal distribution*.

Lognormal distribution

Random variable X is log-normally distributed if $\log X$ is normally distributed.

Question: Some lognormal distribution looks normal so how do we check if data follows normal or lognormal?



Unit Outcomes

- 1) Understand curve and surface parametrization
- 2) Understand Riemannian metric tensors
- 3) Compute Jacobian determinant

Self assessment questions

- 1) Can you work out lecture materials using exponential map.
- 2) Given two surfaces, compute the Jacobian determinant of area change
- 2) Given displacement vector field, compute the Jacobian determinant.

Research problem (*Hard*). Compute the surface normal vector of a given surface mesh.