PROJECT 19: Boltzmann Tilting

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The project is intentionally flexible and methodologically focused; students may use *any* dataset, including data already used by other students. However, the corpus callosum 2D probability map data may be the easiest starting points.

Goal

The goal of this project is to develop an accelerated permutation testing framework for efficiently estimating extremely small p-values. Instead of relying on uniform permutation sampling, students will reformulate permutation inference using a Boltzmann distribution

$$q_\lambda(\pi) = \frac{\exp(\lambda T(\pi))}{\sum_{\pi'} \exp(\lambda T(\pi'))},$$

which increases the sampling frequency of large values of the test statistic $T(\pi)$ over permutations π—rare events in the tail region—thereby enabling faster and more stable estimation of small tail probabilities.

Description

Students will reformulate classical permutation testing as a rare-event probability estimation problem and introduce Boltzmann tilting as a strategy to accelerate convergence. They will demonstrate how tilted sampling over the permutation space can still recover the original uniform-permutation p-value. The project will first develop the method in a one-sample framework using the t-statistic and the maximum t-statistic (Chung et al. 2019), and will then extend it to the two-sample permutation setting on the chosen dataset.

Students will further examine the computational–statistical trade-off inherent in tilted sampling, recognizing that overly aggressive biasing toward extreme permutations (through the tilting parameter λ) can increase estimator variability and reduce stability. The project will therefore emphasize achieving a principled balance between computational acceleration and statistical reliability while preserving valid permutation inference.

Students are expected to compare the performance of the accelerated method against standard uniform permutation testing as well as the transposition test (Chung et al. 2019) studied in class. The comparison will focus on computational time, required sample size for stable estimation, estimator variability, and accuracy in small p-value regimes.

Finally, students will explore strategies for combining online permutation testing with Boltzmann tilting to accelerate small p-value estimation while maintaining estimator stability and valid permutation inference.

Learning Outcomes

By completing this project, students will understand permutation testing as a rare-event probability estimation problem rather than merely a resampling procedure. They will gain experience implementing and analyzing accelerated resampling methods, learn the role of the Boltzmann distribution in rare-event simulation, understand how valid inference can be preserved under distribution tilting.

Bibliography

Chung, M., Xie, L., Huang, S.-G., Wang, Y., Yan, J. & Shen, L. (2019), 'Rapid acceleration of the permutation test via transpositions', *International Workshop on Connectomics in Neuroimaging* **11848**, 42–53.