Laplace Transform

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Abstract. The Laplace transform is a fundamental analytical tool for characterizing linear dynamical systems and smoothing operators through their spectral response. We present the Laplace transform as a principled framework for defining and interpreting low–pass filtering and exponential smoothing in dynamic data. By formulating smoothing operators as solutions to linear differential equations, the Laplace transform establishes a direct correspondence between time–domain dynamics and frequency–domain attenuation, yielding physically and statistically interpretable parameters such as decay rates and characteristic time scales. This provides the mathematical foundation for Laplace Transform based smoothing and related filtering techniques for dynamic data. The MATLAB codes and sample data used in the paper can be downloaded from https://github.com/laplcebeltrami/BMI768/tree/main/laplacetransform.

1 Laplace Transform in One Dimension

Let $f(t)$ be a real–valued function defined for $t \geq 0$ that is piecewise continuous and of at most exponential growth. The one–dimensional Laplace transform of f is defined as (Courant & Hilbert 1953, Hassani 1991, Widder 1945)

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt, \qquad s \in \mathbb{R}.$$

The Laplace transform measures the behavior of a time–domain signal $f(t)$ by comparing it against exponentially decaying test functions e^{-st}. For large values of s, the exponential term decays rapidly, so the transform emphasizes the early–time behavior of $f(t)$.

The Laplace transform takes a time–domain function $f(t)$ as input and produces a new function $\mathcal{L}\{f\}(s)$ defined on new variable s. Thus, the Laplace transform represents the original function in a different coordinate system, where the argument s controls the rate of exponential decay and determines how strongly contributions from different time scales are emphasized. [1]

For smaller values of s, the decay is slower, allowing contributions from later times to influence the integral. In this sense, s controls how strongly the distant

[1] This is intuition you must have. The essence of Laplace transform in plain English.

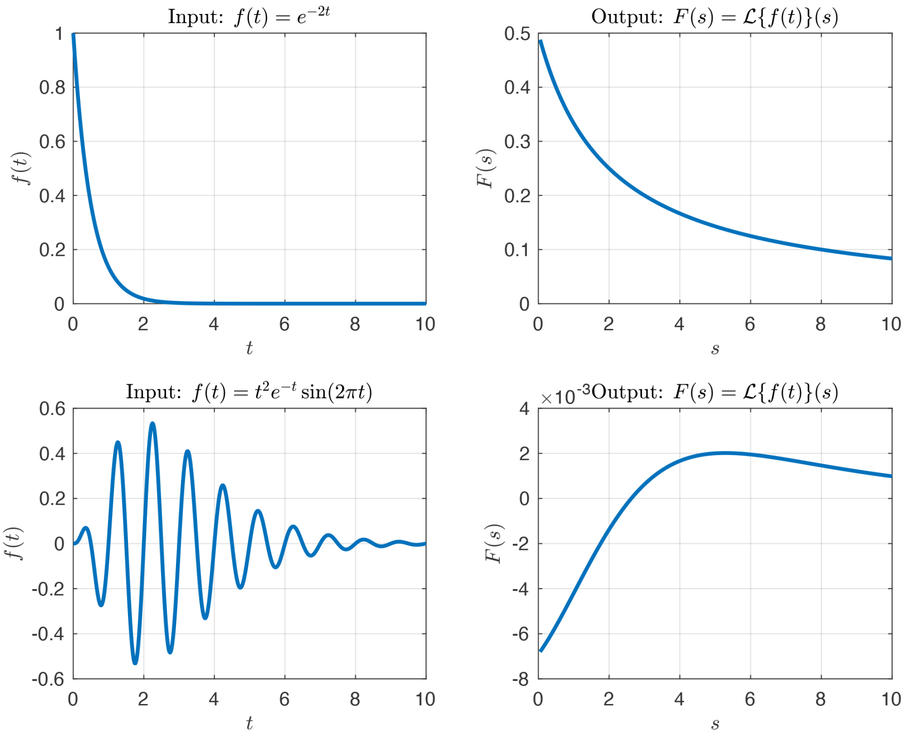

Fig. 1: Laplace transform. Time domain input $f(t)$ (left) and Laplace domain output $F(s) = \mathcal{L}\{f(t)\}(s)$ (right). **Top** shows a simple exponentially decaying signal $f(t) = e^{-2t}$, whose Laplace transform computed by `laplace()` is $F(s) = 1/(s+2)$. **Bottom** shows a more complex signal $f(t) = t^2 e^{-t} \sin(2\pi t)$ and its transform.

past is discounted, and $F(s)$ encodes how the signal accumulates under exponential forgetting. Unlike the Fourier transform (Courant & Hilbert 1953, Hassani 1991), which decomposes a signal into oscillatory components, the Laplace transform emphasizes exponential growth and decay, thereby revealing stability properties and transient behavior. This makes it particularly well suited for analyzing the persistence and stability of dynamical systems.

