The Waisman Laboratory for Brain Imaging and Behavior

University of Wisconsin SCHOOL OF MEDICINE AND PUBLIC HEALTH

Topology for Image Analysis

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Spectral geometry

Trees

Segmentation

Tubular structures

Clustering

TDA

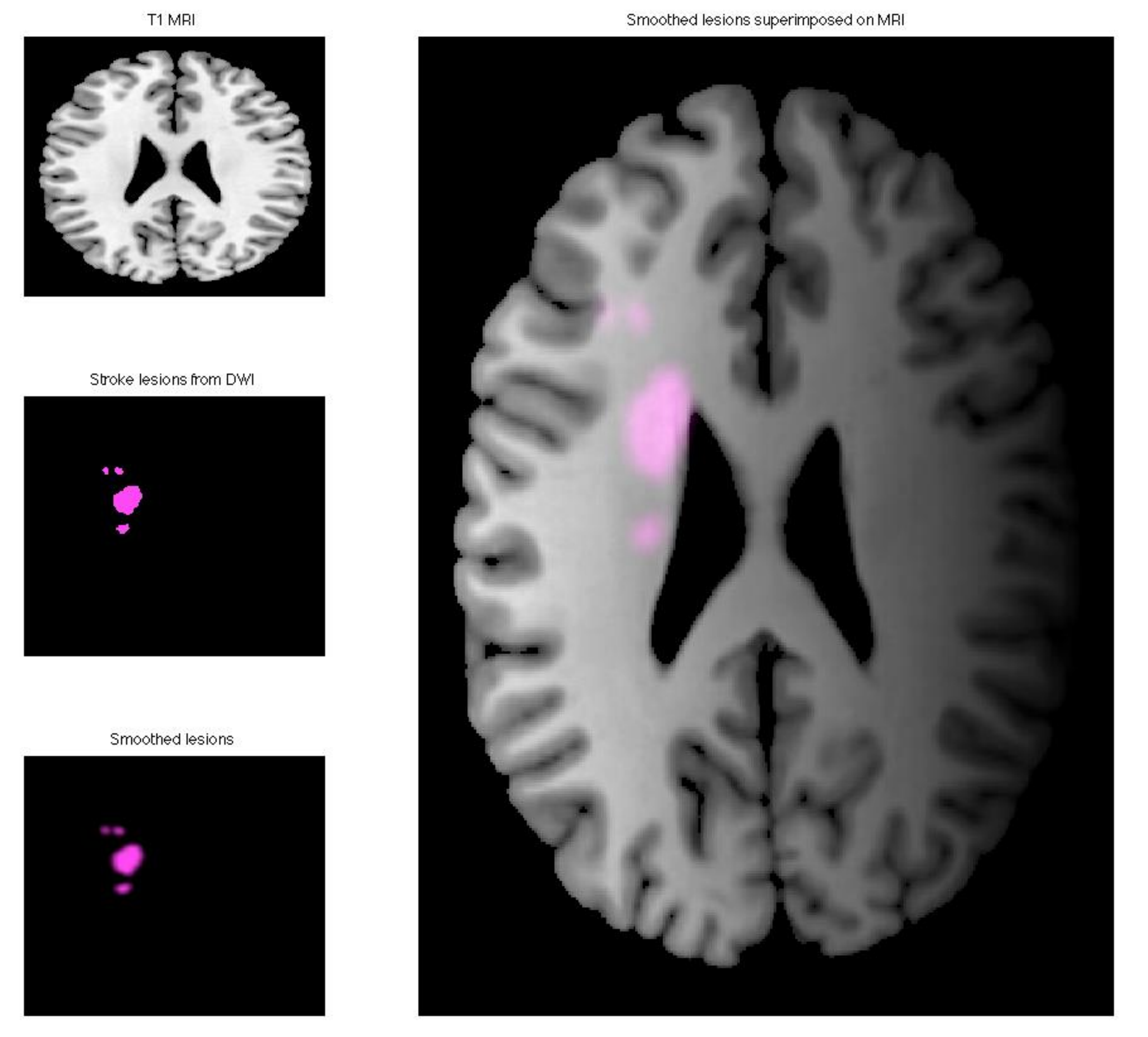
Persistent homology

Topological constraints

Topology correction

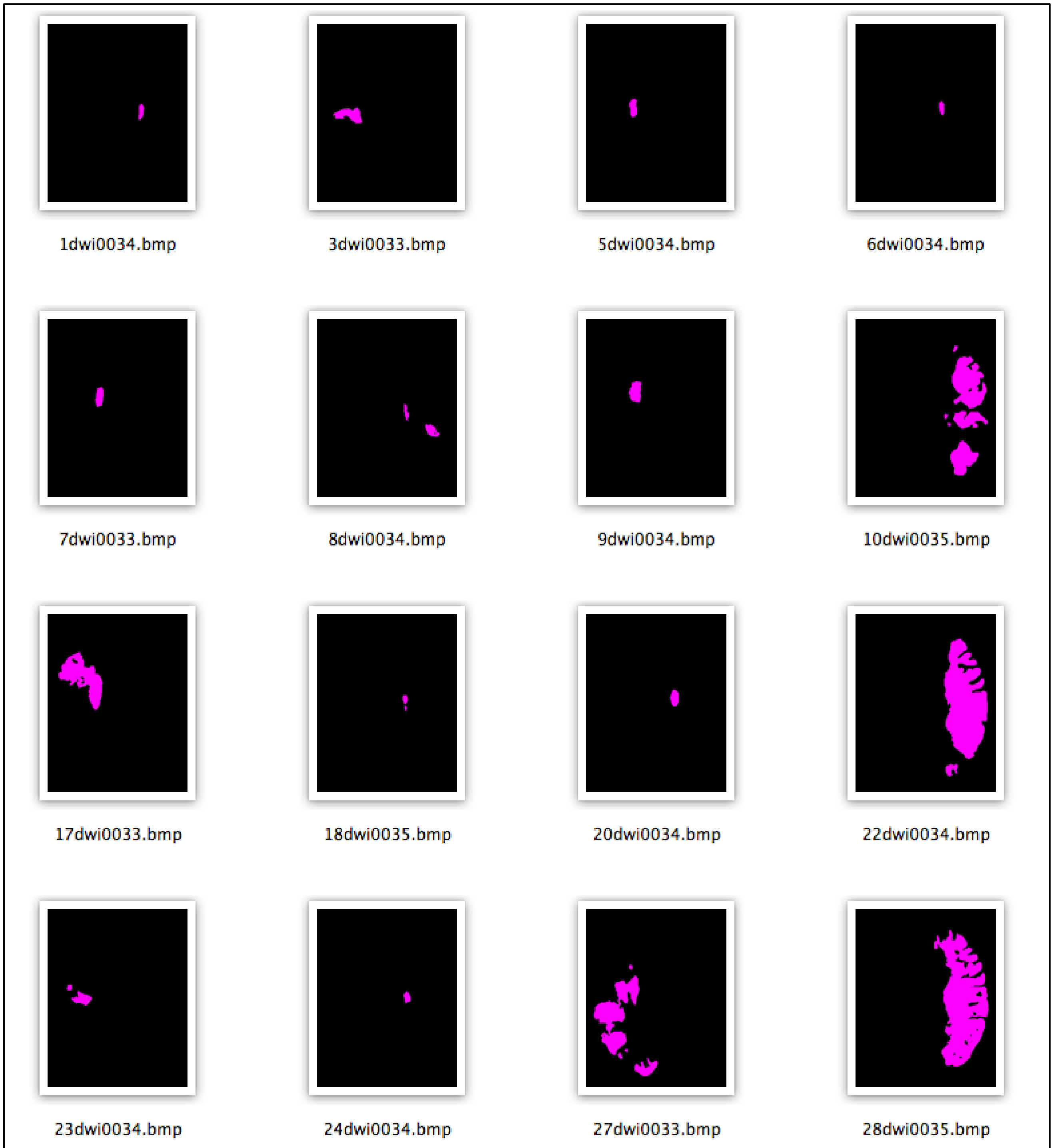
Why we need topology in image segmentation?

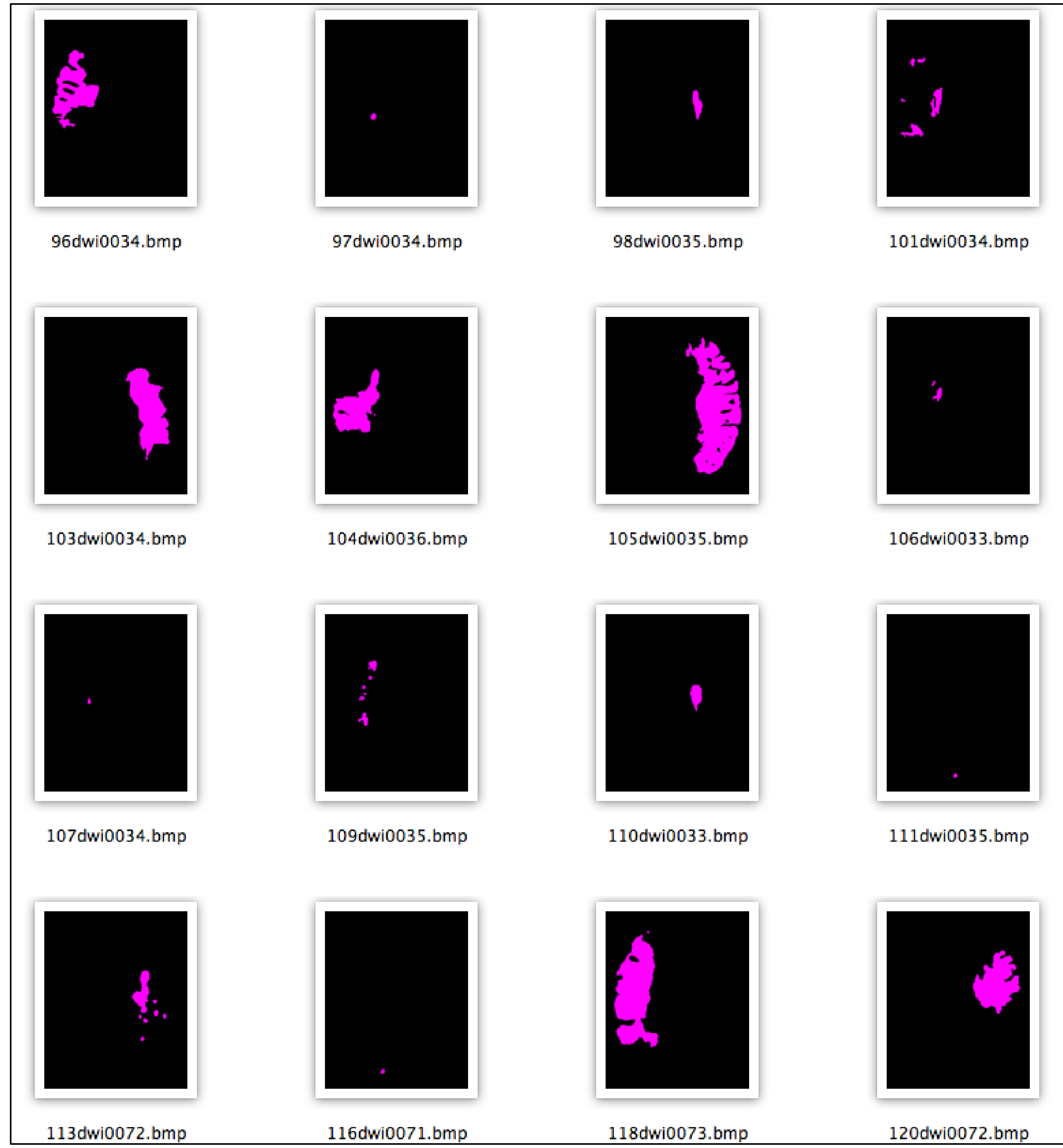
Image segmentation is a topological operation of making an image into multiple disjoint regions. The goal of segmentation is to simplify and/or change the representation of an image (functional data) into discrete states.



A stroke patient with dysphagia in a diffusion tensor image

Stroke patients with dysphagia





Group 0: patients who got improved after one month (n1=58)

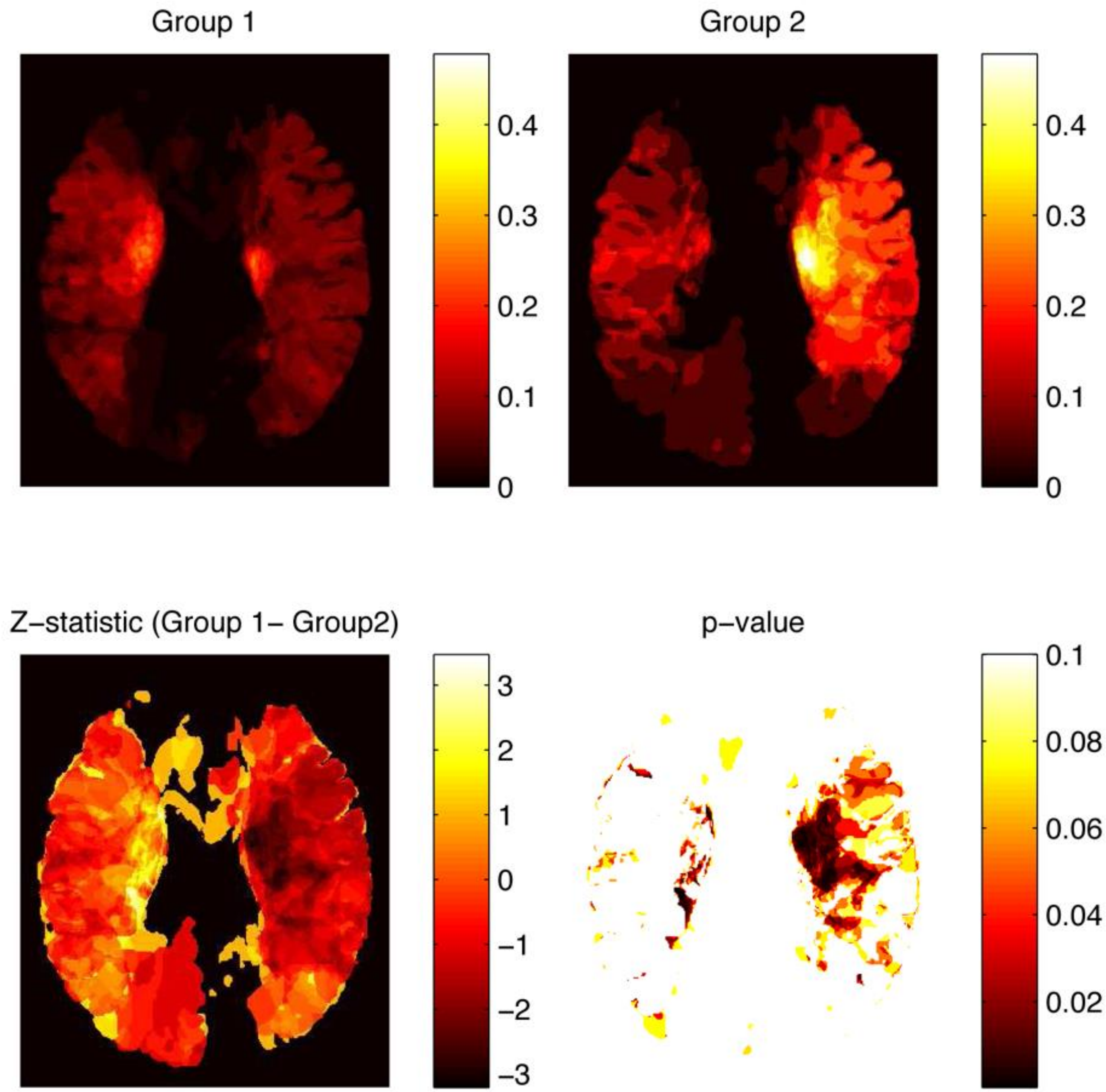
Group 1: patients who didn't get better (n2=23)

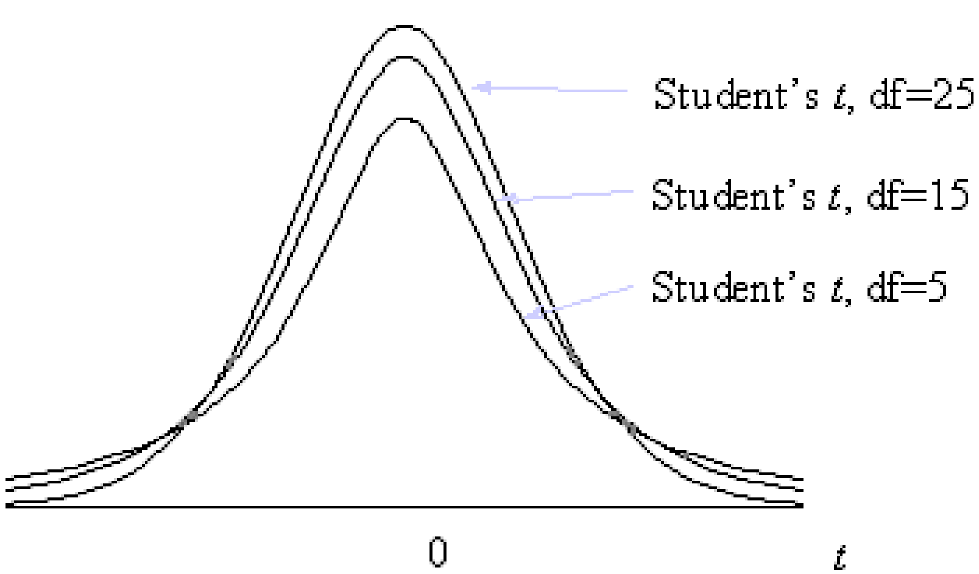
Question: How we discriminate the groups?

