Creating a chiral QCD model in FeynRules

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In the following, we will construct a nonlinear chiral Lagrangian which describes the Goldstone bosons of QCD. The theory is based upon an $SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R$ chiral symmetry, under which left- and right-handed components transform differently, which is spontaneously broken to $SU(3)_V \times U(1)_B$. The description of the pseudoscalar mesons $\pi, K,$ and η through Goldstone boson dynamics is possible without the use of perturbative QCD [1]. The small masses attributed to these mesons come about as a result of the violation of chiral symmetry, via a mass term introduced into the Lagrangian.

1 The QCD Lagrangian

The fermionic QCD Lagrangian, written in terms of four-component Dirac fields,

$$\mathcal{L}_{QCD} = \bar{\psi}_i (i\not{D} - m_q)\psi_i, \tag{1}$$

has a chiral symmetry in the massless case. We can consider this Dirac spinor instead as being composed of two Weyl spinors, where Weyl fermions are related to ordinary four component fermions ψ by [2]

$$\psi_L \equiv P_+\psi, \psi_R \equiv P_-\psi, \tag{2}$$

where the projection operators P_+ and P_- are defined as [2]

$$P_\pm \equiv (1 \pm \gamma_5)/2. \tag{3}$$

Left- and right-chiral fermion fields are chirally symmetric under Lorentz transformations, implying that they are irreducible representations. It then follows that [2]

$$\mathcal{L}(\psi) = i\bar{\psi}\not{\partial}\psi = i\bar{\psi}_L\not{\partial}\psi_L + i\bar{\psi}_R\not{\partial}\psi_R, \tag{4}$$

indicating separate symmetry transformations for the different chirality components.

2 Goldstone bosons

Goldstone's theorem states that there shall arise a massless scalar boson whenever there is a spontaneously broken global symmetry, written $G \to H$. This boson is called a Goldstone Boson (GB) or Nambu-Goldstone Boson (NGB). The degenerate minimum of the potential in a spontaneously broken symmetry forms a valley, and the Goldstone bosons parametrise motions along this valley; they span the coset G/H [3]. Their movement is described by the broken generators (of course, the unbroken generators of G will annihilate this vacuum by definition).

We can write a Lagrangian for these GBs by using chiral perturbation theory. When writing an effective Lagrangian like this, the heavier degrees of freedom are integrated out, and usually live at the cutoff, but if they are weakly coupled they may appear below the cutoff [3]. Looking ahead to the Composite Higgs models, where we will go next, the heavier degrees of freedom live far away at high energy scales. In QCD, the cutoff is on the order of a GeV.

Spontaneous symmetry breaking refers only to the axial sector of the charges, where the symmetry group $SU(3)_L \times SU(3)_R$ is spontaneously broken to the the flavour group $SU(3)_V$. The V refers to the vector nature of the corresponding currents. Given that the initial symmetry has sixteen preserved generators, and the broken theory has eight, we expect eight Goldstone bosons. The chiral symmetry will then undergo an additional explicit breaking, in order to allow the eight NGBs to acquire masses [5].

3 Chiral QCD Lagrangians

We will consider the global symmetry group G to be an $SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R$ chiral symmetry. Assume that G is spontaneously broken to $SU(3)_V \times U(1)_B$ by some vacuum expectation value (vev) Σ_0, where $SU(3)_V$ is a symmetry which acts on left and right handed fields identically, and B represents the baryon number conservation symmetry. The breaking of the $U(1)$ axial symmetry is achieved via the anomaly terms discussed in Section 4.3.

We will parametrise the Goldstone bosons in our theory by [3]

$$U = e^{i\pi^a T^a/f}, \tag{5}$$

where f is some decay constant (to be discussed later), a labels the broken generators T^a, and π^a/f defines the angle in the broken direction under which Σ_0 rotates when it is acted upon by this matrix. This rotation is written [3]

$$\Sigma = U[\Sigma_0] = U\Sigma_0 \tag{6}$$

for Σ_0 in the fundamental of G. In fact, $\Sigma_0 = f\mathbb{I}_{3\times3}$ is in the bifundamental of $SU(3)_L \times SU(3)_R$, since the formation of the $\langle\bar{q}q\rangle$ condensate in the QCD vacuum is responsible for the breaking of the chiral symmetry [1]. The rotation for a bifundamental is written [3]

