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0.1 Theoretical behaviour of power-law fluids

The viscosity of a power-law fluid with power index n and flow consistency index K varies with shear rate, $\dot{\gamma}$, according to the function

$$\mu(\dot{\gamma}) = K\dot{\gamma}^{n-1}$$

For any power-law fluid, the generalized Reynolds number is (Metzner 1955):

$$Re = \frac{\rho D^n \bar{u}^{2-n}}{8^{n-1}K}$$

Where D is the characteristic length, $\bar{u}$ is the average velocity, and ρ is the fluid density.

0.1.1 Bounded by parallel plates

Consider a planar flow, a laminar flow between infinite parallel plates separated by width a in the y-direction, and with the axis $y = 0$ equidistant from each plate. Then the axial velocity profile is

$$u(y) = \frac{1}{1 + 1/n}\left(\frac{dP}{dL}\frac{1}{K}\right)^{1/n}\left\{\left(\frac{a}{2}\right)^{1+1/n} - |y|^{1+1/n}\right\}$$

where $\frac{dP}{dL}$ is the pressure drop along the flow. The volumetric flow rate is determined by integration between the plates across a unit of depth.

$$\begin{aligned} Q &= 1 \cdot \int_{-a/2}^{a/2} u(y)\, dy \\ &= \frac{a}{2 + 1/n}\left(\frac{dP}{dL}\frac{1}{K}\right)^{1/n}\left(\frac{a}{2}\right)^{1+1/n} \end{aligned}$$

Normalizing $u(y)$ by Q gives

$$\frac{u(y)}{Q} = \frac{1}{a}\frac{2 + 1/n}{1 + 1/n}\left\{1 - \left(\frac{2|y|}{a}\right)^{1+1/n}\right\}$$

which for a given n, a has a maximum at $y = 0$, corresponding to a maximum velocity u_m.

$$\frac{u_m}{Q} = \frac{1}{a}\left(\frac{2+1/n}{1+1/n}\right)$$

The average velocity $\bar{u}$ is

$$\begin{aligned}\bar{u} = \frac{Q}{1\cdot a} &= \frac{1}{2+1/n}\left(\frac{dP}{dL}\frac{1}{K}\right)^{1/n}\left(\frac{a}{2}\right)^{1+1/n}\\ &= \left(\frac{1+1/n}{2+1/n}\right)u_m\end{aligned}$$

For infinite parallel plates, the characteristic length is $2a$ where a is the distance between the plates, so the generalized Reynolds number has the form

$$\begin{aligned}Re &= \frac{\rho\left(2a\right)^n\bar{u}^{2-n}}{\left(2^3\right)^{n-1}K} = \frac{\rho 2^{3-2n}a^n\bar{u}^{2-n}}{K}\\ &= \frac{\rho 2^{3-2n}\left(Q/a\cdot 1\right)^{2-n}}{K} = \frac{\rho 2^{3-2n}Q^{2-n}}{Ka^{2-2n}}\end{aligned}$$

For a Newtonian fluid ($n = 1$, $K = \mu$),

$$Re = \frac{2\rho Q}{\mu}$$

0.1.2 Bounded axisymmetrically

For an axial flow bounded symmetrically at radius $r = R$, there is a laminar profile.

$$u\left(r\right) = \frac{1}{1+1/n}\left(\frac{1}{2}\frac{dP}{dL}\frac{1}{K}\right)^{1/n}\left(R^{1+1/n} - r^{1+1/n}\right)$$

The volumetric flow rate is determined by integration over all radiuses:

$$\begin{aligned}Q &= \int_0^R 2\pi r\cdot u\left(r\right)dr\\ &= \frac{\pi}{3+1/n}\left(\frac{1}{2}\frac{dP}{dL}\frac{1}{K}\right)^{1/n}R^{3+1/n}\end{aligned}$$