Two sample test without smoothing

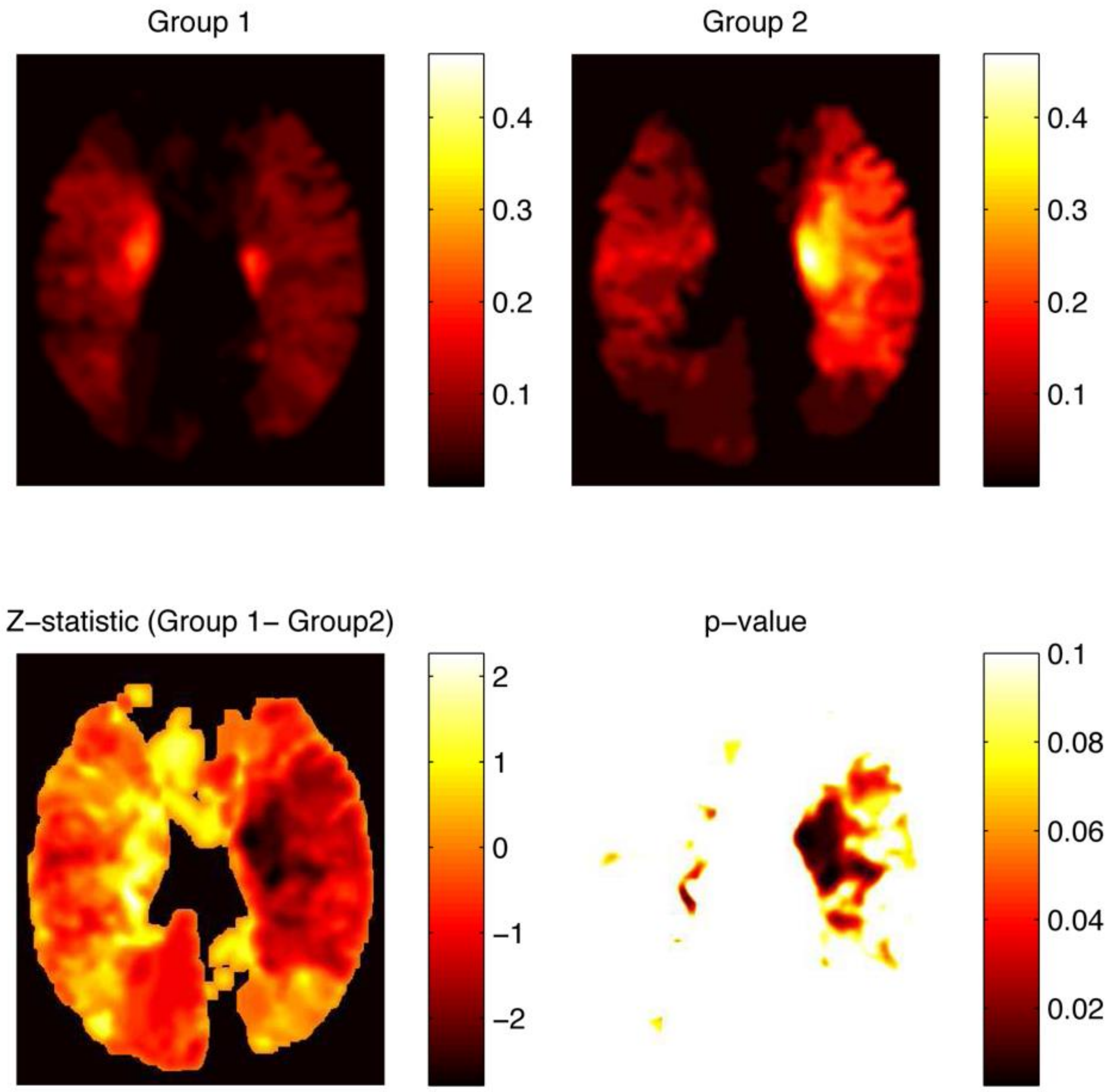
$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \quad \sim t_{n1+n2\text{-}2}$$

$$s^2 = \frac{\sum_{i=1}^{n_1}(x_i - \bar{x}_1)^2 + \sum_{j=1}^{n_2}(x_j - \bar{x}_2)^2}{n_1 + n_2 - 2}$$





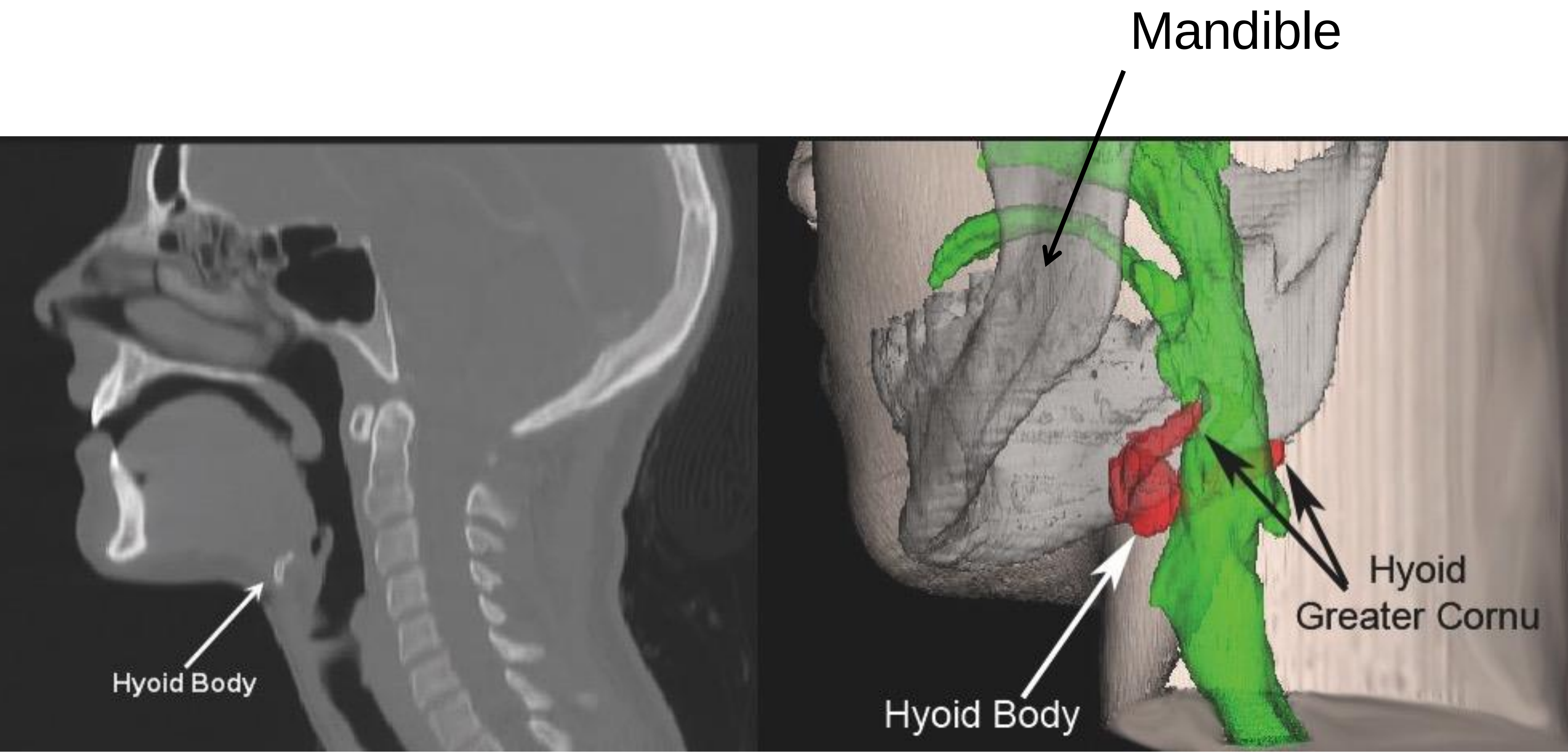
Two sample test with smoothing



Euler characteristics

How do we really check if topological defects in images are corrected without seeing images?

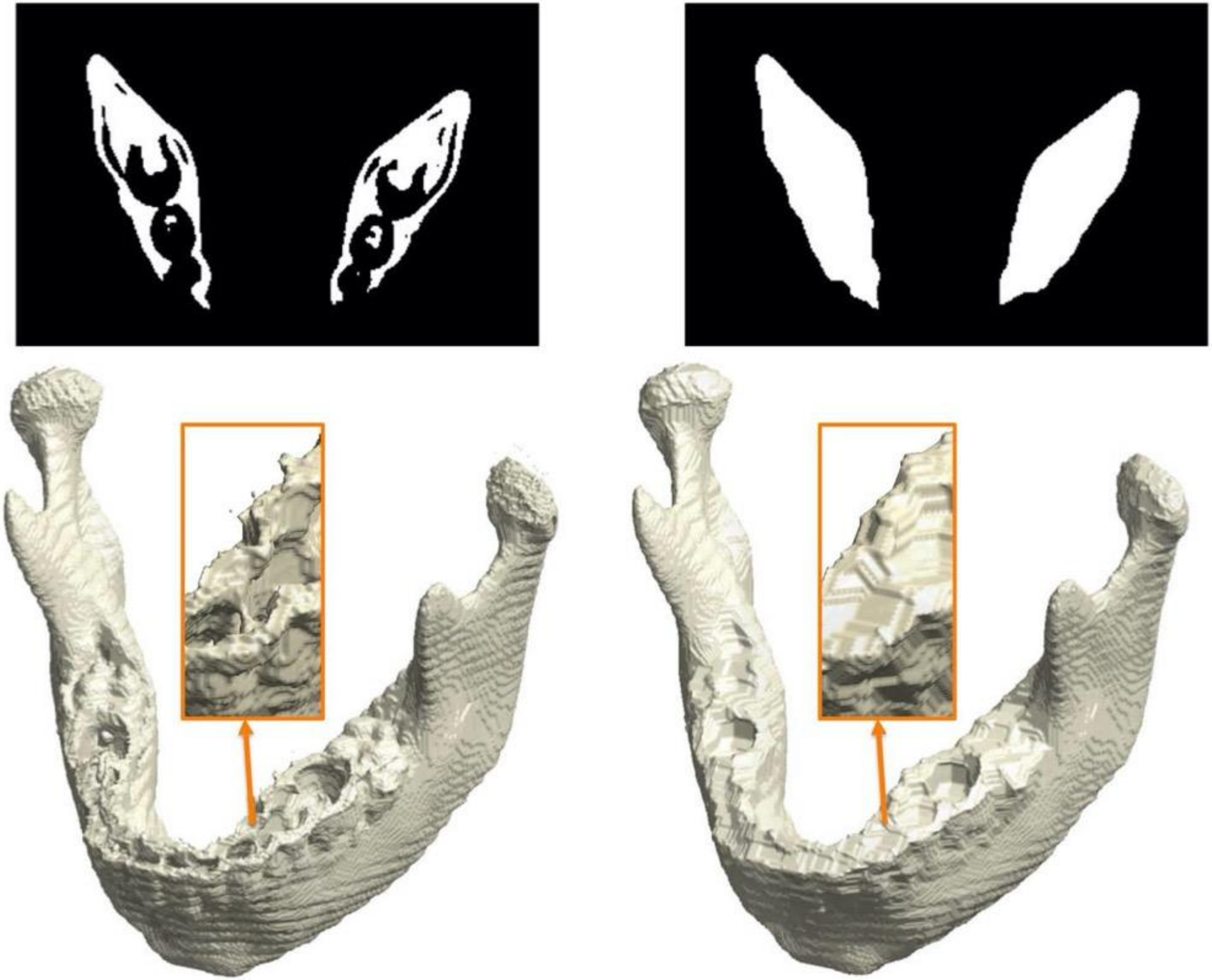
Head and neck CT image



Semiautomatic histogram thresholding in ANALYZE package

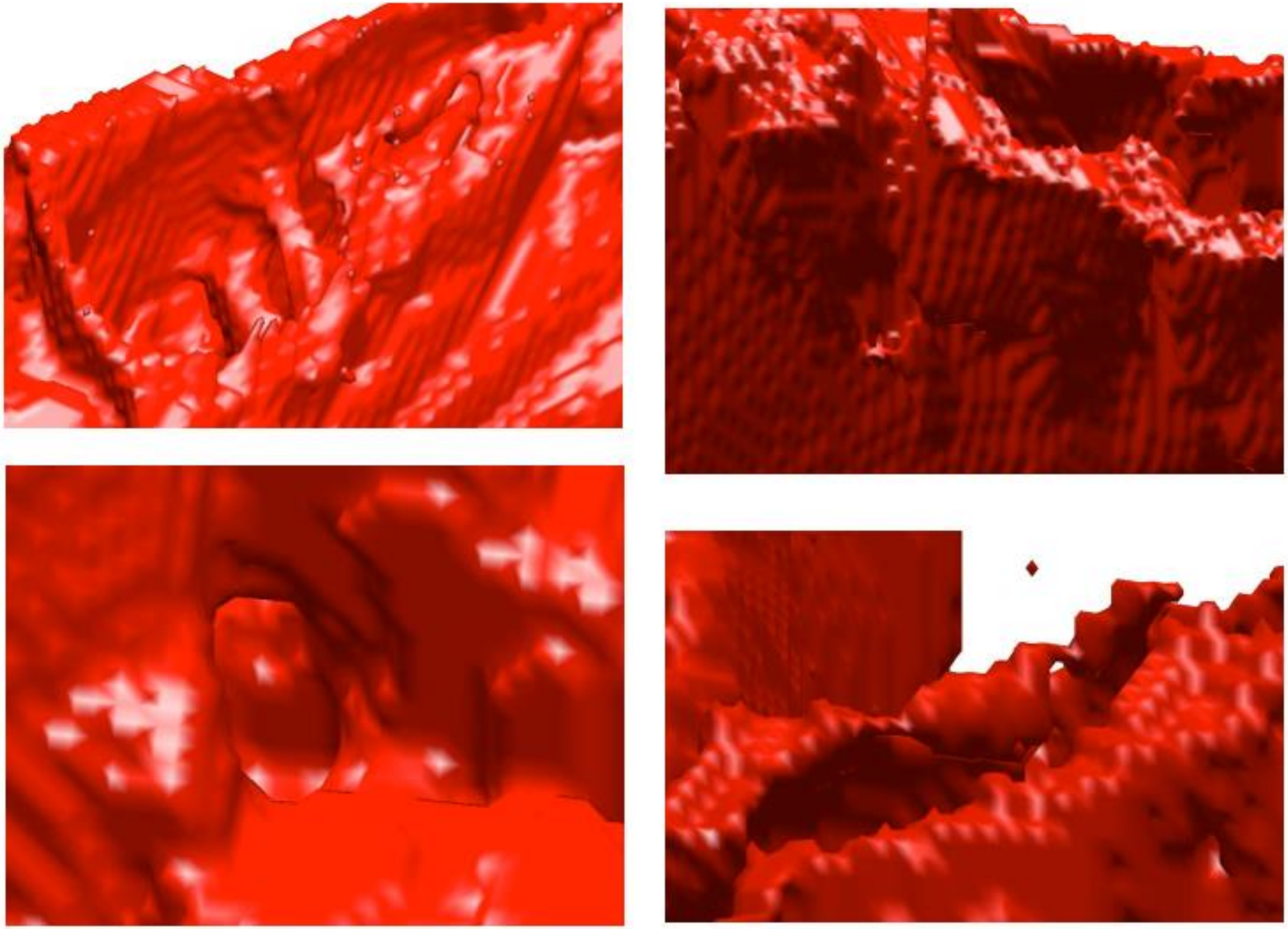
→ Automatic pipeline using ANTS-based template matching

