T1. $\xi \sim R[0,\theta]_\theta$, $\vec{x}_n = (x_1, \dots x_n)$ – выборка

$M[\xi] = \frac{\theta}{2}$, $M[\xi^2] = \int_0^\theta x^2 \cdot \frac{1}{\theta}\, dx = \frac{1}{\theta} \cdot \frac{x^3}{3}\Big|_0^\theta = \frac{\theta^2}{3}$

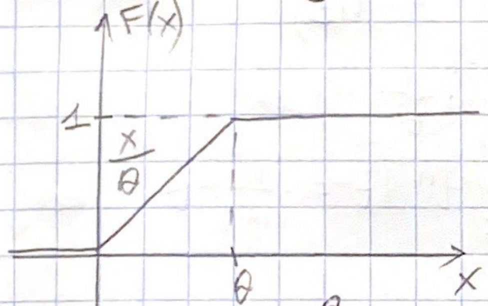
а) $\tilde{\theta}_1 = 2\bar{x}$

$M[\tilde{\theta}] = M\left[2 \cdot \frac{1}{n}\sum_{i=1}^n x_i\right] = \frac{2}{n}\sum_{i=1}^n M[x_i] = 2M\xi = \theta$ –
– несмещ.

$D[\tilde{\theta}_1] = \frac{4}{n^2} D\left[\sum_{i=1}^n x_i\right] = \frac{4}{n} D\xi = \frac{4}{n}\left(\frac{\theta^2}{4} - \frac{\theta^2}{3}\right) = \frac{\theta^2}{3n} \to$
$\xrightarrow[n\to\infty]{} 0$ по дост. условию состоятельна

$\tilde{\theta}_2 = x_{min}$

$\xi \sim F(x)$ независимы, знаем, что $y = \min(\xi_1, \dots \xi_n) \sim 1-(1-F(y))^n$



$\varphi(y) = \Phi'(y) = +n(1-F(y))^{n-1} \cdot p(y)$, где

$p(y) = \frac{1}{\theta}\ (0;\theta)$

$M[\tilde{\theta}_2] = \int_0^\theta y \cdot \varphi(y)\, dy = \int_0^\theta \left(y n (1-F(y))^{n-1} \cdot \frac{1}{\theta}\right) dy = \left\{t = \frac{y}{\theta}\right\} =$
$= \int_0^1 \underbrace{t n (1-t)^{n-1}}_{B(2,n)} \cdot \theta\, dt = n\theta \cdot B(2,n) = n\theta \cdot \frac{\Gamma(2)\cdot\Gamma(n)}{\underbrace{\Gamma(n+2)}_{(n+1)\cdot n \cdot \Gamma(n)}} = \frac{\theta}{n+1}$ –
– смещ., тогда возьмём $\tilde{\theta}_2' = (n+1)\tilde{\theta}_2 = (n+1)x_{min}$ – несмещ.

$M[\tilde{\theta}_2^2] = \int_0^\theta y^2 \cdot n\left(1-\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}\, dy = \int_0^1 t^2 (1-t)^{n-1}\, \theta^2 n\, dt =$
$= n\theta^2 \cdot B(3,n) = n\theta^2 \cdot \frac{\Gamma(3)\cdot\Gamma(n)}{\underbrace{\Gamma(n+3)}_{(n+2)(n+1)\cdot n\cdot\Gamma(n)}} = \frac{2\theta^2}{(n+2)(n+1)}$

$D[\tilde{\theta}_2] = \frac{2\theta^2}{(n+2)(n+1)} - \frac{\theta^2}{(n+1)^2} = \theta^2 \frac{2n+2-n-2}{(n+2)(n+1)^2} = \theta^2 \frac{n}{(n+2)(n+1)^2}$

$D[\tilde{\theta}_2'] = D[(n+1)\tilde{\theta}_2] = (n+1)^2 \cdot \theta^2 \frac{n}{(n+2)(n+1)^2} = \theta^2 \frac{n}{n+2} \xrightarrow[n\to\infty]{} \theta^2 \neq 0$
– по достаточному условию несост.

