Integração por partes:

$$\boxed{\int f(x)\,g(x)\ \mathrm{d}x \;=\; F(x)\,g(x) \;-\; \int F(x)\,g'(x)\ \mathrm{d}x}$$

$$\int_a^b \underset{\text{(P)}}{f(x)}\,\underset{\text{(D)}}{g(x)}\ \mathrm{d}x \;=\; \Big[\ F(x)\,g(x)\ \Big]_a^b - \int_a^b F(x)\,g'(x)\ \mathrm{d}x$$

Integrais "quase imediatos": $\quad u = g(x)$

$$\int u'\, u^p \,\mathrm{d}x = \frac{u^{p+1}}{p+1} + C\,, \quad C \in \mathbb{R},\ p \neq -1$$

$$\int \frac{u'}{u} \,\mathrm{d}x = \ln|u| + C\,, \quad C \in \mathbb{R}$$

$$\int u' \operatorname{sen} u \,\mathrm{d}x = -\cos u + C\,, \quad C \in \mathbb{R}$$

$$\int u' \cos u \,\mathrm{d}x = \operatorname{sen} u + C\,, \quad C \in \mathbb{R}$$

$$\int u' \sec^2 u \,\mathrm{d}x = \operatorname{tg} u + C\,, \quad C \in \mathbb{R}$$

$$\int u' \operatorname{cosec}^2 u \,\mathrm{d}x = -\operatorname{cotg} u + C\,, \quad C \in \mathbb{R}$$

$$\int \frac{u'}{\sqrt{1-u^2}} \,\mathrm{d}x = \operatorname{arcsen}(u) + C\,, \quad C \in \mathbb{R}$$

$$\int \frac{u'}{1+u^2} \,\mathrm{d}x = \operatorname{arctg}(u) + C\,, \quad C \in \mathbb{R}$$

$$\int u' \sec(u) \operatorname{tg}(u) \,\mathrm{d}x = \sec u + C\,, \quad C \in \mathbb{R}$$

$$\int u' \operatorname{cosec}(u) \operatorname{cotg}(u) \,\mathrm{d}x = -\operatorname{cosec} u + C\,, \quad C \in \mathbb{R}$$

$$\int u'\, e^u \,\mathrm{d}x = e^u + C\,, \quad C \in \mathbb{R}$$

$$\int u'\, a^u \,\mathrm{d}x = \frac{a^u}{\ln a} + C\,, \quad C \in \mathbb{R}\,,\ a \in \mathbb{R}^+ \setminus \{1\}$$

Primitivas de produtos de $sen(\alpha x)$ e $cos(\beta x)$: $\quad \alpha \neq \beta$

- $\int sen(\alpha x)\, cos(\beta x)\, \mathrm{d}x$
- $\int sen(\alpha x)\, sen(\beta x)\, \mathrm{d}x$
- $\int cos(\alpha x)\, cos(\beta x)\, \mathrm{d}x$

Podemos usar a integração por partes duas vezes consecutivas, ou, em alternativa, usar as fórmulas trigonométricas

$$sen(A+B) = sen(A)cos(B) + cos(A)sen(B) \qquad cos(A+B) = cos(A)cos(B) - sen(A)sen(B)$$
$$sen(A-B) = sen(A)cos(B) - cos(A)sen(B) \qquad cos(A-B) = cos(A)cos(B) + sen(A)sen(B)$$

e deduzir:

- $sen(A)\, cos(B) = \frac{1}{2}\,(\, sen(A+B) + sen(A-B)\,)$
- $cos(A)\, cos(B) = \frac{1}{2}\,(\, cos(A+B) + cos(A-B)\,)$
- $sen(A)\, sen(B) = \frac{1}{2}\,(-cos(A+B) + cos(A-B)\,)$