Topology correction in segmentation

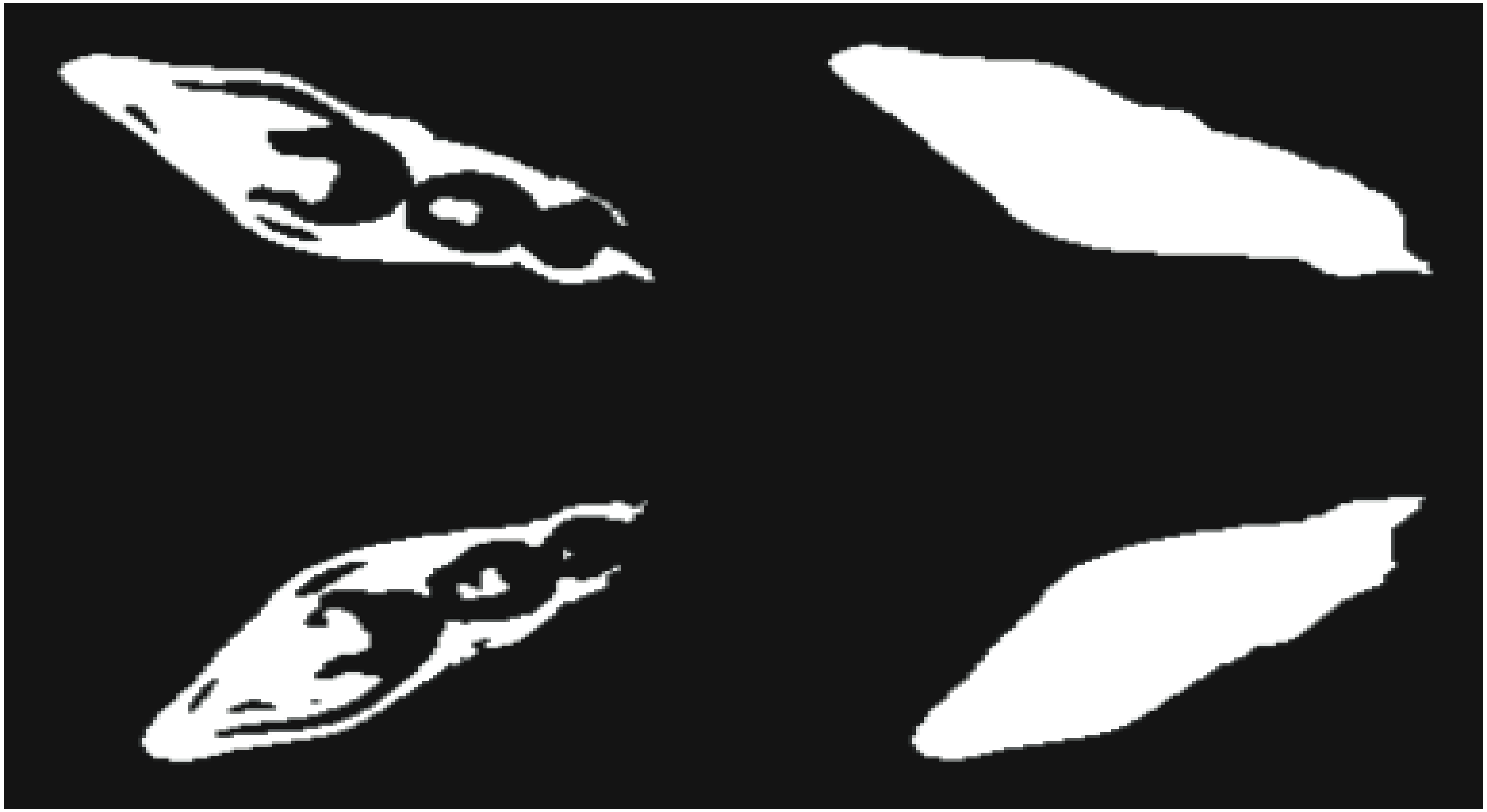


Hole & handles
corrected using
Euler characteristic

Chung et al. 2015 Medical Image Analysis. 22:63-76

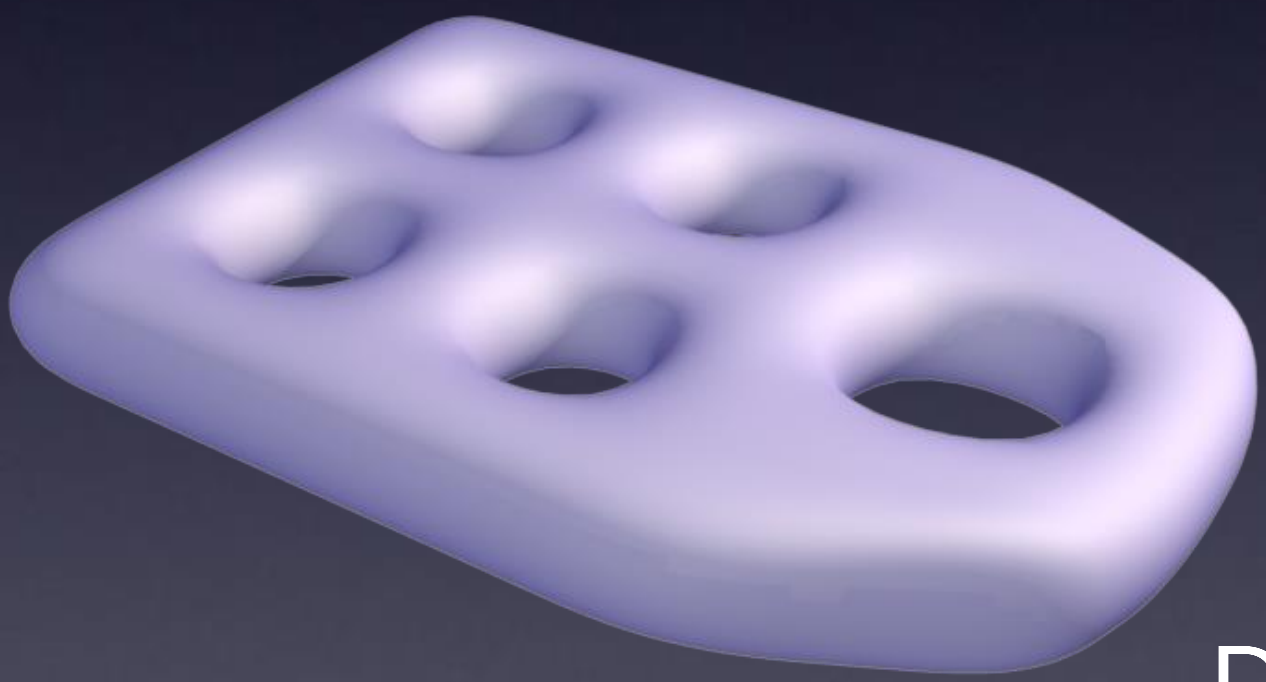


Volume rendering of a mandible from CT
Holes and handles in binary volume



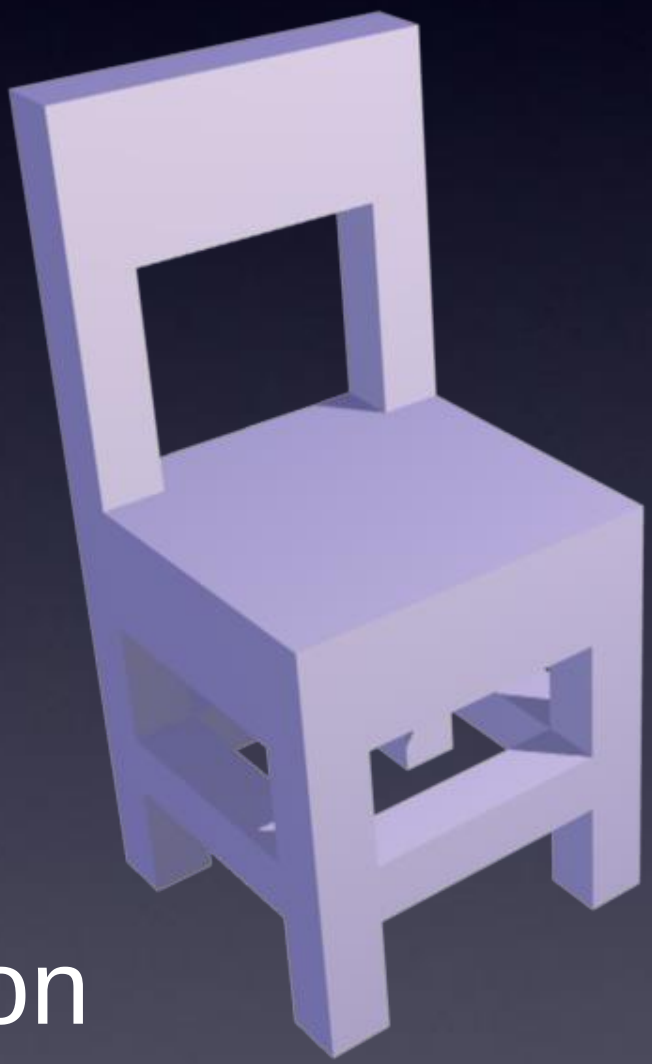
By checking the Euler characteristic of the binary volume of a mandible, holes in the binary volume can be detected. This process is necessary to make the mandible binary volume to be topologically equivalent to a solid sphere.

Genus = the number of holes in a surface or an object.



5 holes

Deformation invariant



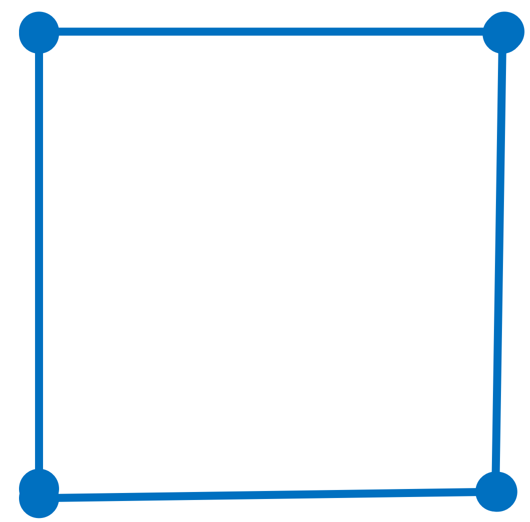
Matt Black

Euler characteristic with convex polyhedrons

Polyhedron is a solid object with polygonal faces, edges and nodes.

Euler characteristic: EC = N – E + F - … .

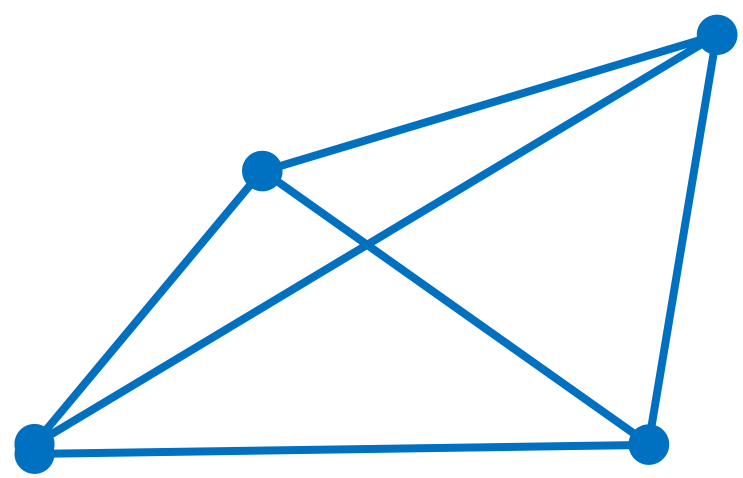
EC = N – E + F
= 4 – 4 + 1
= 1



If there is no face

EC = N – E
= 4 – 4
= 0

EC = N – E + F -V
= 4 – 6 + 4 - 1
= 1



If there is no volume

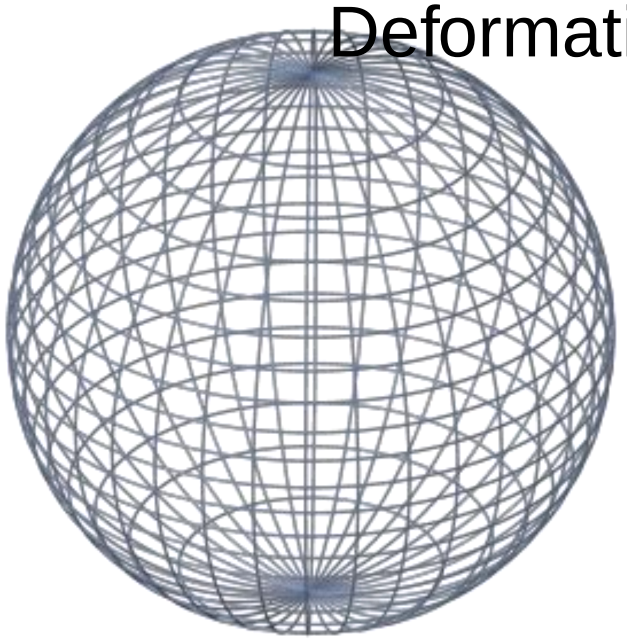
EC = N – E + F
= 4 – 6 + 4
= 2

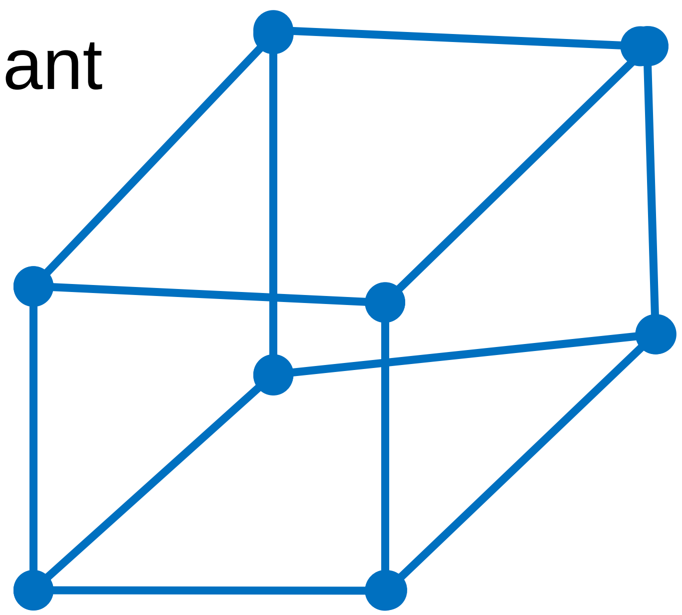
Question: Check if we have EC = #dimension - #hole

EC is an approximate measure of data dimension!

Computing Euler characteristic of 3D objects

Topologically
equivalent
Deformation-invariant



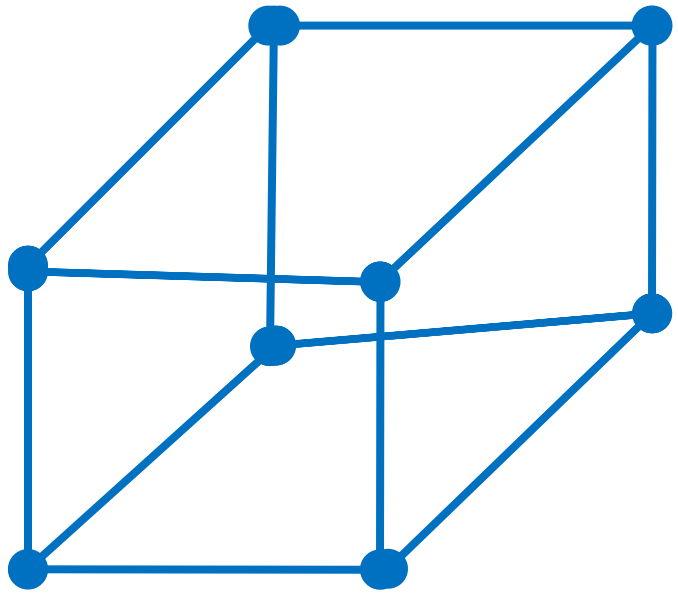


Sphere

$EC = N - E + F$

$= 8 - 12 + 6$

$= 2$



Solid ball

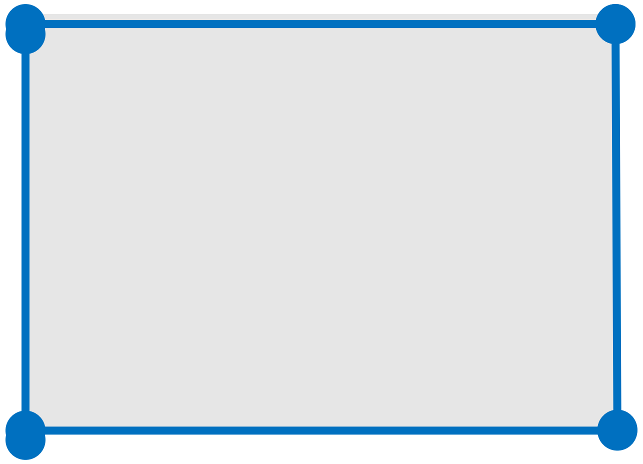
$EC = N - E + F - V$

$= 8 - 12 + 6 - 1$

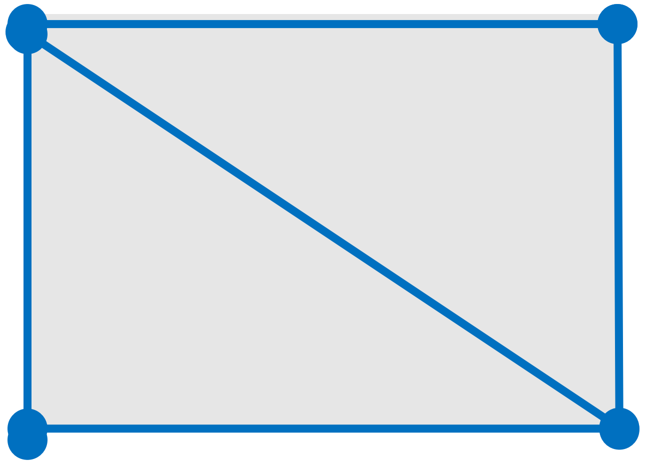
$= 1$

Computing Euler characteristic by parts

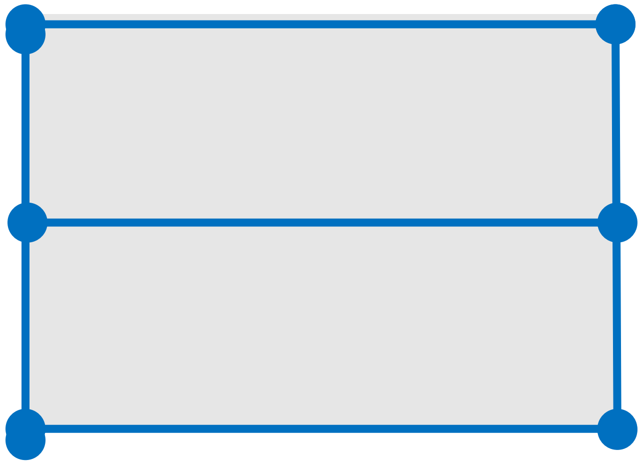
Cover an object with polyhedrons



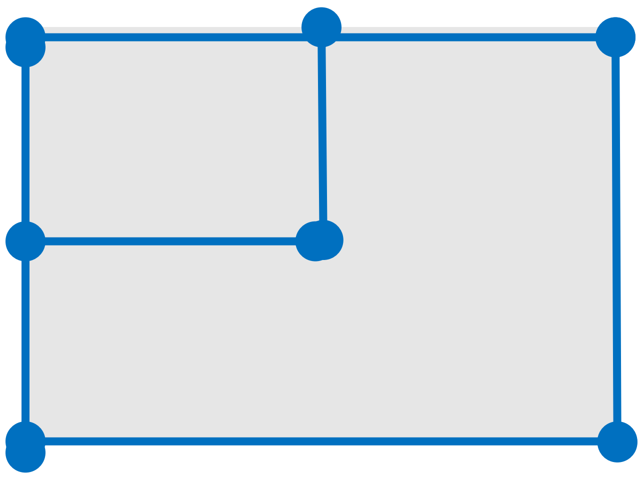
EC = N – E + F
=4 – 4 + 1
= 1



EC = N – E + F
=4 – 5 + 2
= 1



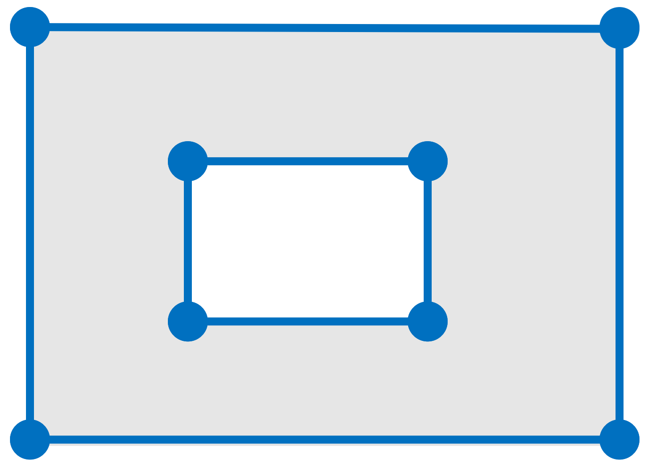
EC = N – E + F
=6 – 7 + 2
= 1



EC = N – E + F
=7 – 8 + 2
= 1

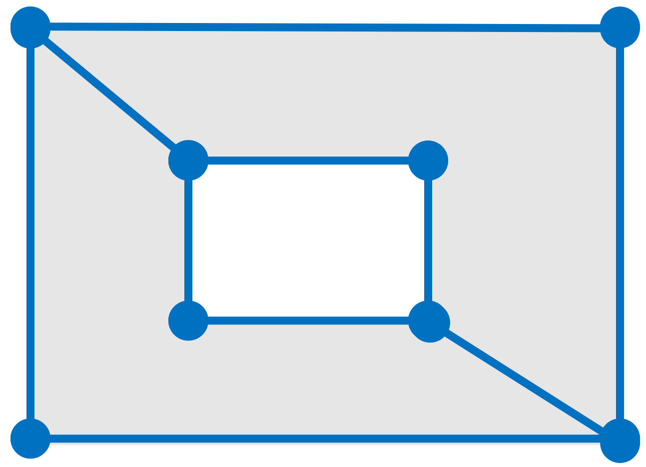
Computing Euler characteristic by parts

Cover an object
with polyhedrons

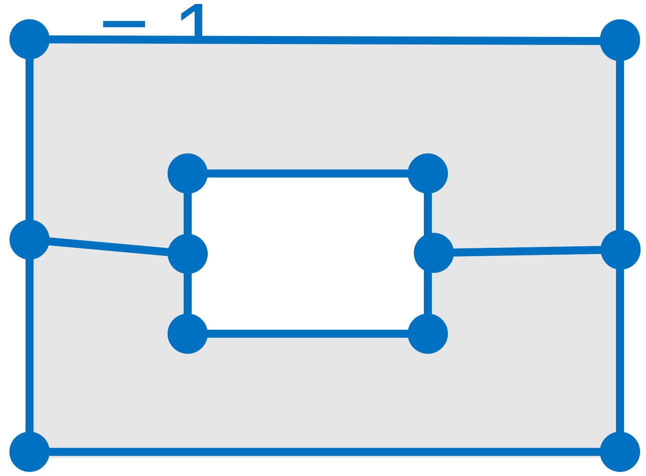


Incorrect
computation

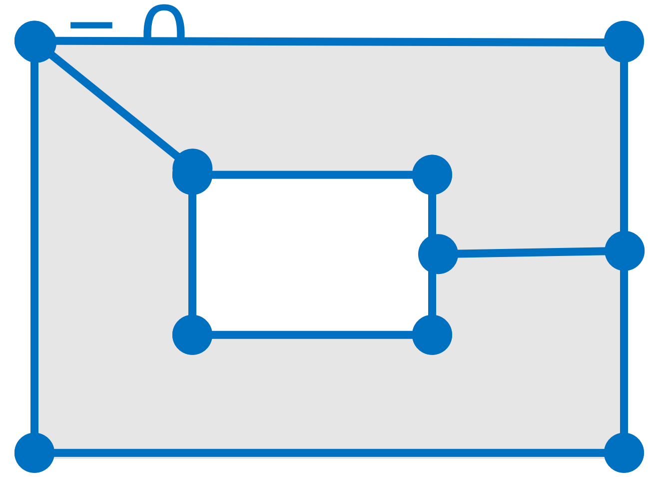
EC = N – E + F
=8 – 8 + 1
= 1



EC = N – E + F
= 8 – (8+2) + (1+1)
= 0

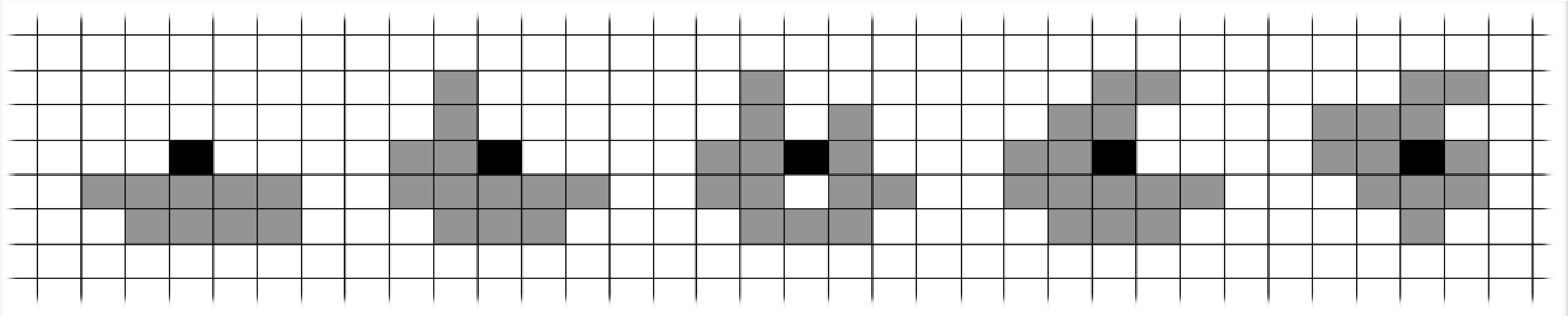


EC = N – E + F
=(8+4) – (8+6) + (1+1)