проверим по опр. состоятельности: $\tilde{\theta}_2' \xrightarrow[p]{} \theta$

$$P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0 \quad \forall \varepsilon > 0$$

$$P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \geq P(\tilde{\theta}_2' \geq \theta + \varepsilon) = P((n+1)\tilde{\theta}_2 \geq \theta + \varepsilon) =$$

$$= P\left(\tilde{\theta}_2 \geq \frac{\theta+\varepsilon}{n+1}\right) = P\left(x_1 \geq \frac{\theta+\varepsilon}{n+1}, \ldots, X_n \geq \frac{\theta+\varepsilon}{n+1}\right) =$$

$$= \prod_{i=1}^{n} P\left(x_i \geq \frac{\theta+\varepsilon}{n+1}\right) = \left(P\left(\xi \geq \frac{\theta+\varepsilon}{n+1}\right)\right)^n = \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n =$$

$$= \left(1 - \frac{\theta+\varepsilon}{(n+1)\theta}\right)^n \xrightarrow[n\to\infty]{} e^{-1-\frac{\varepsilon}{\theta}} > 0 \text{ - противоречие} \Rightarrow$$

$\Rightarrow$ оценка не явл состоятельной

$\underline{\tilde{\theta}_3 = X_{max}}$

$$\xi \sim F(x),\ \max(\xi_1, \ldots \xi_n) \sim (F(z))^n$$

$$\Psi(z) = n \cdot (F(z))^{n-1} \cdot p(z) = n \cdot \left(\frac{z}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} \quad (0, \theta)$$

$$M[\tilde{\theta}_3] = \int_0^\theta z \cdot n \left(\frac{z}{\theta}\right)^{n-1} \frac{1}{\theta} dz = n\theta \int_0^1 t^n dt = \frac{n}{n+1}\theta \text{ - смещ} \Rightarrow$$

$$\Rightarrow \tilde{\theta}_3' = \frac{n+1}{n} X_{max} \text{ - несмещ.}$$

$$M[\tilde{\theta}_3^2] = \int_0^\theta z^2 \cdot n \left(\frac{z}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} dz = \int_0^1 n\theta^2 t^{n+1} dt = \frac{n}{n+2}\theta^2$$

$$D[\tilde{\theta}_3] = \theta^2 \left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right) = \theta^2 \frac{n^3 + 2n^2 + n - n^3 - 2n^2}{(n+2)(n+1)^2} =$$

$$= \frac{n\theta^2}{(n+2)(n+1)^2}$$

$$D[\tilde{\theta}_3'] = \frac{(n+1)^2}{n^2} \cdot D[\tilde{\theta}_3] = \frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0 \text{ - состоят.}$$

$\underline{\tilde{\theta}_4 = X_{min} + X_{max}}$

$$M[\tilde{\theta}_4] = M[\tilde{\theta}_2] + M[\tilde{\theta}_3] = \frac{\theta}{n+1} + \frac{\theta}{n+1} \cdot n = \theta \text{ - несмещ.}$$

$$D[\tilde{\theta}_4] = D[\tilde{\theta}_2] + D[\tilde{\theta}_3] + 2\operatorname{cov}(\tilde{\theta}_2; \tilde{\theta}_3)$$

$$\operatorname{cov}(\tilde{\theta}_2; \tilde{\theta}_3) = M(\tilde{\theta}_3 \cdot \tilde{\theta}_2) - M(\tilde{\theta}_3) \cdot M(\tilde{\theta}_2)$$

ка
юятельна

$= 2M\xi = \theta$ — несмещ.

$= \frac{\theta^2}{12} + \frac{1}{(n-1)} \cdot \frac{\theta^2}{12} =$

$\xrightarrow{p} \frac{\theta}{2}$ (по ЗБЧ Хинчина)

$\frac{1}{n-1} \sum_{k=2}^{n} X_k \xrightarrow{p} X_1 + \frac{\theta}{2}$ — не яв. сост.