For Newtonian ($n = 1$) fluids, this is the Hagen-Poiseuille equation. The average velocity is

$$\bar{u} = \frac{Q}{\pi R^2} = \frac{1}{3 + 1/n} \left(\frac{1}{2} \frac{dP}{dL} \frac{1}{K} \right)^{1/n} R^{1+1/n}$$

and so taking $u_m = u(0)$,

$$\frac{\bar{u}}{u_m} = \frac{1 + 1/n}{3 + 1/n}$$

The characteristic length is $2R$, so the generalized Reynolds number is

$$\begin{aligned} Re &= \frac{\rho \left(2R\right)^n \bar{u}^{2-n}}{\left(2^3\right)^{n-1} K} = \frac{\rho 2^{3-2n} R^n \bar{u}^{2-n}}{K} \\ &= \frac{\rho 2^{3-2n} R^n \left({}^{Q}\!/\!{}_{\pi R^2}\right)^{2-n}}{K} = \frac{\rho 2^{3-2n} R^{3n-4} \left(Q/\pi\right)^{2-n}}{K} \end{aligned}$$

For a Newtonian fluid,

$$Re = \frac{\rho D \bar{u}}{\mu}$$

0.1.3 Bounded in a square cross-section

In a *square channel* with side length a, the characteristic length is a.

$$\begin{aligned} Re &= \frac{\rho a^n \bar{u}^{2-n}}{8^{n-1} K} \\ &= \frac{\rho a^n \left({}^{Q}\!/\!{}_{a^2}\right)^{2-n}}{8^{n-1} K} = \frac{\rho Q^{2-n}}{8^{n-1} K a^{4-3n}} \end{aligned}$$

For a Newtonian fluid,

$$Re = \frac{\rho Q}{\mu a}$$

0.1.4 Dimensionless form

Normalizing u by u_m in gives a single dimensionless profile,

$$\begin{aligned} u^* &= \frac{u}{u_m} \\ &= 1 - \left(d^*\right)^{1+1/n} \qquad : d^* \in \left\{ r^* = \frac{r}{R}, y^* = \frac{y}{a/2} \right\} \end{aligned}$$

so that the shape of the planar and axisymmetric profiles is the same.

0.2 Similitude of branched microchannel simulations and experiments

The condition of similarity between the simulated and physical microchannel:

$$Re_{sim} = Re_{phys} = Re$$

0.2.1 Units and geometry

Each branch has a square cross-section, with side length $a_{phys} = 60\,\mu\text{m}$. The merged channel has width $120\,\mu\text{m}$, and depth $60\,\mu\text{m}$. Simulation units for length, time, and mass are given as $\mathbb{L}$, $\mathbb{T}$, and $\mathbb{M}$.

Scaling factors

The length scaling factor is

$$\mathcal{L} = \frac{60.0\,\mathbb{L}}{60.0\,\mu\text{m}} = 1.0\,\mathbb{L}/\mu\text{m}$$

The time scaling factor can be determined from the simulation and physical velocities and the length scaling factor:

$$\begin{aligned}\mathcal{T} &= \frac{\left(Q/a^2\right)_{phys}}{\bar{u}_{sim}}\mathcal{L} \\ &= \frac{\left(5.0\,\mu\text{l}\,\text{h}^{-1}\right)\left(2.778\times 10^5\mu\text{m}^3\,\text{s}^{-1}\text{h}\,\mu\text{l}^{-1}\right)}{60.0\,\mu\text{m}}\left(\frac{1.0\,\mathbb{L}/\mu\text{m}}{1.0\,\mathbb{L}/\mathbb{T}}\right) \\ &= 385.8\,\mathbb{T}/\text{s}\end{aligned}$$

0.2.2 2D (Planar) Simulations

The length scaling factor is

$$\mathcal{L} = \frac{60.0\,\mathbb{L}}{60.0\,\mu\text{m}} = 1.0\,\mathbb{L}/\mu\text{m}$$

$$\begin{aligned}\mathcal{T} &= \frac{\left(Q/a^2\right)_{phys}}{\bar{u}_{sim}}\mathcal{L} \\ &= \frac{\left(5.0\,\mu\text{l}\,\text{h}^{-1}\right)\left(2.778\times 10^5\mu\text{m}^3\,\text{s}^{-1}\text{h}\,\mu\text{l}^{-1}\right)}{60.0\,\mu\text{m}}\left(\frac{1.0\,\mathbb{L}/\mu\text{m}}{1.0\,\mathbb{L}/\mathbb{T}}\right) \\ &= 385.8\,\mathbb{T}/\text{s}\end{aligned}$$

