Preparação para o 1º miniteste
Folha de exercícios 2

1. $S(x) = \sum_{n=1}^{+\infty} \frac{\cos(nx)}{n\sqrt{n+1}}$

a) $\frac{\cos(nx)}{n\sqrt{n+1}} < \frac{1}{n\sqrt{n+1}} = a_n$

$\frac{1}{n\sqrt{n+1}} \sim \frac{1}{n^{3/2}}$

$\sum_{n=1}^{+\infty} \frac{1}{n^{3/2}}$ é uma série convergente (série harmónica com $\alpha = 3/2 > 1$)

$\lim_{n \to +\infty} \frac{\frac{1}{n\sqrt{n+1}}}{\frac{1}{n^{3/2}}} = \lim_{n \to +\infty} \frac{n^{3/2}}{n\sqrt{n+1}} = \lim_{n \to +\infty} \frac{\cancel{n}\sqrt{n}}{\cancel{n}\sqrt{n+1}} =$

$= \lim_{n \to +\infty} \sqrt{\frac{n}{n+1}} = \lim_{n \to +\infty} \sqrt{\frac{n}{n}} = \lim_{n \to +\infty} \sqrt{1} = 1 \in \mathbb{R}^+$,

logo $\sum_{n=1}^{+\infty} \frac{1}{n\sqrt{n+1}}$ tem a mesma natureza que $\sum_{n=1}^{+\infty} \frac{1}{n^{3/2}}$.

Por isso, $\sum_{n=1}^{+\infty} \frac{1}{n\sqrt{n+1}}$ é convergente

Pelo critério de Weierstrass, como $S(x) < a_n$, sendo a_n uma série convergente de termos positivos, então $S(x)$ é uniformemente convergente.

Critério de Weierstrass

Consideremos a série de funções $\sum_{n=1}^{+\infty} f_n$, com as funções f_n definidas em D. Se $|f_n(x)| \leq a_n$, $\forall n \geq n_0$, $\forall x \in D$, e a série numérica de termos não negativos $\sum_{n=1}^{+\infty} a_n$ é convergente, então a série $\sum_{n=1}^{+\infty} f_n$ é uniformemente convergente em D.

b) Como propriedade de uma série de funções uniformemente convergente, então $S(x)$ é contínua em $\mathbb{R}$.

6.
a) $\sum_{n=1}^{+\infty} \frac{nx^n}{2^n}$, $a_n = \frac{n}{2^n}$

$$R = \frac{1}{L}, \quad L = \lim_{n \to +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \to +\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} =$$

$$= \lim_{n \to +\infty} \frac{\cancel{2^n}(n+1)}{2^{n+1} \cdot n} = \lim_{n \to +\infty} \frac{n+1}{2n} =$$

$$= \lim_{n \to +\infty} \frac{\cancel{n}}{2\cancel{n}} = \frac{1}{2}$$

$$R = \frac{1}{1/2} = 2$$

b) $f(x) = \sum_{n=1}^{+\infty} \frac{n}{2^n} x^n$, $-2 < x < 2$

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n, \quad |x| < 1$$

$$\sum_{n=1}^{+\infty} \frac{n}{2^n} x^n = \sum_{n=1}^{+\infty} n \left(\frac{x}{2}\right)^n = \sum_{n=1}^{+\infty} n y^n$$

substituição : $y = \frac{x}{2}$

$$\frac{1}{1-y} = \sum_{n=0}^{+\infty} y^n, \quad |y| < 1$$

$$\frac{1}{1-\left(\frac{x}{2}\right)} = \sum_{n=0}^{+\infty} \left(\frac{x}{2}\right)^n, \quad |x| < 2$$

$$\frac{1}{\frac{2-x}{2}} = \sum_{n=0}^{+\infty} \left(\frac{x}{2}\right)^n, \quad |x| < 2$$

$$\frac{2}{2-x} = \sum_{n=0}^{+\infty} \left(\frac{x}{2}\right)^n; \quad |x| < 2$$

$$\left(\frac{2}{2-x}\right)' = \left(\sum_{n=0}^{+\infty} \left(\frac{x}{2}\right)^n\right)', \quad |x| < 2$$

$$\frac{2}{(2-x)^2} = \sum_{n=0}^{+\infty} n\left(\frac{x}{2}\right)^{n-1} \cdot \frac{1}{2} \quad , \quad |x|<2$$

$$\frac{2}{(2-x)^2} = \sum_{n=1}^{+\infty} \frac{n}{2^{n-1+1}} \times x^{n-1} \quad , \quad |x|<2$$

$$\frac{2}{(2-x)^2} = \sum_{n=1}^{+\infty} \frac{n}{2^n} \times x^{n-1} \quad , \quad |x|<2$$

$$\frac{2x}{(2-x)^2} = x\sum_{n=1}^{+\infty} \frac{n}{2^n} \cdot x^{n-1} \quad , \quad |x|<2$$

$$\frac{2x}{(2-x)^2} = \sum_{n=1}^{+\infty} \frac{n}{2^n} \cdot x^n \quad , \quad |x|<2$$

Séries de potências de algumas funções

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n \, , \forall x \in \left]-1,1\right[$$

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \, , \forall x \in \mathbb{R}$$

$$\operatorname{sen} x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \, , \forall x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!} \, , \forall x \in \mathbb{R}$$

$$\operatorname{senh} x = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!} \, , \forall x \in \mathbb{R}$$

$$\cosh x = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \, , \forall x \in \mathbb{R}$$

7.