$$K(y,z) = \begin{cases} F^n(z) - (F(z)-F(y))^n, & z \geq y \\ F^n(z), & z < y \end{cases}$$

$$\varkappa(y,z) = \frac{\partial^2 K}{\partial y\, \partial z} = \frac{\partial}{\partial z}\left(n(F(z)-F(y))^{n-1} \cdot F'(y)\right) = n(n-1)(F(z) -$$

$$- F(y))^{n-2} \cdot F'(y) \cdot F'(z) \quad (z \geq y)$$

$$M(\tilde{\theta}_3 \cdot \tilde{\theta}_2) = \iint\limits_{-\infty}^{+\infty} yz \cdot \varkappa(y,z)\, dy\, dz = \int\limits_0^\theta dz \int\limits_0^z yz \cdot n(n-1)\left(\frac{z-y}{\theta}\right)^{n-2} \cdot \frac{1}{\theta^2} dy =$$

$$= \left\{ t = \frac{y}{z} \right\} = (*)$$

найдём $\int\limits_0^1 t z^2 \cdot n(n-1)\left(\frac{z}{\theta} - \frac{tz}{\theta}\right)^{n-2} \cdot \frac{1}{\theta^2} z\, dt = n(n-1)\int\limits_0^1 t z^2 \cdot$

$$\cdot \left(\frac{z}{\theta}\right)^{n-2} (1-n)^{n-2} \cdot \frac{1}{\theta^2} z\, dt = n(n-1)\frac{z^{n+1}}{\theta^n} \underbrace{\int\limits_0^1 t(1-t)^{n-2} dt}_{B(2,n-1) = \frac{\Gamma(n-1)}{\Gamma(n+1)} = \frac{\Gamma(n-1)}{n(n-1)\Gamma(n-1)}} =$$

$$= \frac{z^{n+1}}{\theta^n}$$

$$(*) = \int\limits_0^\theta \frac{z^{n+1}}{\theta^n} dz = \frac{1}{\theta^n} \frac{z^{n+2}}{n+2}\Big|_0^\theta = \frac{\theta^2}{n+2}$$

$$\operatorname{cov}(\tilde{\theta}_2, \tilde{\theta}_3) = \frac{\theta^2}{n+2} - \frac{\theta^2}{(n+1)^2} \cdot n = \theta^2 \frac{1}{(n+2)(n+1)^2}$$

$$D[\tilde{\theta}_4] = \theta^2\left(\frac{n}{(n+1)^2(n+2)} + \frac{n}{(n+1)^2(n+2)} + 2 \cdot \frac{1}{(n+1)^2(n+2)}\right) =$$

$$= \frac{2\theta^2}{(n+1)(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow$$ оценка состоятельна

$$\tilde{\theta}_5 = x_1 + \frac{1}{(n-1)} \sum_{k=2}^n x_k$$

$M[\tilde{\theta}_5] = M[x_1] + \frac{1}{n-1}(n-1) M[x_2] = 2M\xi = \theta$ – несмещ.

$$D[\tilde{\theta}_5] = D[x_1] + \frac{1}{(n-1)^2} \sum_{k=2}^n D\xi = \frac{\theta^2}{12} + \frac{1}{(n-1)} \cdot \frac{\theta^2}{12} =$$

$$= \frac{\theta^2}{12}\left(1 + \frac{1}{n-1}\right) \xrightarrow[n\to\infty]{} 0$$

по определению: $\tilde{\theta}_5 = \underbrace{X_1}_{\xrightarrow{p} X_1} + \overbrace{\frac{1}{n-1}\sum_{k=2}^n X_k}^{\xrightarrow{p} \frac{\theta}{2} \text{ (по ЗБЧ Хинчина)}} \xrightarrow{p} X_1 + \frac{\theta}{2}$ – не явл. сост... сост.

12. a) $\alpha_1 = \int_{-\infty}^{+\infty} x\, p(x)\,dx = \int_0^{+\infty} x\cdot e^{-x}\,dx = \Gamma(2) = 1$

$$\mu_3 = \int_0^{+\infty} (x-1)^3\cdot e^{-x}\,dx = \frac{1}{e}\int_0^{+\infty} (x-1)^3 e^{-(x-1)}\,d(x-1) =$$

$$= \frac{1}{e}\Gamma(4) = \frac{1}{e}3\cdot\Gamma(3) = \frac{1}{e}6$$

c) $p(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$\cancel{\bar{p}(\,) = \frac{1}{h}\int_{-\infty}^{+\infty} p(x)\cdot q(\qquad = \frac{1}{h}\int_{-1}^{1} e^{-x}\cdot\frac{3}{4}(1-x^2)\,dx =}$$

