0.1 Theoretical behaviour of power-law fluids

The viscosity of a power-law fluid with power index n and flow consistency index K varies with shear rate, $\dot{\gamma}$, according to the function

$$\mu(\dot{\gamma}) = K\dot{\gamma}^{n-1}$$

For any power-law fluid, the generalized Reynolds number is (Metzner 1955):

$$Re = \frac{\rho D^n \bar{u}^{2-n}}{8^{n-1}K}$$

Where D is the characteristic length, $\bar{u}$ is the average velocity, and ρ is the fluid density.

0.1.1 Bounded by parallel plates

Consider a planar flow, a laminar flow between infinite parallel plates separated by width a in the y-direction, and with the axis $y = 0$ equidistant from each plate. Then the axial velocity profile is

$$u\left(y\right) = \frac{1}{1+1/n}\left(\frac{dP}{dL}\frac{1}{K}\right)^{1/n}\left\{\left(\frac{a}{2}\right)^{1+1/n} - \left|y\right|^{1+1/n}\right\}$$

where $\frac{dP}{dL}$ is the pressure drop along the flow. The volumetric flow rate is determined by integration between the plates across a unit of depth.

$$\begin{aligned} Q &= 1 \cdot \int_{-a/2}^{a/2} u\left(y\right) dy \\ &= \frac{a}{2+1/n}\left(\frac{dP}{dL}\frac{1}{K}\right)^{1/n}\left(\frac{a}{2}\right)^{1+1/n} \end{aligned}$$

Normalizing $u\left(y\right)$ by Q gives

$$\frac{u\left(y\right)}{Q} = \frac{1}{a}\frac{2+1/n}{1+1/n}\left\{1 - \left(\frac{2\left|y\right|}{a}\right)^{1+1/n}\right\}$$

which for a given n, a has a maximum at $y = 0$, corresponding to a maximum velocity u_m.

$$\frac{u_m}{Q} = \frac{1}{a}\left(\frac{2 + 1/n}{1 + 1/n}\right)$$

The average velocity $\bar{u}$ is

$$\begin{aligned}\bar{u} = \frac{Q}{1 \cdot a} &= \frac{1}{2 + 1/n}\left(\frac{dP}{dL}\frac{1}{K}\right)^{1/n}\left(\frac{a}{2}\right)^{1+1/n} \\ &= \left(\frac{1 + 1/n}{2 + 1/n}\right)u_m\end{aligned}$$

For infinite parallel plates, the characteristic length is $2a$ where a is the distance between the plates, so the generalized Reynolds number has the form

$$\begin{aligned}Re &= \frac{\rho\,(2a)^n\,\bar{u}^{2-n}}{(2^3)^{n-1}\,K} = \frac{\rho 2^{3-2n} a^n \bar{u}^{2-n}}{K} \\ &= \frac{\rho 2^{3-2n}\,(Q/a \cdot 1)^{2-n}}{K} = \frac{\rho 2^{3-2n} Q^{2-n}}{K a^{2-2n}}\end{aligned}$$

For a Newtonian fluid ($n = 1$, $K = \mu$),

$$Re = \frac{2\rho Q}{\mu}$$

0.1.2 Bounded axisymmetrically

For an axial flow bounded symmetrically at radius $r = R$, there is a laminar profile.

$$u\,(r) = \frac{1}{1 + 1/n}\left(\frac{1}{2}\frac{dP}{dL}\frac{1}{K}\right)^{1/n}\left(R^{1+1/n} - r^{1+1/n}\right)$$

The volumetric flow rate is determined by integration over all radiuses:

$$\begin{aligned}Q &= \int_0^R 2\pi r \cdot u\,(r)\,dr \\ &= \frac{\pi}{3 + 1/n}\left(\frac{1}{2}\frac{dP}{dL}\frac{1}{K}\right)^{1/n} R^{3+1/n}\end{aligned}$$

