Exercício 16:

$$y''' + y' = \underbrace{\operatorname{sen} x}_{b(x)}$$

- Solução homogénea associada

$$y''' + y' = 0$$

$$p(r) = r^3 + r = 0 \Leftrightarrow r(r^2+1) = 0$$

$$\underline{r = 0} \quad \vee \quad \underline{r = \pm i}$$

$$\text{SFS: } \{1, \cos(x), \operatorname{sen}(x)\}$$

Solução geral da homogénea

$$\boxed{y_h = C_1 + C_2\cos(x) + C_3 \operatorname{sen} x, \quad C_1, C_2, C_3 \in \mathbb{R}}$$

- Solução particular

$$1.\operatorname{sen} x = P_m(x)\, e^{\alpha x} \operatorname{sen}(\beta x)$$

onde $P_m(x) = 1$ (grau zero) $\rightarrow$ $m = 0$

$\alpha = 0$, $\beta = 1$

Como $\alpha + \beta i = 0 + 1i = i$ é raiz do polinômio característico então $K = 1$

Logo, a solução particular da EDO é da forma

$$y_p = x^1 e^{0x} \left[\underset{\text{grau zero}}{A(x)} \cos(x) + \underset{\text{grau zero}}{B(x)} \operatorname{sen}(x) \right]$$

3

$$y_p = x\,(A\cos(x) + B\,\text{sen}\,x) = \underline{A\,x\cos(x) + Bx\,\text{sen}(x)}$$

A, B constantes a determinar

$$y_p' = \underbrace{-Ax\,\text{sen}\,x}\, + \underbrace{A\cos(x)} + \underbrace{Bx\cos(x)} + \underbrace{B\,\text{sen}(x)}$$

$$y_p'' = -Ax\cos x\, \underbrace{-A\,\text{sen}\,x}\, \underbrace{-A\,\text{sen}\,x}\, - Bx\,\text{sen}(x) + \underline{B\cos x} + \underline{B\cos(x)}$$

$$y_p'' = \underbrace{-2A\,\text{sen}\,x} + 2B\cos x\, \underbrace{-Ax\cos(x)}\, - Bx\,\text{sen}\,x$$

$$y_p''' = \underline{-2A\cos x}\, \underline{-2B\,\text{sen}\,x} + Ax\,\text{sen}\,x\, \underline{-A\cos x} - Bx\cos x\, \underline{-B\,\text{sen}\,x}$$

$$y_p''' = -3A\cos x - 3B\,\text{sen}\,x + Ax\,\text{sen}\,x\, \underline{-Bx\cos x}$$

$$y_p''' + y_p' = \operatorname{sen} x$$

$$\underbrace{-3A\cos x}\ \underline{-3B\operatorname{sen} x}\ \overbrace{\cancel{+Ax\operatorname{sen} x}\ \cancel{-Bx\cos x}}^{y'''}\ \cancel{-Ax\operatorname{sen} x}\ \underline{+A\cos x}$$

$$\underbrace{\cancel{+Bx\cos x}\ \underline{+B\operatorname{sen} x}} = \operatorname{sen} x$$

$$-2A\cos x - 2B\operatorname{sen} x = 1.\operatorname{sen} x$$

$$\boxed{A=0}$$

$$-2B = 1$$

$$\boxed{B = -\frac{1}{2}.}$$

$$\boxed{y_p = -\frac{1}{2}x\operatorname{sen} x}$$

Solução geral da ED

$$y = y_h + y_p$$

$$y = C_1 + C_2\cos(x) + C_3\operatorname{sen} x - \frac{1}{2}x\operatorname{sen} x$$

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5

$\boxed{\text{Exercicio 17}}$ ← TPC.

Transformada de Laplace

Exemplo 1 $f: [0,+\infty[\to \mathbb{R}, \quad f(t) = 1.$

$$\mathcal{L}\{f\}(s) = \int_0^{+\infty} e^{-st} \underbrace{f(t)}_{1} dt = \int_0^{+\infty} e^{-st} dt$$

$$\int_0^{+\infty} e^{-st} dt = \lim_{b\to+\infty} \int_0^b e^{-st} dt = \lim_{b\to+\infty} \left(-\frac{e^{-st}}{s} \right) \Bigg|_0^b$$

$$= \lim_{b\to+\infty} \frac{-e^{-sb}}{s} + \frac{1}{s} = \underline{\lim_{b\to+\infty}} \frac{-e^{-sb}+1}{s} = \begin{cases} \frac{1}{s}, & s>0 \\ +\infty, & s<0 \end{cases}$$

7

Para $s>0$ o integral impróprio é convergente, para $s<0$ é divergente.