$$\Sigma = U_L \Sigma_0 U_R^\dagger. \tag{7}$$

Under $SU(3)_V$, (where $U_L = U_R$), $\Sigma = \Sigma_0$, annihilating the vacuum. Under the axial transformations, where $U_L = U_R^\dagger$, the Goldstone bosons contained in the matrix exponentiation acquire a vev and we get [3]

$$\Sigma = e^{i\pi^a T^a/f}\Sigma_0 e^{i\pi^a T^a/f} = e^{2i\pi^a T^a/f} \tag{8}$$

The chiral generators are associated with Goldstone bosons; here they are the Standard Model octet of pseudoscalar mesons $\pi^0, \pi^+, \pi^-, K^+, K^-, K^0, \bar{K^0}$, and η. The η' forms the final part of the $SU(3)_L \times SU(3)_R$ nonet. The pions (including the π^0) form an adjoint **3** of $SU(2)$, and the kaon pairs each form a fundamental **2** representation of $SU(2)$. The η and η' are singlets.

The Goldstone bosons transform non-linearly under the broken (axial) symmetry, leading to a shift symmetry [4]

$$\pi^a \to \pi^a + \epsilon^a. \tag{9}$$

This is the reason we are not able to write mass terms for the Goldstone bosons; non-derivative terms are forbidden.

3.1 The η' meson

Quarks belong to the fundamental **3** representation of $SU(N_c)$ (and antiquarks to the $\bar{\mathbf{3}}$), and gluons to the adjoint reprenentation [5]. Mesons composed of these quarks are therefore described by a direct product $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} + \mathbf{1}$.

A rotation in flavour space, where we interchange quark flavours, rotates the particles within the pseudoscalar octet into each other. If $SU(3)$ were a perfect symmetry, η_0 and η_8 would be degenerate. However, we in fact find some mixing between these two states, which yield the physical states η and η'. The singlet state, $\eta_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$, is exempt from this mixing. In fact, since the flavour symmetry is not exact, the physical η and η' mesons which form part of this nonet are a linear combination of η_1 and $\eta_8 = \frac{1}{\sqrt{6}} = (u\bar{u} + d\bar{d} - 2s\bar{s})$, using [6]

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} \eta \\ \eta' \end{pmatrix}. \tag{10}$$

The η' is predominantly the singlet state; this mixing angle is determined by perturbation theory [6]. It turns out that the η' is heavy and is therefore not expected to emerge as a Goldstone boson. This is due to an anomaly, which is described in Section 4.3.

3.2 Non-linear realisation

Consider expressing the pseudoscalar $SU(3)_V$ octet of the Goldstone bosons π_a as a unitary matrix, [7]

$$\Pi = \pi_a(x)T_a = \frac{\sqrt{2}}{2}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}. \tag{11}$$

We write [2]

$$\Sigma = e^{2i\Pi/f}. \tag{12}$$

The (pion) decay constant f is determined by the magnitude of the vev Σ_0 which breaks G to H.

4 Lagrangian

The first non-trivial term of the Lagrangian goes like [3]

$$\frac{f^2}{4}\mathrm{Tr}\left[(\partial_\mu\Sigma)^\dagger\partial^\mu\Sigma\right]. \tag{13}$$

If we expand this we get

$$\frac{f^2}{4}\mathrm{Tr}\left[\partial_\mu(\mathbb{I} + \frac{1}{f}2i\pi^a T^a - \frac{4}{f^2}(\pi^a T^a)^2 - \frac{4i}{f^3}(\pi^a T^a)^3 + ..)^2\right] \tag{14}$$

$$= \frac{f^2}{4}\mathrm{Tr}\left[\frac{-4}{f^2}\partial_\mu\pi^a T^a\partial_\mu\pi^b T^b + \frac{16}{f^4}\partial_\mu\pi^a T^a\partial_\mu\pi^b T^b\pi^a T^a\pi^b T^b - \frac{16}{f^6}\partial_\mu\pi^a T^a\partial_\mu\pi^b T^b(\pi^a T^a)^2(\pi^b T^b)^2..)\right] \tag{15}$$