Branch 1 (Newtonian – Water) The upper channel contains water flowing at 5.0 µl h^{-1}. The density and viscosity are taken for pure water at 25 °C.

$$
\begin{aligned}
Re &= \frac{\rho Q}{\mu a} \\
&= \frac{\left(997\,\mathrm{kg\,m^{-3}}\right)\left(5.0\,\mathrm{\mu l\,h^{-1}}\right)\left(2.778\times10^{-13}\,\mathrm{h\,\mu l^{-1} m^{3}\,s^{-1}}\right)}{\left(8.9\times10^{-4}\,\mathrm{Pa\,s}\right)\left(6.0\times10^{-5}\,\mathrm{\mu m}\right)} \\
&= 0.02593 \approx 0.026
\end{aligned}
$$

In the simulated planar case, this corresponds to a viscosity of:

$$
\begin{aligned}
\mu_{sim} &= \frac{\rho\,(2a)\,\bar{u}}{Re} \\
&= \frac{\left(1.0\,\mathbb{M}/\mathbb{L}^3\right)\left(2\cdot 60.0\,\mathbb{L}\right)\left(1.0\,\mathbb{L}/\mathbb{T}\right)}{0.02593} \\
&= 4628\,\mathbb{M}/\mathbb{L}\mathbb{T}
\end{aligned}
$$

Branch 2 (Non-Newtonian – Polyox)

The second branch contains Polyox WSR-301 flowing at 5.0 µl h^{-1}. The density is assumed to be the same as for water. The power-law model parameters for the 0.3% Polyox solution were estimated experimentally as $K = 0.02519\,\mathrm{Pa\,s}$ and $n = 0.7859$.

$$
\begin{aligned}
Re &= \frac{\rho Q^{2-n}}{8^{n-1} K a^{4-3n}} \\
&= \frac{\left(997\,\mathrm{kg\,m^{-3}}\right)\left\{\left(5.0\,\mathrm{\mu l\,h^{-1}}\right)\left(2.778\times10^{-13}\,\mathrm{h\,\mu l^{-1} m^{3}\,s^{-1}}\right)\right\}^{2-0.7859}}{8^{0.7859-1}\left(0.02519\,\mathrm{Pa\,s}\right)\left(6.0\times10^{-5}\,\mathrm{\mu m}\right)^{4-3(0.7859)}} \\
&= 2.13\times10^{-3}
\end{aligned}
$$

$$
\begin{aligned}
K_{sim} &= \frac{\rho 2^{3-2n} a^{n} \bar{u}^{2-n}}{Re} \\
&= \frac{\left(1.0\,\mathbb{M}/\mathbb{L}^3\right) 2^{3-2(0.7859)}\left(2\cdot 60.0\,\mathbb{L}\right)^{0.7859}\left(1.0\,\mathbb{L}/\mathbb{T}\right)^{2-0.7859}}{2.13\times10^{-3}} \\
&= 31\,550\,\mathbb{M}/\mathbb{L}\mathbb{T}^{2-0.7859}
\end{aligned}
$$

Note that the power law model always retains proper dimensions for viscosity:

$$
\mu\left[\frac{\mathbb{M}}{\mathbb{L}\mathbb{T}}\right] = K\left[\frac{\mathbb{M}}{\mathbb{L}\mathbb{T}^{2-n}}\right]\left(\dot{\gamma}\left[\mathbb{T}^{-1}\right]\right)^{n-1}
$$

For the 1.0% Polyox solution, the power-law parameters are $K = 1.316\,\mathrm{Pa\,s}$ and $n = 0.5355$, and $Re = 1.09\times10^{-4}$ and $K_{sim} = 4.52\times10^{5}$.