Probability and Statstics

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Three things cannot be long hidden: the sun, the moon, and the truth

Buddha

Textbooks

- **Probability & Statistics for Engineers & Scientists**, Ninth Edition, Ronald E. Walpole, Raymond H. Myer
- **Elementary Statistics: Picturing the World,** 6th Edition, Ron Larson and Betsy Farber
- **Elementary Statistics**, 13th Edition, Mario F. Triola

Reference books

- **Probability Demystified**, Allan G. Bluman
- **Schaum's Outline of Probability and Statistics**
- **MATLAB Primer**, Seventh Edition
- **MATLAB Demystified** by McMahon, David

References

Readings for these lecture notes:

- Probability & Statistics for Engineers & Scientists, Ninth edition, Ronald E. Walpole, Raymond H. Myer

- Probability Demystified, Allan G. Bluman

- An Introduction with Statistical Applications, Second Edition, John J. Kinney

These notes contain material from the above three books.

Conditional Probability

A **conditional probability** is the probability of an event occurring, given that another event has already occurred. The conditional probability of event *B* occurring, given that event *A* has occurred, is denoted by **P(B|A)** and is read as "probability of *B*, given *A*."

Example: Two cards are selected in sequence from a standard deck of 52 playing cards. Find the probability that the second card is a queen, given that the first card is a king. (Assume that the king is not replaced.)

Solution

Because the first card is a king and is not replaced, the remaining deck has 51 cards, 4 of which are queens. So,

P(B|A) = $\frac{4}{51}$ = 0.078.

Independent and Dependent Events [1]

Two events A and B are **independent** if and only if **$P(B|A) = P(B)$ or $P(A|B) = P(A)$,** assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

OR

Two events, A and B, are said to be **independent** if the fact that **event A** occurs does not affect the probability that **event B** occurs.

Independent and Dependent Events [2]

Two events are **independent** when the occurrence of one of the events does not affect the probability of the occurrence of the other event. Two events *A* and *B* are independent when

$P(B|A) = P(B)$ or $P(A|B) = P(A)$

Events that are **not independent** are **dependent.**

Independent and Dependent Events [3]

To determine whether *A* and *B* are independent, first calculate **P(B)** , the probability of ***event B***. Then calculate **P(B|A)**, the probability of ***B, given A***. If the values **are equal**, then the events are **independent**. If ***P*(B) $\neq$ P(B|A),** then *A* and *B* are dependent events.

Independent and Dependent Events [4]

Example 1: If a coin is tossed and then a die is rolled, the **outcome of the coin in no way affects** or changes the probability of the outcome of the die.

Example 2: Selecting a card from a deck, replacing it, and then selecting a second card from a deck. The outcome of the first card, as long as it is **replaced**, has no effect on the probability of the outcome of the second card.

Classifying Events as Independent or Dependent

Example Determine whether the events are independent or dependent.

1. Selecting a king (*A*) from a standard deck of 52 playing cards, not replacing it, and then selecting a queen (*B*) from the deck

2. Tossing a coin and getting a head (*A*), and then rolling a six-sided die and obtaining a 6 (*B*)

3. Driving over 85 miles per hour (*A*), and then getting in a car accident (*B*)

Solution

1. **$P(B|A) = \frac{4}{51}$** and ***$P(B) = \frac{4}{51}$***. The occurrence of *A* changes the probability of the occurrence of B, so the events are dependent.

2. **$P(B|A) = \frac{1}{6}$** and **$P(B) = \frac{1}{6}$**. The occurrence of *A* does not change the probability of the occurrence of *B*, so the events are independent.

3. Driving over 85 miles per hour increases the chances of getting in an accident, so these events are dependent.

Independent and Dependent Events [4]

Two events A and B are **independent** if and only if **$P(B|A) = P(B)$** or **$P(A|B) = P(A)$,** assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**

OR

When the occurrence of the first event in some way changes the probability of the occurrence of the second event, the two events are said to be **dependent**.