Potências inteiras (positivas) de senos ou cosenos

Usar a fórmula do Binómio de Newton: $(1+B)^n = \sum_{i=0}^{n} \binom{n}{i} B^i$

$$\int \operatorname{sen}^n x \, dx =$$

$$\begin{array}{ll} (n \text{ par}:) & \left\{ \begin{array}{l} \int (\operatorname{sen}^2 x)^{\frac{n}{2}} \, dx = \int \left(\frac{1}{2}(1-\cos 2x)\right)^{\frac{n}{2}} \, dx = \frac{1}{2^{\frac{n}{2}}} \int (1-\cos 2x)^{\frac{n}{2}} \, dx \\ \\ \int \operatorname{sen}^{(n-1)} x \operatorname{sen} x \, dx = \int (\operatorname{sen}^2 x)^{\frac{n-1}{2}} \operatorname{sen} x \, dx = \int (1-\cos^2 x)^{\frac{n-1}{2}} \operatorname{sen} x \, dx \end{array} \right. \\ (n \text{ ímpar}:) & \end{array}$$

$$\int \cos^n x \, dx =$$

$$\begin{array}{ll} (n \text{ par}:) & \left\{ \begin{array}{l} \int (\cos^2 x)^{\frac{n}{2}} \, dx = \int \left(\frac{1}{2}(1+\cos 2x)\right)^{\frac{n}{2}} \, dx = \frac{1}{2^{\frac{n}{2}}} \int (1+\cos 2x)^{\frac{n}{2}} \, dx \\ \\ \int \cos^{(n-1)} x \cos x \, dx = \int (\cos^2 x)^{\frac{n-1}{2}} \cos x \, dx = \int (1-\operatorname{sen}^2 x)^{\frac{n-1}{2}} \cos x \, dx \end{array} \right. \\ (n \text{ ímpar}:) & \end{array}$$

$$\boxed{\operatorname{sen}^2 x = \frac{1}{2}(1-\cos 2x)} \qquad \boxed{\cos^2 x = \frac{1}{2}(1+\cos 2x)}$$

Potências inteiras (positivas) da secante

$$\int sec\, x \,\mathrm{d}x = \int \sec x \frac{\sec x + \operatorname{tg} x}{\sec x + \operatorname{tg} x}\,\mathrm{d}x = \int \frac{\sec^2 x + \sec x \operatorname{tg} x}{\sec x + \operatorname{tg} x}\,\mathrm{d}x = \ln|\sec x + \operatorname{tg} x| + C$$

$$\int sec^2 x\,\mathrm{d}x = \operatorname{tg} x + C$$

$$\boxed{\int u' sec\, u\,\mathrm{d}x = \ln|\sec u + \operatorname{tg} u| + C}$$

$$\int sec^3 x\,\mathrm{d}x = \int \underset{(P)}{sec^2 x}\ \underset{(D)}{\sec x}\,\mathrm{d}x = \cdots = \frac{1}{2}\operatorname{tg} x \sec x + \frac{1}{2}\ln|\sec x + \operatorname{tg} x| + C$$

$$\int sec^n x\,\mathrm{d}x = \int \underset{(D)}{sec^{n-2} x}\ \underset{(P)}{\sec^2 x}\,\mathrm{d}x \qquad \text{(integração por partes)}$$

Depois da Int. por partes usar $\boxed{1 + \operatorname{tg}^2 x = \sec^2 x}$

Potências inteiras (positivas) da tangente

$$\int \operatorname{tg} x \, dx = -\ln|\cos x| + C$$

$$\int \operatorname{tg}^2 x \, dx = \int \sec^2 x - 1 \, dx = \operatorname{tg} x + x + C$$

$$\int \operatorname{tg}^3 x \, dx = \int \operatorname{tg}^2 x \operatorname{tg} x \, dx = \int (\sec^2 x - 1) \operatorname{tg} x \, dx = \int \underbrace{\sec^2 x}_{u'} \underbrace{\operatorname{tg} x}_{u} \, dx - \int \operatorname{tg} x \, dx =$$

$$= \frac{\operatorname{tg}^2 x}{2} + \ln|\cos x| + C$$

$$\int tg^n x \, dx =$$

$$\begin{cases} (n \text{ par}:) & \int (\operatorname{tg}^2 x)^{\frac{n}{2}} \, dx = \int (\sec^2 x - 1)^{\frac{n}{2}} \, dx \\ (n \text{ ímpar}:) & \int \operatorname{tg}^{(n-1)} x \operatorname{tg} x \, dx = \int (\operatorname{tg}^2 x)^{\frac{n-1}{2}} \operatorname{tg} x \, dx = \int (\sec^2 x - 1)^{\frac{n-1}{2}} \operatorname{tg} x \, dx \end{cases}$$

$$\int \sec^n x \operatorname{tg} x \, dx = \int \sec^{n-1} \sec x \operatorname{tg} x \, dx = \frac{\sec^n x}{n} + C, \quad C \in \mathbb{R}$$

$$\boxed{1 + \operatorname{tg}^2 x = \sec^2 x}$$

$$\int f(x)\,\mathrm{d}x$$

Substituição:

