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0.1 Theoretical behaviour of power-law fluids

The viscosity of a power-law fluid with power index n and flow consistency index K varies with shear rate, $\dot{\gamma}$, according to the function

$$\mu(\dot{\gamma}) = K\dot{\gamma}^{n-1}$$

For any power-law fluid, the generalized Reynolds number is (Metzner 1955):

$$Re = \frac{\rho D^n \bar{u}^{2-n}}{8^{n-1} K}$$

Where D is the characteristic length, \bar{u} is the average velocity, and ρ is the fluid density.

0.1.1 Bounded by parallel plates

Consider a planar flow, a laminar flow between infinite parallel plates separated by width a in the y -direction, and with the axis $y = 0$ equidistant from each plate. Then the axial velocity profile is

$$u(y) = \frac{n}{n+1} \left(\frac{dP}{dL} \frac{1}{K} \right)^{1/n} \left\{ \left(\frac{a}{2} \right)^{1+1/n} - |y|^{1+1/n} \right\}$$

where $\frac{dP}{dL}$ is the pressure drop along the flow. The volumetric flow rate is determined by integration between the plates across a unit of depth.

$$\begin{aligned} Q &= 1 \cdot \int_{-a/2}^{a/2} u(y) dy \\ &= \frac{a \cdot n}{2n+1} \left(\frac{dP}{dL} \frac{1}{K} \right)^{1/n} \left(\frac{a}{2} \right)^{1+1/n} \end{aligned}$$

Normalizing $u(y)$ by Q gives

$$\frac{u(y)}{Q} = \frac{1}{a} \frac{2n+1}{n+1} \left\{ 1 - \left(\frac{2|y|}{a} \right)^{1+1/n} \right\}$$

which for a given n , a has a maximum at $y = 0$, corresponding to a maximum velocity u_m .

$$\frac{u_m}{Q} = \frac{1}{a} \frac{2n+1}{n+1}$$

The average velocity \bar{u} is

$$\begin{aligned} \bar{u} &= \frac{Q}{1 \cdot a} = \frac{n}{2n+1} \left(\frac{dP}{dL} \frac{1}{K} \right)^{1/n} \left(\frac{a}{2} \right)^{1+1/n} \\ &= \frac{n+1}{2n+1} u_m \end{aligned}$$

For infinite parallel plates, the characteristic length is $2a$ where a is the distance between the plates, so the generalized Reynolds number has the form

$$\begin{aligned} Re &= \frac{\rho (2a)^n \bar{u}^{2-n}}{(2^3)^{n-1} K} = \frac{\rho 2^{3-2n} a^n \bar{u}^{2-n}}{K} \\ &= \frac{\rho 2^{3-2n} (Q/a \cdot 1)^{2-n}}{K} = \frac{\rho 2^{3-2n} Q^{2-n}}{K a^{2-2n}} \end{aligned}$$

For a Newtonian fluid ($n = 1$, $K = \mu$),

$$Re = \frac{2\rho Q}{\mu}$$

0.1.2 Bounded axisymmetrically

For an axial flow bounded symmetrically at radius $r = R$, there is a laminar profile.

$$u(r) = \frac{1}{1+1/n} \left(\frac{1}{2} \frac{dP}{dL} \frac{1}{K} \right)^{1/n} \left(R^{1+1/n} - r^{1+1/n} \right)$$

The volumetric flow rate is determined by integration over all radiuses:

$$\begin{aligned} Q &= \int_0^R 2\pi r \cdot u(r) dr \\ &= \frac{\pi}{3+1/n} \left(\frac{1}{2} \frac{dP}{dL} \frac{1}{K} \right)^{1/n} R^{3+1/n} \end{aligned}$$

For Newtonian ($n = 1$) fluids, this is the Hagen-Poiseuille equation. The average velocity is

$$\bar{u} = \frac{Q}{\pi R^2} = \frac{1}{3 + 1/n} \left(\frac{1}{2} \frac{dP}{dL} \frac{1}{K} \right)^{1/n} R^{1+1/n}$$

and so taking $u_m = u(0)$,

$$\frac{\bar{u}}{u_m} = \frac{1 + 1/n}{3 + 1/n}$$

The characteristic length is $2R$, so the generalized Reynolds number is

$$\begin{aligned} Re &= \frac{\rho (2R)^n \bar{u}^{2-n}}{(2^3)^{n-1} K} = \frac{\rho 2^{3-2n} R^n \bar{u}^{2-n}}{K} \\ &= \frac{\rho 2^{3-2n} R^n (Q/\pi R^2)^{2-n}}{K} = \frac{\rho 2^{3-2n} R^{3n-4} (Q/\pi)^{2-n}}{K} \end{aligned}$$

For a Newtonian fluid,

$$Re = \frac{\rho D \bar{u}}{\mu}$$

0.1.3 Bounded in a square cross-section

In a *square channel* with side length a , the characteristic length is a .

$$\begin{aligned} Re &= \frac{\rho a^n \bar{u}^{2-n}}{8^{n-1} K} \\ &= \frac{\rho a^n (Q/a^2)^{2-n}}{8^{n-1} K} = \frac{\rho Q^{2-n}}{8^{n-1} K a^{4-3n}} \end{aligned}$$

For a Newtonian fluid,

$$Re = \frac{\rho Q}{\mu a}$$

0.1.4 Dimensionless form

Normalizing u by u_m in gives a single dimensionless profile,

$$\begin{aligned} u^* &= \frac{u}{u_m} \\ &= 1 - (d^*)^{1+1/n} \quad : d^* \in \left\{ r^* = \frac{r}{R}, y^* = \frac{y}{a/2} \right\} \end{aligned}$$

so that the shape of the planar and axisymmetric profiles is the same.

