Transformada de Laplace

Definição

Seja $f: [0, +\infty[\to \mathbb{R}$, chama-se transformada de Laplace de f à função F definida por

$$F(s) = \int_0^{+\infty} e^{-st} f(t)\, dt$$

nos pontos $s \in \mathbb{R}$ em que este integral impróprio é convergente

ou $\mathcal{L}\{f\}(s)$ ou $\mathcal{L}\{f(t)\}(s)$ ou $\mathcal{L}\{f\}$

Exemplo 1

$f: [0, +\infty[\to \mathbb{R}$, $f(t) = 1$

$$\mathcal{L}\{f\}(s) = \int_0^{+\infty} e^{-st} \underbrace{f(t)}_{1}\, dt = \int_0^{+\infty} e^{-st}\, dt$$

$$\int_0^{+\infty} e^{-st}\, dt = \lim_{b \to +\infty} \int_0^b e^{-st} = \lim_{b \to +\infty} \left(- \frac{e^{-st}}{s} \right) \Big|_0^b =$$

$$= \lim_{b \to +\infty} - \frac{e^{-sb}}{s} + \frac{1}{s} = \lim_{b \to +\infty} \frac{-e^{-st} + 1}{b} =$$

$$= \begin{cases} \frac{1}{s}, & s > 0 \\ +\infty, & s < 0 \end{cases}$$

Para $s > 0$, o integral impróprio é convergente, para $s < 0$ é divergente

$s = 0$

$$\int_0^{+\infty} e^{-0t}\, dt = \int_0^{+\infty} dt = \lim_{b \to +\infty} \int_0^b dt = \lim_{b \to +\infty} t \Big|_0^b$$

$$= \lim_{b \to +\infty} b = +\infty$$

Para $s = 0$, o integral impróprio é divergente

Assim,

$$\mathcal{L}\{1\}(s) = \frac{1}{s}, \text{ para } s > 0$$

Transformada de Laplace Inversa

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$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$\mathcal{L}\{\text{sen}(\overset{a}{2}t)\} = \frac{2(a)}{s^2+2^2}$ (a^2)

$(f*g)(t) =$

$\int_0^t f(\tau)\, g(t-\tau)\, dt$

Exemplo: $\mathcal{L}^{-1}\left\{\frac{2}{(s-1)(s^2+4)}\right\} =$

$= \mathcal{L}^{-1}\left\{\frac{1}{s-1} \cdot \frac{2}{(s^2+4)}\right\} =$

$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} * \mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)}\right\} =$

$= e^{t} * \text{sen}(2t) =$

$= \int_0^t \text{sen}(2\tau)\cdot e^{t-\tau}\, d\tau$

$u = e^{t-\tau}$ $\quad du = e^{(t-\tau)}\cdot(-1)$

$dv = \text{sen}(2\tau)$ $\quad v = -\frac{\cos(2\tau)}{2}$

$= -\frac{e^{(t-\tau)}}{2}\cos(2\tau)\Big|_0^t - \frac{1}{2}\int_0^t \cos(2\tau)\cdot e^{(t-\tau)}\, d\tau$

$= -\frac{e^{(t-\tau)}}{2}\cos(2\tau)\Big|_0^t - \frac{1}{2}\left[e^{(t-\tau)}\cdot\frac{\text{sen}(2\tau)}{2}\Big|_0^t + \int_0^t \frac{1}{2}\,\text{sen}(2\tau)\cdot e^{(t-\tau)}\right.$

$\frac{5}{4}\int_0^t \text{sen}(2\tau)\, e^{(t-\tau)}\, d\tau = -\frac{\cos(2t)}{2} + \frac{e^t}{2} - \frac{1}{2}\cdot\frac{\text{sen}(2t)}{2}$

$\int_0^t \text{sen}(2\tau)\, e^{(t-\tau)}\, d\tau = -\frac{2}{5}\cos(2t) + \frac{2}{5}e^{t} - \frac{1}{5}\text{sen}(2t)$