The Waisman Laboratory for Brain Imaging and Behavior

University of Wisconsin SCHOOL OF MEDICINE AND PUBLIC HEALTH

BMI/STAT-768

Statistical Methods for Medical Image Analysis

Moo K. Chung
Department of Biostatistics and Medical Informatics
University of Wisconsin-Madison

https://github.com/laplcebeltrami/BMI768

Geometric Data Analysis

Geometric Data Analysis (GDA) is a field that focuses on understanding and interpreting the geometric structures inherent in complex datasets. It leverages mathematical and statistical techniques to explore properties such as shape, curvature, and topology, enabling the identification of patterns and relationships within the data. GDA encompasses methods like manifold learning, principal component analysis, and dimensionality reduction, which help in visualizing high-dimensional data in more interpretable forms.

Information geometry

Data embedding

Covariance matrix

Data manifolds

GDA

quantifies the shape of data

Symmetric positive definite matrices

Statistics on manifolds

PCA

Spectral clustering

Functional-PCA

Spectral geometry

Unit Objectives

1) Understand curve and surface parametrization

2) Understand Riemannian metric tensors

3) Compute Jacobian determinant

Use the *exponential map* example in understanding these concepts

Reference

Widely used basic textbook on differential geometry

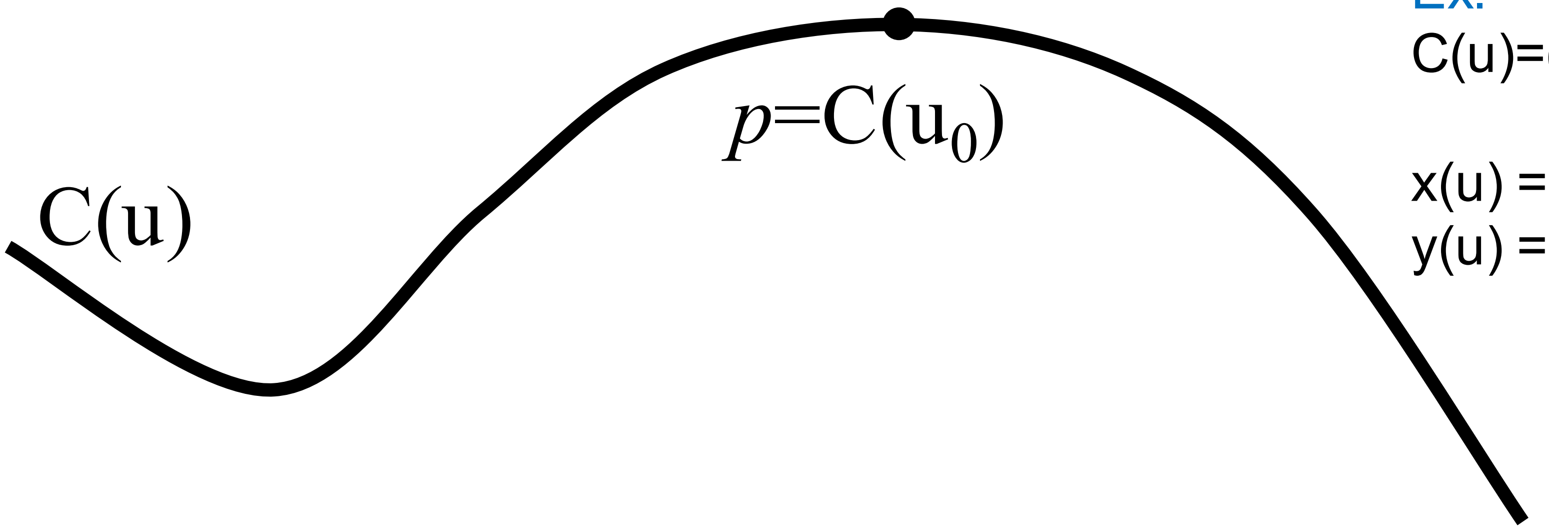
Differential Geometry of Curves and Surfaces

Manfredo P. do Carmo

Instituto de Matematica Pura e Aplicada (IMPA)
Rio de Janeiro, Brazil

Differential Geometry of a Curve

Point p on the curve at u_0



Smooth curves can be parameterized.

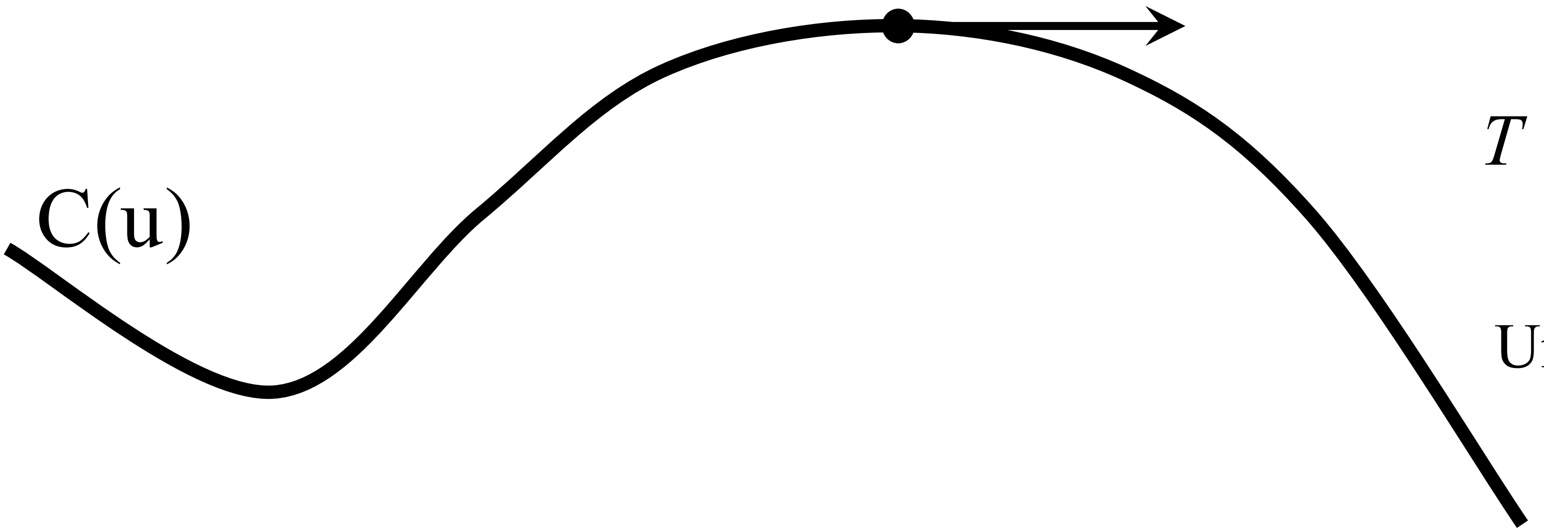
Ex.
C(u)=(x(u),y(u))

x(u) = cos(u)
y(u) = sin(u)

Tangent and normal vectors

Tangent T to the curve at u_0

$$C_u = \frac{\partial C(u)}{\partial u}$$



$$T = \frac{C_u}{\|C_u\|}$$

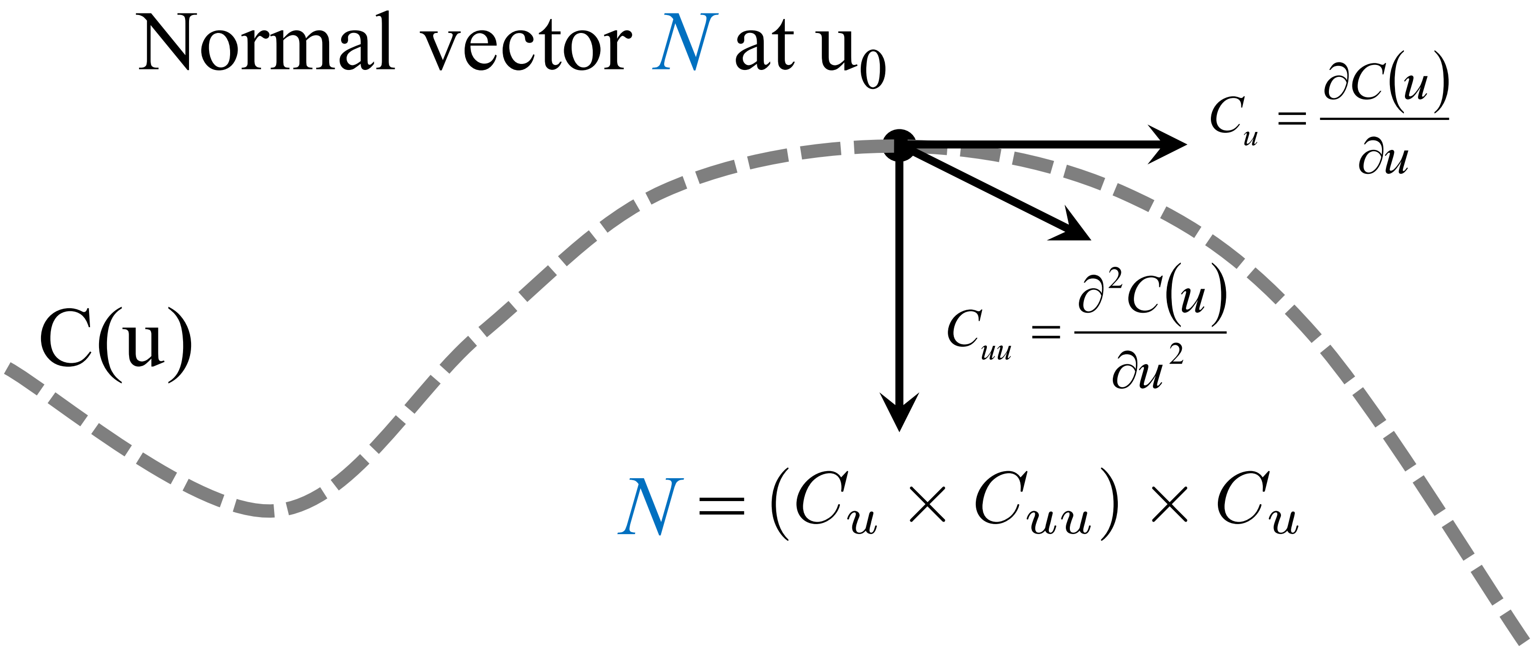
Unit tangent

Ex.

x(u) = cos(u)

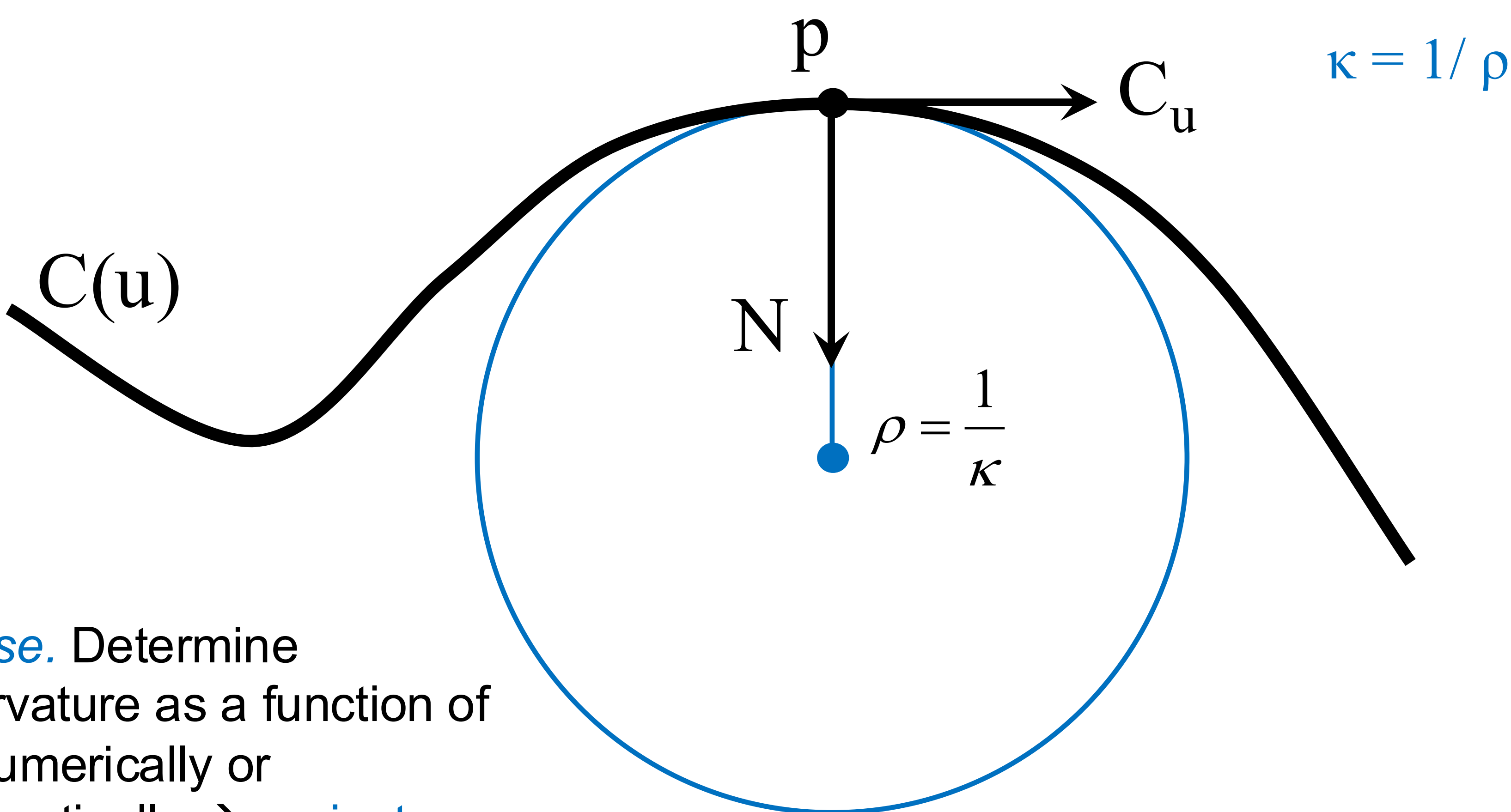
y(u) = sin(u)