For Newtonian ($n = 1$) fluids, this is the Hagen-Poiseuille equation. The average velocity is

$$\bar{u} = \frac{Q}{\pi R^2} = \frac{1}{3 + 1/n}\left(\frac{1}{2}\frac{dP}{dL}\frac{1}{K}\right)^{1/n} R^{1+1/n}$$

and so taking $u_m = u(0)$,

$$\frac{\bar{u}}{u_m} = \frac{1 + 1/n}{3 + 1/n}$$

The characteristic length is $2R$, so the generalized Reynolds number is

$$\begin{aligned} Re &= \frac{\rho\left(2R\right)^n \bar{u}^{2-n}}{\left(2^3\right)^{n-1} K} = \frac{\rho 2^{3-2n} R^n \bar{u}^{2-n}}{K} \\ &= \frac{\rho 2^{3-2n} R^n \left({}^{Q}\!/_{\pi R^2}\right)^{2-n}}{K} = \frac{\rho 2^{3-2n} R^{3n-4} \left(Q/\pi\right)^{2-n}}{K} \end{aligned}$$

For a Newtonian fluid,

$$Re = \frac{\rho D \bar{u}}{\mu}$$

0.1.3 Bounded in a square cross-section

In a *square channel* with side length a, the characteristic length is a.

$$\begin{aligned} Re &= \frac{\rho a^n \bar{u}^{2-n}}{8^{n-1} K} \\ &= \frac{\rho a^n \left({}^{Q}\!/_{a^2}\right)^{2-n}}{8^{n-1} K} = \frac{\rho Q^{2-n}}{8^{n-1} K a^{4-3n}} \end{aligned}$$

For a Newtonian fluid,

$$Re = \frac{\rho Q}{\mu a}$$

0.1.4 Dimensionless form

Normalizing u by u_m in gives a single dimensionless profile,

$$\begin{aligned} u^* &= \frac{u}{u_m} \\ &= 1 - \left(d^*\right)^{1+1/n} \qquad : d^* \in \left\{ r^* = \frac{r}{R}, y^* = \frac{y}{a/2} \right\} \end{aligned}$$

so that the shape of the planar and axisymmetric profiles is the same.

0.2 Similitude of branched microchannel simulations and experiments

The condition of similarity between the simulated and physical microchannel:

$$Re_{sim} = Re_{phys} = Re$$

0.2.1 Units and geometry

Each branch has a square cross-section, with side length $a_{phys} = 60\,\mu\mathrm{m}$. The merged channel has width 120 µm, and depth 60 µm. Simulation units for length, time, and mass are written as $\mathbb{L}$, $\mathbb{T}$, and $\mathbb{M}$.

Scaling factors

The length scaling factor is a simple ratio:

$$\mathcal{L} = \frac{60.0\,\mathbb{L}}{60.0\,\mu\mathrm{m}} = 1.0\,{}^{\mathbb{L}}\!/\!{}_{\mu\mathrm{m}}$$

The time scaling factor can be determined from the simulation and physical velocities and the length scaling factor:

$$\begin{aligned}
\mathcal{T} &= \frac{\left({}^{Q}\!/\!{}_{a^2}\right)_{phys}}{\bar{u}_{sim}}\mathcal{L} \\
&= \frac{\left(5.0\,\mu\mathrm{l\,h^{-1}}\right)\left(2.778\times10^{5}\mu\mathrm{m^3\,s^{-1}h\,\mu l^{-1}}\right)}{60.0\,\mu\mathrm{m}}\left(\frac{1.0\,{}^{\mathbb{L}}\!/\!{}_{\mu\mathrm{m}}}{1.0\,{}^{\mathbb{L}}\!/\!{}_{\mathbb{T}}}\right) \\
&= 385.8\,{}^{\mathbb{T}}\!/\!{}_{\mathrm{s}}
\end{aligned}$$

0.2.2 2D (Planar) Simulations

Branch 1 (Newtonian – Water)

The upper channel contains water flowing at $5.0\,\mu\mathrm{l\,h^{-1}}$. The density and viscosity are taken for pure water at 25 °C.