EC = N – E + F
=(8+2)–(8+4) +(1+1)
= 0

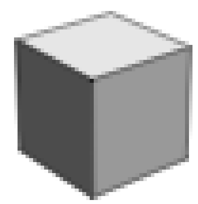
Iterative computation (online) of Euler characteristi



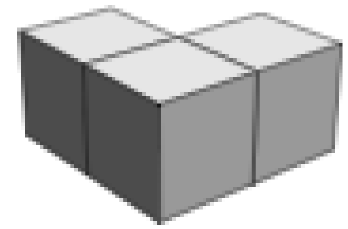
# neighbors	1	2	2	3	4
# Vertex	2	1	0	0	0
# Edge	3	2	2	1	0
# Face	1	1	1	1	1
EC change	0	0	-1	0	1

Question: Given binary image, write an iterative algorithm

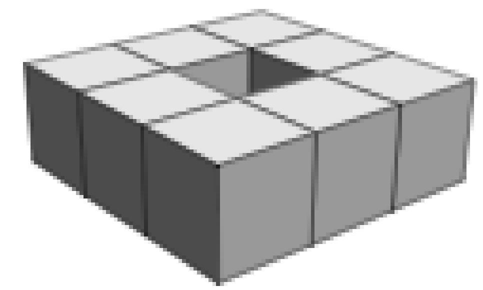
Computing Euler characteristic in 3D image



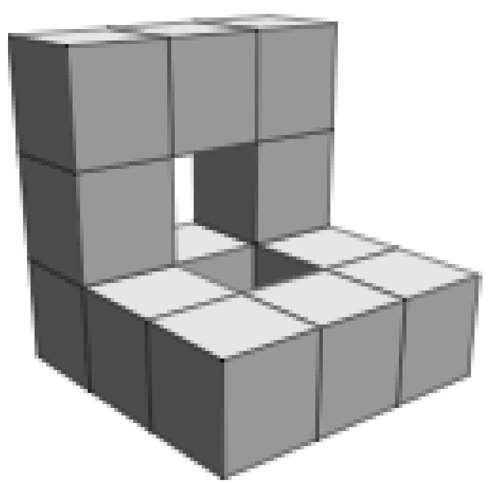
(a) 8-12+6-1=1



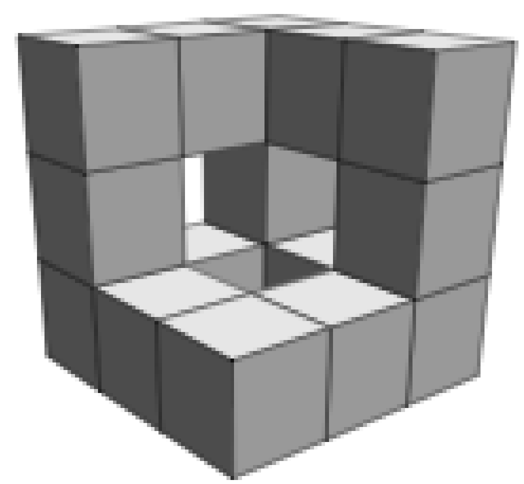
(b) 16-28+16-3=1



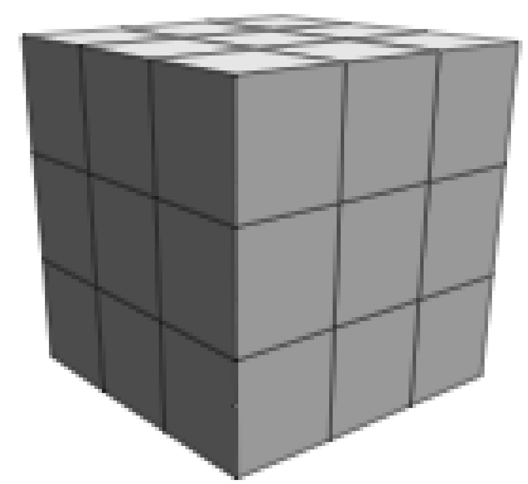
(c) 32-64+40-8=0



(d) 48-100+64-13=-1



(e) 56-120+78-16=-2

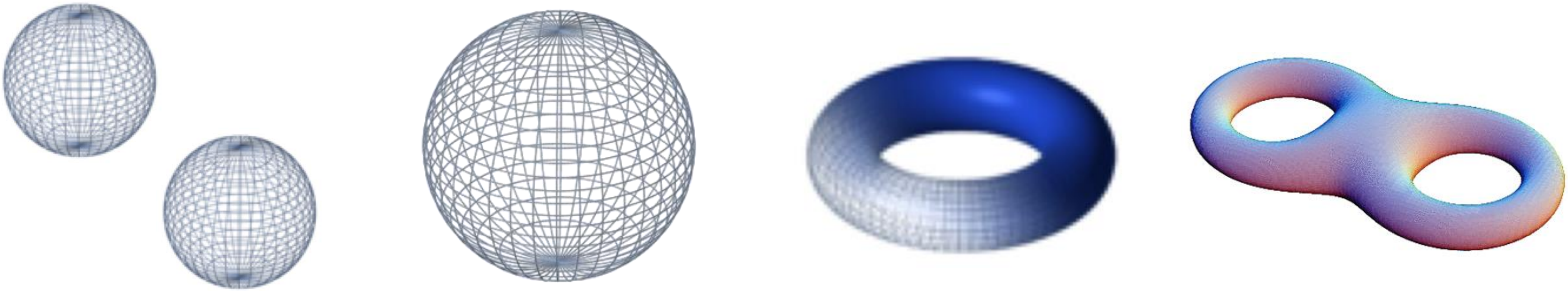


(f) 64-144+108-26=2

Partition search region into voxels.

EC = # vertices - # edges + # faces - # volume

Euler characteristic: most widely used topological invariant



4 2 0 -2

For an object with n-handles, EC = $2-2n$

Question: prove the statement

Expected Euler characteristic/Betti numbers

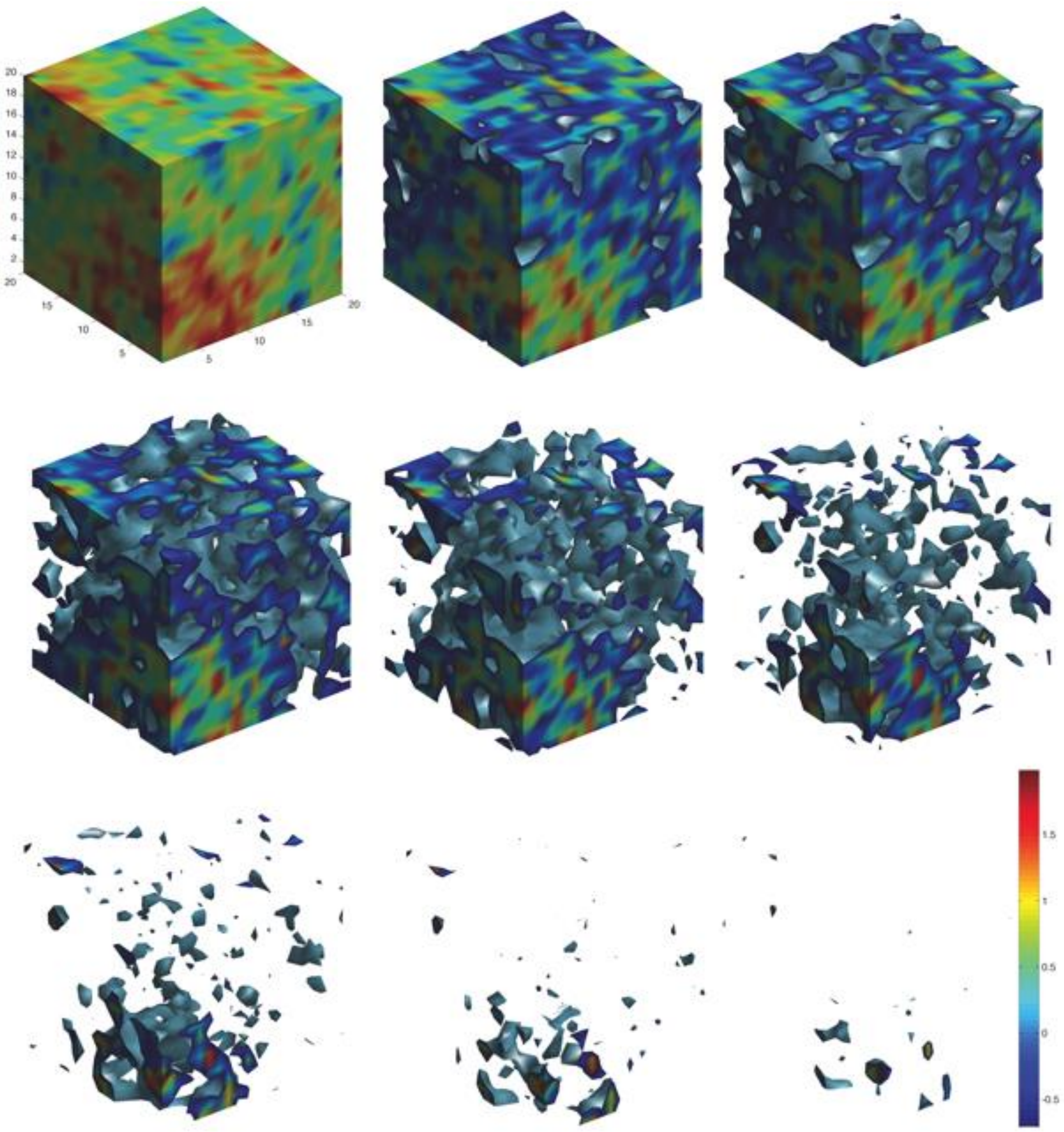
Random field, stochastic process

$$P\Big(\sup_{x\in\mathbb{M}} T(x) > h\Big)$$

$$A_h = \{x \in \mathbb{M} : T(x) > h\}$$

$$P\Big(\sup_{x\in\mathbb{M}} T(x) > h\Big) = \mathbb{E}\chi(A_h)$$

$$\chi(A_h) = \sum_j (-1)^j \beta_j(A_h)$$



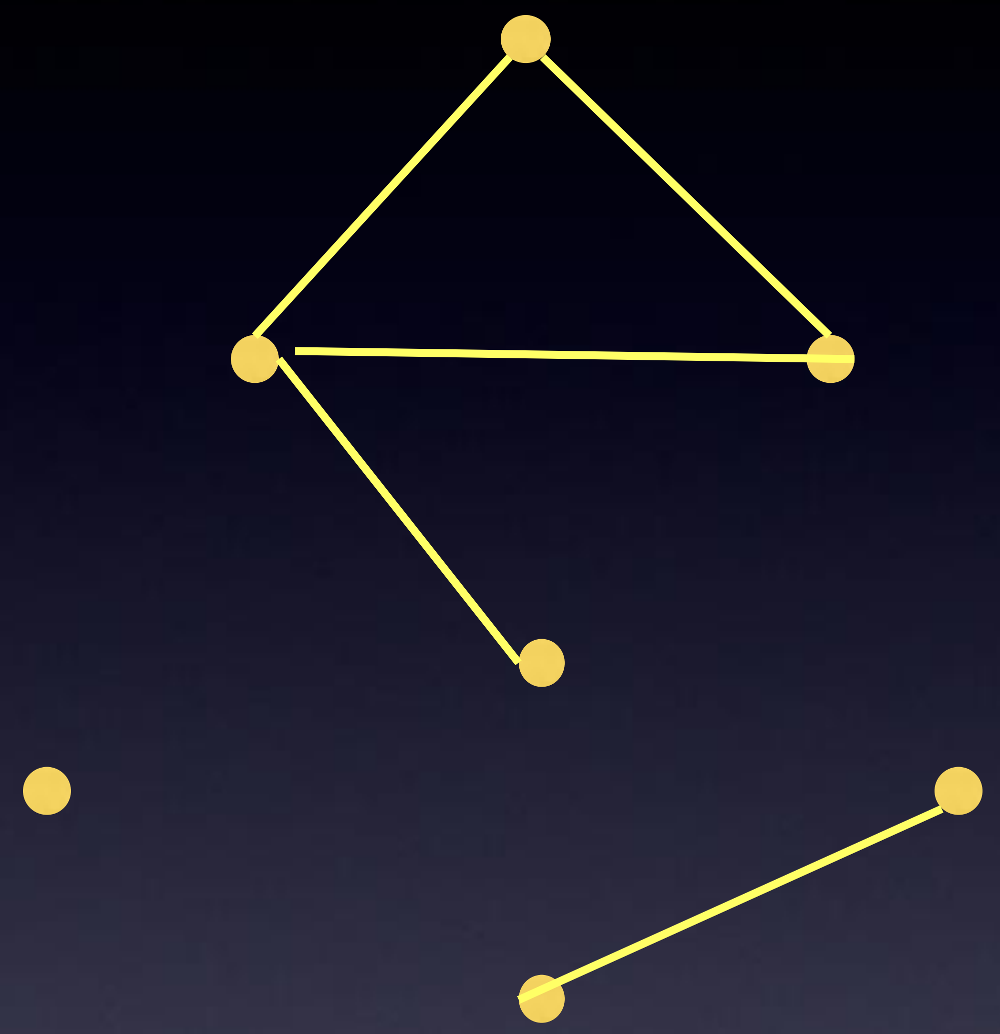
Milor 1963 Morse theory

Adler, 1994 The geometry of random fields

Worsley et al., 1996 Human Brain Mapping

Betti numbers β_i

of i-dimensional holes/loops

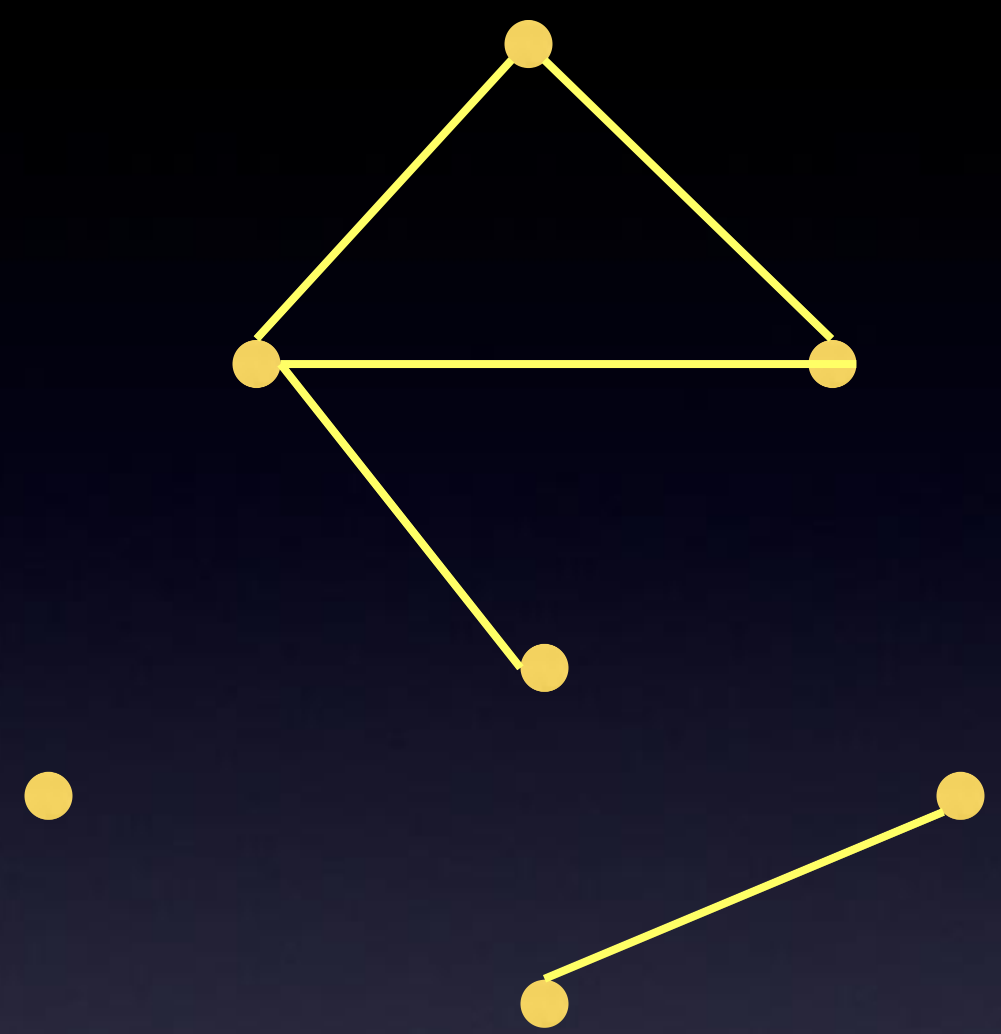


β_0 = # of connected components = 3

β_1 = # of cycles = 1

Euler characteristic: $\chi = 3 - 1 = 2$

numbers β_i # of i-dimensional holes/loops



β_0= # of connected components = 3

β_1= # of 1D holes = 1

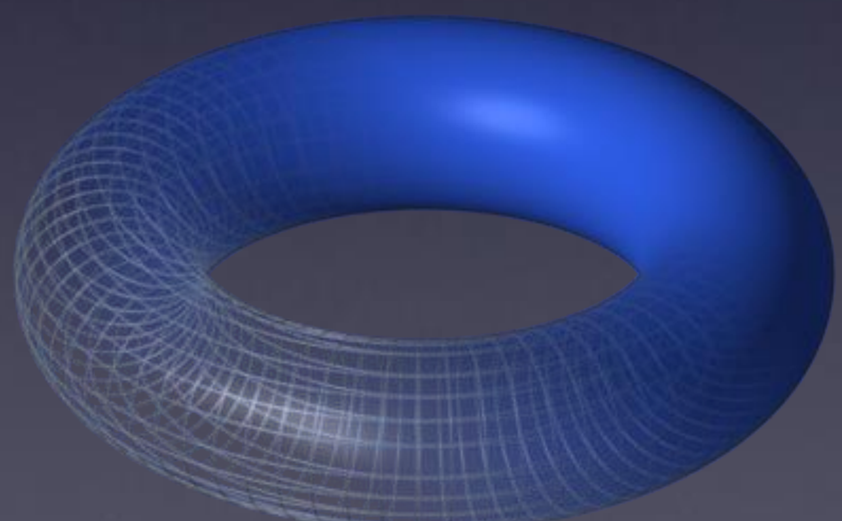
β_2= # of 2D cavities =0

Betti-number representation:

(3,1,0,0,...)