Here are a few examples of the Laplace transform computed and displayed in MATLAB (Fig. 1). For input signal

$$f(t) = e^{-2t}, \qquad t \geq 0,$$

the corresponding Laplace transform is

$$F(s) = \int_0^\infty e^{-st} e^{-2t}\, dt = \frac{1}{s+2}.$$

It can be computed using MATLAB's symbolic language:

```
syms t s
f = exp(-2*t);
F = laplace(f, t, s);

tvals = 1:0.01:10;
fvals = subs(f, t, tvals);
plot(tvals, fvals)
```

A more complex example, shown in Fig. 1 (bottom row), is $f(t) = t^2 e^{-t}\sin(2\pi t)$, whose Laplace transform is given symbolically in Matlab as

$$F(s) = \frac{4\pi(2s+2)^2}{\left((s+1)^2+4\pi^2\right)^3} - \frac{4\pi}{\left((s+1)^2+4\pi^2\right)^2}.$$

A simple but instructive example is given by the scalar identity

$$\int_0^\infty e^{-t(s-a)}\,dt = \frac{1}{s-a}, \qquad s > a. \tag{1}$$

Setting $a = -2$ yields the exponentially decaying function

$$f(t) = e^{-2t}, \qquad t \geq 0,$$

whose Laplace transform is

$$F(s) = \int_0^\infty e^{-st}e^{-2t}\,dt = \int_0^\infty e^{-t(s+2)}\,dt = \frac{1}{s+2}.$$

This example, shown in Fig. 1 (bottom), illustrates how the Laplace transform maps exponential decay in time to a simple rational function of the transform variable s.

In general, when data are too complex to admit a simple analytic representation, we approximate the underlying signal by projecting it onto a structured family of basis functions $\{\phi_k\}$ whose Laplace transforms are analytically known. Suppose that $f(t)$ is represented as

$$f(t) \;\approx\; \sum_{k=1}^{K} c_k\,\phi_k(t).$$

Then we have

$$\mathcal{L}\{f\}(s) \;\approx\; \sum_{k=1}^{K} c_k\,\mathcal{L}\{\phi_k\}(s).$$

The coefficients c_k serve as compact, task–relevant features, enabling downstream modeling, prediction, or inference to be carried out directly in the coefficient space rather than on the raw signal.

Exponentials $\phi_k(t) = e^{-\alpha_k t}$ are the most natural basis (Widder 1945), as their Laplace transforms are obtained in closed form as