Normal vector of a curve



1D curve has only one curvature

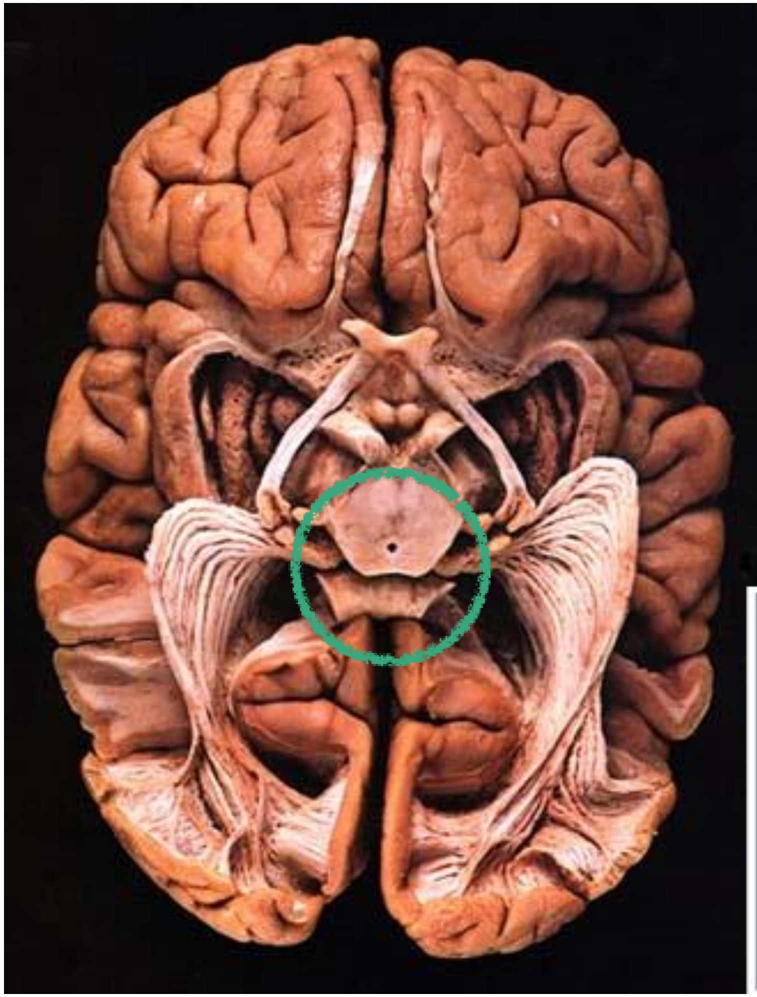
Curvature κ at u_0 and the radius ρ osculating circle

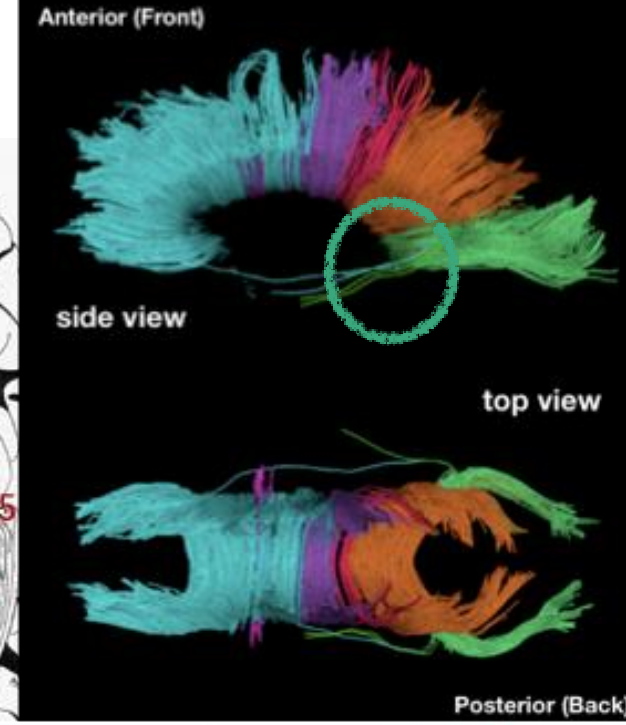


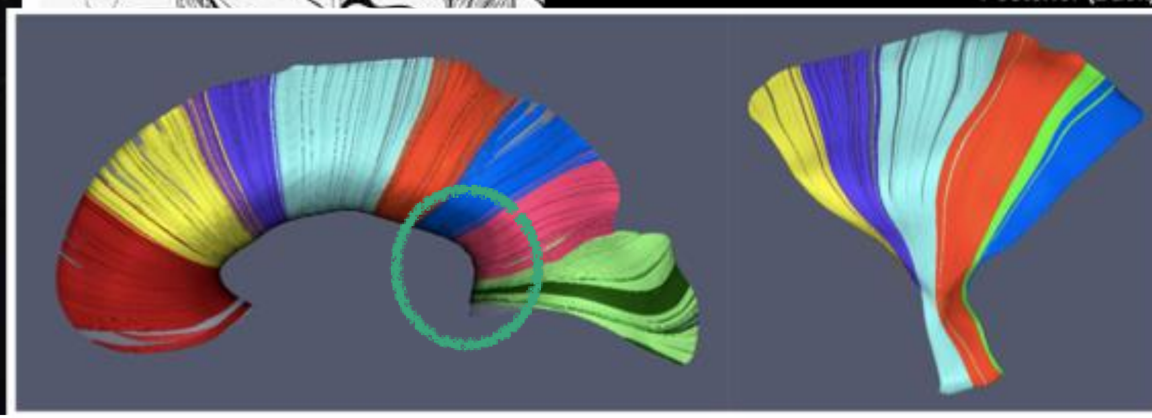
Exercise. Determine the curvature as a function of C(u) numerically or mathematically → project

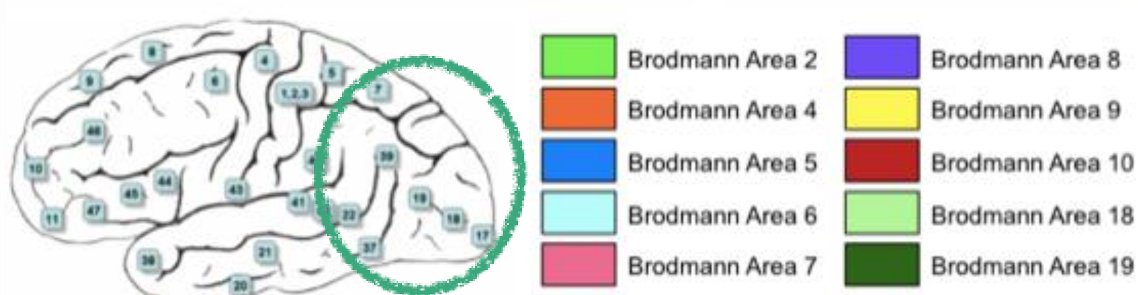
White matter fibers

James Gee
Univ. Penn









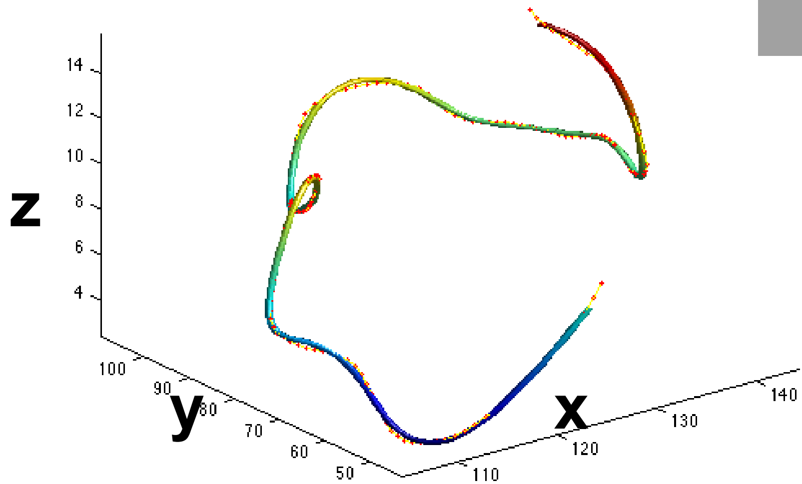
Fibers passing through the splenium of the corpus callosum

Cosine series representation

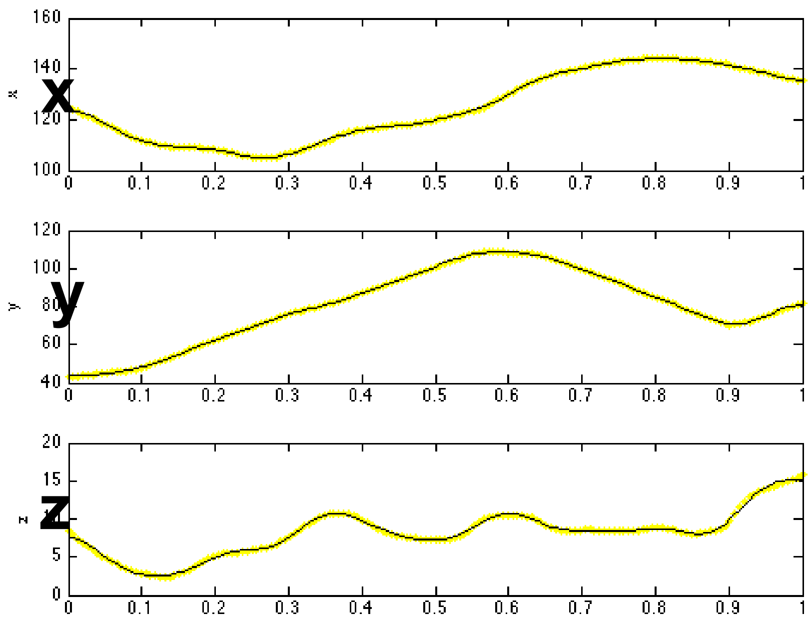
Cover art
Chung, M.K. 2012
Computational Neuroanatomy: The Methods, World Scientific Press

Matlab visualization

White matter fiber tract data



parameterization



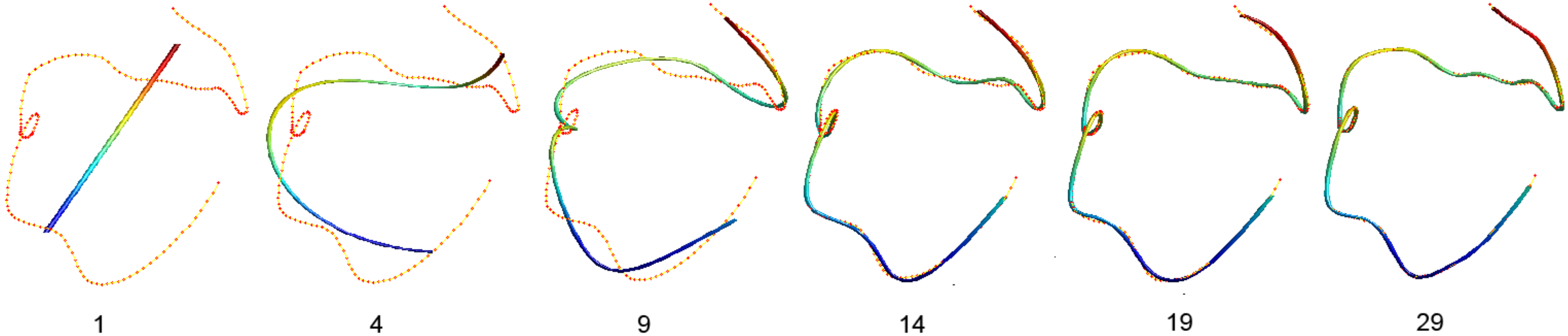
88.1799	56.6336	5.7367
-12.4775	-11.2552	-2.0791
2.4336	-15.4428	-0.4021
4.3956	2.2733	-0.9354
-0.0106	-0.0674	0.6999
2.1773	-2.4194	-0.1176
0.5808	0.8390	1.2942
0.0615	-0.1893	0.1188
-0.2629	0.7524	0.1089
0.7909	-0.7276	-0.1901
0.5458	0.6236	0.6939
0.4295	-0.4337	0.2185
0.2150	0.4157	0.0254
0.1584	-0.1973	0.0762
-0.1557	0.2466	-0.1086
0.0632	-0.0978	-0.0208
0.0389	-0.0143	-0.0284
-0.0014	-0.1193	0.1970
0.0004	0.0129	-0.0198
0.1342	0.0002	0.0260