$$\cancel{= \frac{1}{h}\int_{-1}^{1}\frac{3}{4}e^{-x}\,dx - \frac{1}{h}\int_{-1}^{1}\frac{3}{4}x^2e^{-x}\,dx = \frac{-3}{4h}e^{-x}\Big|_{-1}^{1} + \frac{3}{4h}\Big(x^2e^{-x}\Big|_{-1}^{1} +}$$

$$\cancel{+ \int_{-1}^{1} 2xe^{-x}\,dx\Big) = -\frac{3}{4h}\left(\frac{1}{e}-e\right) + \frac{3}{4h}\left(\frac{1}{e}-e\right) +}$$

$$\cancel{\frac{3}{4h}\Big(-2xe^{-x}\Big|_{-1}^{1} + 2\int_{-1}^{1} e^{-x}\,dx\Big) = \frac{3}{4h}\left(-2\left(\frac{1}{e}+e\right) - 2\left(\frac{1}{e}-e\right)\right) =}$$

$$\cancel{= \frac{4}{e}\cdot\frac{3}{4h} = \frac{3}{he} = 1,76}$$

$s = 1,341 \Rightarrow h = 0,629$

$$\bar{p}(z) = \frac{1}{h}\int_{-\infty}^{+\infty} p(x)\cdot q\left(\frac{z-x}{h}\right)dx = \frac{1}{h}\int_{-1}^{1} e^{-x}\cdot\frac{3}{4}\left(1-\frac{(z-x)^2}{h}\right)dx =$$

$$= \frac{3}{4h}\int_{-1}^{1} e^{-x}\,dx - \frac{3}{4h^2}\int_{-1}^{1}(z-x)^2\cdot e^{-x}\,dx =$$

$$= \frac{-3}{4h}\left(\frac{1}{e}-e\right) - \frac{3}{4h^2}\left(\int_{-1}^{1} z^2e^{-x}\,dx - 2z\int_{-1}^{1} x\cdot e^{-x}\,dx + \int_{-1}^{1} x^2e^{-x}\,dx\right) =$$

$$= -\frac{3}{4h}\left(\frac{1}{e}-e\right) + \frac{3z^2}{4h^2}\left(\frac{1}{e}-e\right) + \frac{3}{4h^2}\Big(2zxe^{-x}\Big|_{-1}^{1} - 2z\int_{-1}^{1} e^{-x}\,dx -$$

$$- x^2e^{-x}\Big|_{-1}^{1} + \int_{-1}^{1} 2xe^{-x}\,dx\Big) = \frac{3}{4h}\left(\frac{1}{e}-e\right)\left(z^2\cdot\frac{1}{h}-1\right) + \frac{3z}{2h^2}\left(\frac{1}{e}+e\right) -$$

$$+\frac{3z}{2h^2}\left(\frac{1}{e}-e\right)-\frac{3}{4h^2}\left(\frac{1}{e}-e\right)-\frac{3}{2h^2}\left(\frac{1}{e}+e\right)+\frac{3}{2h^2}\left(\frac{1}{e}-e\right)=$$

$$=\frac{3}{4h}\left(\frac{1}{e}-e\right)\left(\frac{z^2}{h}-1\right)+\left(\frac{1}{e}+e\right)\left(\frac{3z}{2h^2}-\frac{3}{2h^2}\right)+\left(\frac{1}{e}-e\right)\cdot$$

$$\cdot\left(\frac{3z}{2h^2}-\frac{3}{4h^2}+\frac{3}{2h^2}\right)=\frac{3}{4h^2}\left((x-z)(x-z+2)-h+2\right)e^{-x}\Big|_{-1}^{1}=$$

$$=\frac{3}{4h^2}\left(\frac{1}{e}\left((1-z)(3-z)-h+2\right)-e\left((-1-z)(1-z)-h+2\right)\right)=$$

$$=\frac{3}{4h^2}\left(-\frac{h}{e}+\frac{2}{e}+\frac{3}{e}-\frac{z}{e}-\frac{3z}{e}+\frac{z^2}{e}+eh-2e+e-ze+\right.$$

$$\left.ze-z^2e\right)=\frac{3}{4h^2}\left(z^2\left(\frac{1}{e}-e\right)+z\left(-\frac{4}{e}\right)+\frac{5}{e}-\frac{h}{e}+eh-e\right)=$$

