вероятности отвергнуть H_0

T10. $H_0: p_0(x) = \begin{cases} 1, & x \in (0;1) \\ 0, & x \notin (0;1) \end{cases}$

$H_1: p_1(x) = \begin{cases} \dfrac{e^{1-x}}{e-1}, & x \in (0;1) \\ 0, & x \notin (0;1) \end{cases}$

а) $n = 1$, α

$$l = \frac{p_1}{p_0} = \frac{e^{1-x}}{e-1} \geq c \Rightarrow e^{-x} \geq \tilde{c} \Rightarrow G: x \leq B$$

$$P(x \leq B \mid H_0) = \alpha = \int_0^B 1\, dx \Rightarrow B = \alpha$$

$G: x \leq \alpha$ – критическая обл-ть

$\alpha_1 = P(H_1 \mid H_0) = \alpha$ – ошибка I-го рода

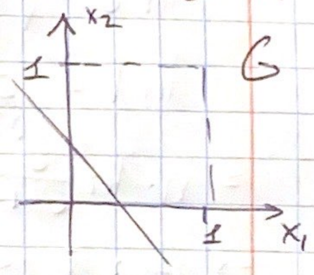
$$W = P(x \leq \alpha \mid H_1) = \int_0^\alpha \frac{e}{e-1} e^{-x}\, dx = \frac{e}{e-1} e^{-x} \Big|_\alpha^0 =$$

$= \frac{e}{e-1}(1-e^{-\alpha})$ – мощность критерия

$\alpha_2 = 1 - W = 1 - \frac{e}{e-1}(1-e^{-\alpha})$ – ошибка IIго рода

в) $n=2, \alpha$

$$l = \frac{L_1}{L_0} = \frac{e^{1-x_1} \cdot e^{1-x_2}}{(e-1)^2} \geq c \Rightarrow e^{-x_1-x_2} \geq B \Rightarrow G: x_1+x_2 \leq A$$



$$P(x_1+x_2 \leq A \mid H_0) = \iint\limits_G 1\cdot 1 \, dx_1 dx_2 = \alpha \Rightarrow$$

$$\Rightarrow \frac{A^2}{2} = \alpha \Rightarrow A = \sqrt{2\alpha}$$

$G: x_1+x_2 \leq \sqrt{2\alpha}$ – критическая обл-ть

$\alpha_1 = P(H_1|H_0) = \alpha$ – ошибка Iго рода

$$W = P(x_1+x_2 \leq \sqrt{2\alpha} \mid H_1) = \int\limits_0^{\sqrt{2\alpha}} dx_1 \int\limits_0^{\sqrt{2\alpha}-x_1} \left(\frac{e}{e-1}\right)^2 \cdot e^{-x_1-x_2} dx_2 =$$

$= \left(\frac{e}{e-1}\right)^2 \left(1 - e^{-\sqrt{2\alpha}} - \sqrt{2\alpha}\, e^{-\sqrt{2\alpha}}\right)$ – мощность критерия

$\alpha_2 = 1 - W$ – ошибка IIго рода

е) n, α (асимптотический критерий)

$$l = \frac{L_1}{L_0} = \prod_{i=1}^{n} p_1(x) \geq c$$

$$\ln l = \sum_{i=1}^{n} \underbrace{p_1(x_i)}_{\eta_i}$$

ЦПТ: $\dfrac{\sum\limits_{i=1}^{n} \eta_i - n \cdot M[\eta_i]}{\sqrt{n D[\eta_i]}} \to N(0;1)$

при справедливости H_0: $M[\eta_i] = M[\ln \frac{e}{e-1} e^{-x_i}] =$

$= M[\ln \frac{e}{e-1} - x_i] = \ln \frac{e}{e-1} - \frac{1}{2}$

$D[\eta_i] = D[\ln \frac{e}{e-1} - x_i] = D[x_i] = \frac{1}{12}$

$\alpha = P(l \geq c \mid H_0) = P(\ln l \geq \ln c \mid H_0) =$

$$= P\Bigg(\underbrace{\frac{\ln l - n(\ln\frac{e}{e-1} - \frac{1}{2})}{\sqrt{n/12}}}_{\to N(0,1)} \geq \underbrace{\frac{\ln c - n(\ln\frac{e}{e-1} - \frac{1}{2})}{\sqrt{n/12}}}_{A \approx u_{1-\alpha}}\Bigg) = \alpha$$

$$\ln c = \sqrt{\frac{n}{12}} \cdot u_{1-\alpha} + n \cdot \ln\frac{e}{e-1} - \frac{n}{2}$$

$$\ln l = \sum_{i=1}^{n} \ln \frac{e}{e-1} \cdot e^{-x_i} = \sum_{i=1}^{n} \left(\ln\frac{e}{e-1} - x_i\right) =$$

$$= n \cdot \ln\frac{e}{e-1} - n\bar{x} \geq \sqrt{\frac{n}{12}}\, u_{1-\alpha} + n \cdot \ln\frac{e}{e-1} - \frac{n}{2} \Rightarrow$$