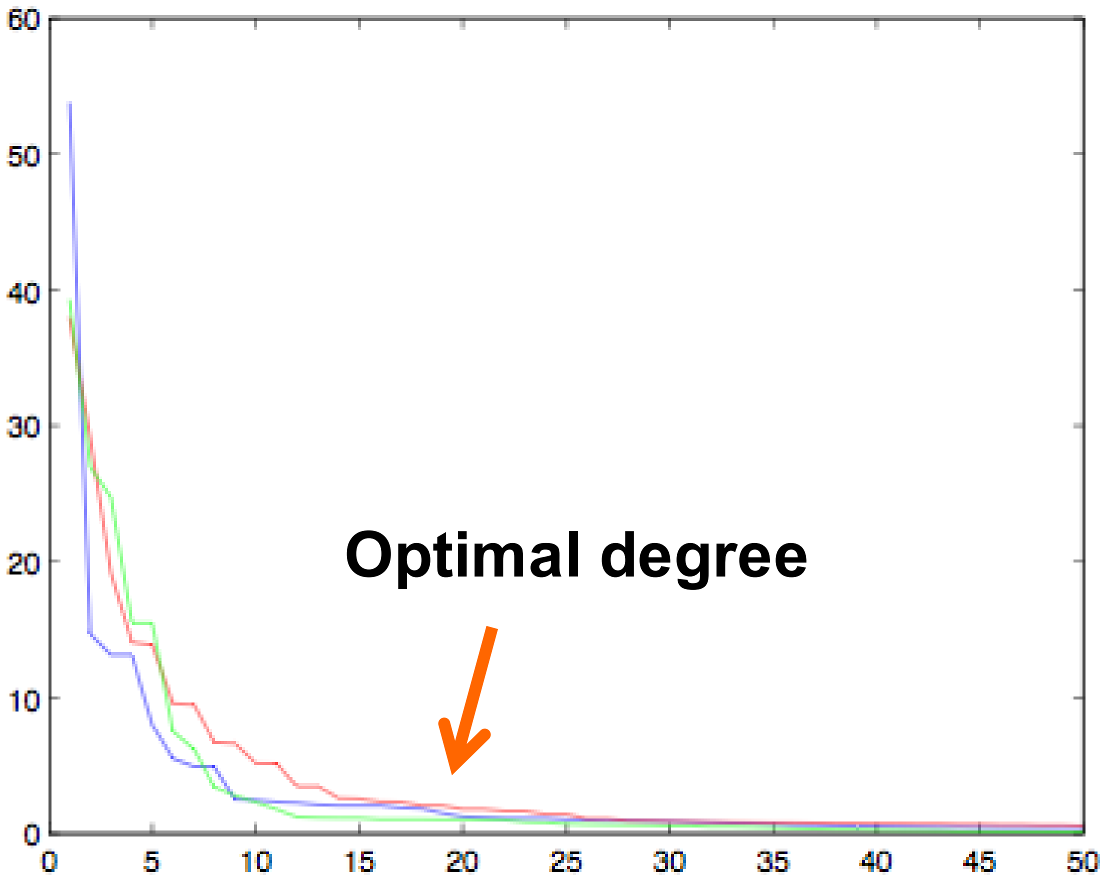
Any tract can be compactly parameterized with only 60 coefficients. **Why?**

basis expansion

$$(x, y, z)' = \sum_{l=0}^{19} \beta_l \cos(l\pi t)$$

Cosine series representation





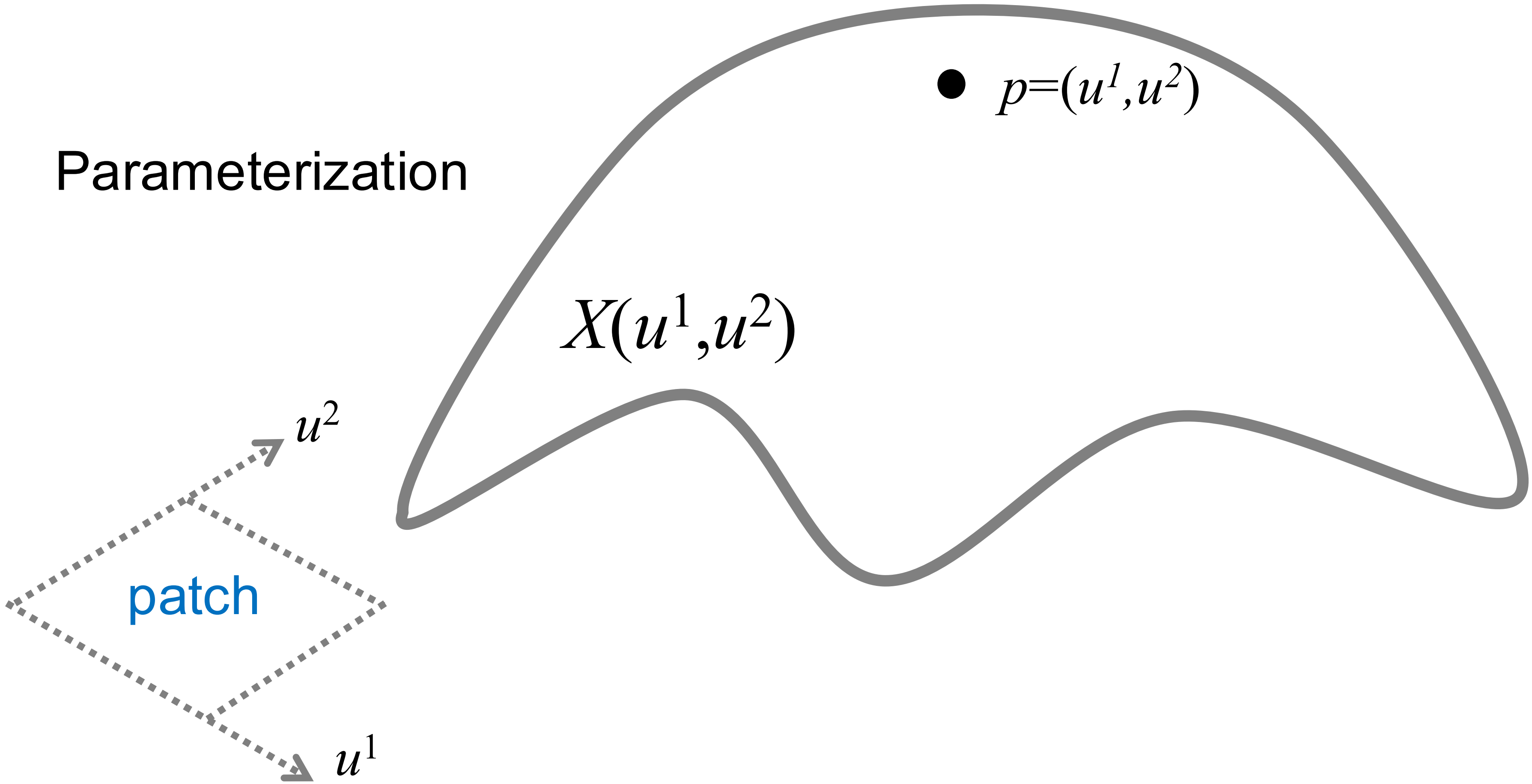
The optimal degree chosen using the forward model selection.

Exercise. Compute Curvature and display on top of 3D curve.

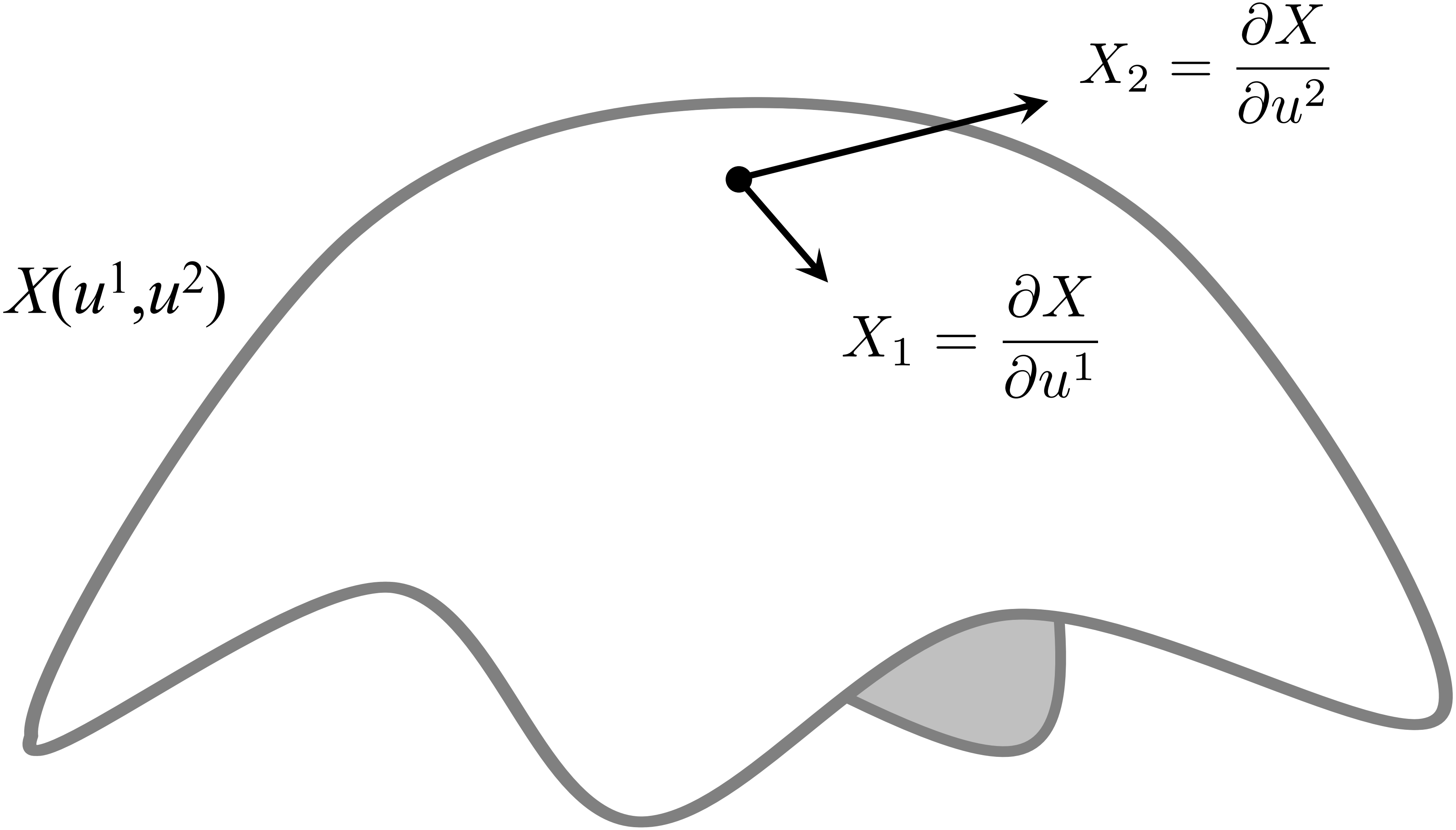
Differential Geometry of a Surface

Point p on the surface

Parameterization



Tangent vectors



Example: quadratic polynomial surface

$$x = u^1$$
$$y = u^2$$
$$z = \beta_0 + \beta_1 u^1 + \beta_2 u^2 + \beta_3 u^1 u^2 + \beta_4 (u^1)^2 + \beta_5 (u^2)^2$$

$$X(u^1, u^2) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u^1 \\ u^2 \\ \beta_0 + \beta_1 u^1 + \cdots + \beta_5 (u^2)^2 \end{pmatrix}$$

$$\frac{\partial X}{\partial u^1} = \begin{pmatrix} 1 \\ 0 \\ \beta_1 + \beta_3 u^2 + 2\beta_4 u^1 \end{pmatrix}$$

$$\frac{\partial X}{\partial u^2} = \begin{pmatrix} 0 \\ 1 \\ \beta_2 + \beta_3 u^1 + 2\beta_5 u^2 \end{pmatrix}$$