$x = u(t)$, u = função invertível e dif. nalgum int. J com $u(J) \subset D_f$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u'(t) \;\Leftrightarrow\; \mathrm{d}x = u'(t)\,\mathrm{d}t$$

$$\int f(x)\,\mathrm{d}x \;=\; \int f(u(t))\,u'(t)\,\mathrm{d}t \;=\; H(t) + C$$

Reverter a substituição:

$$t = u^{-1}(x)$$

$$\int f(x)\,\mathrm{d}x \;=\; H(u^{-1}(x)) + C$$

Regras para substituição: $\int f(x)\,\mathrm{d}x$

$f(x)$ contém		Substituição
$\sqrt[k]{a+bx}$	$\rightsquigarrow$	$\sqrt[k]{a+bx} = t \quad (t \geq 0 \text{ se } k \text{ par})$
$\sqrt{a^2+x^2}$	$\rightsquigarrow$	$x = a \operatorname{tg} t$
$\sqrt{a^2-x^2}$	$\rightsquigarrow$	$x = a \operatorname{sen} t$
$\sqrt{x^2-a^2}$	$\rightsquigarrow$	$x = a \sec t$
$\sqrt{ax^2+bx+c}$	$\rightsquigarrow$	$x + \frac{b}{2a} = z$

$$\boxed{ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2+K\right]}\,, \quad K = \frac{c}{a} - \frac{b^2}{4a^2}$$

Potências inteiras (positivas) de senos ou cosenos

Usar a fórmula do Binómio de Newton: $(1+B)^n = \sum_{i=0}^{n} \binom{n}{i} B^i$

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$$\int \cos^n x \, dx =$$

$$\begin{cases} (n \text{ par}:) & \int (\cos^2 x)^{\frac{n}{2}} \, dx = \int \left(\frac{1}{2}(1+\cos 2x)\right)^{\frac{n}{2}} \, dx = \frac{1}{2^{\frac{n}{2}}} \int (1+\cos 2x)^{\frac{n}{2}} \, dx \\ (n \text{ ímpar}:) & \int \cos^{(n-1)} x \cos x \, dx = \int (\cos^2 x)^{\frac{n-1}{2}} \cos x \, dx = \int (1-\operatorname{sen}^2 x)^{\frac{n-1}{2}} \cos x \, dx \end{cases}$$

$$\boxed{\operatorname{sen}^2 x = \frac{1}{2}(1-\cos 2x)} \qquad \boxed{\cos^2 x = \frac{1}{2}(1+\cos 2x)}$$

Potências inteiras (positivas) da tangente

$$\int \operatorname{tg} x \, dx = -\ln|\cos x| + C$$

$$\int \operatorname{tg}^2 x \, dx = \int \sec^2 x - 1 \, dx = \operatorname{tg} x + x + C$$

$$\int \operatorname{tg}^3 x \, dx = \int \operatorname{tg}^2 x \operatorname{tg} x \, dx = \int (\sec^2 x - 1) \operatorname{tg} x \, dx = \int \underbrace{\sec^2 x}_{u'} \underbrace{\operatorname{tg} x}_{u} \, dx - \int \operatorname{tg} x \, dx =$$