Euler characteristic:

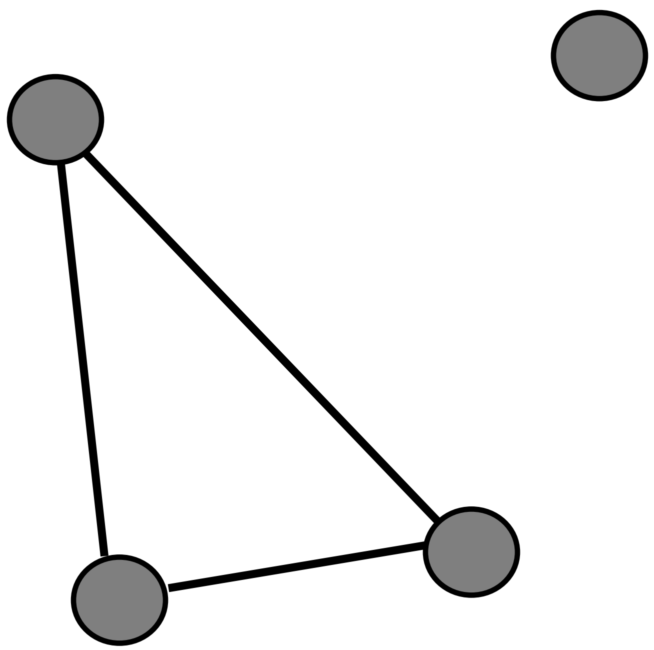
$$\chi = \beta_0 - \beta_1 = 2$$

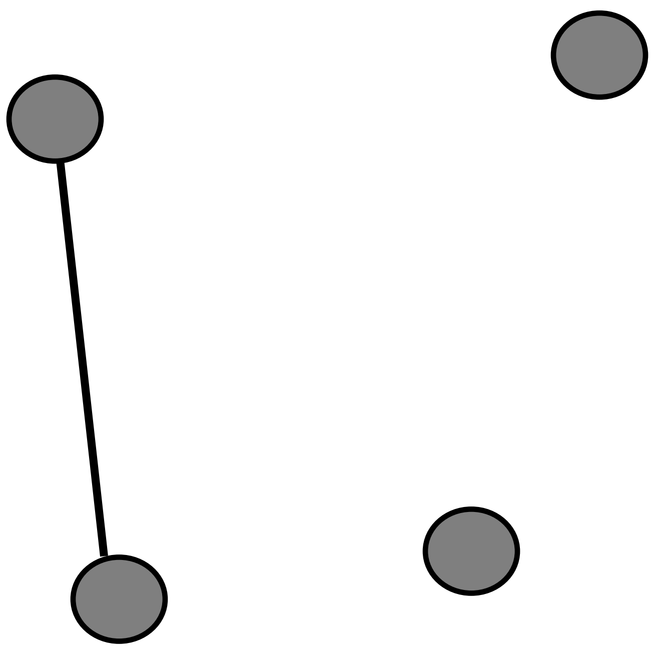


$$\beta_0 = 1, \beta_1 = 2, \beta_2 = 1$$

(1,2,1,0,0,...)

Betti numbers in graphs and networks





$\beta_0 = 2$

$\beta_1 = 1$

$$\beta_0 - \beta_1 = 1$$

$p = 4$

$q = 3$

$$p - q = 1$$

$\beta_0 = 3$

$\beta_1 = 0$

$$\beta_0 - \beta_1 = 3$$

$p = 4$

$q = 1$

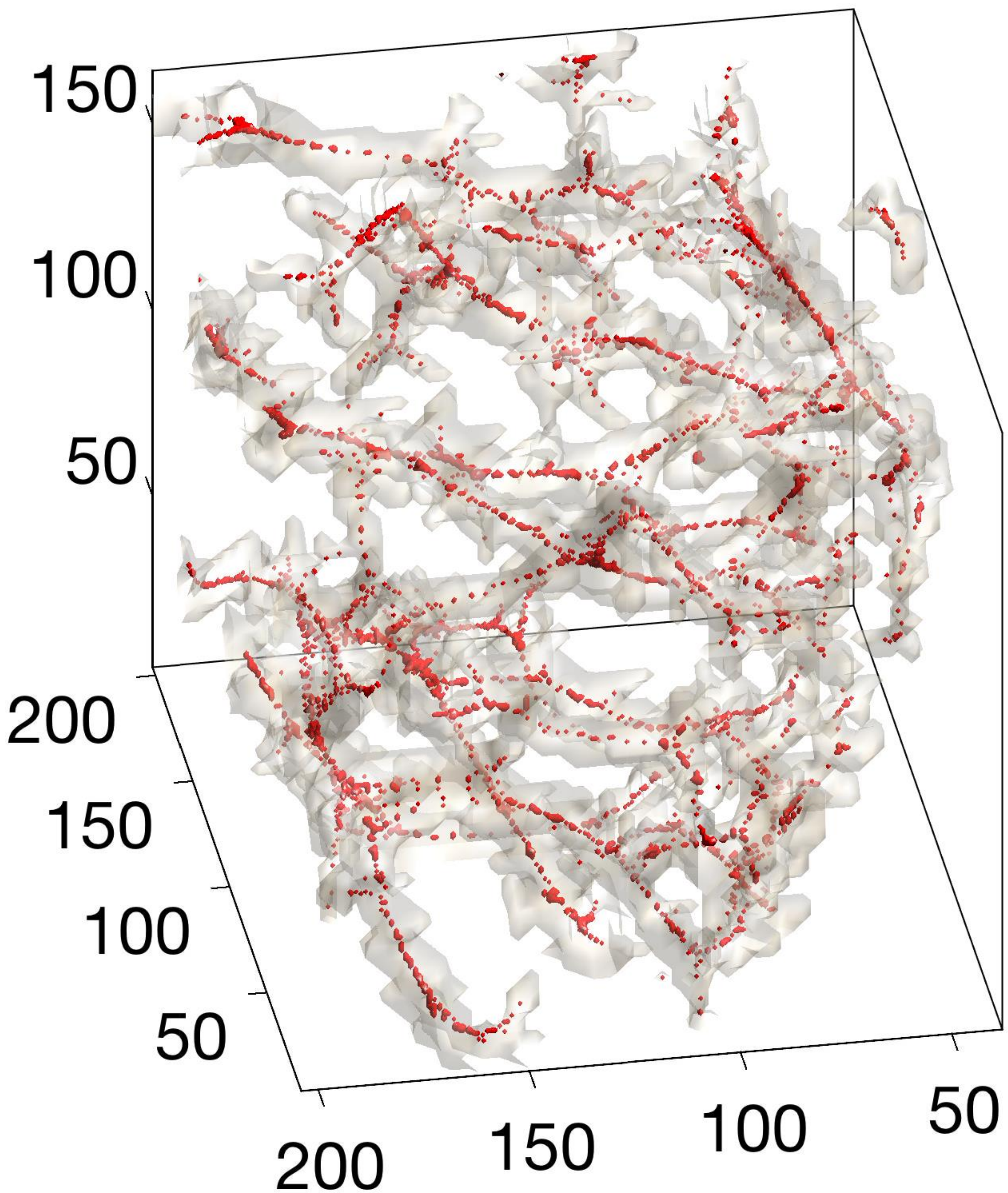
$$p - q = 3$$

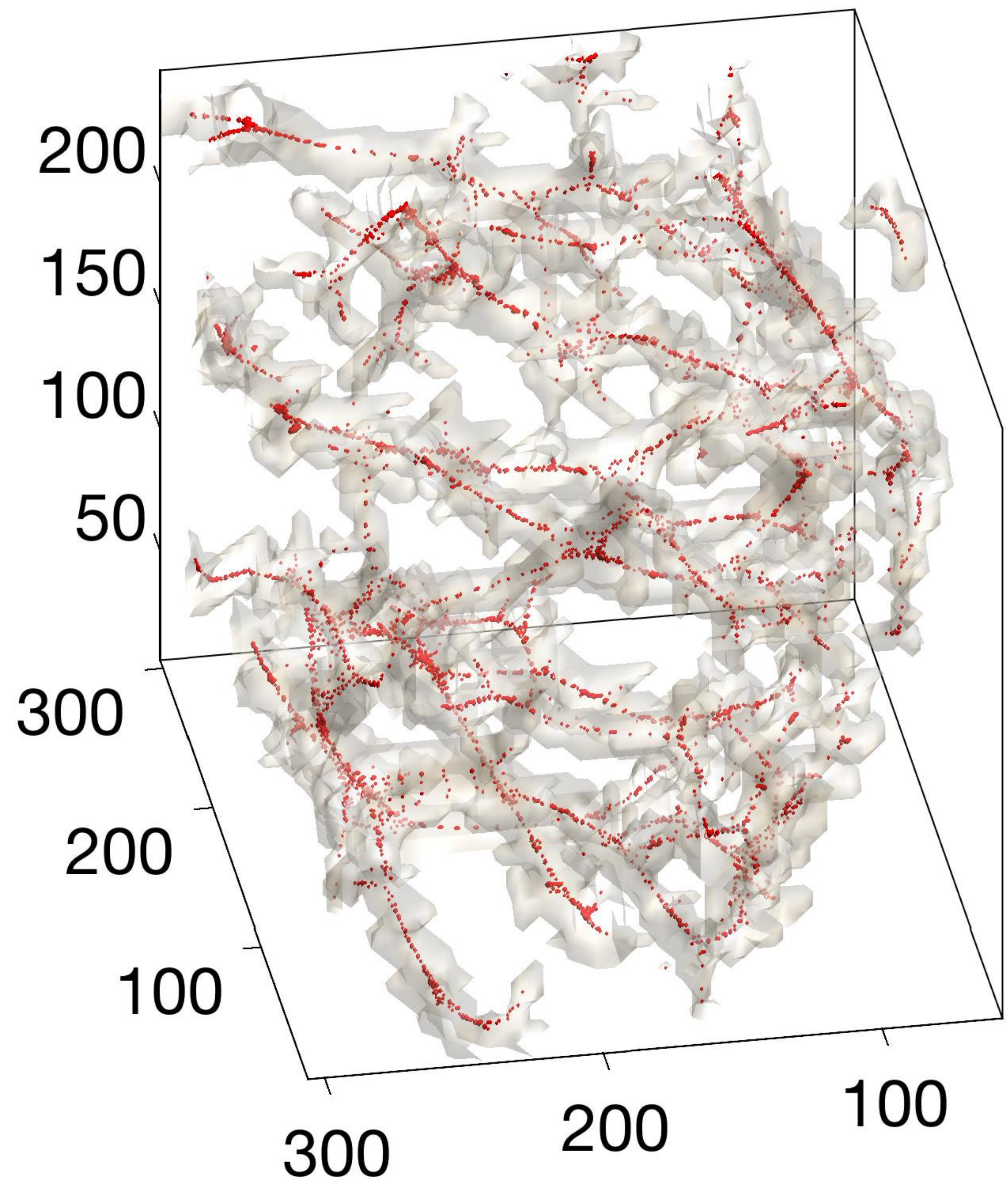
How to compute the number of cycles in big network data?

How many cycles in the network?