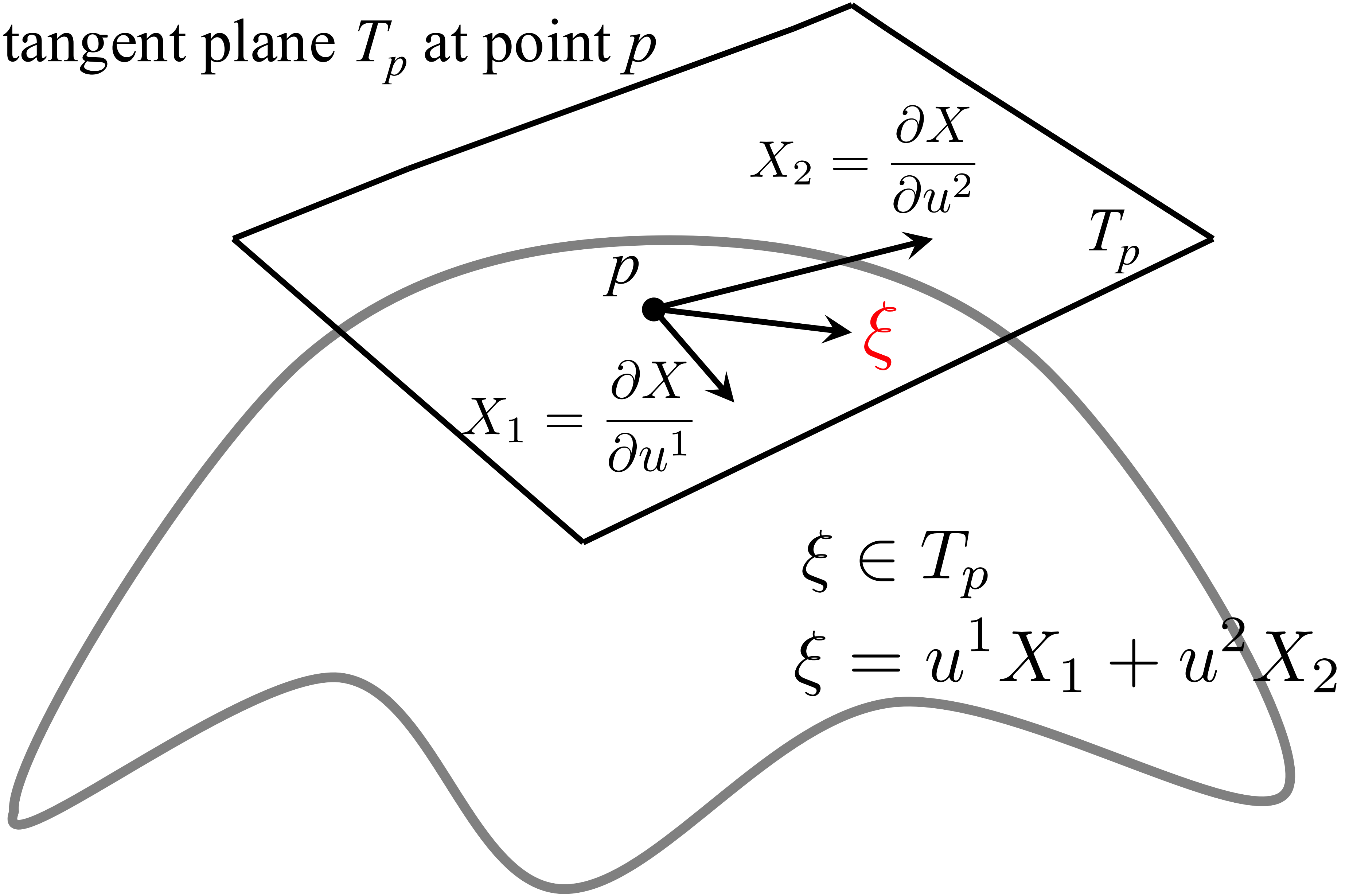
At origin $(u^1, u^2) = (0, 0)$,

$$\frac{\partial X}{\partial u^1} = \begin{pmatrix} 1 \\ 0 \\ \beta_1 \end{pmatrix}$$

$$\frac{\partial X}{\partial u^2} = \begin{pmatrix} 0 \\ 1 \\ \beta_2 \end{pmatrix}$$

Tangent plane

tangent plane T_p at point p



First Fundamental Form

Differential form

$$d\xi = du^1 X_1 + du^2 X_2$$

$$d\xi^2 =^d \langle d\xi, d\xi \rangle = \sum_{i,j} \langle X_i, X_j \rangle du^i du^j$$

Riemannian metric

Metric tensor

$$g_{ij} = \langle X_i, X_j \rangle \dashrightarrow g = (g_{ij})$$

Example: quadratic surface

$$x = u^1$$
$$y = u^2$$
$$z = \beta_0 + \beta_1 u^1 + \beta_2 u^2 + \beta_3 u^1 u^2 + \beta_4 (u^1)^2 + \beta_5 (u^2)^2$$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ \beta_1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 1 \\ \beta_2 \end{pmatrix}$$

$$g_{11} = \langle X_1, X_1 \rangle = \|X_1\|^2 = 1 + \beta_1^2$$

$$g_{12} = \langle X_1, X_2 \rangle = \beta_1 \beta_2$$

$$g_{22} = \langle X_2, X_2 \rangle = \|X_2\|^2 = 1 + \beta_2^2$$

$$\rightarrow \quad g = \begin{pmatrix} 1+\beta_1^2 & \beta_1\beta_2 \\ \beta_1\beta_2 & 1+\beta_2^2 \end{pmatrix}$$

positive definite symmetric

Riemannian metric tensor

Riemannian manifold (*M,g*) is a smooth manifold M with inner product g on the the tangent space $T_p(M)$. *At each point p, the metric varies smoothly in such a way that* if X_1 and X_2 are differentiable tangent vector, g is also a smooth function. The family of such inner products is called a Riemannian metric tensor.

If the metric doesn't vary, it's just the boring Euclidean space.

Riemannian metric tensor

Question. If g_1, g_2, ...g_n are metrics, under what condition $c_1\, g_1 + c_2\, g_2 + \ldots\ldots + c_n g_n$ is metric as well?

Area element

parameter space *N* → manifold *M*

Area of manifold M = $\int_M d\mu(p) = \int_N \sqrt{detg}\, du^1 du^2$

Local surface area element = Jacobian determinant

$$J = \sqrt{\det g}$$

measures the amount of area with respect to the unit patch area

Exercise: Show surface area is invariant under parameterization

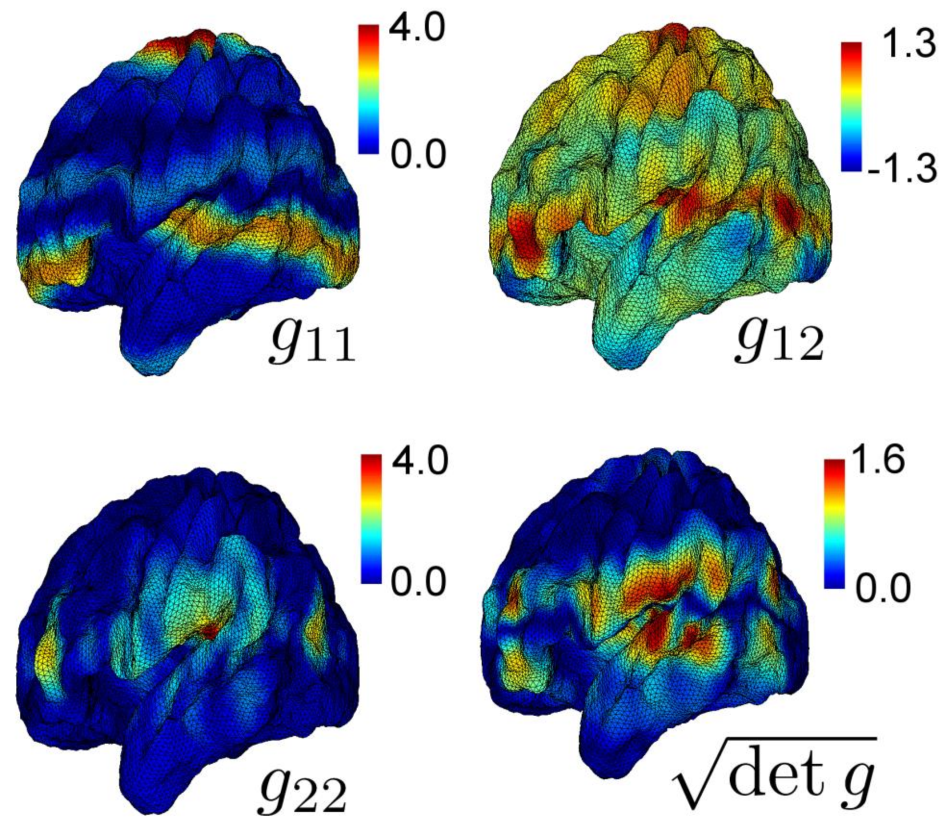
Tensor-based morphometry

Surface parameterization:

$$u = (u^1, u^2) \rightarrow X(u)$$

Riemannian metric tensor:

$$g_{ij} = \left\langle \frac{\partial X}{\partial u^i}, \frac{\partial X}{\partial u^j} \right\rangle$$



Chung et al. 2003 NeuroImage 18:198-213

Chung et al. 2008. IEEE Transactions on Medical Imaging 27:1143-1151

Jacobian matrix and its determinant

$$\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$$

$$\mathbf{x} = \{x_1, \cdots, x_n\} \in \mathbb{R}^n$$

Jacobian matrix

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1} & \cdots & \frac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Not symmetric

Jacobian determinant $\det \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$

Can we equate?

$$\det \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \sqrt{\det g}$$

Relation between Jacobian matrix and metric tensors

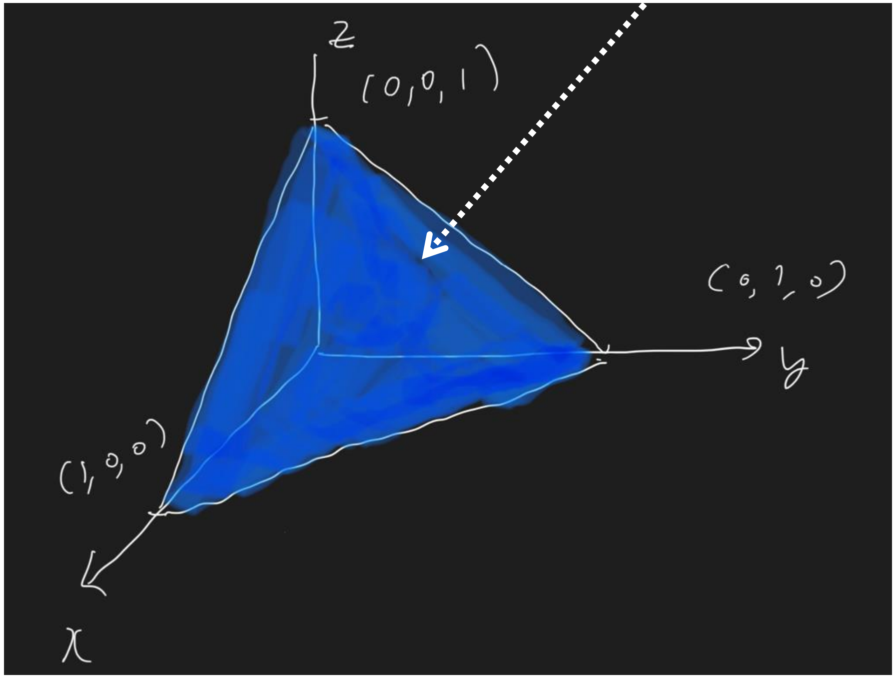
The 3×2 Jacobian matrix J of mapping from parameter space $\mathcal{N}$ to cortical surface $\mathcal{M}$ is given by $J = (\partial_\theta \hat{\nu}, \partial_\varphi \hat{\nu})$. The Riemannian metric tensors are $g = (g_{ij}) = J^t J$. The component is given by $g_{ij} = \partial_i \widehat{\nu} \cdot \partial_j \widehat{\nu}$ with the vector inner product $\cdot$. The Riemannian metric tensors measure the amount of deviation of a cortical surface from a flat Euclidean plane. If the cortical surface is flat, we obtain $g_{ij} = \delta_{ij}$, the identity matrix. The Riemannian metric tensors enable us to compute the local *area element* $\sqrt{\det g}$. The area element measures the amount of the transformed area in $\mathcal{M}$ of the unit area in the parameterized space $\mathcal{N}$ via the mapping ν. Fig. 8 shows the estimation of the metric tensors for a subject. Using the area element, the total surface area of $\mathcal{M}$ can be written as