Independent and Dependent Events [5]

Example 1: Suppose a card is selected from a deck and not replaced, and a second card is selected. In this case, the probability of 1 selecting any specific card on the first draw is **52**, but since this card is not replaced, the probability of selecting any other specific card on the second 1 draw is **51**, since there are only 51 cards left.

Example 2: Drawing a ball from an urn, not replacing it, and then drawing a second ball.

First Multiplication Rule [1]

- Before explaining the first multiplication rule, consider the Example of tossing two coins. The sample space is **HH, HT, TH, TT**. From classical probability theory, it can be determined that the probability of getting two heads is $\mathbf{1/4}$.

- However, there is another way to determine the probability of getting two heads. In this case, the probability of getting a head on the first toss is $\mathbf{1/2}$, and the probability of getting a head on the second toss is also $\mathbf{1/2}$.

First Multiplication Rule [2]

- So the probability of getting two heads can be determined by multiplying $^{1}/_{2} \times {}^{1}/_{2} = {}^{1}/_{4}$.

Multiplication Rule I [1]

Multiplication Rule I: For two **independent** events A and B,

$P(A \text{ and } B) = P(A) \times P(B).$

In other words, when two independent events occur in sequence, the probability that both events will occur can be found by multiplying the probabilities of each individual event.

The word **“and”** is the key word and means that both events occur in sequence and to multiply.

Multiplication Rule I [2]

Example: A coin is tossed and a die is rolled. Find the probability of getting a **tail on the coin** and a **5 on the die**.

Solution:

Let A be the event of getting a tail on the coin

P(A) = $^{1}/_{2}$ = **0.5 (or 50%)**

Let B be the event of getting a 5 on the die

P(B) = $^{1}/_{6}$ = **0.1667 (or 16.67 %)**

Since A and B are **independent** events, so

P(A and B) = P(A) × P(B)

= $^{1}/_{2} \times {}^{1}/_{6} = {}^{1}/_{12}$

= **0.0833 (or 8.33 %)**

Multiplication Rule 1 [3]

The previous Example can also be solved using **classical probability**. Recall that the sample space for tossing a coin and rolling a die is

S ={H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

n(S) = 12

Let **A** be the event of getting a **"T5"**

A = {T5}

n(A) = 1

P(A) = $^{1}/_{12}$

= 0.0833 (or 8.33 %)

Multiplication Rule 1 [4]

Example: An urn contains **2 red balls**, **3 green balls**, and **5 blue balls**. A ball is selected at random and its color is noted. Then it is **replaced** and **another ball** is selected and its color is noted. Find the probability of each of these:

a. Selecting **2 blue balls**

b. Selecting a **blue ball** and then a **red ball**

c. Selecting a **green ball** and then a **blue ball**

Solution

Let **R** be an event of getting a red ball

Let **G** be an event of getting a green ball

Let **B** be an event of getting a blue ball

P(**R**) = $^{2}/_{10}$, P(**G**) = $^{3}/_{10}$, P(**B**) = $^{5}/_{10}$

Since events are independent, so

a. P(**B** and **B**) = P(**BB**) = P(**B**) x P(**B**) = $^{5}/_{10}$ x $^{5}/_{10}$ = $^{1}/_{4}$

= **0.25 (or 25%)**

b. P(**B** and **R**) = P(**B**) x P(**R**) = $^{5}/_{10}$ x $^{2}/_{10}$ = $^{1}/_{10}$

= **0.10 (or 10 %)**

c. P(**G** and **B**) = P(**G**) x P(**B**) = $^{3}/_{10}$ x $^{5}/_{10}$ = $^{3}/_{20}$

= **0.15 (or 15 %)**

Multiplication Rule 1 [5]

Example: A die is tossed **3 times**. Find the probability of getting **three 6s**.