$$= \frac{f^2}{4}\mathrm{Tr}\left[\frac{-4}{2f^2}\partial_\mu\Pi\partial_\mu\Pi + \frac{8}{f^4}\partial_\mu\Pi\partial_\mu\Pi\Pi\Pi\Pi - \frac{8}{f^6}\partial_\mu\Pi\partial_\mu\Pi(\Pi)^2(\Pi)^2 + ..)\right] \tag{16}$$

$$= \mathrm{Tr}\left[\frac{1}{2}\partial_\mu\Pi\partial_\mu\Pi + \frac{2}{f^2}\partial_\mu\Pi\partial_\mu\Pi\Pi^2 - \frac{2}{f^4}\partial_\mu\Pi\partial_\mu\Pi(\Pi)^4 + ..)\right] \tag{17}$$

where we used the normalisation of generators $\mathrm{Tr}T^aT^b = \frac{1}{2}\delta_{ab}$ [3]. We obtain pion kinetic terms, as well as 4 pion interaction terms and higher order interactions.

4.1 Gauging the theory

We need to add the electromagnetic interactions; noting that electromagnetism breaks $SU(3)_L \times SU(3)_R$, we write the charge operator as [2]

$$Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}. \tag{18}$$

We then modify the QED covariant derivative [4]

$$\bar{\psi}i\not{D}\psi \to \bar{\psi}_L(\partial^\mu - ieA^\mu Q_L)\psi_L + \bar{\psi}_R(\partial^\mu - ieA^\mu Q_R)\psi_R, \tag{19}$$

where in order to maintain $SU(3)_L \times SU(3)_R$ invariance we must allow [4]

$$Q_L \to LQ_L L^\dagger, \quad Q_R \to RQ_R R^\dagger. \tag{20}$$

Note that, while Q_L and Q_R are numerically identical, they work differently as symmetry breaking parameters, acting on left and right handed quarks respectively. We then write our QED covariant derivative as

$$D_\mu\Sigma = \partial_\mu\Sigma + ieQ_L A^\mu\Sigma - ieA^\mu\Sigma Q_R. \tag{21}$$

However, we also need to add the SM W and Z gauge bosons, so the above derivative is not sufficient. Instead we write [3]

$$D_\mu\Sigma = \partial_\mu\Sigma - \frac{igW^a_\mu}{2}\tau^a\Sigma - i\frac{g'}{6}B_\mu. \tag{22}$$

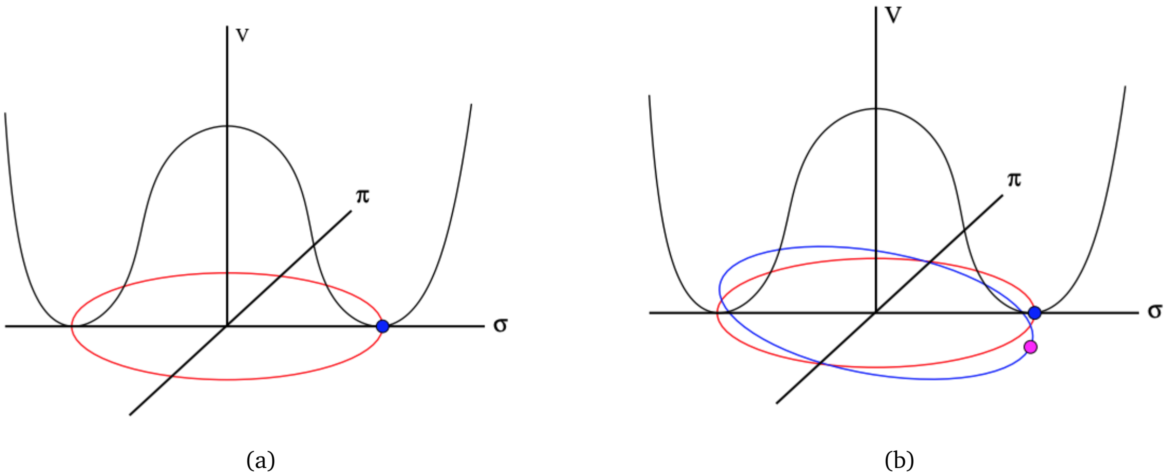

Figure 1: The effective potential [8].