$$= \frac{\operatorname{tg}^2 x}{2} + \ln|\cos x| + C$$

$$\int tg^n x \, dx =$$

$$\begin{array}{ll} (n \text{ par}:) & \left\{ \begin{array}{l} \displaystyle\int (\operatorname{tg}^2 x)^{\frac{n}{2}} \, dx = \int (\sec^2 x - 1)^{\frac{n}{2}} \, dx \\ \\ \displaystyle\int \operatorname{tg}^{(n-1)} x \operatorname{tg} x \, dx = \int (\operatorname{tg}^2 x)^{\frac{n-1}{2}} \operatorname{tg} x \, dx = \int (\sec^2 x - 1)^{\frac{n-1}{2}} \operatorname{tg} x \, dx \end{array} \right. \\ (n \text{ ímpar}:) & \end{array}$$

$$\int \sec^n x \operatorname{tg} x \, dx = \int \sec^{n-1} x \sec x \operatorname{tg} x \, dx = \frac{\sec^n x}{n} + C, \quad C \in \mathbb{R}$$

$$\boxed{1 + \operatorname{tg}^2 x = \sec^2 x}$$

Potências inteiras (positivas) da secante

$$\int sec\,x\,\mathrm{d}x = \int \sec x \frac{\sec x + \operatorname{tg} x}{\sec x + \operatorname{tg} x}\,\mathrm{d}x = \int \frac{\sec^2 x + \sec x \operatorname{tg} x}{\sec x + \operatorname{tg} x}\,\mathrm{d}x = \ln|\sec x + \operatorname{tg} x| + C$$

$$\int sec^2 x\,\mathrm{d}x = \operatorname{tg} x + C$$

$$\int sec^3 x\,\mathrm{d}x = \int \underset{\text{(P)}}{sec^2 x}\,\underset{\text{(D)}}{\sec x}\,\mathrm{d}x = \cdots = \frac{1}{2}\operatorname{tg} x \sec x + \frac{1}{2}\ln|\sec x + \operatorname{tg} x| + C$$

$$\int sec^n x\,\mathrm{d}x = \int \underset{\text{(D)}}{sec^{n-2} x}\,\underset{\text{(P)}}{\sec^2 x}\,\mathrm{d}x \qquad \text{(integração por partes)}$$

Depois da Int. por partes usar $\boxed{1 + \operatorname{tg}^2 x = \sec^2 x}$

Decomposição de uma fracção própria em fracções simples

Vamos apresentar, sem demonstração, o processo de decomposição de uma fracção própria $\dfrac{R(x)}{D(x)}$ numa soma de fracções simples.

1. Decompõe-se o polinómio $D(x)$ em factores irredutíveis [2] .
2. Seja

$$D(x) = d(x-\alpha_1)^{r_1}(x-\alpha_2)^{r_2}\cdots(x-\alpha_n)^{r_n}(x^2+\beta_1 x+\gamma_1)^{s_1}(x^2+\beta_2 x+\gamma_2)^{s_2}\cdots(x^2+\beta_m x+\gamma_m)^{s_m} \ ,$$

com $d \in \mathbb{R}\setminus\{0\}$, a decomposição do polinómio $D(x)$ em factores irredutíveis.

2.1 A cada factor da decomposição de $D(x)$ em factores irredutíveis do tipo

$$(x-\alpha)^r$$

corresponde, na decomposição da fracção $\dfrac{R(x)}{D(x)}$ em fracções simples, uma soma do tipo

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_{r-1}}{(x-\alpha)^{r-1}}+\frac{A_r}{(x-\alpha)^r} \ ,$$

onde $A_1, A_2, \cdots, A_{r-1}, A_r$ são constantes reais a determinar.

2.2 A cada factor da decomposição de $D(x)$ em factores irredutíveis do tipo

$$(x^2+\beta x+\gamma)^s$$

corresponde, na decomposição da fracção $\dfrac{R(x)}{D(x)}$ em fracções simples, uma soma do tipo

$$\frac{B_1x+C_1}{x^2+\beta x+\gamma}+\frac{B_2x+C_2}{(x^2+\beta x+\gamma)^2}+\cdots+\frac{B_{s-1}x+C_{s-1}}{(x^2+\beta x+\gamma)^{s-1}}+\frac{B_sx+C_s}{(x^2+\beta x+\gamma)^s} \ ,$$

onde $B_1, B_2, \cdots, B_{s-1}, B_s$ e $C_1, C_2, \cdots, C_{s-1}, C_s$ são constantes reais a determinar.