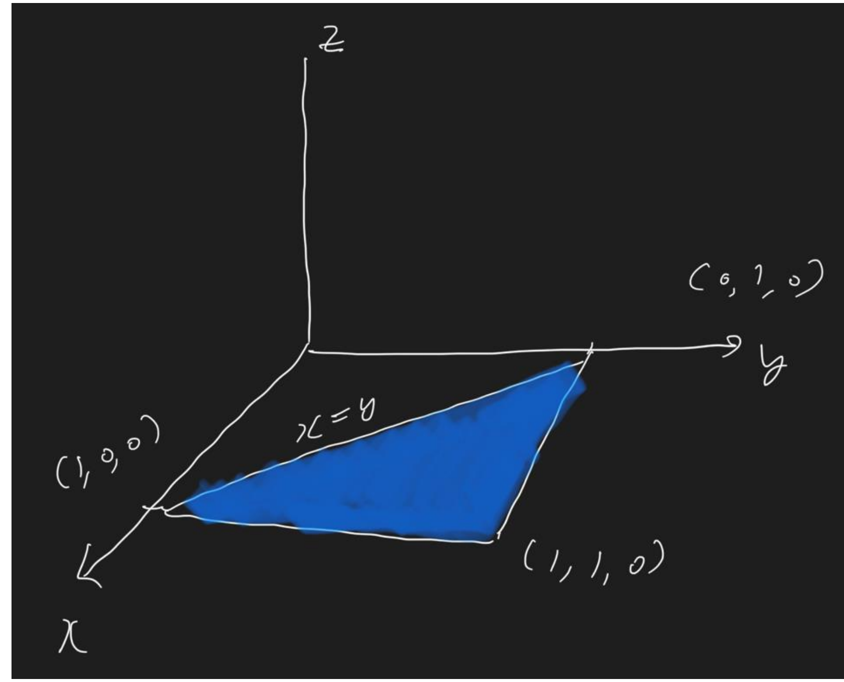
$$\mu(\mathcal{M}) = \int_0^{2\pi} \int_0^{\pi} \sqrt{\det g}(\theta, \varphi) \, d\theta \, d\varphi.$$

Chung, et al. 2008. . Tensor-based cortical surface morphometry via weighed spherical harmonic representation. *IEEE TMI* 27:1143-1151

Exercise: probability distribution on 2-simplex

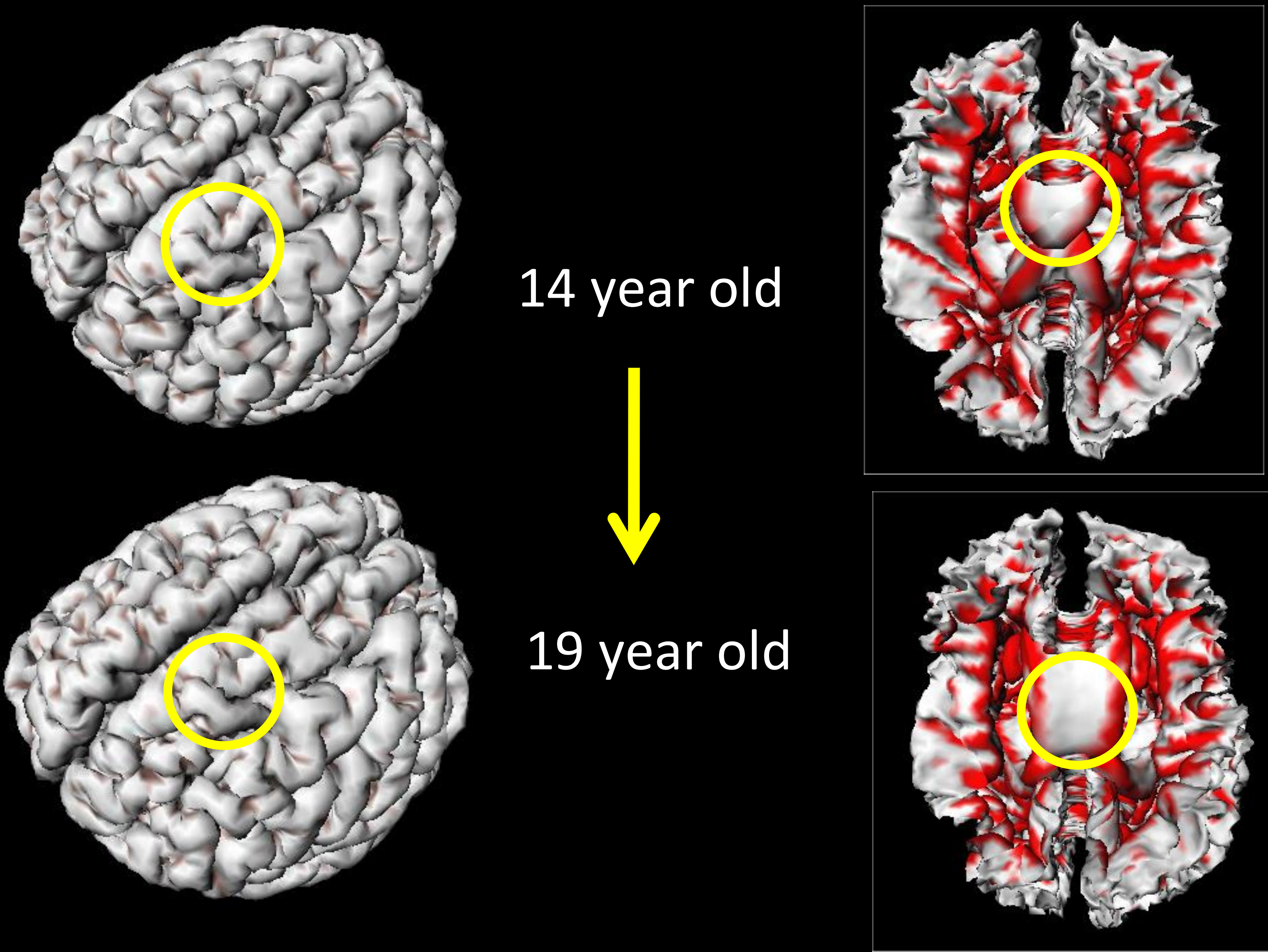
$$f(x, y, z)$$



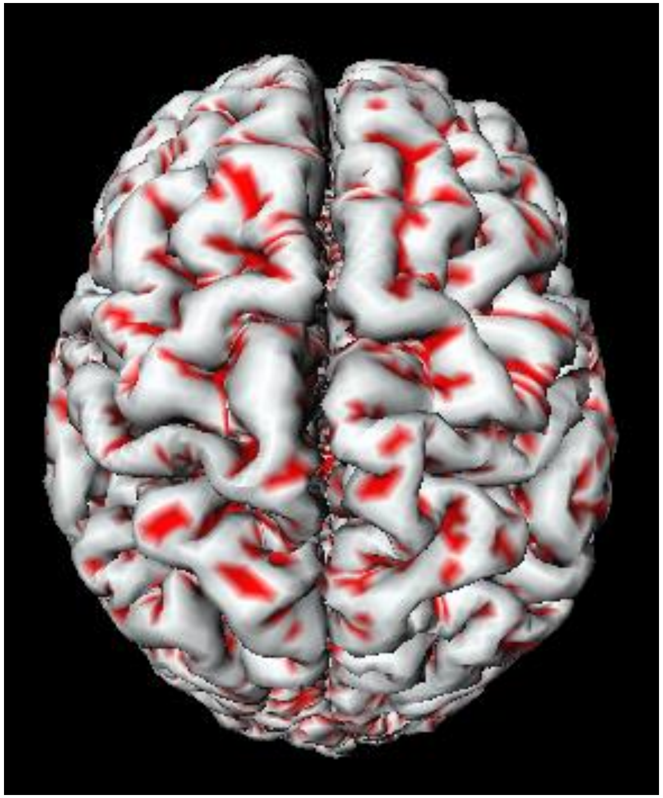


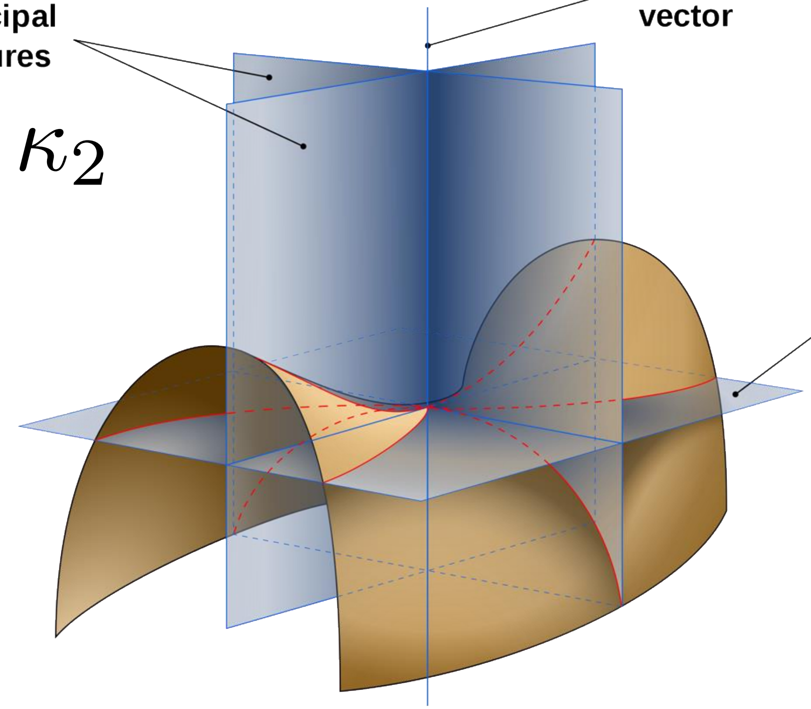
Dirichlet distribution on a triangle domain

How to quantify brain surface growth and atrophy?



Mean and Gaussian curvatures





$$\kappa_1, \kappa_2$$

[Chung et al. 2003 CVPR Tensor-based surface modeling and analysis]

mean curvature

Gaussian curvature

Matlab demo

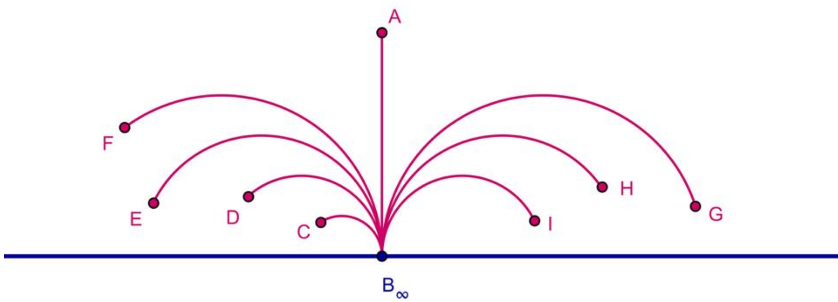
Hyperbolic space

Non-Euclidean space with constant negative Gaussian curvature.
Ex. Saddle surface

Poincare half-plane

$$\{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

Metric: $$\frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$$



Parallel lines

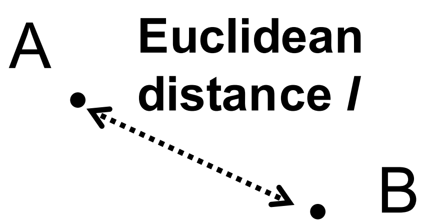
Poincare disk

$$\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < 1\}$$

$$r^2 = \|x\|_2^2 = x_1^2 + \cdots x_n^2$$

You can have L2-norm as well. Why?