Tree data

Lung blood vessel trees from CT

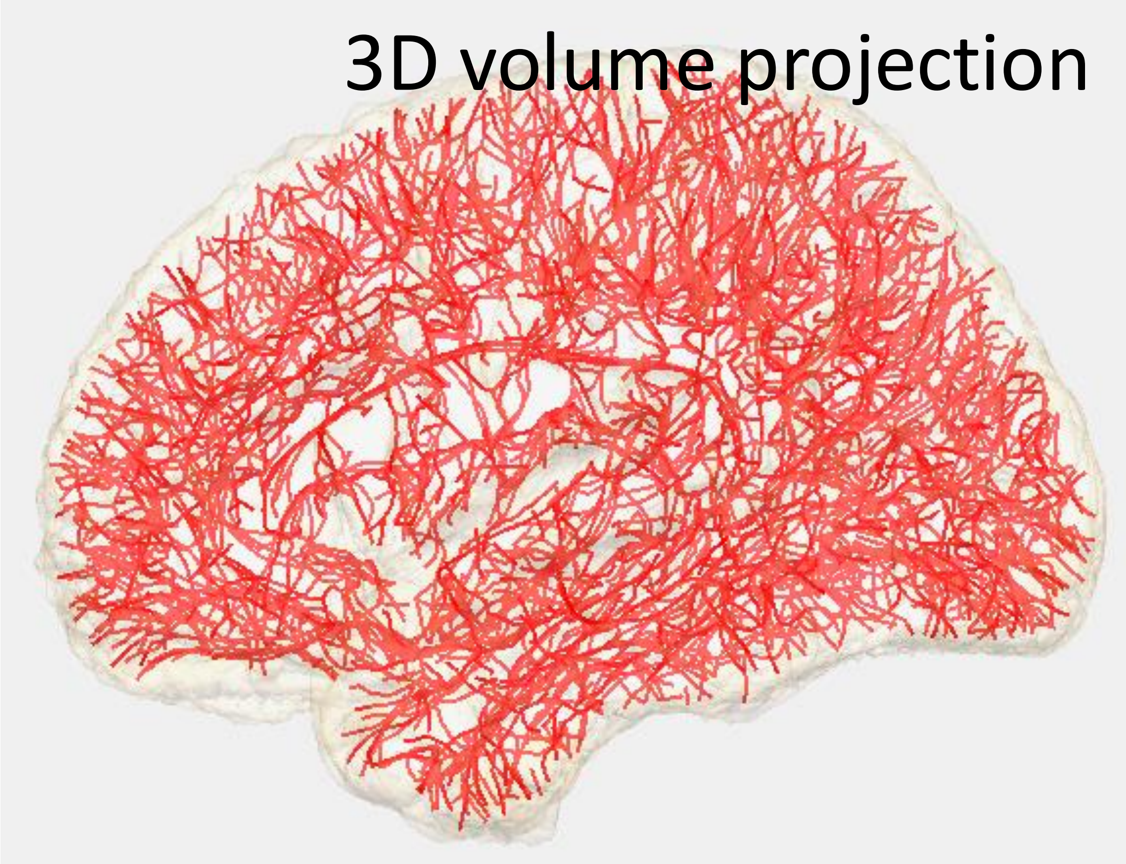




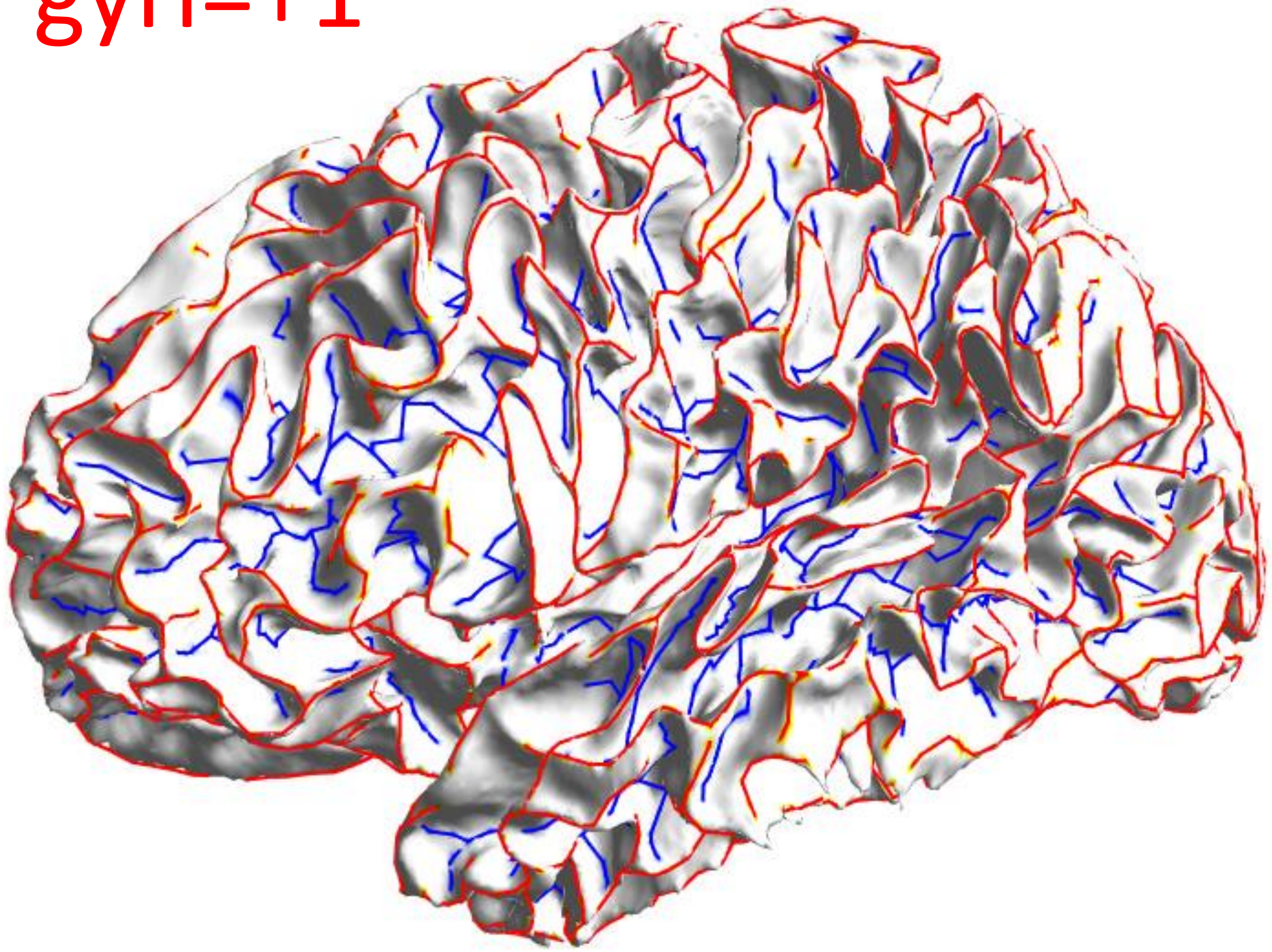
Chung et al. 2018 EMBC

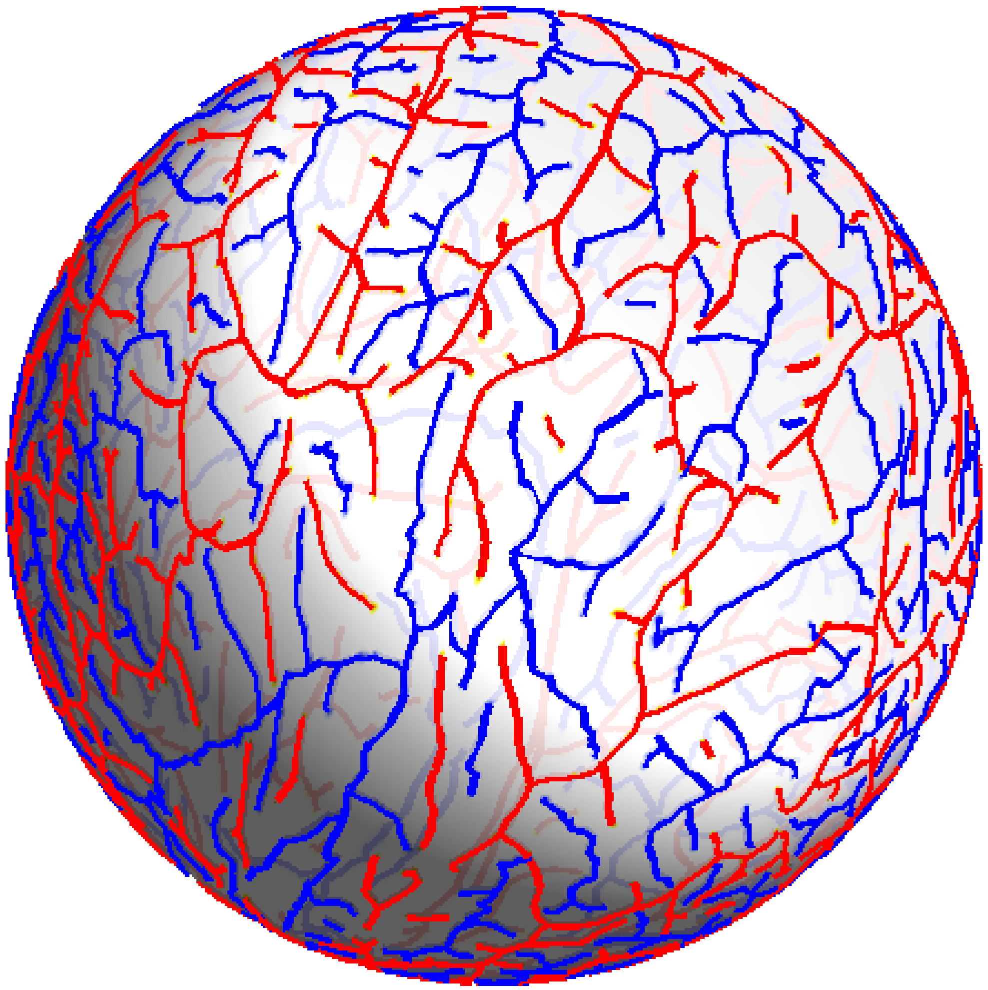
Chung et al. 2019, Mathematics of Shapes and

Sulcal and gyral trees of brain from MRI



sulci = -1
gyri=+1





White matter surface

Trees on manifold

Topology of tree

Euler characteristic

Betti numbers

Surface mesh data (2-simplices)

Marching cubes algorithm

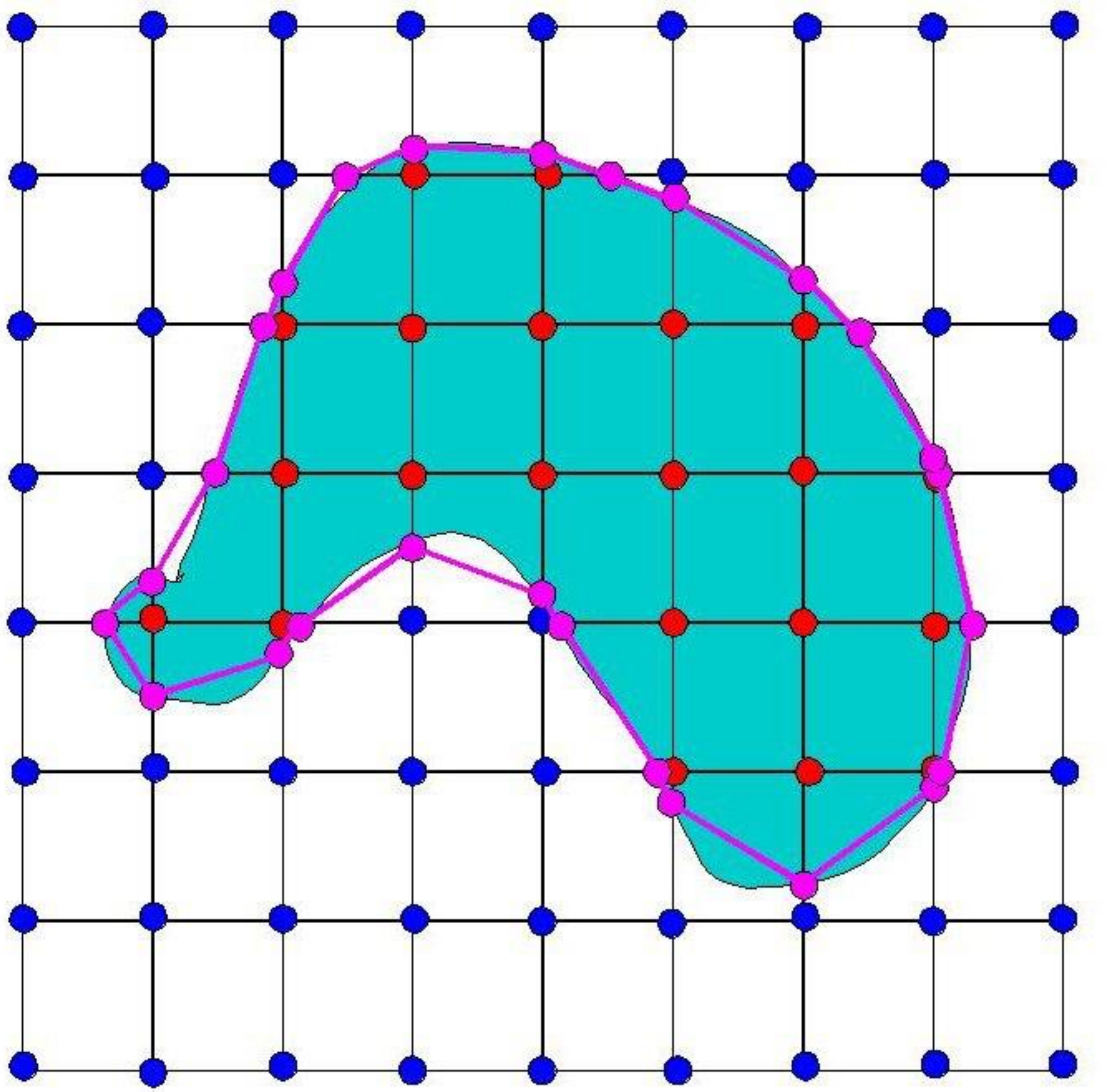
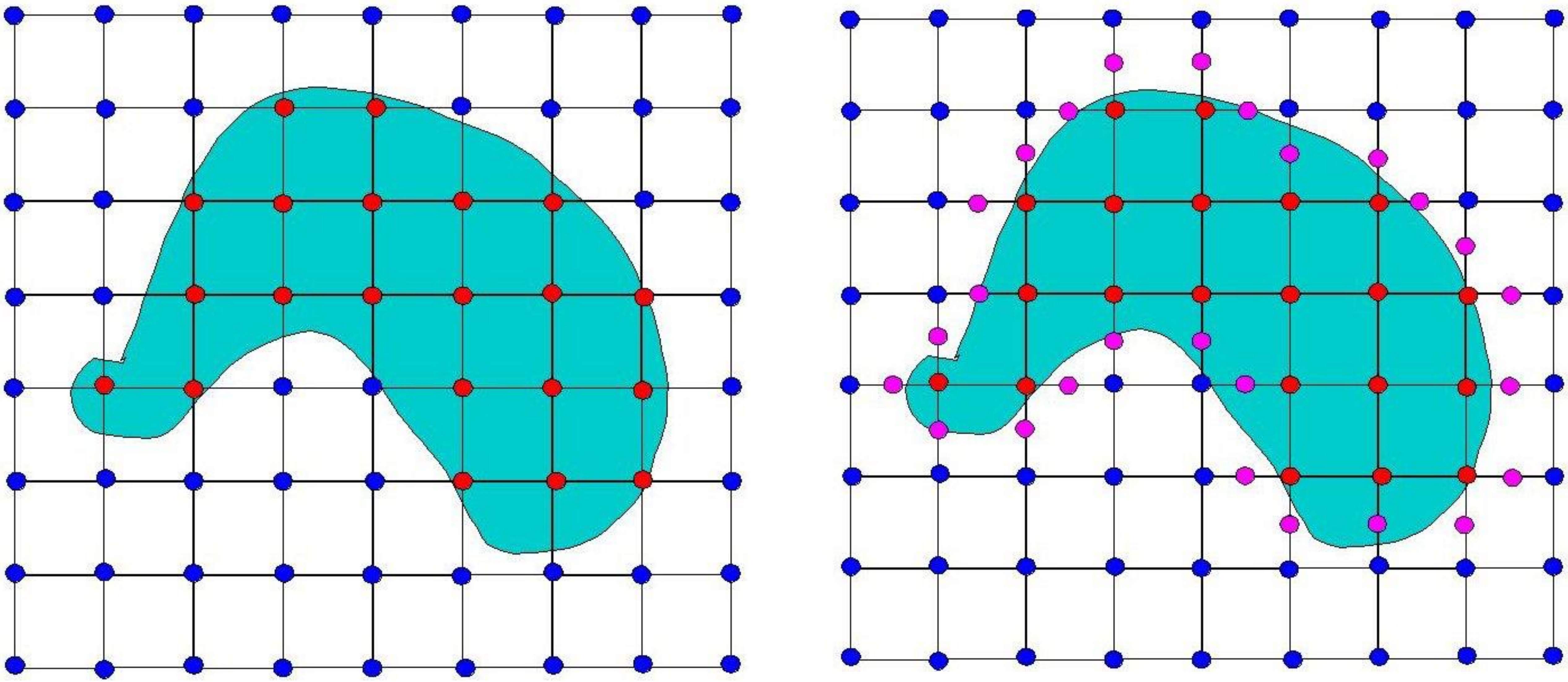


Image intensity values are classified into two parts: in & out

The marching cubes algorithm will extract the boundary of binary segmentation

Marching cubes algorithm



Depending on the amount of partial vacuuming, we connect the centers of cubes

Marching cubes algorithm in 3D

Cell consists of 8 voxel values:

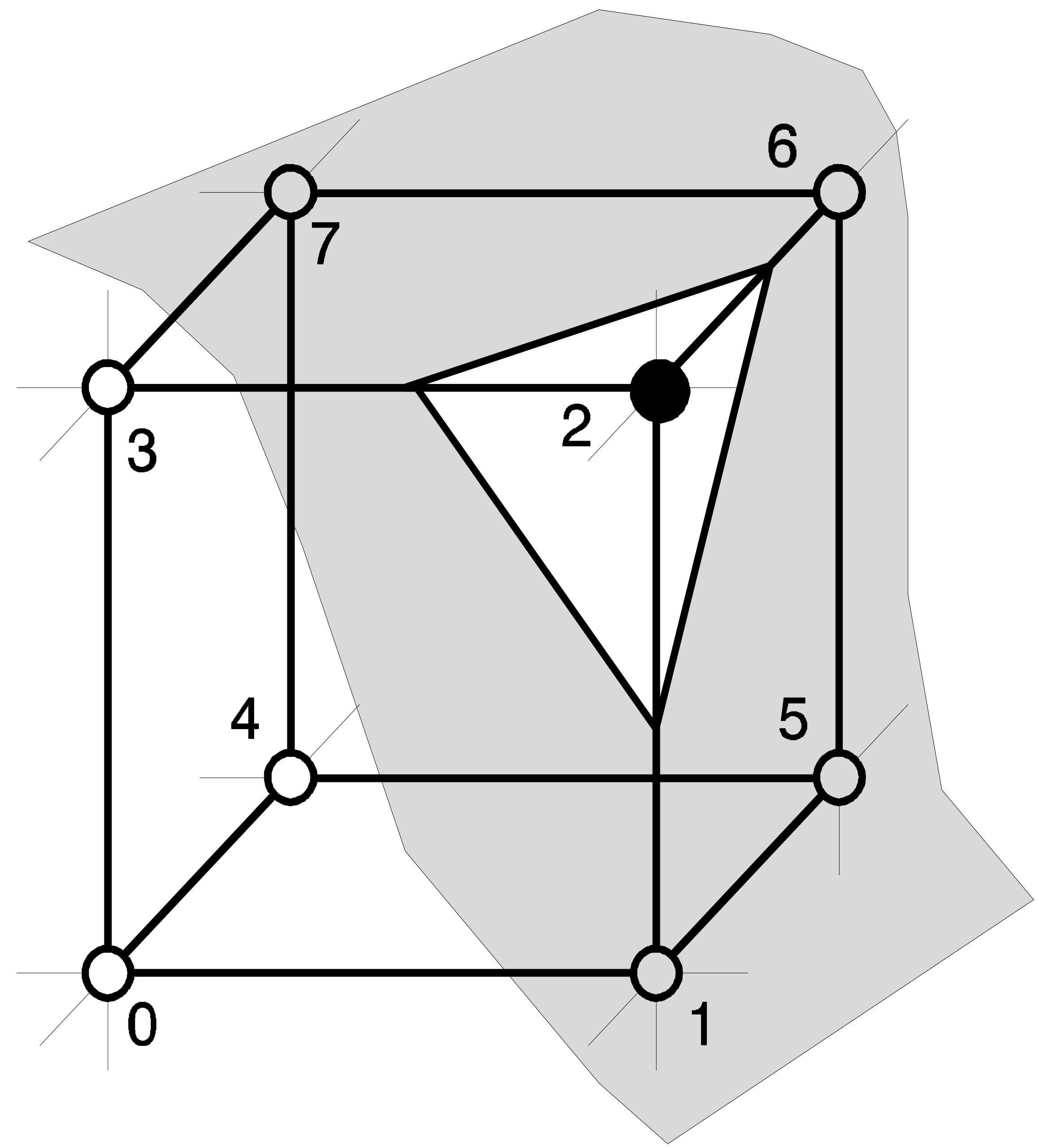
1. Classify each vertex as inside or outside

2. Build an index

3. Compute edge list from table[index]

4. Determine vertex coordinates

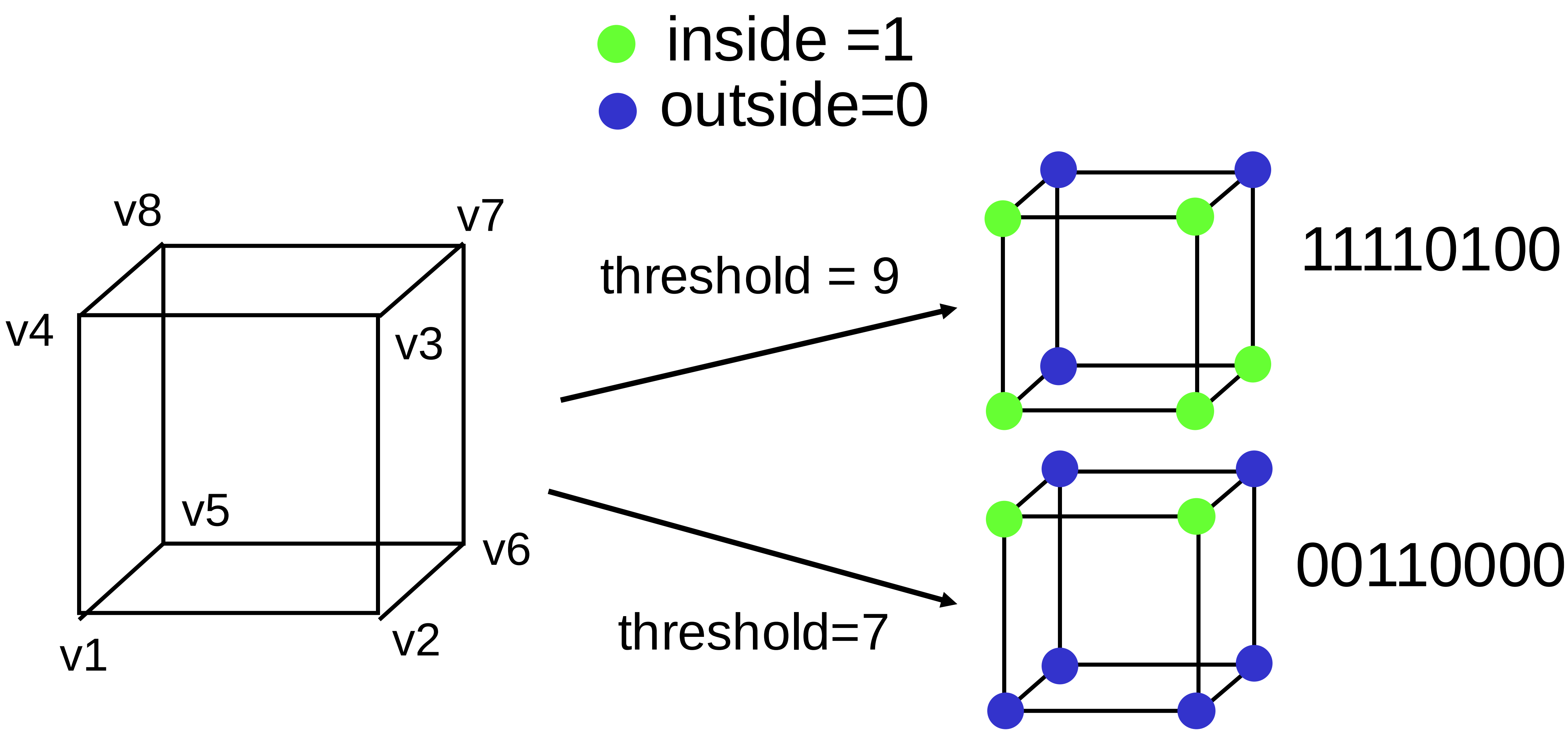
4. Go to next cell



0	0	0	0	0	1	0	0

Marching Cubes

Use the binary labeling of each voxel vertex to create an index



Index:

