

# PROJECT 20: Topology Preserving Flows

Moo K. Chung

University of Wisconsin-Madison, USA

[mkchung@wisc.edu](mailto:mkchung@wisc.edu)

The project is intentionally flexible and methodologically focused; students may choose and use any dataset, including data already used by other students, although Moving-MNIST will likely serve as the most convenient and visually interpretable option.

## Goal

Classical intensity-based smoothing methods do not preserve topology: for example, excessive diffusion of the digit “0” can fill its central hole after thresholding, destroying the loop structure. Rather than diffusing image intensities, this project reformulates the problem in terms of 2D vector fields (flows) derived from data. Such flows may arise from temporal differences in Moving-MNIST frames or from embedded time series where vectors describe local state evolution. The objective is to design diffusion on these flows that reduces noise while preserving their intrinsic topological structure.

## Methods

Given flow  $X$ , we perform diffusion on the associated flow field on the components of Hodge decomposition (Anand & Chung 2024). In two dimensions,

$$X = \nabla u + \nabla^\perp v,$$

where  $u$  is the scalar potential (gradient component) and  $v$  is the stream function (curl component). Here the differential operators are defined as follows:

$$\nabla u = \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix}, \quad \nabla^\perp v = \begin{pmatrix} -\partial_y v \\ \partial_x v \end{pmatrix}.$$

The operator  $\nabla^\perp$  denotes the rotated gradient, which in two dimensions produces a vector field orthogonal to  $\nabla v$ . Consequently, we have

$$\nabla \cdot (\nabla^\perp v) = 0.$$

Diffusion on flows is defined via the continuous 1-Hodge Laplacian,

$$\frac{\partial X}{\partial t} = -\mathcal{L}_1 X,$$

where

$$\mathcal{L}_1 X = \nabla(\nabla \cdot X) + \nabla^\perp(\nabla^\perp \cdot X).$$

Here

$$\nabla \cdot X = \partial_x X_1 + \partial_y X_2, \quad \nabla^\perp \cdot X = \partial_x X_2 - \partial_y X_1,$$

so that  $\nabla^\perp \cdot X$  is the scalar curl of  $X$ .

Substituting the Hodge decomposition  $X = \nabla u + \nabla^\perp v$  into the diffusion equation yields

$$\frac{\partial}{\partial t}(\nabla u + \nabla^\perp v) = -\mathcal{L}_1(\nabla u + \nabla^\perp v).$$

Using the identities

$$\mathcal{L}_1(\nabla u) = \nabla(\Delta u), \quad \mathcal{L}_1(\nabla^\perp v) = \nabla^\perp(\Delta v),$$

we obtain decoupled evolution equations for the scalar potentials:

$$\frac{\partial u}{\partial t} = -\Delta u, \quad \frac{\partial v}{\partial t} = -\Delta v.$$

Thus diffusion of the vector field reduces to classical heat diffusion applied separately to the gradient and curl generators.

## Description

Students will theoretically investigate why the proposed diffusion is topology preserving and in what precise sense this preservation holds. They will then implement the diffusion framework on flows derived from both in-class data and synthetic examples with known structure (e.g., circular flows). After computing the Hodge decomposition, students will diffuse the scalar generators and reconstruct the evolved flow field. Through analysis and numerical experiments, they will demonstrate how the method smooths noise while maintaining circulation patterns and topological invariants.

Students will further investigate how the smoothed flows can be incorporated into statistical inference on a collection of flows, such as estimating mean flows, statistically comparing groups, or assessing variability while preserving topological structure.

## Learning Outcomes

By completing this project, students will understand the Hodge decomposition of two-dimensional vector fields into gradient and curl components and the role of the 1-Hodge Laplacian in defining diffusion on flows. They will learn how diffusion acts separately on these components, how circulation structures are represented in the curl part, and in what sense the evolution preserves or alters topological features. Students will gain experience implementing flow-based diffusion and applying persistent homology to quantify topology. Finally, they will develop insight into how topology-aware smoothing can support statistical inference on collections of two-dimensional flows.

## Bibliography

Anand, D. & Chung, M. (2024), Hodge-decomposition of brain networks, *in* ‘2024 IEEE International Symposium on Biomedical Imaging (ISBI)’, IEEE, pp. 1–5.