We will then have terms like [3]

$$\frac{f^2}{4}\mathrm{Tr}\left[(D_\mu\Sigma)^\dagger D^\mu\Sigma\right] \supset \frac{g^2f^2}{4}W_\mu^+W^{-\mu}+ \tag{23}$$

which mirror the SM Lagrangian terms, except the vev is replaced by the pion decay constant.

The full covariant derivative is then

$$D_\mu\Sigma = \partial_\mu\Sigma - igW_\mu^a\tau^a\Sigma - i\frac{g'}{6}B_\mu \tag{24}$$

4.2 Symmetry breaking terms (the mass terms!)

Up until now, we had been considering an ideal world where all quarks are massless in order to obtain chiral symmetry (whereas for flavour symmetry we just require the same masses). We shall break the symmetry using the mass matrix [2]

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{25}$$

We can imagine that this matrix M transforms as a bifundamental of $SU(3)_L \times SU(3)_R$, and write down the invariants, after which we freeze the spurion M to its diagonal terms [3]. The leading invariant is must contain $M(\Sigma+\Sigma^\dagger)$, so we find the term [3],

$$\Delta\mathcal{L} = \mu^3\mathrm{Tr}\left[M(\Sigma+\Sigma^\dagger)\right], \tag{26}$$

where μ is a dimensionful constant. Expanding this, we find that the linear terms cancel and we have

$$\Delta\mathcal{L} = \mu^3\mathrm{Tr}\left[M^2(\frac{-4}{f^2}(\Pi_aT_a)^2)\right] \tag{27}$$

In Figure 1, the double well (or Mexican hat) potential is depicted, with two possible minima for the vacuum. Chiral symmetry is broken when the quark condensate obtains a specific value in this minimum state. The GB parametrise directions along the inequivalent minima. As seen in Figure 1(b), adding in small masses for the quarks tilts the potential, selecting one minimum as the true vacuum and yielding a mass for the GB.

4.3 Anomaly term

The η' is a flavour $SU(3)$ singlet, and is far heavier than the other Goldstone bosons; its mass cannot be attributed to the small masses of the quarks. The axial anomaly, which explains this phenomenon, is due to the interaction of a current with external fields, leading to a violation of symmetry of interaction.

In QCD, the external axial, or chiral, anomaly violates the conservation of the axial current of light quarks, $j_{\mu 5}$, upon interaction with external photons or gluons. While the flavour octet axial current is conserved in QCD before the symmetry is broken by the quark masses, the singlet axial current is not conserved, where [9]

$$j_{\mu 5}^{(0)} = \sum_q \bar{\psi}_q \gamma_\mu \gamma_5 \psi_q. \tag{28}$$

The divergence of the current, upon interations with gluons, is then [9]

$$\partial_\mu j_{mu5}^{(0)} = 3\frac{\alpha_s N_c}{4\pi} G_{\mu\nu}^n \tilde{G}_{\mu\nu}^n. \tag{29}$$

The same reasoning holds for the interaction with photons, where [9]

$$\partial_\mu j_{mu5}^{(0)} = \frac{e^2 e_q^2 N_c}{8\pi^2} F_{\alpha\beta}^n \tilde{F}_{\alpha\beta}^n. \tag{30}$$

Since the $U(1)_A$ symmetry is broken upon quantisation, there is in fact no symmetry to be broken by the QCD vacuum and hence no associated Goldstone boson. This non-conservation of the current means that it cannot be assumed that the η and η' mesons acquire masses only through the introduction of the quarks masses [1]. For these reasons, the η' is allowed to be heavy.