Metric: $$4\frac{r^2}{(1-r^2)^2}$$



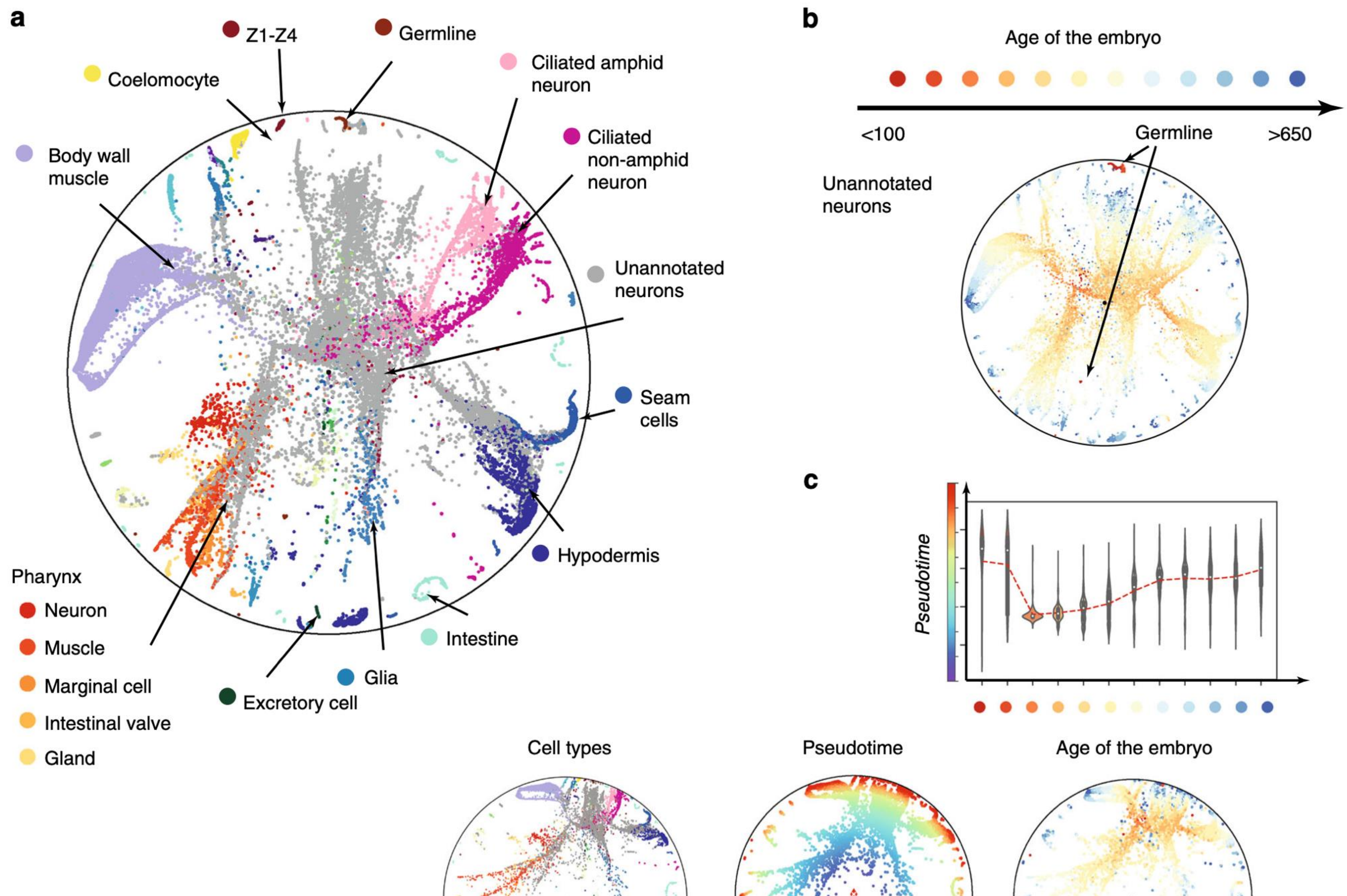
$$dist(A,B) = ln(\frac{1+l}{1-l}) = 2artanh(l)$$

Inverse hyperbolic function

Exercise: *Compute the distance in the Poincare disk*

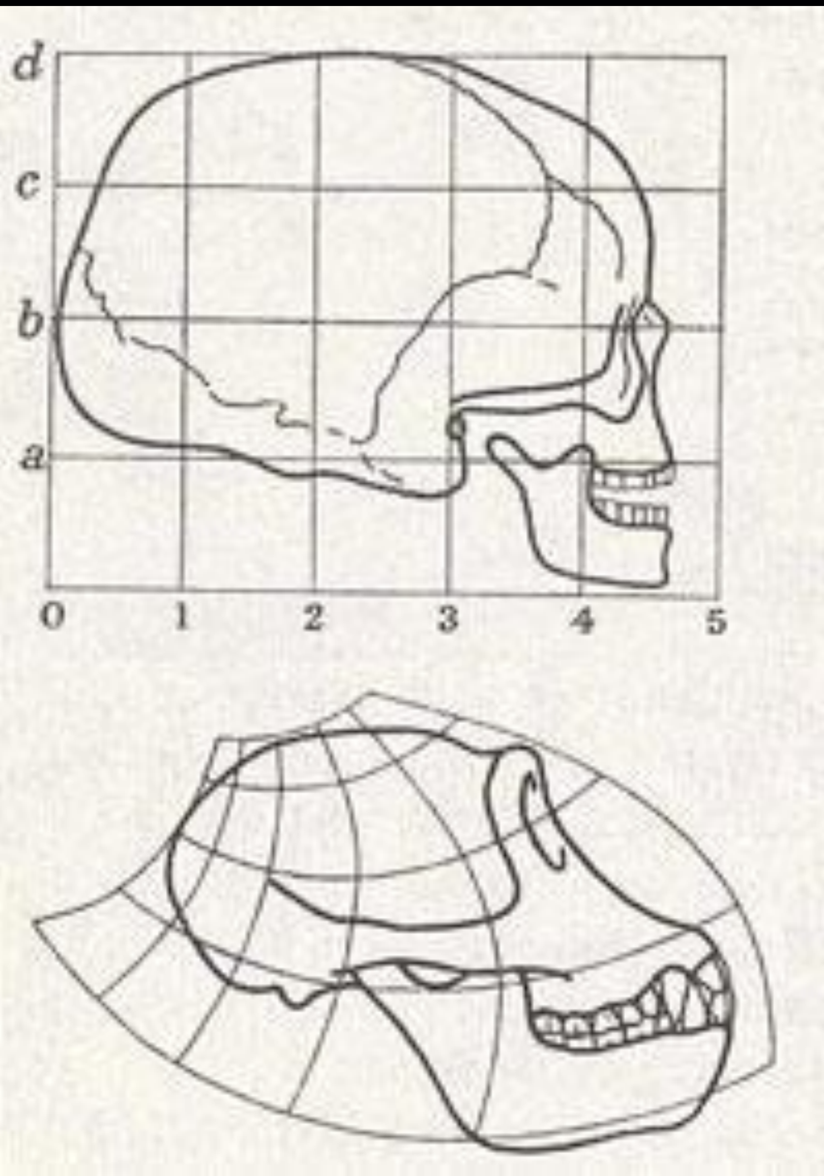
Embedding graphs to Poincare disk

NATURE COMMUNICATIONS | https://doi.org/10.1038/s41467-020-16822-4 ARTICLE



Deformable shape model

D’Arcy Thompson 1860-1948



figuratively speaking, the ‘

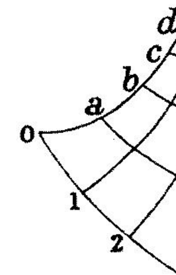
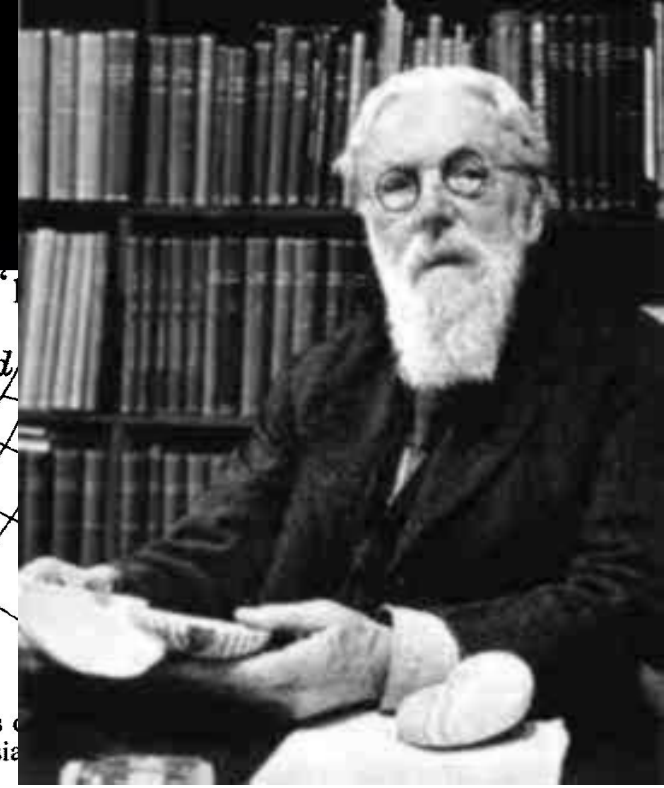


Fig. 178. Co-ordinates
the Cartesia



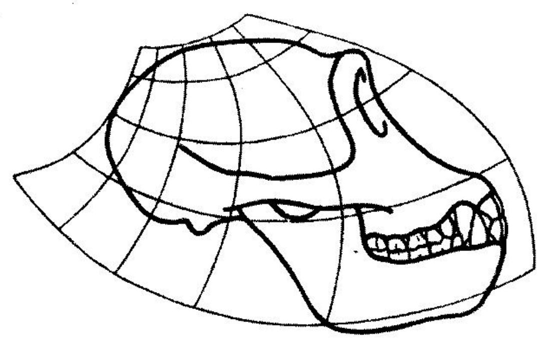


Fig. 179. Skull of chimpanzee.

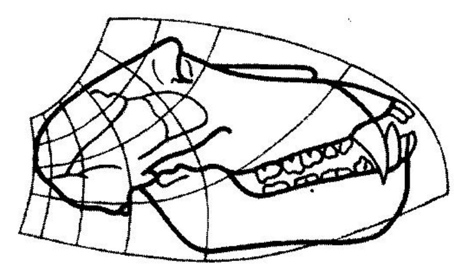
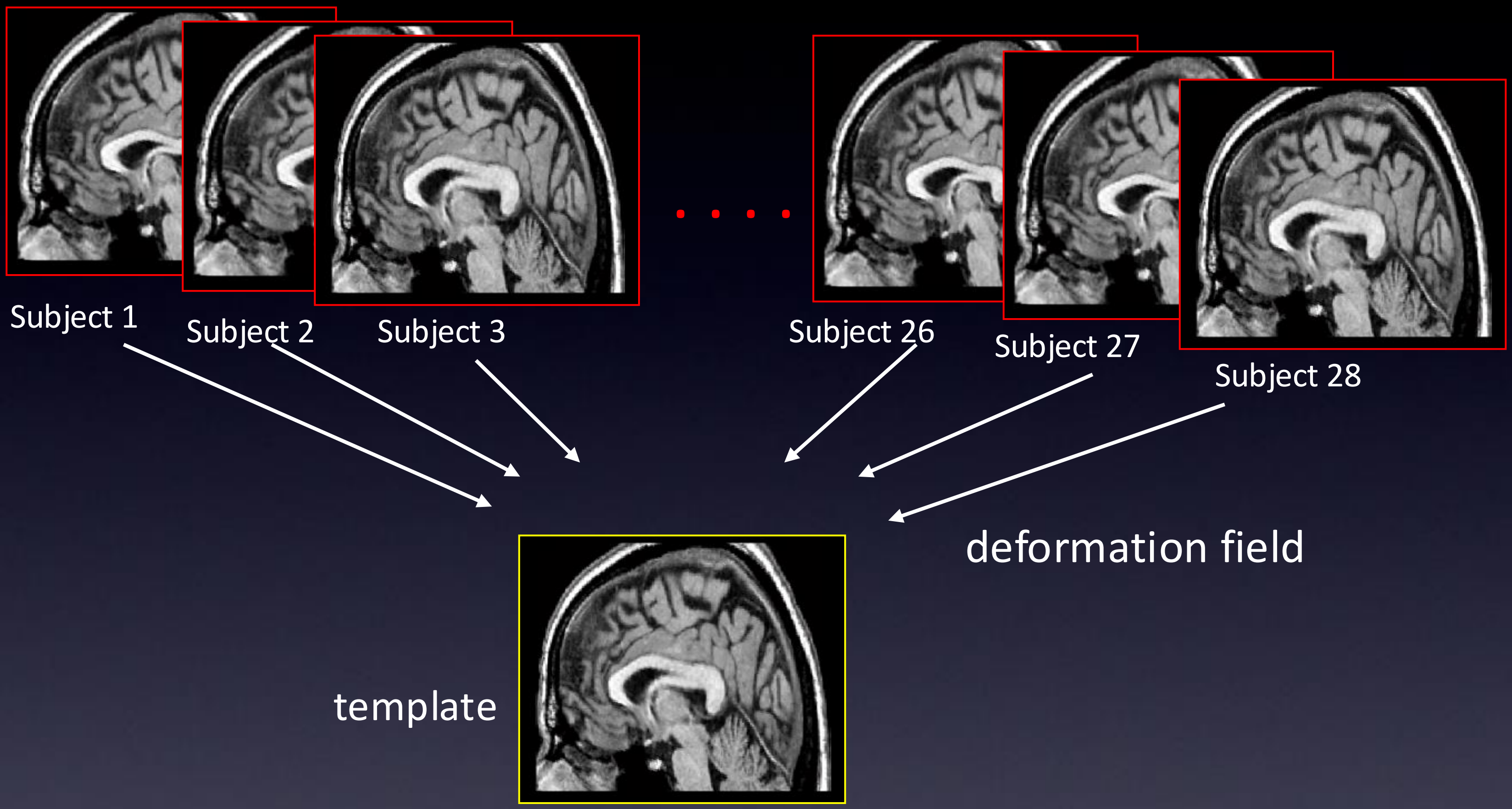