$$=\frac{3}{4h^2}\left(-2{,}35z^2-1{,}47z+0{,}6\right)$$

d) ЦПТ: $\dfrac{\frac{1}{n}\sum x_i-M\xi}{\sqrt{D\xi}}\sqrt{n}\rightsquigarrow N(0;1)$

$$M\xi=\int_0^{+\infty}x\cdot e^{-x}dx=-x\cdot e^{-x}\Big|_0^{+\infty}+\int_0^{+\infty}e^{-x}dx=e^0-e^{+\infty}=1$$

$$M\xi^2=\int_0^{+\infty}x^2e^{-x}dx=-x^2e^{-x}\Big|_0^{+\infty}+2\int_0^{+\infty}xe^{-x}dx=2$$

$$\frac{\overline{X}-1}{1}5\rightsquigarrow N(0;1)\Rightarrow\overline{X}\rightsquigarrow N\left(1;\frac{1}{25}\right)$$

f) $g_{\xi_i\,и\,\xi_j}=n(n-1)C_{n-i-1}^{y-i-1}p(x)p(y)\cdot F^{i-1}(x)\cdot(F(y)-F(x))^{y-i-1}\cdot$

$$\cdot(1-F(y))^{n-j}=600\cdot C_{24-i}^{y-i-1}e^{-x-y}(1-e^{-x})^{i-1}\cdot(e^{-x}-e^{-y})^{y-i-1}e^{-24y}=$$

$$=600\,C_{24-i}^{y-i-1}e^{-x-25y}(1-e^{-x})^{i-1}(e^{-x}-e^{-y})^{y-i-1},\quad x>0,\ y>0,\ i<j$$

13. $p(x) = \begin{cases} \frac{e^{-x/\theta}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad \theta > 0$ $\quad n = 3$

$$M\xi = \int_0^{+\infty} \frac{x}{\theta} e^{-\frac{x}{\theta}} dx = \left\{ t = \frac{x}{\theta} \right\} = \theta \int_0^{+\infty} t e^{-t} dt = \theta \cdot \Gamma(2) = \theta$$

$$M\xi^2 = \int_0^{+\infty} \frac{x^2}{\theta} \cdot e^{-\frac{x}{\theta}} dx = \frac{1}{\theta} \cdot (x^2 + 2\theta x + 2\theta^2) e^{-\frac{x}{\theta}} \cdot (-\theta) \Big|_0^{+\infty} =$$

$$= 2\theta^2 - 0 = 2\theta^2$$

$$D\xi = 2\theta^2 - \theta^2 = \theta^2$$

$$\tilde{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} x_i = \overline{X}$$

a) $M[\tilde{\theta}_1] = \frac{1}{n} \cdot n M\xi = \theta$ - несмещ.

b) ~~для начала проверим состоятельность~~

$$D[\tilde{\theta}_1] = \frac{1}{n^2} \cdot n \cdot D\xi = \frac{\theta^2}{n} \xrightarrow[n \to \infty]{} 0 \Rightarrow \text{состоятельна}$$

$$I(\theta) = -M\left[\frac{\partial^2 \ln p}{\partial \theta^2}\right]$$

$$\ln e^{-\frac{x}{\theta}}/\theta = -\frac{x}{\theta} - \ln \theta$$

$$\frac{\partial}{\partial \theta} \left(\frac{x}{\theta^2} - \frac{1}{\theta} \right) = \frac{-2x}{\theta^3} + \frac{1}{\theta^2}$$

$$I(\theta) = -M\left[\frac{-2x}{\theta^3} + \frac{1}{\theta^2}\right] = \frac{2}{\theta^3} M\xi - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

$$g(\theta) = g(\tilde{\theta}_1) \Rightarrow g'(\theta) = 1$$

$$D[\tilde{\theta}_1] \geq \frac{g'^2(\theta)}{n \cdot I(\theta)} \Rightarrow \theta^2 \geq \frac{1}{n \cdot \frac{1}{\theta^2}} \Rightarrow \theta^2 \geq \frac{\theta^2}{n} \Rightarrow$$