Solution

Let **A** be the event of getting a **'6'**

P(A) = $^{1}/_{6}$

Since events are independent, so

P(**A** and **A** and **A**) = **P(A) x P(A) x P(A)**

$= {}^{1}/_{6} \times {}^{1}/_{6} \times {}^{1}/_{6}$

$= {}^{1}/_{216}$ **(= 0.0046 or 0.4600 %)**

OR

P(AAA) $= {}^{1}/_{6} \times {}^{1}/_{6} \times {}^{1}/_{6}$

$= {}^{1}/_{216}$

= 0.0046 **(or 0.4600 %)**

Multiplication Rule 1 [6]

Example: A box of transistors has **four good transistors** mixed up with **two bad transistors**. A production worker, in order to sample the product, chooses two transistors at random, the first chosen transistor being **replaced** before the second transistor is chosen. What is the probability that **both transistors** are good?

Solution

If the events are

A : the first transistor chosen is good

B : the second transistor chosen is good

$P(A) = \frac{4}{6}$

$P(B) = \frac{4}{6}$

$$P(A \cap B) = P(A) \times P(B)$$
$$= (\frac{4}{6})((\frac{4}{6})$$
$$= \frac{4}{9}$$

= 0.4444 (or 44.44%)

Multiplication Rule II [1]

- When two sequential events are **dependent**, a slight variation of the multiplication rule is used to find the probability of both events occurring.

- For Example, when a card is selected from an ordinary deck of **52 cards** the probability of getting a specific card is $^{\mathbf{1}}/_{\mathbf{52}}$, but the probability of getting a specific card on the second draw is $^{\mathbf{1}}/_{\mathbf{51}}$ since 51 cards remain.

Example: Two cards are selected from a deck and the first card is **not replaced**. Find the probability of getting **two kings**.

Solution

P(two kings) = $^{4}/_{52}$ x $^{3}/_{51}$

= $^{12}/_{2652}$

= $^{1}/_{221}$

= **0.0045 (or 0.45 %)**

Multiplication Rule II [2]

- When the two events A and B are **dependent, the probability that the second event B occurs after the first event A has already occurred is written as P(B|A).**

- This does not mean that B is divided by A; rather, it means and is read as **"the probability that event B occurs given that event A has already occurred."**

- **P(B|A)** also means the **conditional probability** that event B occurs given event A has occurred.

Multiplication Rule II [3]

- The probability of an event ***B*** occurring when it is known that some event ***A*** has occurred is called a **conditional probability** and is denoted by ***P*(*B*|*A*).**

- The symbol ***P*(*B*|*A*)** is usually read **"the probability that *B* occurs given that *A* occurs"**

OR

- simply "the probability of *B*, given *A*."

Multiplication Rule II [4]:

When two events are **dependent**, the probability of both events occurring is **P(A and B)= P(A) x P(B|A)**

Example: A box contains **24 toasters**, **3** of which are **defective**. If **two toasters** are selected and tested, find the probability that **both are defective** (assume toasters are not replaced).

Solution

Let **D_1** be the event that **first toaster** is **defective**.

Let **D_2** be the event that **second toaster** is **defective**.

$P(D_1 \text{ and } D_2) = P(D_1) \times P(D_2|D_1)$

$= {}^{3}/_{24} \times {}^{2}/_{23}$

$= {}^{1}/_{8} \times {}^{2}/_{23}$

$= {}^{1}/_{92}$

= **0.0109 (or 1.0870 %)**

Multiplication Rule II [5]:

When two events are **dependent**, the probability of both events occurring is **P(A and B) = P(A) x P(B|A)**

Multiplication Rule II [6]:

Example: Two cards are drawn **without replacement** from a deck of 52 cards. Find the probability that **both are queens**.

Solution

Let **Q_1** be the event that the **first card is a queen.**

Let **Q_2** be the event that the **second card is a queen**.