Fig. 180. Skull of baboon.

diagram ... In Fig. 180
I have sh ... oon, and it
is obviou ... order, and
differs or ... ion.[1] These
anthropo ... another by

On Growth and Form
D’Arcy Thompson

Tensor-based morphometry framework

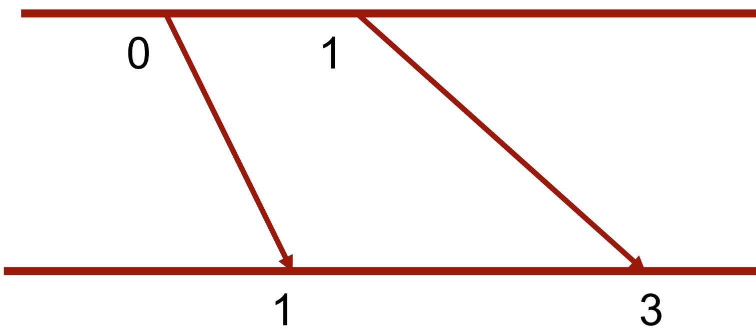


MRIs will be warped into a template and anatomical differences can be compared at a common reference frame.

Jacobian determinant in Euclidean space

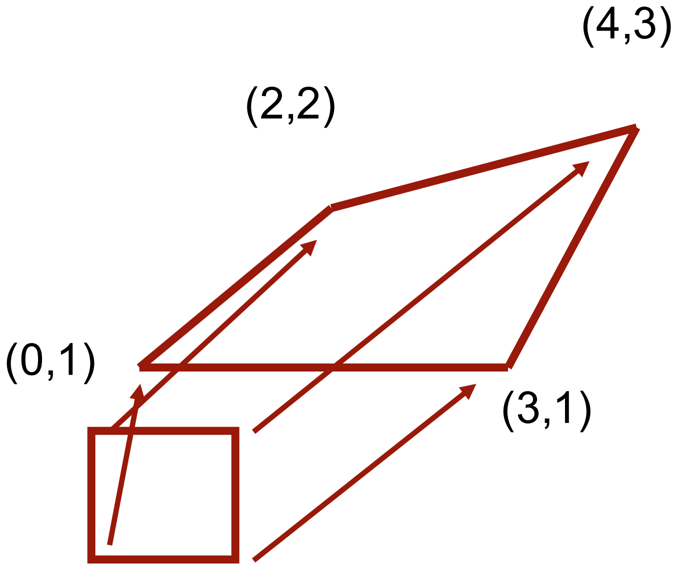
- 1D:

$$x' = 2x + 1$$
$$J(x) = 2$$

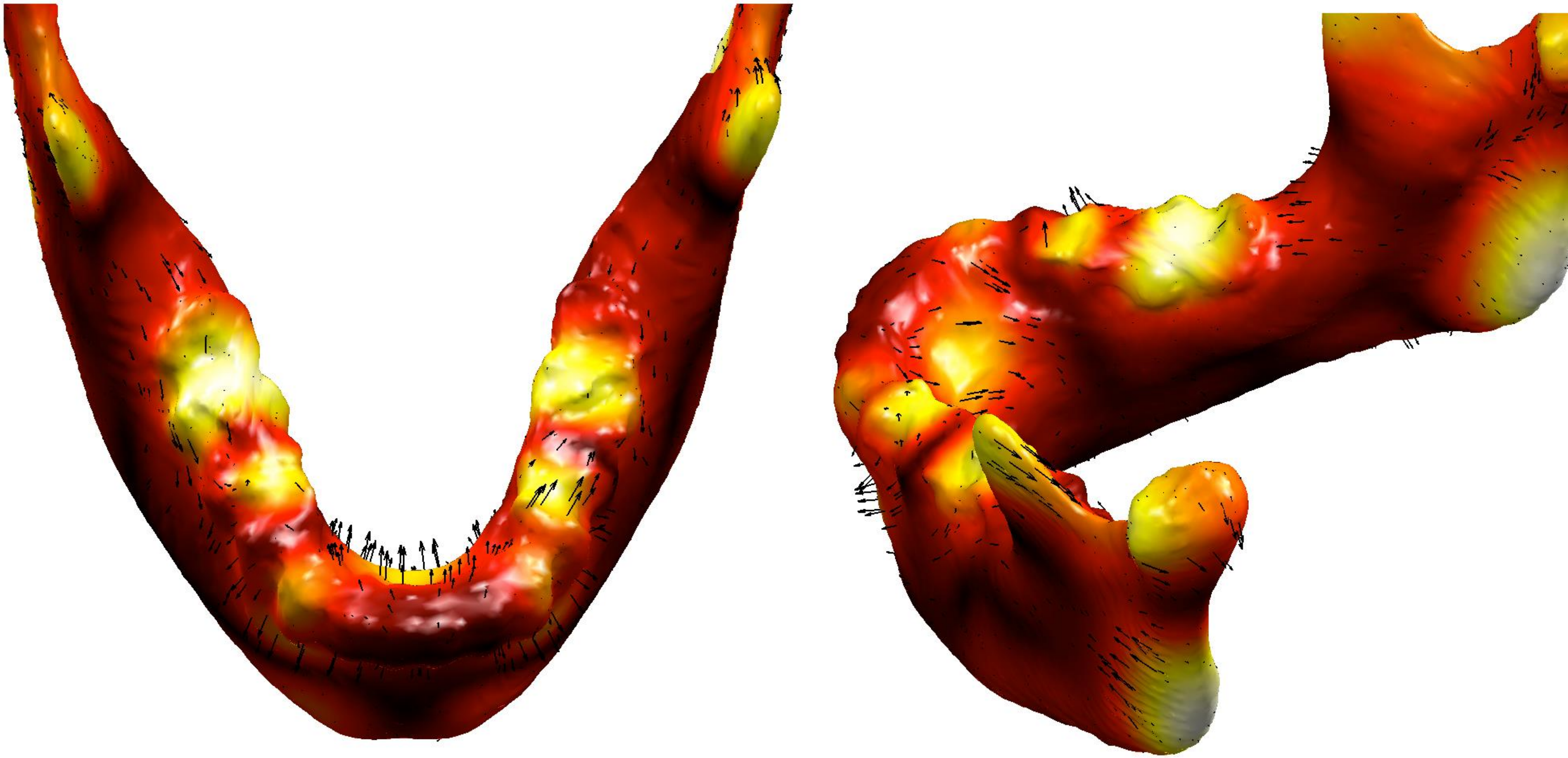


- 2D:

$$x' = 2x + y + 1$$
$$y' = x + 2y$$
$$J(x, y) = 4 - 1 = 3$$



3D Displacement vector field on surface template



Computing Jacobian determinant via metric tensors

$$d_1, d_2, d_3 = d(x_1, x_2, x_3)$$

target position Initial position

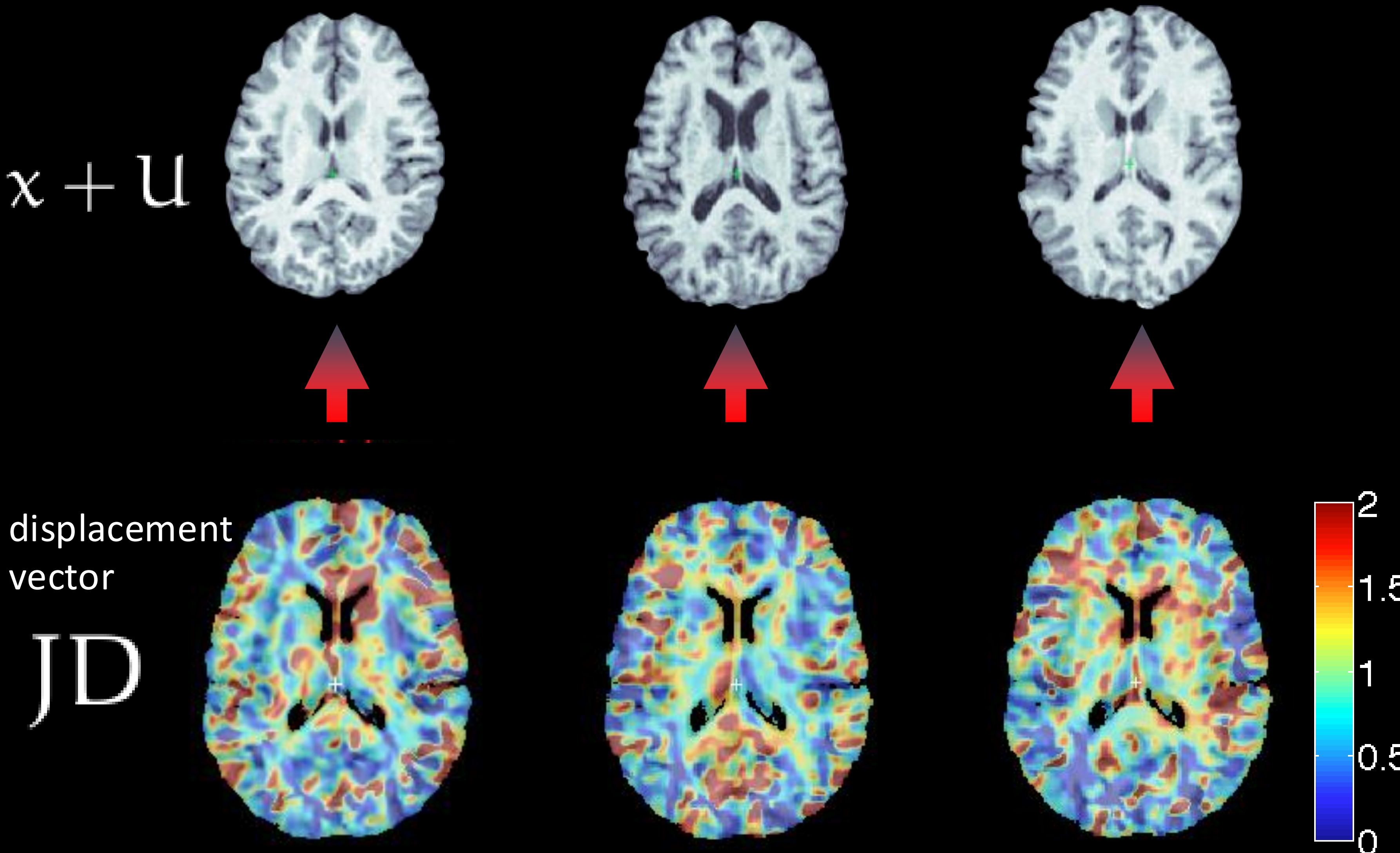
$$U(x_1, x_2, x_3) = d(x_1, x_2, x_3) - (x_1, x_2, x_3)$$