$\Rightarrow$ оценка явл. эфф. по Крамеру-Рао

$$\tilde{\theta}_2 = \frac{x_{min} + x_{max}}{2}$$

a) $\tilde{\theta}_{min} = x_{min} \sim 1 - (1 - F(y))^n$

$$\varphi = n(1 - F(y))^{n-1} \cdot p(y)$$

$$M[\tilde{\theta}_{min}] = \int_0^{+\infty} yn(1-(1-e^{-y/\theta}))^{n-1}\cdot\frac{1}{\theta}e^{-\frac{y}{\theta}}\,dy =$$

$$= \int_0^{\infty} \frac{yn}{\theta} e^{-\frac{ny}{\theta}}\,dy = \frac{\theta}{n}\cdot\Gamma(2) = \frac{\theta}{n}$$

$$D[\tilde{\theta}_{min}] = M[\tilde{\theta}^2_{min}] - M^2[\tilde{\theta}_{min}] = \frac{2\theta^2}{n^2} - \frac{\theta^2}{n^2} = \frac{\theta^2}{n^2}$$

$$\tilde{\theta}_{max} = X_{max} \sim (F(y))^n$$

$$= n(F(\ ))^{n-1}\cdot p(\ )$$

$$M[\tilde{\theta}_{max}] = \int_0^{\infty} y\cdot n(1-e^{-\frac{y}{\theta}})^2 \frac{e^{-\frac{y}{\theta}}}{\theta}\,dy = \int_0^{\infty} \frac{yn}{\theta}(1-2e^{-\frac{y}{\theta}} + e^{-\frac{2y}{\theta}})e^{-\frac{y}{\theta}}\,dy = \int_0^{\infty}\frac{yn}{\theta}e^{-\frac{y}{\theta}}\,dy - 2\int_0^{\infty}\frac{yn}{\theta}e^{-\frac{2y}{\theta}}\,dy + \int_0^{\infty}\frac{yn}{\theta}e^{-\frac{3y}{\theta}}\,dy = \frac{n}{\theta}\theta^2 - \frac{2n}{\theta}\cdot\frac{\theta^2}{4} + \frac{n}{\theta}\cdot\frac{\theta^2}{9} = \frac{11}{6}\theta$$

$$M[\tilde{\theta}^2_{max}] = \int_0^{\infty}\frac{y^2n}{\theta}e^{-\frac{y}{\theta}}\,dy - \int_0^{\infty}\frac{2y^2n}{\theta}e^{-\frac{2y}{\theta}}\,dy + \int_0^{\infty}\frac{y^2n}{\theta}e^{-\frac{3y}{\theta}}\,dy = 2\theta^2 n - n\frac{\theta^2}{2} + \frac{2n\theta^2}{27} = \frac{85}{54}\theta^2 n$$

$$D[\tilde{\theta}_{max}] = \frac{85}{18}\theta^2 - \frac{131}{36}\theta^2 = \frac{49}{36}\theta^2$$

$$M[\tilde{\theta}_2] = \frac{1}{2}\left(\frac{\theta}{n} + \frac{11}{6}\theta\right) = \frac{1}{2}\cdot\frac{13\theta}{6} = \frac{13}{12}\theta$$ — смещ => θ

$$\Rightarrow \tilde{\theta}_2' = \frac{12}{13}\tilde{\theta}_2$$

b) $D[\tilde{\theta}_2] = \frac{1}{4}D[\tilde{\theta}_{min}] + \frac{1}{4}D[\tilde{\theta}_{max}] + \frac{2}{4}cov(\tilde{\theta}_{min};\tilde{\theta}_{max})$

$$K(y,z) = \begin{cases} F(z)^n, & y > z \\ F^n(z) - (F(z)-F(y))^n, & y \le z \end{cases} \Rightarrow$$

$$\Rightarrow \varkappa = \begin{cases} 0, & y > z \\ n(n-1)(F(z)-F(y))^{n-2}\cdot F'(z)\cdot F'(y), & y \le z \end{cases}$$

$$M(\tilde{\theta}_{min};\tilde{\theta}_{max}) = \int_{-\infty}^{+\infty} yz\cdot\varkappa(y,z)\,dy\,dz =$$

$$= 6\int_0^{+\infty}dz\int_0^{z} yz(e^{-\frac{y}{\theta}} - e^{-\frac{z}{\theta}})\,e^{-\frac{z+y}{\theta}}\frac{1}{\theta^2}\,dy =$$