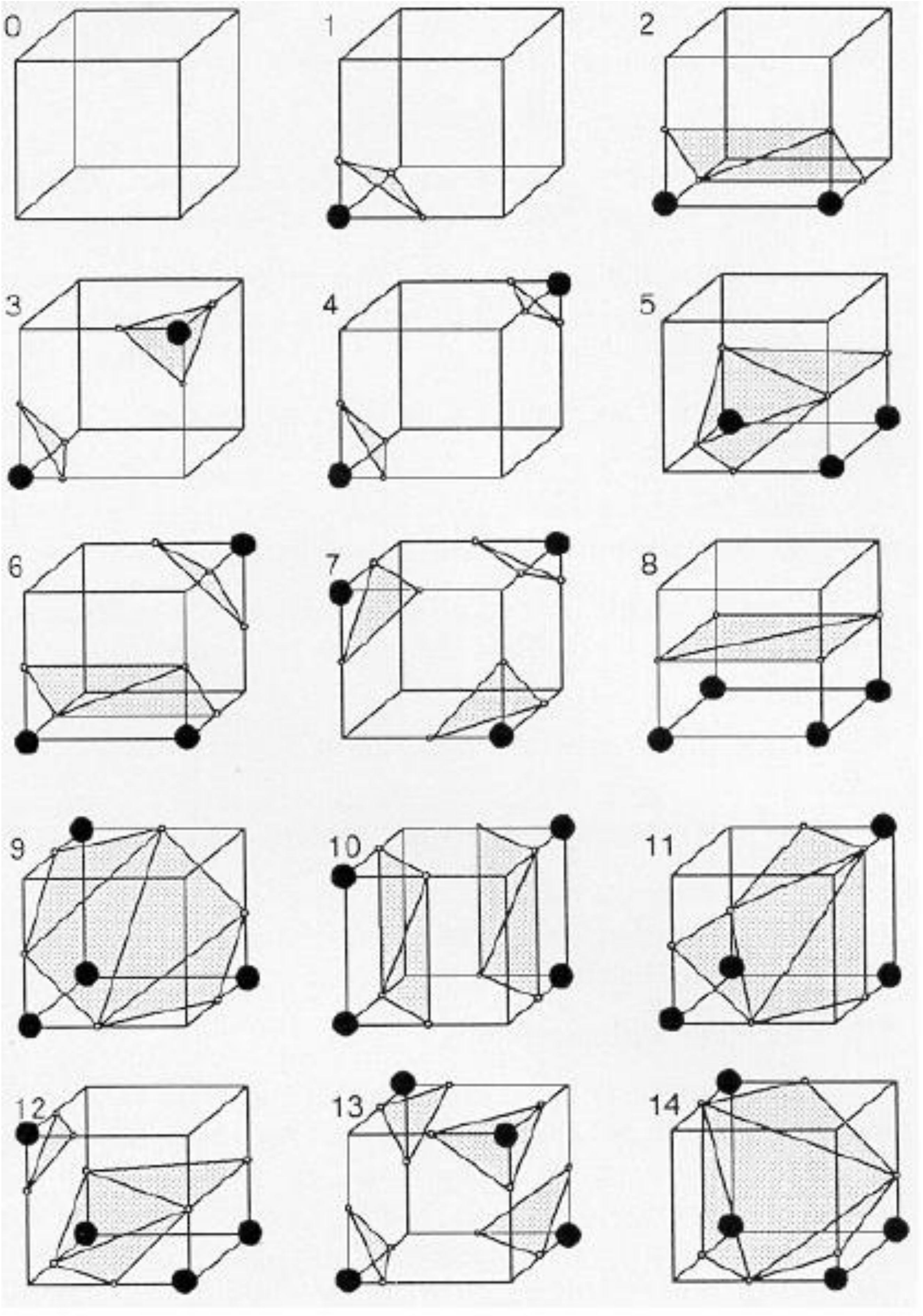
v1	v2	v3	v4	v5	v6	v7	v8

Marching Cubes

All 256 cases can be derived from 1+14=15 base cases due to symmetries

Invented by Carl Crawford (GE's Medical Systems Business Group). Expert on CT reconstruction.

Table



Marching cubes in MATLAB

```
FV =
isosurface(X,Y,Z,V,ISOVALUE)
```

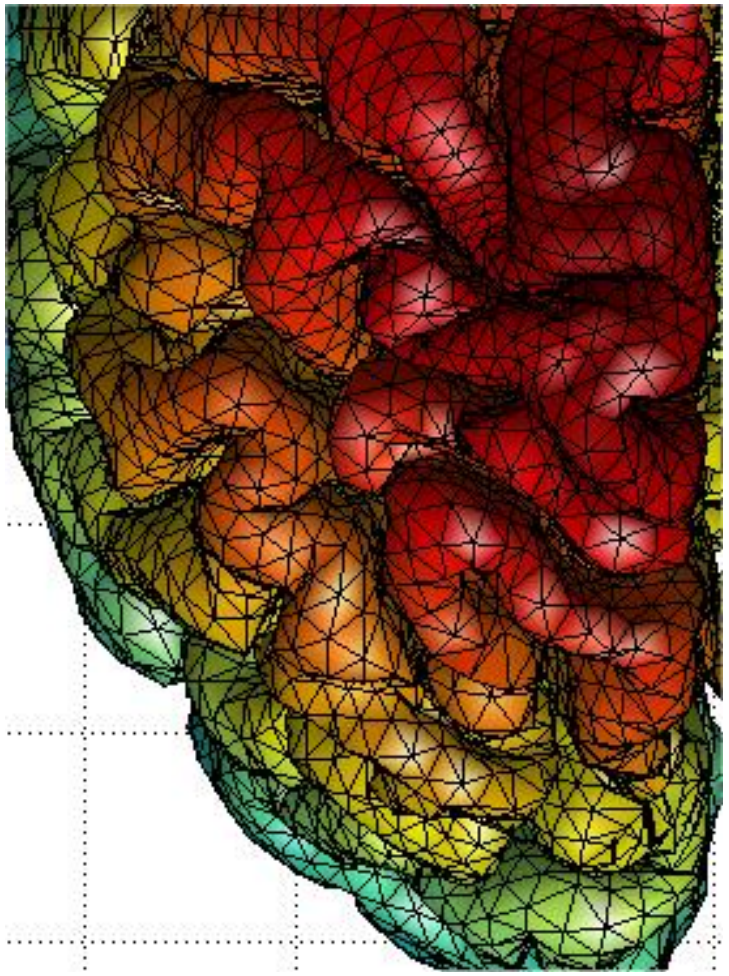
computes isosurface geometry for data V at isosurface value ISOVALUE.

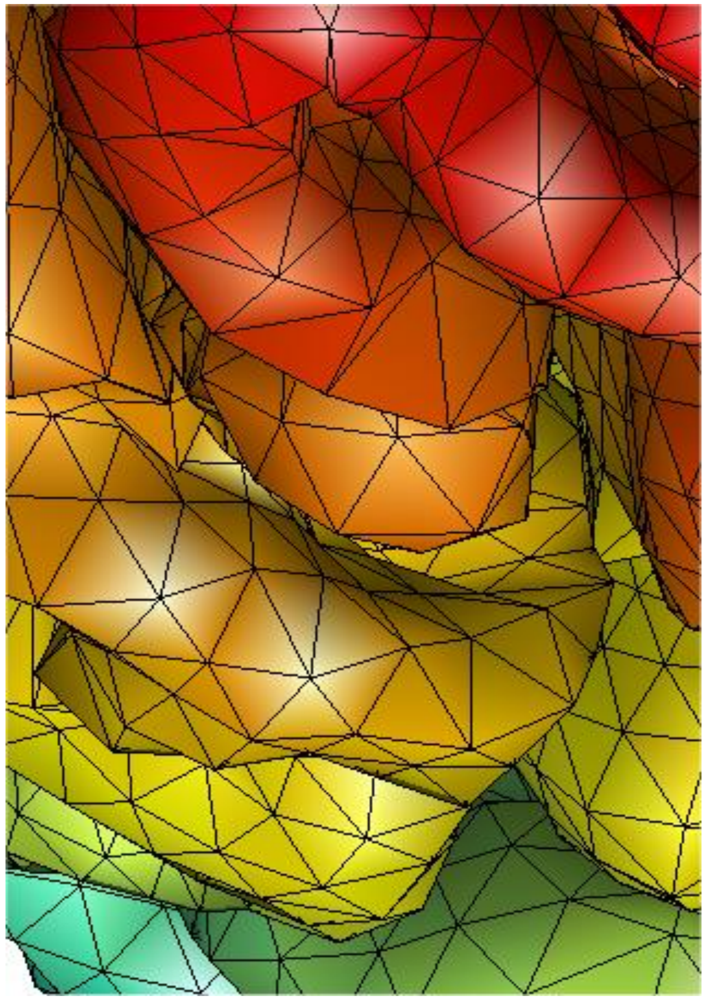
Optional: arrays (X,Y,Z) specify the coordinates at which the data V is given. The struct FV contains the faces and vertices of the isosurface and can be passed directly to the PATCH command.

It is based on simpler version of marching cubes.

Output of isosurface.m
Triangle mesh data structure

Basis of most surface rendering tools for 3D computer games: as 3D Max Studio, Maya





Data structure for triangle mesh

```
>>surf =
    vertices: [1282x3 double]
       faces: [2560x3 double]
```

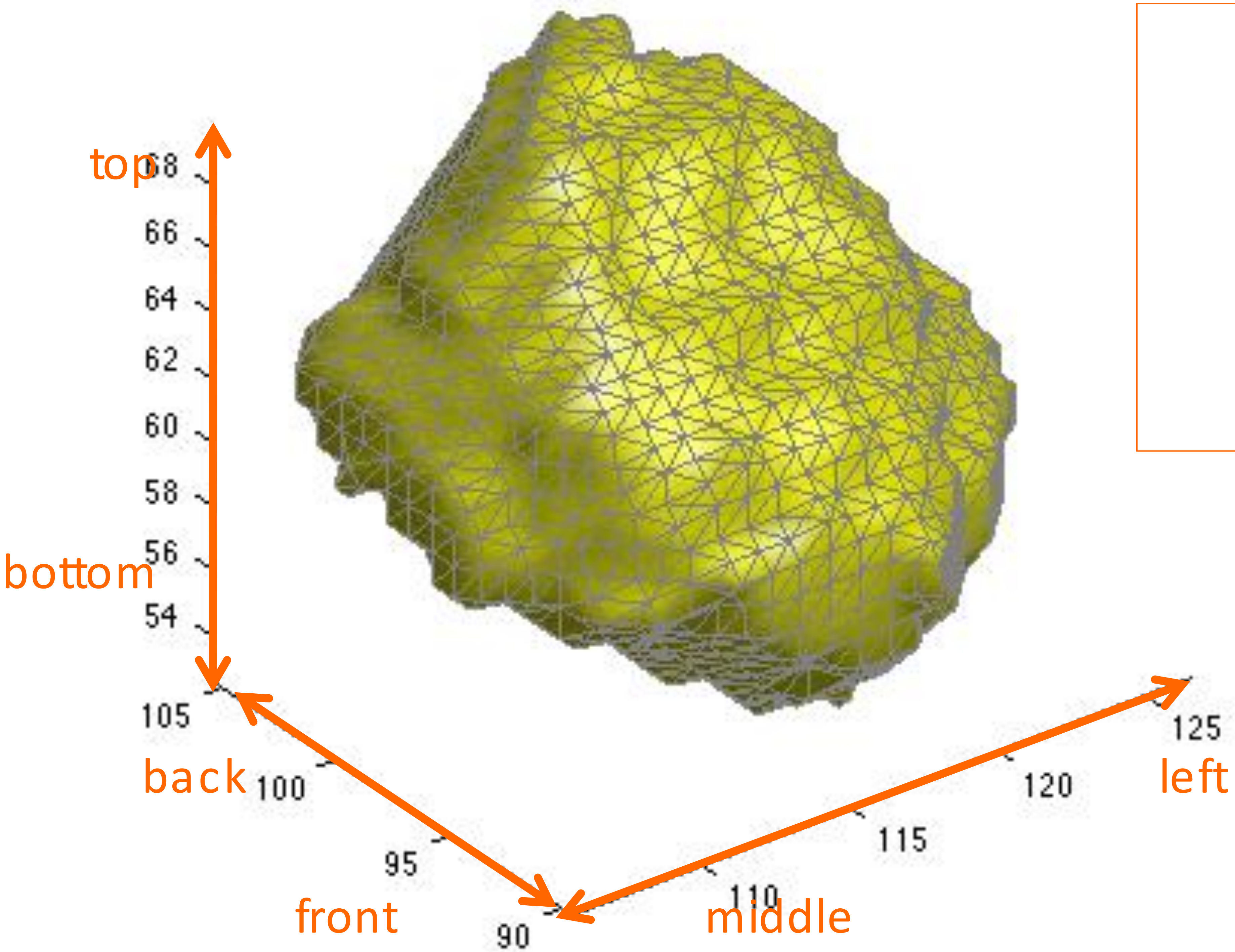
structured array

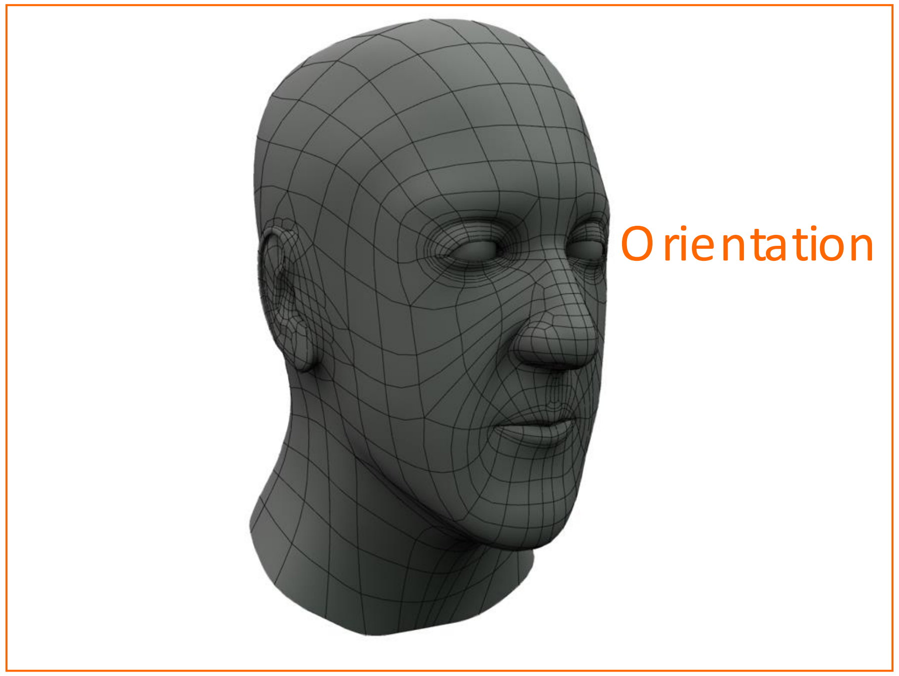
```
>>surf.faces
ans =
          1          2          3
          1          4          2
          1          3          5
         ...
```

```
>> surf.vertices
ans =
   75.0000   93.0000   51.5050
   74.5050   93.0000   52.0000
   75.0000   92.5050   52.0000
    ...
```

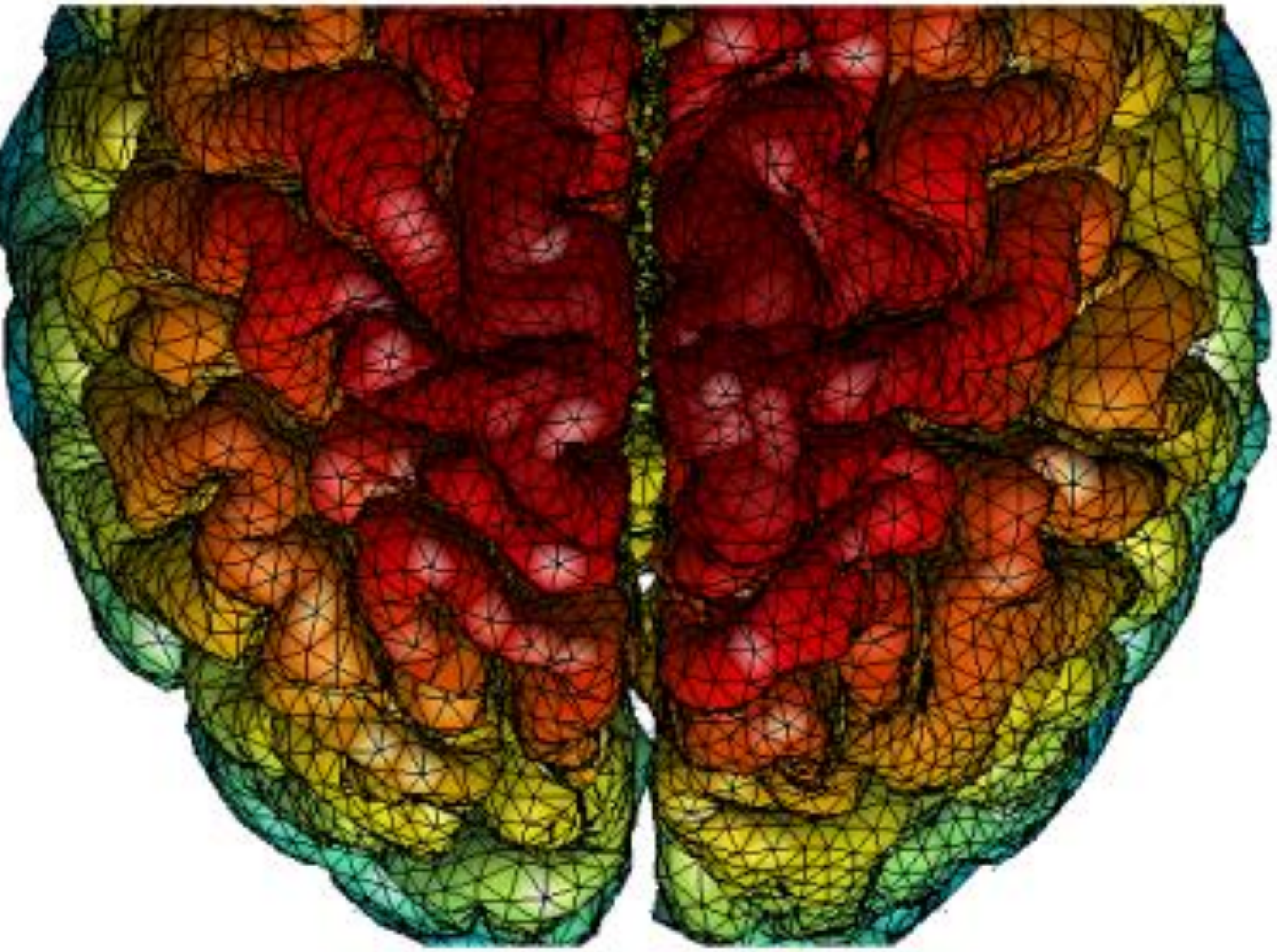
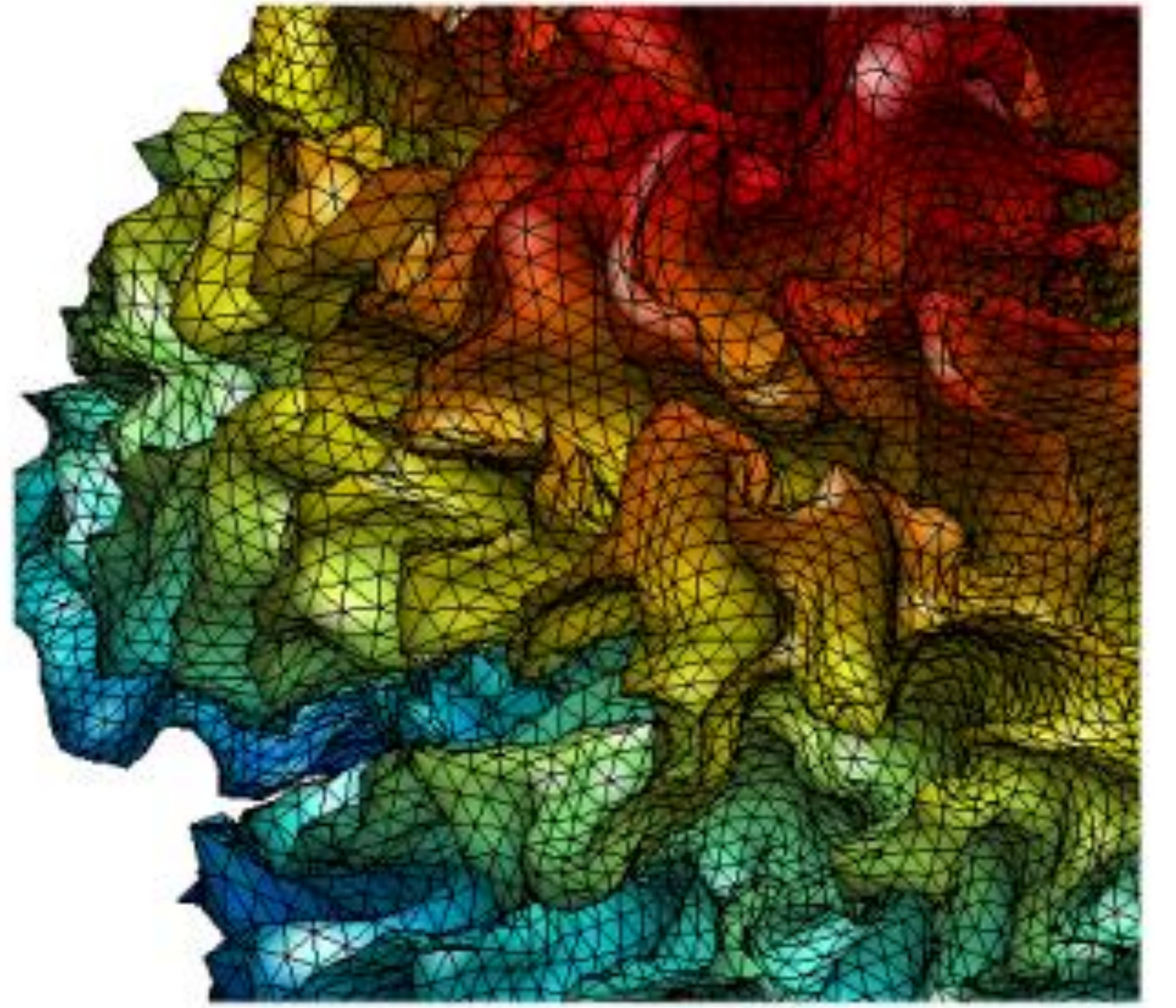
vertex coordinates