Displacement vector field

Jacobian determinant

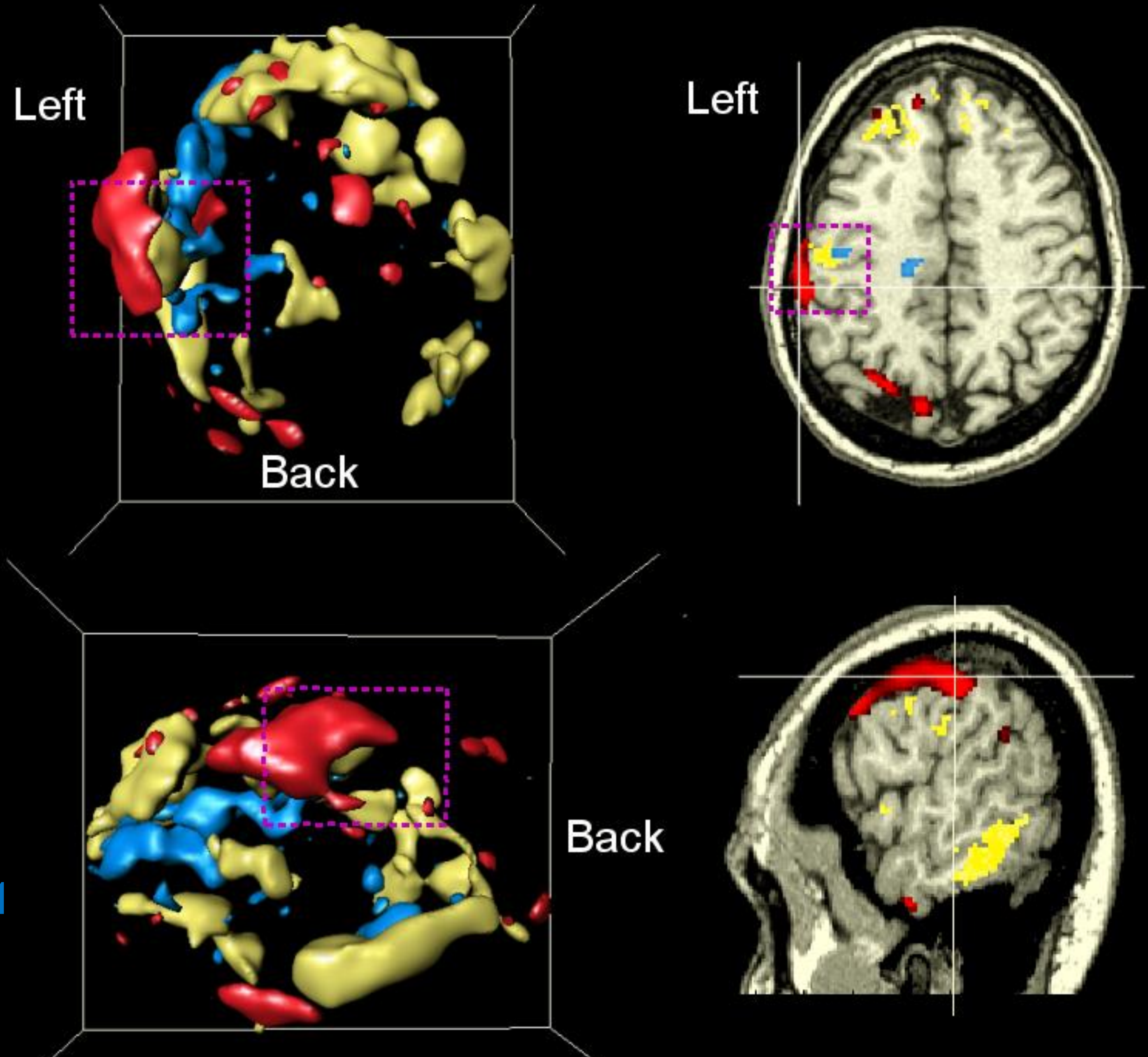
$$J(x) = \det \frac{\partial d(x)}{\partial x} = \det \left[\frac{\partial d_j}{\partial x_i} \right]$$

Jacobian determinant in 3D

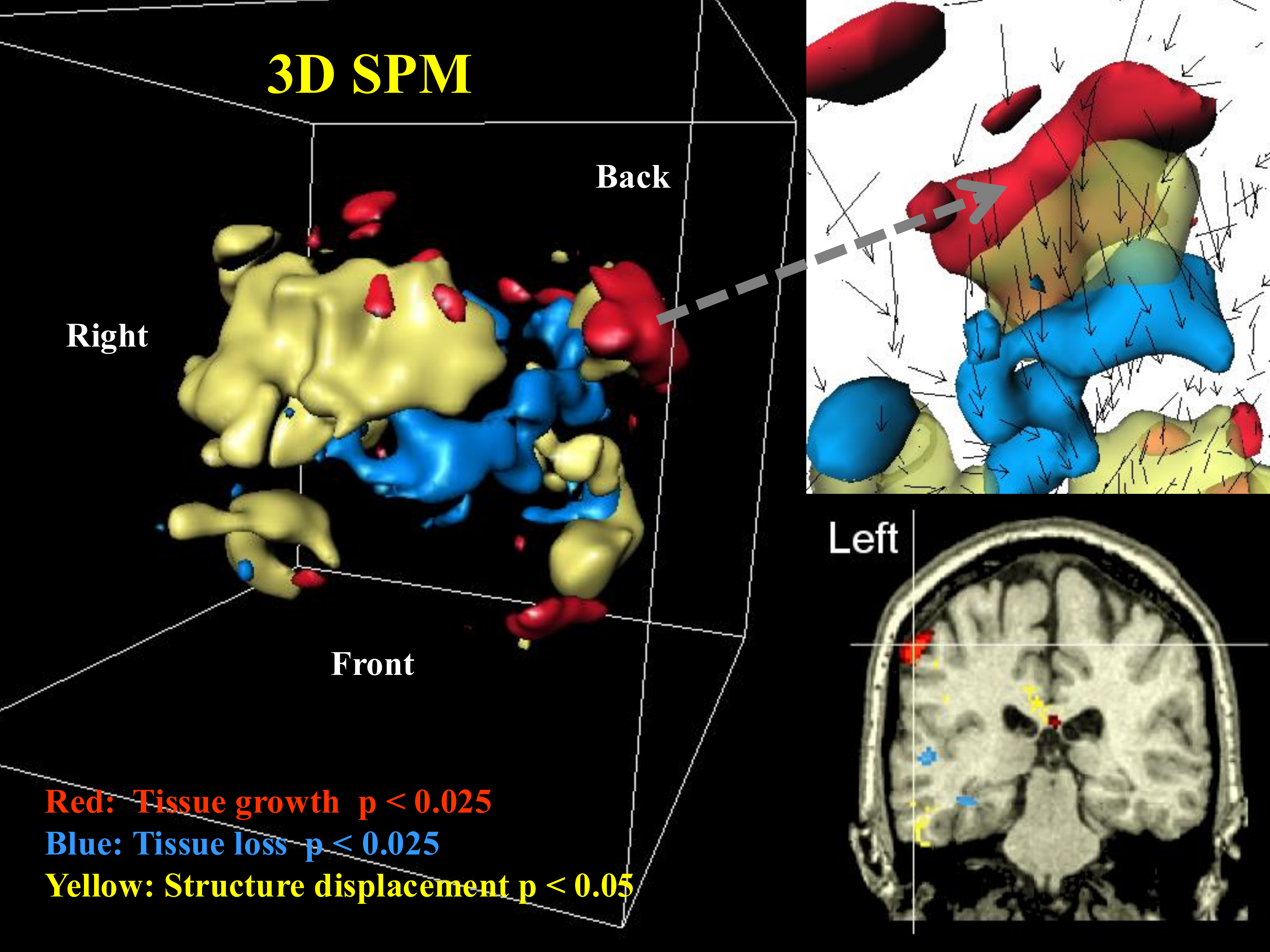


S.-G. Kim, Seoul National University

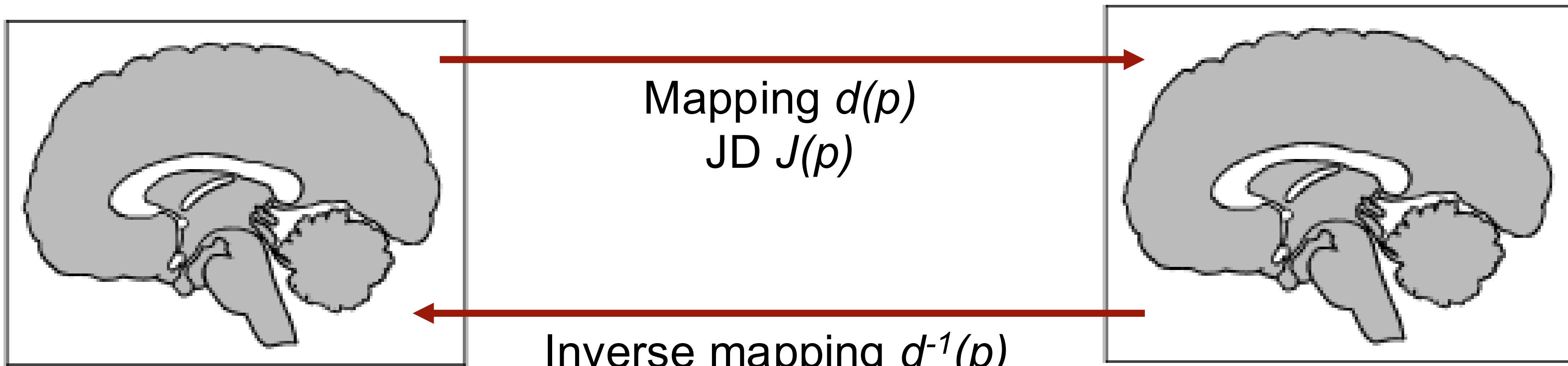
Brain growth in children



Chung et al., 2001
NeuroImage
14:595-606


3D SPM
Back
Right
Front
Left
Red: Tissue growth p < 0.025
Blue: Tissue loss p < 0.025
Yellow: Structure displacement p < 0.05

Statistical properties of Jacobian determinant



Subject 1

Subject 2

- $J(p) > 0$ for one-to-one mapping
- $J(p) > 1$ volume increase; $J(p) < 1$ volume decrease
- Due to symmetry, the statistical distribution of J(p) and 1/J(p) should be *identical*.

Leow A. et al. 2005 IPMI

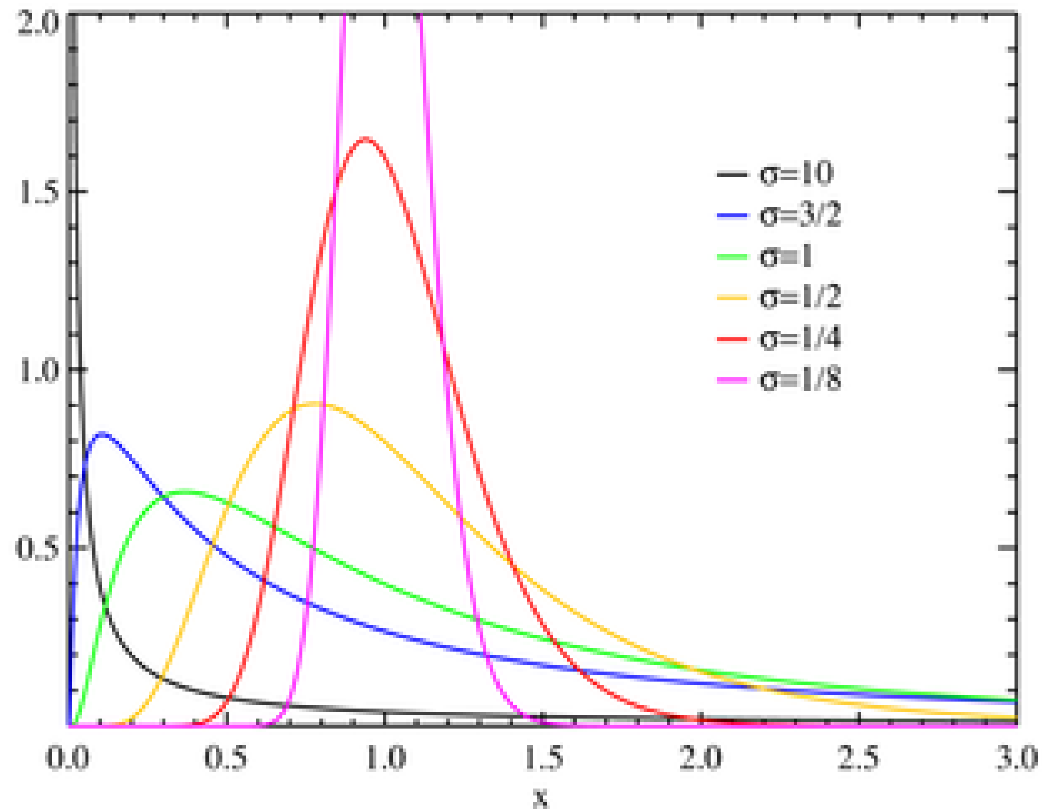
Lognormality of JD

- Domain $-\infty < \log J(p) < \infty$
- If *J(p)=1*, $\log J(p) = 0$
- Symmetry: $\log\left[J^{-1}(p)\right] = -\log J(p)$
- These 3 properties show that JD can be modeled as *lognormal distribution*.

Lognormal distribution

Random variable X is log-normally distributed if *log X* is normally distributed.

Question: Some lognormal distribution looks normal so how do we check if data follows normal or lognormal?



Unit Outcomes

1) Understand curve and surface parametrization

2) Understand Riemannian metric tensors

3) Compute Jacobian determinant

Self assessment questions

1) Can you work out lecture materials using exponential map.

2) Given two surfaces of the same subject, compute the Jacobian determinant of area change

2) Given displacement vector field, compute the Jacobian determinant.

Research problem (*Hard*). Compute the surface normal vector of a given surface mesh.