$\Rightarrow G: \bar{x} \leq \frac{1}{2} - \frac{u_{1-\alpha}}{\sqrt{12n}}$ – критическая обл-ть

$\alpha_1 = \alpha$ – ошибка I-го рода

найдём мощность критерия:

$$W = P\left(\bar{x} \leq \frac{1}{2} - \frac{u_{1-\alpha}}{\sqrt{12n}} \,\Big|\, H_1\right)$$

вспомним т. Фишера: $\frac{\bar{x} - M\xi}{\sqrt{D\xi}}\sqrt{n} \to N(0;1)$

при справедливости H_1: $M\xi = \int_0^1 \frac{e}{e-1} e^{-x} \cdot x\, dx = \frac{e-2}{e-1}$

$$M\xi^2 = \int_0^1 \frac{e}{e-1} x^2 e^{-x} dx = \frac{2e-5}{e-1}$$

$$D\xi = \frac{e^2 - 3e + 1}{(e-1)^2}$$

$$W = P\Bigg(\underbrace{\frac{\bar{x} - \alpha_1}{\sqrt{D\xi}} \cdot \sqrt{n}}_{\to N(0;1)} \leq \underbrace{\frac{\left(\frac{1}{2} - \frac{u_{1-\alpha}}{\sqrt{12n}}\right) - \alpha_1}{\sqrt{D\xi}} \cdot \sqrt{n}}_{B}\Bigg) =$$

$$= \int_{-\infty}^{B} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx$$

$\alpha_2 = 1 - W$ – ошибка II-го рода

d) $n,\ \alpha,\ G: x_{min} < c$

$$P(x_{min} < c \mid H_0) = \alpha$$

$$F_{min}(x) = 1 - (1 - F(x))^n \Rightarrow \alpha = 1 - (1 - F_0(c))^n$$

$$F_0(c) = 1 - \sqrt[n]{1-\alpha}$$

$$c = 1 - \sqrt[n]{1-\alpha}$$

если $x_{min} < (1 - \sqrt[n]{1-\alpha}) \Rightarrow$ отвергаем H_0

$$W = P(x_{min} < c \mid H_1) = F_{min}(c) = 1 - (1 - F_1(c))^n =$$

$$= 1 - \left(1 - \frac{e}{e-1}\left(1 - e^{\sqrt[n]{1-\alpha} - 1}\right)\right)^n$$

$\alpha_1 = \alpha$, $\alpha_2 = 1 - W$

T11

грани	1	2	3	4
случаи	6	6	4	8

$n = 24$

$$H_0: \xi \sim \frac{1}{4}\delta(x-1) + \frac{1}{4}\delta(x-2) + \frac{1}{6}\delta(x-3) + \frac{1}{3}\delta(x-4) = p_0(x)$$

$$H_1: \xi \sim \frac{1}{4}\delta(x-1) + \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x-3) + \frac{1}{4}\delta(x-4) = p_1(x)$$

$n = 2,\ \alpha = 0{,}2$

$$l = \frac{L_1}{L_0} = \frac{p_1(x_1)\cdot p_1(x_2)}{p_0(x_1)\cdot p_0(x_2)}$$

	1	2	3	4
1	1	1	3/2	3/4
2	1	1	3/2	3/4
3	3/2	3/2	9/4	9/8
4	3/4	3/4	9/8	9/16

при справедливости H_0:

	1	2	3	4
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{24}$	$\frac{1}{12}$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{24}$	$\frac{1}{12}$
3	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{36}$	$\frac{1}{18}$
4	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{9}$

выберем обл-ть $G: l \geq \frac{3}{2}$

т.к. $P(l \geq c \mid H_0) = \alpha = 0{,}2$,

а при $P(l \geq 1{,}5 \mid H_0) = 0{,}196$

$$\alpha_1 = \sup P(x_i \in G \mid H_0) = \frac{1}{36} + \frac{4}{24} = \frac{7}{36}$$

при справедливости H_1:

	1	2	3	4
1	1/16	..	1/16	..
2	.		1/16	..
3	1/16	1/16	1/16	.
4	..	..	..	..

$$W = \sup P(X_i \in G \mid H_1) = \frac{5}{16}$$

$$\alpha_2 = 1 - W = \frac{11}{16}$$

№15. $p(x) = \begin{cases} \frac{\theta - 1}{x^\theta}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1$