If we simply include a term which breaks the symmetry, we can write [10]

$$\mathcal{L}_{anomaly} = \frac{f^2 a}{6}(i \ln \det U)^2, \tag{31}$$

where a is a constant chosen for convenience. Then, using $\det(\exp \Pi)) = \exp(\mathrm{Tr}(\Pi))$,

$$\mathcal{L}_{anomaly} = \frac{f^2 a}{6}(i \ln \exp \mathrm{Tr}(\Pi))^2 = -\frac{f^2 a}{6}(\mathrm{Tr}(\Pi))^2 \tag{32}$$

4.4 The full chiral Lagrangian

So we have that the chiral Lagrangian is written, for $\Sigma = \exp[2i\Pi/f], \quad \Pi = \pi_a(x) T_a$,

$$\mathcal{L} = \frac{f^2}{4}\mathrm{Tr}\left[(D_\mu\Sigma)^\dagger D^\mu \Sigma\right] + \mu^3 \mathrm{Tr}\left[M(\Sigma + \Sigma^\dagger)\right] - \frac{f^2 a}{6}(\mathrm{Tr}(\Pi))^2 \tag{33}$$

5 FeynRules Implementation

In order to compute the powers of the pion matrix Π we implement the `SquareMatrix` function

The program can't see the SU(2) objects within the pion matrix correctly. Some proposed solutions:

1) write the Pion matrix in terms of SU(2) generators
2) act using `del` and then change to `DC` afterwards.

5.1 1. How to write the Pion matrix

First, I wrote it like this

```
S[11] == {
  ClassName        -> Pion,
  Unphysical       -> True,
  SelfConjugate    -> True,
  Indices          -> {Generation, Generation},
```

```
    Definitions       -> {
          Pion[1,1] -> pi[3]/2 + (Cos[Th] eta + Sin[Th] etap)/(2 Sqrt[3]),
          Pion[1,2] -> Sqrt[2]*pi[1],
          Pion[1,3] -> Sqrt[2]/2*Kap,
          Pion[2,1] -> Sqrt[2]/2*pi[2],
          Pion[2,2] -> -pi[3]/2 + (Cos[Th] eta + Sin[Th] etap)/(2 Sqrt[3]),
          Pion[2,3] -> Sqrt[2]*Ka0,
          Pion[3,1] -> Sqrt[2]/2*Kapbar,
          Pion[3,2] -> Sqrt[2]/2*Ka0bar,
          Pion[3,3] ->  - (Cos[Th] eta + Sin[Th] etap)/Sqrt[3] }
```

But this is 1) not written in a general way, and 2) not enabling the covariant derivative to act properly on the SU(2) objects within the matrix. I can only get the covariant derivative to do something more than a regular derivative if I give the entire pion matrix SU2W, SU2W indices.

The second option is to write the Pion matrix in terms of SU(2) generators, so something like

```
Pion[i_,j_] = (..)pi[k] + (..)Ka[l] + (..)Eta[m]
```

5.1.1 Decomposing the pion matrix into SU(2) parts

Each time we want to do this (in the future) we can use `LieArt` to decompose our matrix into SU(2) (hopefully). For now, SU(3) decomposition into SU(2) is straightforward.

The pion matrix is an $SU(3) \otimes SU(3)$ nonet, where

$$(\mathbf{3}, \bar{\mathbf{3}}) = \mathbf{9} = \mathbf{8} \oplus \mathbf{1}. \tag{34}$$

It is decomposed into an $SU(3)$ octet, and an $SU(3)$ singlet. The octet can be decomposed into $SU(2)$ by

$$\mathbf{8} = \mathbf{3} + \mathbf{2} + \mathbf{2} + \mathbf{1}, \tag{35}$$

where the adjoint **3** is the pions, the two fundamental representations are the K^0 and the $K^{\pm}$, and the singlet is the η_8. The $SU(3)$ singlet is the η_0 which is not a Goldstone.

5.2 del -> DC

I have not have great success with this so far, as the DC function seems to be too complex for straighforward Mathematica replace rules. I have tried several things, including

```
/.del->DC

del[u__, Index[Lorentz, mu]] -> DC[u, Index[Lorentz, mu]]

del[u__, Index[Lorentz, mu]] :> DC[u, Index[Lorentz, mu] ],
f_[del][u__, Index[Lorentz, mu] ] :> f[DC][u, Index[Lorentz, mu] ]

 Lkin := Block[{ii,jj,dkin},
   dkin = fp^2/8 ExpandIndices[ del[ U[ii,jj],mu]] ;
   dkin = dkin/.del[u__, Index[Lorentz, mu]] -> DC[u, Index[Lorentz, mu]]
 ]
```

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