$\underline{s=0}$

$$\int_0^{+\infty} e^{-0t}\,dt = \int_0^{+\infty} dt = \lim_{b\to+\infty} \int_0^b dt = \lim_{b\to+\infty} t\Big|_0^b$$

$$= \lim_{b\to+\infty} b = +\infty .$$

Para $s=0$ o integral impróprio é divergente

Assim

$$\mathcal{L}\{1\}(s) = \frac{1}{s}, \text{ para } s>0 \quad //$$

Exemplo 2:

$g:[0,+\infty[\to\mathbb{R}$

$$g(t)=\begin{cases}1, & t\neq 1,\ t\neq 2\\ 2, & t=1\\ 0, & \text{se } t=2\end{cases}$$

$\mathcal{L}\{g\}=\frac{1}{s},\ s>0$?

$$\mathcal{L}\{g(t)\}(s)=\int_0^{+\infty}e^{-st}\cdot g(t)\,dt=\lim_{b\to+\infty}\underbrace{\int_0^b e^{-st}\,g(t)\,dt}$$

• $s=0$

Como para todo $b>2$ a função $g(t)$ difere da função constante 1 em um número finito de pontos

$$\int_0^b g(t)e^{-0t}\,dt=\int_0^b g(t)\,dt=\int_0^b 1\,dt=t\Big|_0^b=b.$$

Logo, $\lim_{b\to+\infty}b=+\infty$ ∴ o integral impróprio diverge para $s=0$

Se $s \neq 0$,

$$\int_0^b g(t)\, e^{-st}\, dt = \int_0^b e^{-st}\, dt = -\frac{e^{-st}}{s}\Big|_0^b = -\frac{e^{-sb}}{s} + \frac{1}{s}$$

$$\lim_{b \to +\infty} -\frac{e^{-sb}}{s} + \frac{1}{s} = \begin{cases} \frac{1}{s}, & s > 0 \\ +\infty, & s < 0. \end{cases}$$

Portanto, sendo $s \neq 0$ o integral impróprio $\int_0^{+\infty} g(t)\, e^{-st}\, dt$ converge se $s > 0$ e diverge se $s < 0$.

$$\mathcal{L}\{g\} = \frac{1}{s}, \quad s > 0 \quad //$$

Exemplo 3:

$$\mathcal{L}\{e^{at}\}(s), \quad a \in \mathbb{R}$$

$$\underbrace{\int_0^{+\infty} e^{at} e^{-st}\, dt}\; = \int_0^{+\infty} e^{t(a-s)}\, dt = \lim_{b \to +\infty} \int_0^b e^{t(a-s)}\, dt.$$

$$= \lim_{b\to+\infty} \left.\frac{e^{t(a-s)}}{a-s}\right|_0^b = \lim_{b\to+\infty} \frac{e^{b(a-s)}}{a-s} - \frac{1}{a-s}$$

$$= \begin{cases} -\dfrac{1}{a-s}, & a-s<0 \\ +\infty, & a-s>0 \end{cases} \Rightarrow \begin{cases} -\dfrac{1}{a-s}, & s>a \\ +\infty, & s<a. \end{cases}$$

- integral $\int_0^{+\infty} e^{at} e^{-st}\, dt$ converge $s > a$.

$$\mathcal{L}\{e^{at}\} = -\frac{1}{a-s} = \frac{1}{s-a}, \quad \underline{\underline{s > a}}$$

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$$\underbrace{\mathcal{L}\{\cosh(2t) - 2t^3 + \sin(3t)\}}=$$

$$= \mathcal{L}\{\cosh(2t)\} - 2\underline{\mathcal{L}\{t^3\}} + \underbrace{\mathcal{L}\{\sin(3t)\}}$$

- $\mathcal{L}\{\cosh(2t)\} = \dfrac{s}{s^2-4}$, $\underline{s > 2}$
- $\mathcal{L}\{t^3\} = \dfrac{3!}{s^4}$, $\underline{s > 0}$

$\mathcal{L}\{\sin(3t)\} = \dfrac{3}{s^2+9}$, $\underline{s > 0}$

$\longrightarrow$

$$\mathcal{L}\{\cosh(2t) - 2t^3 + \sin(3t)\} =$$

$$= \frac{s}{s^2-4} - \frac{12}{s^4} + \frac{3}{s^2+9}, \quad \underline{\underline{s>2}}$$