$$= \frac{6}{\theta^2}\int_0^{+\infty}dz\int_0^{z} yz(e^{-\frac{z+2y}{\theta}} - e^{-\frac{2z+y}{\theta}})\,dy =$$

$$= \frac{6}{\theta^2}\int_0^{+\infty} z \cdot \left\{ e^{-\frac{z}{\theta}} \cdot \frac{\theta^2}{2}\left(\frac{1}{2} - \frac{1}{2}e^{-\frac{2z}{\theta}} - \frac{\theta}{2}e^{-\frac{2z}{\theta}} \cdot \frac{2z}{\theta^2}\right) - e^{-\frac{2z}{\theta}} \cdot \theta^2 \left(1 - e^{-\frac{z}{\theta}} - \theta e^{-\frac{z}{\theta}} \cdot \frac{z}{\theta^2}\right)\right\} dz = 6\int_0^{+\infty} z\left(\frac{1}{4}e^{-\frac{z}{\theta}} + \frac{3}{4}e^{-\frac{3z}{\theta}} + \frac{z}{2\theta}e^{-\frac{3z}{\theta}} - e^{-\frac{2z}{\theta}}\right) dz =$$

$$= 6\left(\frac{\theta^2}{12} + \frac{\theta^2}{27}\right) = \frac{13}{18}\theta^2$$

$$\operatorname{cov}(\tilde{\theta}_{min}; \tilde{\theta}_{max}) = \frac{13}{18}\theta^2 - \frac{11}{18}\theta^2 = \frac{1}{9}\theta^2$$

$$D[\tilde{\theta}_2] = \frac{1}{4} \cdot \frac{\theta^2}{9} + \frac{1}{4} \cdot \frac{49}{36}\theta^2 + \frac{1}{2} \cdot \frac{1}{9}\theta^2 = \frac{61}{4 \cdot 36}\theta^2$$

$$D[\tilde{\theta}_2'] = \left(\frac{12}{13}\right)^2 \cdot \frac{61}{4 \cdot 36}\theta^2 = \frac{61}{169}\theta^2$$

$$g'(\tilde{\theta}_2') = 2$$

$$\frac{61}{169}\theta^2 \geqslant \frac{2}{3 \cdot \frac{1}{\theta^2}} \Rightarrow \frac{61}{169}\theta^2 \geqslant \frac{2}{3}\theta^2 \Rightarrow$$ не эфф. по Крамеру – Рао

$\underline{\tilde{\theta}_3 = X_{(2)}}$ $(n=3,\ k=2)$

$$\Psi(y) = n \cdot C_{n-1}^{k-1}(F(y))^{k-1}(1-F(y))^{n-k} \cdot p(y) = 3 \cdot 2 \cdot F(y) \cdot (1-F(y)) \cdot p(y)$$

a) $$M[\tilde{\theta}_3] = 6\int_0^{+\infty} y\left(1 - e^{-\frac{y}{\theta}}\right)e^{-\frac{y}{\theta}} \cdot \frac{1}{\theta}e^{-\frac{y}{\theta}} dy = \frac{6}{\theta}\int_0^{+\infty} ye^{-\frac{2y}{\theta}} - ye^{-\frac{3y}{\theta}} dy =$$

$$= \frac{6}{\theta}\left(\frac{\theta^2}{4} - \frac{\theta^2}{9}\right) = \frac{5}{6}\theta$$ – смещ. $\Rightarrow \tilde{\theta}_3' = \frac{6}{5}\theta$ – несмещ.

b) $$M[\tilde{\theta}_3^2] = \frac{6}{\theta}\int_0^{+\infty} y^2 \cdot e^{-\frac{2y}{\theta}} - y^2 \cdot e^{-\frac{3y}{\theta}} dy = \frac{6}{\theta}\left(\frac{2\theta^3}{8} - \frac{2\theta^3}{27}\right) = \frac{14}{18}\theta^2$$

$$D[\tilde{\theta}_3] = \left(\frac{14}{18}\theta^2 - \frac{25}{36}\theta^2\right)\frac{36}{25} = \frac{13}{25}\theta^2$$

$$\frac{13}{25}\theta^2 \geqslant \frac{\theta^2}{3} \Rightarrow$$ эфф. по Крамеру – Рао