0.2 Similitude of branched microchannel simulations and experiments

The condition of similarity between the simulated and physical microchannel:

$$Re_{sim} = Re_{phys} = Re$$

0.2.1 Units and geometry

Each branch has a square cross-section, with side length $a_{phys} = 60 \mu\text{m}$. The merged channel has width $120 \mu\text{m}$, and depth $60 \mu\text{m}$. Simulation units for length, time, and mass are given as \mathbb{L} , \mathbb{T} , and \mathbb{M} .

Scaling factors

The length scaling factor is

$$\mathcal{L} = \frac{60.0 \mathbb{L}}{60.0 \mu\text{m}} = 1.0 \mathbb{L}/\mu\text{m}$$

The time scaling factor can be determined from the simulation and physical velocities and the length scaling factor:

$$\begin{aligned} \mathcal{T} &= \frac{(Q/a^2)_{phys}}{\bar{u}_{sim}} \mathcal{L} \\ &= \frac{(5.0 \mu\text{l h}^{-1}) (2.778 \times 10^5 \mu\text{m}^3 \text{s}^{-1} \text{h} \mu\text{l}^{-1})}{60.0 \mu\text{m}} \left(\frac{1.0 \mathbb{L}/\mu\text{m}}{1.0 \mathbb{L}/\mathbb{T}} \right) \\ &= 385.8 \mathbb{T}/\text{s} \end{aligned}$$

0.2.2 2D (Planar) Simulations

The length scaling factor is

$$\mathcal{L} = \frac{60.0 \mathbb{L}}{60.0 \mu\text{m}} = 1.0 \mathbb{L}/\mu\text{m}$$

$$\begin{aligned} \mathcal{T} &= \frac{(Q/a^2)_{phys}}{\bar{u}_{sim}} \mathcal{L} \\ &= \frac{(5.0 \mu\text{l h}^{-1}) (2.778 \times 10^5 \mu\text{m}^3 \text{s}^{-1} \text{h} \mu\text{l}^{-1})}{60.0 \mu\text{m}} \left(\frac{1.0 \mathbb{L}/\mu\text{m}}{1.0 \mathbb{L}/\mathbb{T}} \right) \\ &= 385.8 \mathbb{T}/\text{s} \end{aligned}$$

Branch 1 (Newtonian – Water) The upper channel contains water flowing at $5.0 \mu\text{l h}^{-1}$. The density and viscosity are taken for pure water at 25°C .

$$\begin{aligned} Re &= \frac{\rho Q}{\mu a} \\ &= \frac{(997 \text{ kg m}^{-3}) (5.0 \mu\text{l h}^{-1}) (2.778 \times 10^{-13} \text{ h } \mu\text{l}^{-1} \text{ m}^3 \text{ s}^{-1})}{(8.9 \times 10^{-4} \text{ Pa s}) (6.0 \times 10^{-5} \mu\text{m})} \\ &= 0.02593 \approx 0.026 \end{aligned}$$

In the simulated planar case, this corresponds to a viscosity of:

$$\begin{aligned} \mu_{sim} &= \frac{\rho (2a) \bar{u}}{Re} \\ &= \frac{(1.0 \text{ M/L}^3) (2 \cdot 60.0 \text{ L}) (1.0 \text{ L/T})}{0.02593} \\ &= 4628 \text{ M/LT} \end{aligned}$$

Branch 2 (Non-Newtonian – Polyox)

The second branch contains Polyox WSR-301 flowing at $5.0 \mu\text{l h}^{-1}$. The density is assumed to be the same as for water. The power-law model parameters for the 0.3% Polyox solution were estimated experimentally as $K = 0.02519 \text{ Pa s}$ and $n = 0.7859$.

$$\begin{aligned} Re &= \frac{\rho Q^{2-n}}{8^{n-1} K a^{4-3n}} \\ &= \frac{(997 \text{ kg m}^{-3}) \{ (5.0 \mu\text{l h}^{-1}) (2.778 \times 10^{-13} \text{ h } \mu\text{l}^{-1} \text{ m}^3 \text{ s}^{-1}) \}^{2-0.7859}}{8^{0.7859-1} (0.02519 \text{ Pa s}) (6.0 \times 10^{-5} \mu\text{m})^{4-3(0.7859)}} \\ &= 2.13 \times 10^{-3} \\ K_{sim} &= \frac{\rho 2^{3-2n} a^n \bar{u}^{2-n}}{Re} \\ &= \frac{(1.0 \text{ M/L}^3) 2^{3-2(0.7859)} (2 \cdot 60.0 \text{ L})^{0.7859} (1.0 \text{ L/T})^{2-0.7859}}{2.13 \times 10^{-3}} \\ &= 31\,550 \text{ M/LT}^{2-0.7859} \end{aligned}$$

Note that the power law model always retains proper dimensions for viscosity:

$$\mu \left[\frac{\text{M}}{\text{LT}} \right] = K \left[\frac{\text{M}}{\text{LT}^{2-n}} \right] (\dot{\gamma} [\text{T}^{-1}])^{n-1}$$

For the 1.0% Polyox solution, the power-law parameters are $K = 1.316 \text{ Pa s}$ and $n = 0.5355$, and $Re = 1.09 \times 10^{-4}$ and $K_{sim} = 4.52 \times 10^5$.