The Waisman Laboratory for Brain Imaging and Behavior

University of Wisconsin SCHOOL OF MEDICINE AND PUBLIC HEALTH

Linear Equations

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Matrix equation

Determined system

$x+z = 4$
$-x+y+z = 4$
$x - y + z = 2$

```
A=[1 0 1
   -1 1 1
   1 -1 1]

b=[4 4 2]'

X=A\b

X =

     1
     2
     3

X=inv(A)*b
```

Backslash \ uses Gaussian elimination

$AX = b$

$X = A^{-1}b$

Matrix A is called invertible (nonsingular) if there exists an *n*-by-*n* square matrix B such that

AB = BA = I_n `= eye (n).`

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

```
>>B= inv(A)
```

assuming `det(A)` is not zero. Matrix inversion is done using Gaussian elimination: $O(n^3)$. The lowest bound: $O(n^2 \log n)$.

Quiz. How to invert really large matrices? You cannot!

Matrix equation

Over-determined system

$$
\begin{aligned}
x \quad\quad &= 4 \\
-x+y &= 4 \\
x-y &= 2
\end{aligned}
$$

Quiz. Mathematically there is no solution. So how to solve this problem?

Least squares estimation (LSE)

We will *approximately* solve

$AX = b$

by minimizing $||AX-b||^2$.

Even if there is no solution in Ax=b, there is the best solution in an optimization sense.

The least squares method grew out of the fields of astronomy as mathematicians sought to provide solutions to the challenges of navigating the Earth's oceans during the Age of Exploration.

$$f(X) = (x-4)^2 + (-x+y-4)^2 + (x-y-2)^2 = (x-4)^2 + (x-y-)^2$$

$$df/dx = 6x - 4y - 4 = 0$$

$$df/dy = -4x + 4y - 4 = 0$$

```
A=[6 -4
   -4 4]
b=[4 4]'

X=A\b

X =


    4.0000
    5.0000
```

Least squares method

How to minimize $||AX-b||^2$?

Let $$f(X) = \|AX - b\|^2$$

Minimum is obtained when the matrix derivative is zero:

$$\frac{df(X)}{dX} = 0$$

The solution is given by $X = (A^TA)^{-1}A^T b$ assuming A^TA is invertible. What if A^TA is not invertible?

Moore-Penrose (peudo) inverse

A pseudoinverse A^- of a matrix A is a generalization of the inverse matrix. The most widely known type of matrix pseudoinverse is the Moore–Penrose pseudoinverse, which was independently described by E. H. Moore in 1920, Arne Bjerhammar in 1951 and Roger Penrose in 1955.

If A^TA is invertible, $A^- = (A^TA)^{-1}A^T$.

In general, A^- is defined as a matrix satisfying
$AA^-A = A$, $A^-AA^- = A^-AA^-$ and AA^- are symmetric.

```
>>pinv(A)
```

Over-determined system

$$
\begin{aligned}
x \quad &= 4 \\
-x+y &= 4 \\
x-y &= 2
\end{aligned}
$$

Quiz. Check if the MATLAB solution is the least squares estimation of the above system.

$$f(X) = (x-4)^2 + (-x+y-4)^2 + (x-y-2)^2$$

```
A=[1 0 0
    -1 1 0
    1 -1 0]

b=[4 4 2]'

pinv(A)
ans =
    1.0000   -0.5000
-0.5000
    1.0000   -0.0000
-1.0000
    0.0000    0.5000
0.5000

X=pinv(A)*b
X =

    4.0000
```

Under-determined system

$-x+y = 4$

Mathematically there are infinitely many solutions.
So how to solve this problem?

Can we still use LSE? Yes.

$f(X)= (-x+y-4)^2$

$df/dx = 2x-2y+8 = 0$

$df/dy = -2x+2y -8 = 0$

```
A=[2 -2
   -2 2]
b=[-8 8]'

X=pinv(A)*b

X =

   -2.0000
    2.0000
```

Question: But why we only get one solution?

Matrix equation

Under-determined system

-x+y = 4

Psudoinverse approach also works.

```
A=[-1 1
   0 0]
b=[4 0]'

X=pinv(A)*b

X =

   -2.0000
    2.0000
```

```
A=[-1 1
   -1 1]
b=[4 4]'

X=pinv(A)*b

X =

   -2.0000
    2.0000
```

```
A=[-1 1
   1 -
1]
b=[4 4]'

X=pinv(A)*b

X =

   -2.0000
    2.0000
```