а) X_n - выборка

$$\text{ОМП}: L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = \frac{(\theta-1)^n}{\left(\prod_{i=1}^{n} x_i\right)^\theta}$$

$$\ln L(\theta) = n \cdot \ln(\theta - 1) - \theta \cdot \ln\left(\prod_{i=1}^{n} x_i\right) = n \cdot \ln(\theta - 1) - \theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta - 1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \tilde{\theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$$

$$\frac{\partial^2 \ln L(\theta)}{\partial \theta^2} = \frac{-n}{(\theta-1)^2} < 0 \Rightarrow \max$$

б) $\int_1^{\sim} \frac{(\theta - 1)}{x^\theta} dx = \frac{1}{2}$ - медиана

$$\int_1^x \frac{\theta - 1}{x^\theta} dx = 1 - x^{1-\theta} = \frac{1}{2} \Rightarrow x^{1-\theta} = \frac{1}{2} \Rightarrow x = 2^{\frac{1}{\theta - 1}}$$

$$\frac{f(\tilde{\theta}) - f(\theta)}{\sigma} \sqrt{n} \rightsquigarrow N(0;1) \qquad f(\theta) = 2^{\frac{1}{\theta-1}}$$

$$I(\theta) = -M\left[\frac{\partial^2 \ln p}{\partial \theta^2}\right]$$

$$\ln p = \ln(\theta - 1) - \theta \cdot \ln x$$

$$\frac{\partial \ln p}{\partial \theta} = \frac{1}{\theta - 1} - \ln x \Rightarrow \frac{\partial^2 \ln p}{\partial \theta^2} = \frac{-1}{(\theta-1)^2} \Rightarrow$$

$$\Rightarrow I(\theta) = \frac{1}{(\theta-1)^2} \Rightarrow \frac{2^{\frac{1}{\tilde{\theta}-1}} - 2^{\frac{1}{\theta-1}}}{\ln 2 \cdot 2^{1/\tilde{\theta}-1}} (\tilde{\theta} - 1)\sqrt{n} \rightsquigarrow N(0;1)$$

$\gamma = 0,95 \Rightarrow -1,96 < \dfrac{2^{1/\tilde{\theta}-1} - med}{\ln 2 \cdot 2^{1/\tilde{\theta}-1}} (\tilde{\theta}-1)\sqrt{n} < 1,96$

$$-2^{\frac{1}{\tilde{\theta}-1}} - \frac{1,96 \cdot \ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\sqrt{n}(\tilde{\theta}-1)} < -med < \frac{1,96 \cdot \ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\sqrt{n}(\tilde{\theta}-1)} - 2^{\frac{1}{\tilde{\theta}-1}}$$

$$\Rightarrow med \in \left(2^{\frac{1}{\tilde{\theta}-1}} - \frac{1,96 \cdot \ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\sqrt{n}(\tilde{\theta}-1)} ; 2^{\frac{1}{\tilde{\theta}-1}} + \frac{1,96 \cdot \ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\sqrt{n}(\tilde{\theta}-1)}\right)$$

c) априорная пл-ть распр. параметра

$$p(y) = \begin{cases} e^{1-y}, & y \geq 1 \\ 0, & y < 1 \end{cases}$$

$$p(\theta, \vec{x}_n) = \frac{P(\vec{x}_n | \theta) \cdot P(\theta)}{P(\vec{x}_n)} = \begin{cases} \dfrac{e^{1-\theta} \cdot \prod p(x_i; \theta)}{p(\vec{x}_n)} \cdot C, & \theta \geq 1 \\ 0, & \theta < 1 \end{cases}$$

$$p(\theta, x_n) = C \cdot L(\theta) \cdot P(\theta)$$

$$\ln p(\theta, x_n) = \ln C + \ln L + \ln P = \ln C + n \cdot \ln(\theta-1) - \theta \sum_{i=1}^{n} \ln x_i + (1-\theta)$$

$$\frac{\partial \ln p}{\partial \theta} = \frac{n}{\theta-1} - 1 - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \tilde{\theta} = \frac{n}{\sum \ln x_i + 1} + 1$$

$$\int_1^{f_1} p(\theta, \vec{x}_n) d\theta = 0,025$$

$$\int_{f_2}^{+\infty} p(\theta, \vec{x}_n) d\theta = 0,025$$

$\Rightarrow I = (f_1; f_2)$ – доверит интервал (по Баесу)

d) $I(\theta) = \dfrac{1}{(\theta-1)^2} \Rightarrow \dfrac{\tilde{\theta}-\theta}{(\tilde{\theta}-1)}\sqrt{n} \sim N(0,1)$, $\tilde{\theta} = \dfrac{n}{\sum \ln x_i} + 1$

$$-1,96 < \frac{\tilde{\theta}-\theta}{\tilde{\theta}-1}\sqrt{n} < 1,96$$

$$\tilde{\theta} - \frac{1,96(\tilde{\theta}-1)}{\sqrt{n}} < \theta < \tilde{\theta} + \frac{1,96(\tilde{\theta}-1)}{\sqrt{n}}$$