2D surface model of left amygdala using marching cubes algorithm





Matlab demo

Euler characteristic for a surface mesh

How to check the topologically defect of a surface?

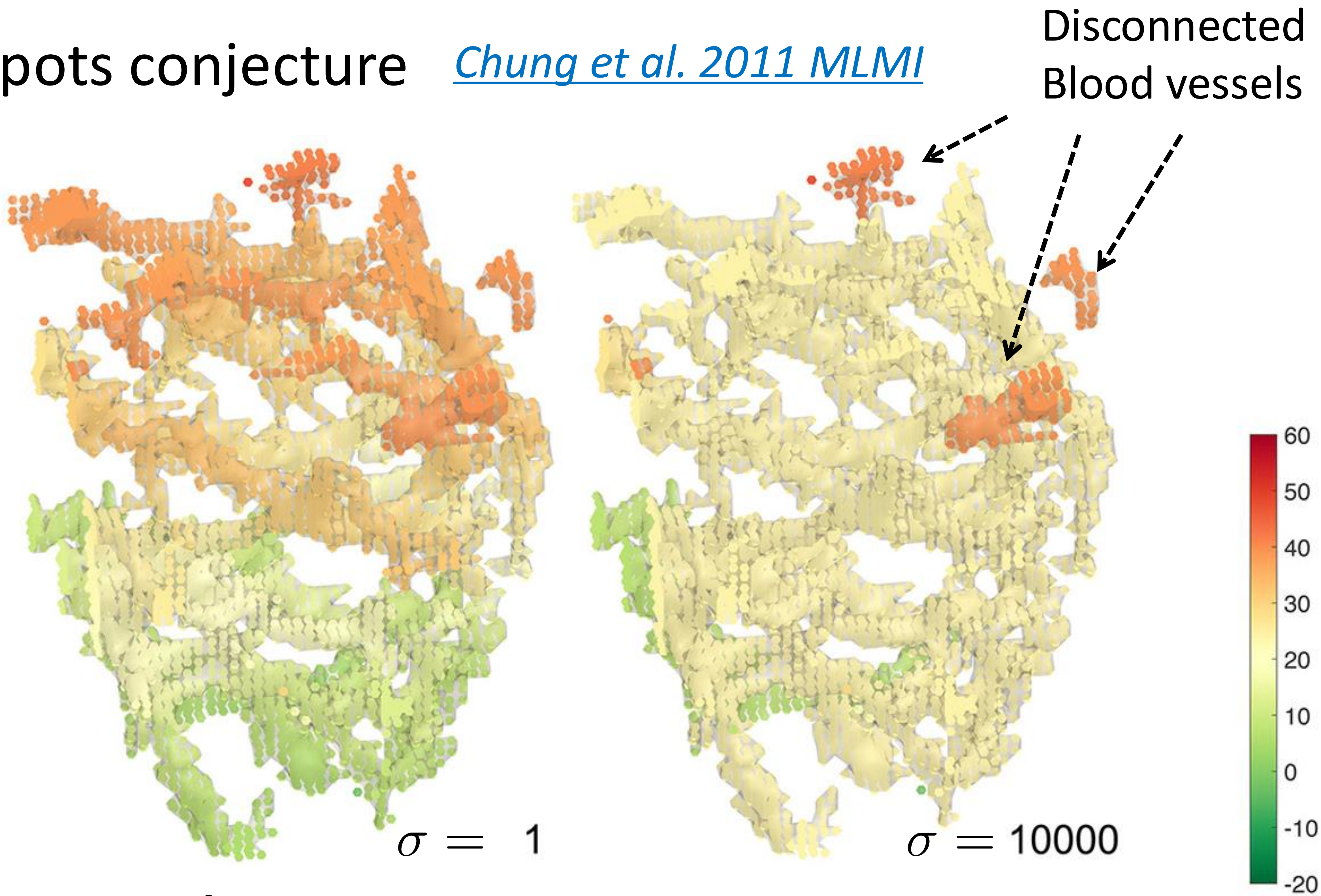
N - E +F = 2 for a surface topologically equivalent to a sphere.

For each triangle, there are three edges. Since two adjacent triangles share the same edge, the total number of edges is E = 3F/2. Hence, we have F=2N – 4 for a closed surface.

Advanced topology problems

Hot spots conjecture

Chung et al. 2011 MLMI



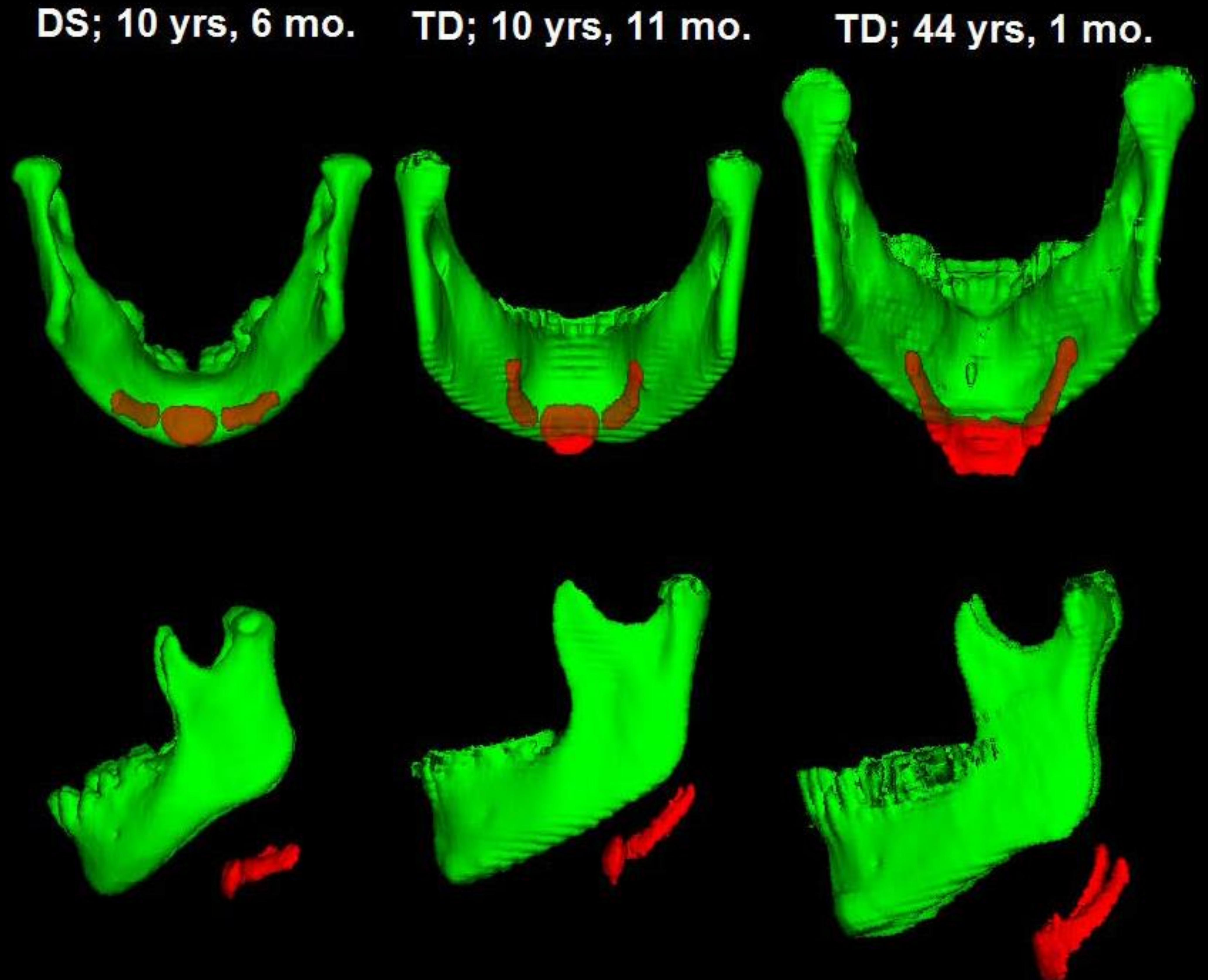
$$K_\sigma * f(p) = \frac{\int_{\mathcal{M}} f(p)\, d\mu(p)}{\mu(\mathcal{M})} + f_1 e^{-\lambda_1 \sigma} \psi_1(p) + R(\sigma, p)$$

Topology term

Diffusion

Mean signal

Topology changing bone fusion

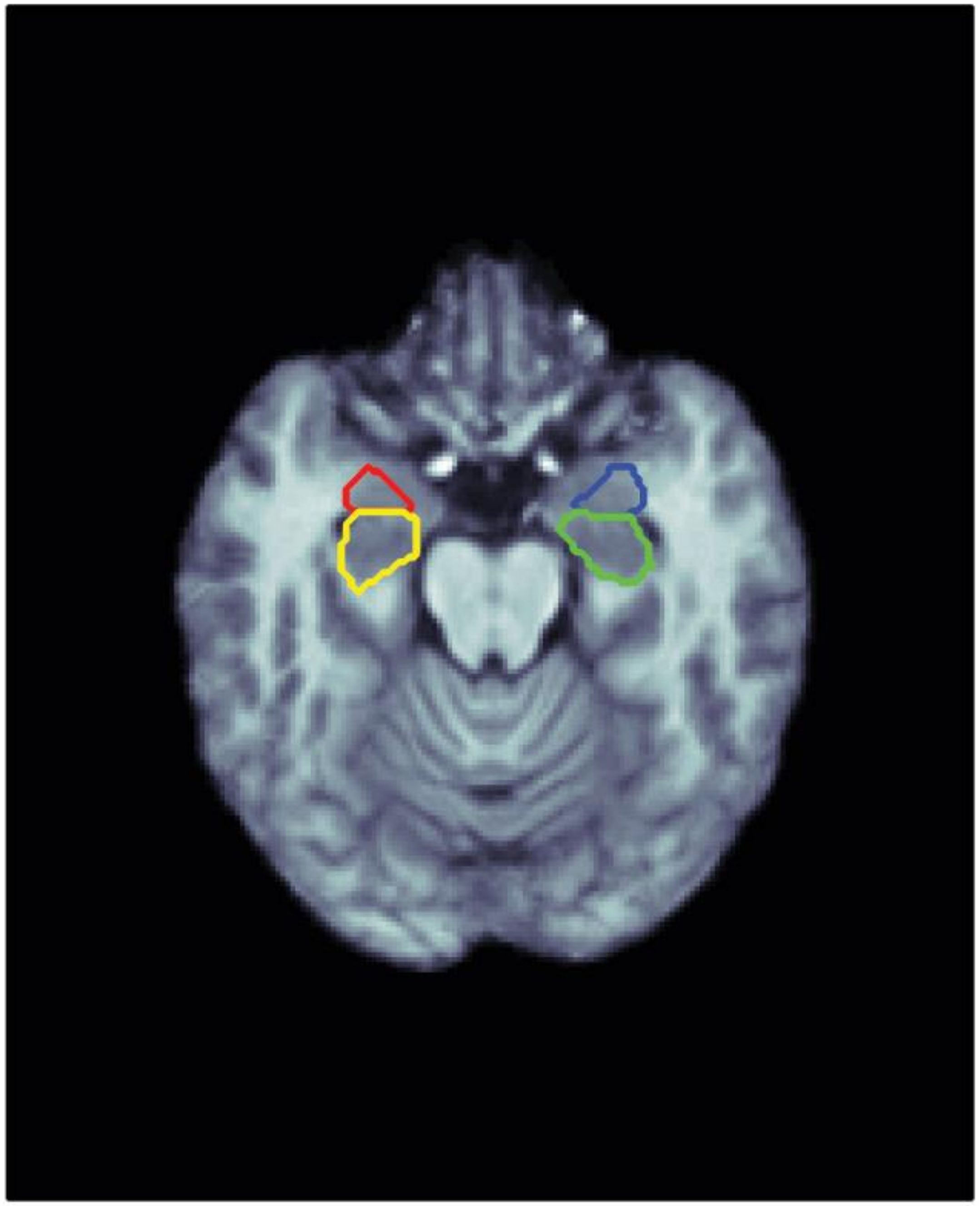


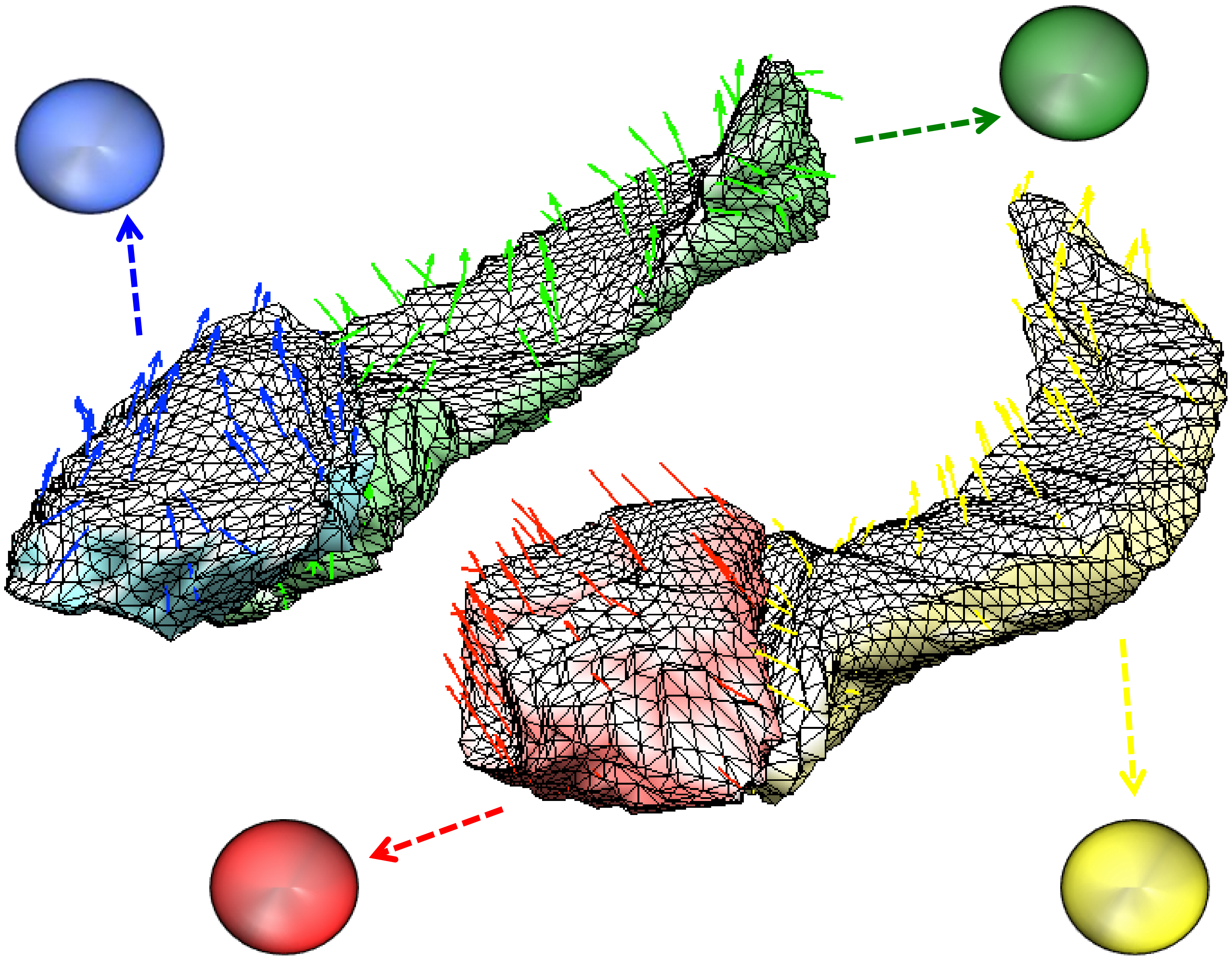
DS: down syndrome TD: typically developing [Chung et al. 2020]

Is it possible to set up a coherent longitudinal growth model for topologically changing structures?

Project: This is nontrivial.

Spherical harmonics (SPHARM) parameterize with respect to a sphere (single connected structure)





How to represent 4 disjoint structures using Hyper-SPHARM basis:
Hosseinbor et al. 2015 Medical Image Analysis