$$\mathcal{L}\{e^{-\alpha_k t}\}(s) = \frac{1}{s+\alpha_k}.$$

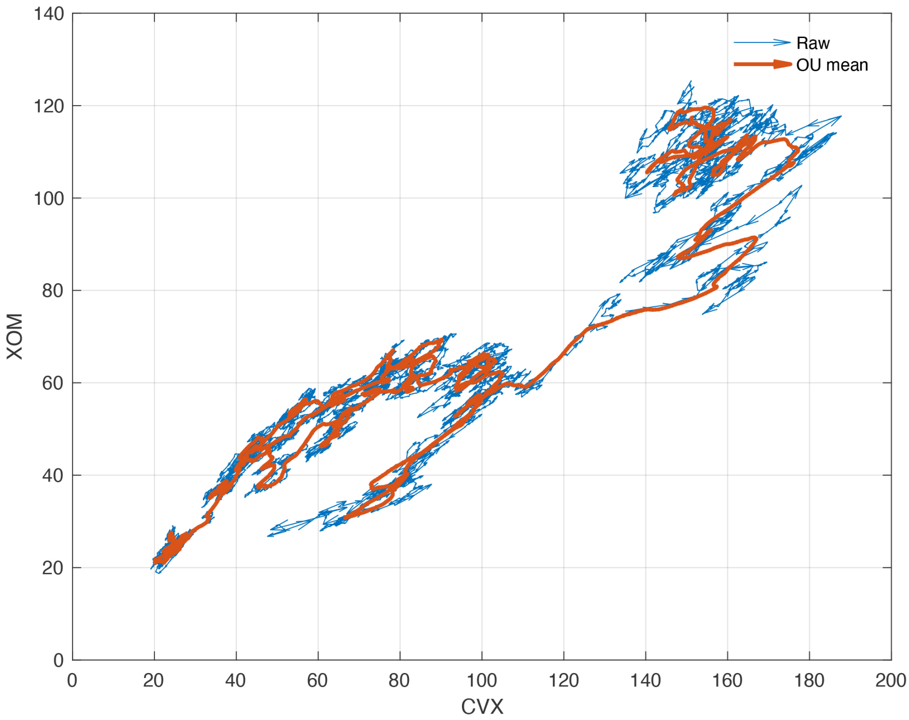

Fig. 2: Time–varying vector field of stock prices and its Ornstein–Uhlenbeck (OU) mean. Daily closing prices of Chevron (CVX) and Exxon Mobil (XOM) are embedded as trajectories in the (CVX, XOM) price plane. Arrows represent one–day price increments $(x_{t+1} - x_t,\ \ y_{t+1} - y_t)$, yielding a time–dependent velocity field. Thin arrows show the raw day–to–day price changes, which are dominated by high–frequency fluctuations. Thick arrows show the Ornstein–Uhlenbeck mean (with $\lambda = 20$, trading–day), revealing persistent co–movement and the dominant drift structure shared by the two stocks.

One of the key advantages of the Laplace transform is that it converts differentiation and integration into algebraic operations in the transform domain.

$$\mathcal{L}\left\{\frac{df}{dt}\right\}(s) = sF(s) - f(0),$$

so differentiation corresponds to multiplication by s if $f(0) = 0$. Similarly, integration becomes division by s:

$$\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\}(s) = \frac{1}{s}F(s).$$

Thus, the Laplace transform replaces calculus operations by algebraic ones, making differential and integral equations through simple multiplication and division.

2 Connection to Dynamical Systems

Consider linear dynamical systems of the form (Ogata 2010)

$$\frac{dy(t)}{dt} + \lambda y(t) = \lambda x(t), \qquad \lambda > 0, \tag{2}$$

relating paired time series $x(t)$ and $y(t)$. Taking the Laplace transform of (2) and assuming zero initial condition $y(0) = 0$ yields

$$Y(s) = \frac{\lambda}{s+\lambda} X(s).$$

Applying the inverse Laplace transform gives the explicit time–domain solution

$$y(t) = \lambda \int_0^t e^{-\lambda(t-\tau)} x(\tau)\, d\tau,$$

showing that $y(t)$ is an exponentially weighted average of past values of $x(t)$. This operation suppresses high–frequency fluctuations while retaining low–frequency structure. The parameter λ controls the exponential decay rate of past information and defines a characteristic time scale $1/\lambda$.

The linear dynamical system (2) smooths y while treating x as a driving input. If we also wish to smooth x using y in an analogous manner, we consider the mirrored system

$$\frac{dx(t)}{dt} + \lambda x(t) = \lambda y(t), \qquad \lambda > 0, \tag{3}$$

which is symmetric to (2) with the roles of x and y exchanged. In the Laplace domain (assuming $x(0) = 0$), this yields

$$X(s) = \frac{\lambda}{s+\lambda} Y(s),$$

and hence

$$x(t) = \lambda \int_0^t e^{-\lambda(t-\tau)} y(\tau)\, d\tau.$$

With unit time step, a forward Euler discretization of (2) gives

$$y_t = (1-\lambda) y_{t-1} + \lambda x_{t-1}, \qquad t \geq 2,$$

and similarly, discretizing (3) yields

$$x_t = (1-\lambda) x_{t-1} + \lambda y_{t-1}, \qquad t \geq 2.$$

Writing $\mathbf{z}_t = (x_t, y_t)^\top$, the coupled recursion can be expressed compactly as

$$\mathbf{z}_t = \begin{pmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{pmatrix} \mathbf{z}_{t-1}.$$

After obtaining the smoothed trajectory $\{(x_t, y_t)\}$, we construct a time–varying vector field in the two–dimensional price space. The tail of each vector is the smoothed price pair (x_t, y_t), and the vector itself is the one–step increment

$$\mathbf{v}_t = \begin{pmatrix} \Delta x_t \\ \Delta y_t \end{pmatrix} = \begin{pmatrix} x_{t+1} - x_t \\ y_{t+1} - y_t \end{pmatrix}.$$

This exponential smoothing suppresses high–frequency fluctuations in the embedded trajectory, yielding the smooth drift structure observed in Fig. 2.

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