$$\begin{aligned}
Re &= \left(\frac{\rho Q}{\mu a}\right)_{phys} \\
&= \frac{\left(997\,\mathrm{kg\,m^{-3}}\right)\left(5.0\,\mu\mathrm{l\,h^{-1}}\right)\left(2.778\times10^{-13}\,\mathrm{h\,\mu l^{-1}m^3\,s^{-1}}\right)}{\left(8.9\times10^{-4}\,\mathrm{Pa\,s}\right)\left(6.0\times10^{-5}\,\mu\mathrm{m}\right)} \\
&= 0.02593 \approx 0.026
\end{aligned}$$

In the simulated planar case, this corresponds to a viscosity of:

$$\begin{aligned}
\mu_s &= \frac{\rho\,(2a)\,\bar{u}}{Re} \\
&= \frac{\left(1.0\,{}^{\mathbb{M}}\!/\!{}_{\mathbb{L}^3}\right)\left(2\cdot 60.0\,\mathbb{L}\right)\left(1.0\,{}^{\mathbb{L}}\!/\!{}_{\mathbb{T}}\right)}{0.02593} \\
&= 4628\,{}^{\mathbb{M}}\!/\!{}_{\mathbb{L}\mathbb{T}}
\end{aligned}$$

Branch 2 (Non-Newtonian – Polyox)

The second branch contains Polyox WSR-301 flowing at $5.0\,\mu\text{l}\,\text{h}^{-1}$. The density is assumed to be the same as for water. The power-law model parameters for the 0.3% Polyox solution were estimated experimentally as $K_p = 0.02519\,\text{Pa}\,\text{s}$ and $n = 0.7859$.

$$\begin{aligned}
Re &= \frac{\rho Q^{2-n}}{8^{n-1} K a^{4-3n}} \\
&= \frac{\left(997\,\text{kg}\,\text{m}^{-3}\right)\left\{\left(5.0\,\mu\text{l}\,\text{h}^{-1}\right)\left(2.778\times10^{-13}\,\text{h}\,\mu\text{l}^{-1}\text{m}^{3}\,\text{s}^{-1}\right)\right\}^{2-0.7859}}{8^{0.7859-1}\left(0.02519\,\text{Pa}\,\text{s}\right)\left(6.0\times10^{-5}\,\text{m}\right)^{4-3(0.7859)}} \\
&= 2.13\times10^{-3}
\end{aligned}$$

$$\begin{aligned}
K_s &= \frac{\left(\rho 2^{3-2n} a^n \bar{u}^{2-n}\right)_s}{Re} \\
&= \left\{\frac{(2a_s)^n}{a_p^{3n-4}}\left(\frac{\rho_s}{\rho_p}\right)\left(\frac{\bar{u}_s}{Q_p}\right)^{2-n}\right\} K_p
\end{aligned}$$

$$= \left\{\frac{(2\cdot 60.0\,\mathbb{L})^n}{\left(6.0\times10^{-5}\,\text{m}\right)^{3n-4}}\left(\frac{1.0\,\mathbb{M}/\mathbb{L}^3}{997\,\text{kg}\,\text{m}^{-3}}\right)\left(\frac{1.0\,\mathbb{L}/\mathbb{T}}{(5.0)\left(2.78\times10^{-13}\right)\,\text{m}^3\,\text{s}^{-1}}\right)^{2-n}\right\} K_p$$

Note that the power law model always retains proper dimensions for viscosity:

$$\boldsymbol{\mu}\left[\frac{\mathbb{M}}{\mathbb{L}\mathbb{T}}\right] = \mathbf{K}\left[\frac{\mathbb{M}}{\mathbb{L}\mathbb{T}^{2-n}}\right]\left(\dot{\gamma}\left[\mathbb{T}^{-1}\right]\right)^{n-1}$$

For the 0.3% Polyox solution, $K_s = 3.15\times10^4\,\mathbb{M}/\mathbb{L}\mathbb{T}^{2-0.7859}$. For the 1.0% Polyox solution, the power-law parameters are $K = 1.316\,\text{Pa}\,\text{s}$ and $n = 0.5355$, and $Re = 1.09\times10^{-4}$ and $K_s = 3.12\times10^5\,\mathbb{M}/\mathbb{L}\mathbb{T}^{2-0.5355}$.