$P(Q_1 \text{ and } Q_2) = P(Q_1) \times P(Q_2 | Q_1)$

$$= {}^{4}/_{52} \times {}^{3}/_{51}$$

$$= {}^{1}/_{221}$$

= **0.0045 (0.4525%)**

Multiplication Rule II [7]:

Example: A box contains **3 orange balls**, **3 yellow** balls, and **2 white balls**. **Three balls** are selected **without replacement**. Find the probability of selecting **2 yellow balls** and a **white ball**.

Solution

Orange balls	Yellow	White balls	Total balls
3	3	2	8

Let Y_1 be the event that the **first ball is yellow**.
Let Y_2 be the event that the **second ball is yellow.**
Let W_3 be the event that the **third ball is white**.

$$P(Y_1 \text{ and } Y_2 \text{ and } W_3) \text{ or } P(Y_1Y_2W_3) = {}^{3}/_{8} \times {}^{2}/_{7} \times {}^{2}/_{6}$$

$$= {}^{12}/_{336}$$

$$= \mathbf{0.0357(or 3.5714\ \%)}$$

Note: **The key word for the multiplication rule is and. It means to multiply.**

Multiplication Rule II [1]:

Example: A box contains 3 orange balls, 3 yellow balls, and 2 white balls. Three balls are selected **without replacement**. Find the probability of selecting **a white ball** and **2 yellow balls**.

Solution

Orange balls	Yellow balls	White balls	Total balls
3	3	2	8

Let **W_1** be the event that the **first ball is white**.
Let **Y_2** be the event that the second **ball is yellow**.
Let **Y_3** be the event that the **third ball is yellow**.

P(W_1 and Y_2 and Y_3) or $P(W_1Y_2Y_3) = {}^{2}/_{8} \times {}^{3}/_{7} \times {}^{2}/_{6}$

$= {}^{12}/_{336}$

= **0.0357 (or3.5714 %)**

Note: The key word for the multiplication rule is **and**. It means to multiply.

Conditional Probability [1]

- Previously, conditional probability was used to find the probability of sequential events occurring when they were **dependent.**

- Recall that **P(B|A)** means the probability of **event B** occurring given that **event A** has already occurred.

- Another situation where **conditional probability** can be used is when **additional information** about an event is known.

- Sometimes it might be known that **some outcomes** in the sample space have **occurred** or that some **outcomes cannot occur.**

Conditional Probability [2]

When conditions are imposed or known on events, there is a possibility that the probability of the certain **event occurring may change.**

Example: A die is rolled; find the probability of getting **a 4 if it is** known that an **even number** occurred when the die was rolled.

Alternative Approach: Conditional Probability [1]

Solution:

If it is known that an even number has occurred, the sample space is

Reduced sample space = {2, 4, 6}

n(S’) = 3

Let **A** be the event of getting a **‘4’**

A = {4}

n(A) = 1

P(A) = $^{1}/_{3}$ = 0.3333 (33.33%)

Sample space of two dice using table

A table can be used for the sample space when two dice are rolled.

	Die 2					
Die 1	1	2	3	4	5	6
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Alternative Approach: Conditional Probability [2]

Example: Two dice are rolled. Find the probability of getting a **sum of 3** **if it** is known that the sum of the spots on the dice was **less than six**.

Solution

Reduced sample space ={(1, 1), (1, 2), (2, 1), (3, 1), (2, 2), (1, 3), (1, 4), (2, 3), (3, 2), and (4, 1)}

n(S’) = 10

Let **A** be the event of getting a **‘sum of 3’**

A = {(1, 2), (2, 1)}, n(A) = 2

P(A) = $^{2}/_{10}$ = $^{1}/_{5}$

or

P(sum of 3|sum less than 6) = $^{2}/_{10}$

= $^{1}/_{5}$ = 0.20 (or 20%)

Alternative Approach: Conditional Probability [3]

The two previous Examples of conditional probability were solved using **classical probability and reduced sample spaces**; however, they can be solved by using the following formula for conditional probability.

Alternative Approach: Conditional Probability [4]

The conditional probability of two events A and B is

$P(A|B) = P(A \text{ and } B)/P(B)$

OR

$$= \frac{P(A \text{ and } B)}{P(B)}$$

P(A and B) means the probability of the outcomes that events **A and B have in common.**

Conditional Probability without reducing the sample space [1]

Example: A die is rolled; find the probability of getting **a 4**, if it is known that an **even number** occurred when the die was rolled.

Solution

S = {1, 2, 3, 4, 5, 6}

Let events are defined as:

A: Getting a **4** on a die

B: An **even number** occur on a die

$\because$ P(A|B) = P(A and B)/P(B)

P(A and B) = $\mathbf{{}^{1}/_{6}}$

P(B) = $\mathbf{{}^{3}/_{6}}$

$$P(A|B) = \frac{\mathbf{P(A\ and\ B)}}{\mathbf{P(B)}} = {}^{1}/_{6} \times {}^{6}/_{3}$$

$$= {}^{1}/_{3} = 0.3333 (\text{or } 33.33\%)$$

Sample space of two dice using table

A table can be used for the sample space when two dice are rolled.

	Die 2					
Die 1	1	2	3	4	5	6
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Conditional Probability without reducing the sample space [2]

Example: Two dice are rolled. Find the probability of getting a sum of 3 if it is known that the sum of the spots on the dice **was less than 6**.

Solution [1]:

$\because$ P(A|B) = P(A ∩ B)/P(B)

Let events are defined as:

A ∩ B: Getting a sum 3 **and** sum of the spots on the dice was less than 6

A: Getting sum of the spots on the dice was 3

B: Getting sum of the spots on the dice was less than 6

A ∩ B = {(2, 1), (1, 2)}

n(A ∩ B) = 2

P(A ∩ B) = $^{2}/_{36} = {}^{1}/_{18}$

= **0.0555 (or 5.55 %)**

Solution [2]:

Let **B** be the event of getting sum of the spots on the dice was **less than 6**

B = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)}

P(B) = $^{10}/_{36}$ = $^{5}/_{18}$ = **0.2777 (or 27.78 %)**

P(A|B) = P(A and B)/P(B)

= $^{1}/_{18} \times {}^{18}/_{5}$ = $^{1}/_{5}$ = **0.2 (or 20 %)**

Alternative approach: Conditional Probability with reducing the sample space

Example: Two dice are rolled. Find the probability of getting a **sum of 3** if it is known that the sum of the spots on the dice **was less than 6**.

Solution

If it is known that the sum of the spots on the dice **was less than 6**

Let reduced sample space = S'

$\Rightarrow S'$ = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)}

n(S') = 10

Let **A** be the event of getting a sum of 3

A = {(2, 1), (1, 2)}

P(A) = $\frac{2}{10}$ = $\frac{1}{5}$ = 0.2 (or 20 %)

Alternative approach: Conditional Probability with reducing the sample space

Example: When two dice were rolled, it is known that the sum was an **even number**. In this case, find the probability that the **sum was 8**.

Solution:

Reduced sample space = **S'**

{(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)}

n(S') = 18

Let**A** be the event of getting a sum of **'8'**

A = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}

n(A) = 5

P(A) = $^{5}/_{18}$ = **0.2777 (27. 78%)**

Example: In a large housing plan, **35%** of the **homes** have a deck **and** a **two-car garage**, and **80%** of the houses have a **two-car garage**. Find the probability that a house has a **deck** given that it has a **two-car garage**.

Solution

Let **D** be the event of getting **deck**

Let **G** be the event of getting **two-car garage**

Given

$P(D \cap G) = 0.35$

$P(G) = 0.80$

P(deck|two-car garage) $= {P(D \cap G)}/{P(G)}$

$= {0.35}/{0.80} = {7}/{16}$

$= 0.4375$ **(or 43.75 %)**