a) $\lim\limits_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$

$$\operatorname{sen} x = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \forall x \in \mathbb{R}$$

$$\frac{\operatorname{sen} x}{x} = \sum_{n=0}^{+\infty} \frac{(-1) x^{2n+1}}{(2n+1)!\, x}, \quad \forall x \in \mathbb{R} \setminus \{0\}$$

$$= \sum_{n=0}^{+\infty} \frac{(-1) x^{2n}}{(2n+1)!}, \quad \forall x \in \mathbb{R} \setminus \{0\}$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = \lim_{x \to 0} \sum_{n=0}^{+\infty} \frac{(-1) x^{2n}}{(2n+1)!}, \quad x \in \mathbb{R} \setminus \{0\}$$

$$= \lim_{x \to 0} \left(1 - \frac{x^2}{6} + \frac{x^4}{120} + \dots + \frac{(-1)^n x^{2n}}{(2n+1)!} \right) =$$

$$= 1 - 0 + 0 - 0 + \dots$$

$$= 1$$

b) $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = 1$

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$e^x - 1 = \left(\sum_{n=0}^{+\infty} \frac{x^n}{n!} \right) - 1 \quad \Leftrightarrow \quad e^x - 1 = \frac{x^0}{0!} - 1 + \sum_{n=1}^{+\infty} \frac{x^n}{n!}$$

$$\Leftrightarrow e^x - 1 = \cancel{1} - \cancel{1} + \sum_{n=1}^{+\infty} \frac{x^n}{n!} \quad \Leftrightarrow \quad \frac{e^x - 1}{x} = \sum_{n=1}^{+\infty} \frac{x^n}{n!\, x}$$

$$\Leftrightarrow \frac{e^x - 1}{x} = \sum_{n=1}^{+\infty} \frac{x^{n-1}}{n!}, \quad x \in \mathbb{R}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \sum_{n=1}^{+\infty} \frac{x^{n-1}}{n!} =$$

$$= 1 + \frac{x}{2} + \frac{x^2}{6} + \dots \frac{x^{n-1}}{n!}$$

$$= 1$$

8.

série de potências $\sum_{n=1}^{+\infty} \frac{(x-3)^n}{n2^n}$

a) Determinar maior subconjunto de IR no qual a série é absolutamente convergente

$R = \frac{1}{L}$, $a_n = \frac{1}{n2^n}$

$L = \lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to +\infty} \frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n2^n}} =$

$= \lim_{n \to +\infty} \frac{n2^n}{(n+1)2^{n+1}} = \lim_{n \to +\infty} \frac{n}{2(n+1)} = \lim_{n \to +\infty} \frac{n}{2n+2}$

$= \frac{1}{2}$ $R = \frac{1}{1/2} = 2$

Intervalo de convergência : $]c - R, c + R[$

centro c da série = 3

$I.C =]3-2, 3+2[=]1, 5[$

Se $x = 1$: $\sum_{n=1}^{+\infty} \frac{(1-3)^n}{n2^n} = \sum_{n=1}^{+\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$

↓

Simplesmente convergente

Se $x = 5$: $\sum_{n=1}^{+\infty} \frac{(5-3)^n}{n2^n} = \sum_{n=1}^{+\infty} \frac{2^n}{n2^n} = \sum_{n=1}^{+\infty} \frac{1}{n}$

↓

Divergente

b) $f'(x) = \left(\sum_{n=1}^{+\infty} \frac{(x-3)^n}{n2^n} \right)' = \sum_{n=1}^{+\infty} \frac{\cancel{n} \cdot (x-3)^{n-1}}{\cancel{n} 2^n} =$

$= \sum_{n=1}^{+\infty} \frac{(x-3)^{n-1}}{2^n}$

$f'(4) = \sum_{n=1}^{+\infty} \frac{1^{n-1}}{2^n} = \sum_{n=1}^{+\infty} \frac{1^n}{2^n} = \sum_{n=1}^{+\infty} \left(\frac{1}{2}\right)^n =$

$= \frac{1/2}{1 - 1/2} = \frac{\frac{1}{2}}{\frac{2-1}{2}} = \frac{1/2}{1/2} = 1$

9. $e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}, \forall x \in \mathbb{R}$

a) representação em série de potências para a função $f(x) = x e^{x^3}$ e indicar o maior subconjunto de $\mathbb{R}$ em que a representação é válida

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \Leftrightarrow e^{x^3} = \sum_{n=0}^{+\infty} \frac{x^{3n}}{n!} \Leftrightarrow$$

$$\Leftrightarrow x e^{x^3} = \sum_{n=0}^{+\infty} \frac{x^{3n+1}}{n!}, x \in \mathbb{R}$$

b) $$\int_0^1 x e^{x^3} = \int_0^1 \sum_{n=0}^{+\infty} \frac{x^{3n+1}}{n!} = \sum_{n=0}^{+\infty} \int_0^1 \frac{x e^{x^3}}{n!} =$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^1 x e^{x^3} = \sum_{n=0}^{+\infty} \frac{1}{n!} \left[\frac{x^{3n+2}}{3n+2} \right]_0^1 =$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} \left[\frac{1^{3n+2}}{3n+2} - \frac{0^{3n+2}}{3n+2} \right] = \sum_{n=0}^{+\infty} \frac{1}{n!(3n+2)}$$

10. $(e^{x^2})' = 2x e^{x^2}, x \in \mathbb{R}$

$$e^{x^2} = \sum_{n=0}^{+\infty} \frac{x^{2n}}{n!}, x \in \mathbb{R}$$

$$\left(e^{x^2}\right)' = \left(\sum_{n=0}^{+\infty} \frac{x^{2n}}{n!}\right)' = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(x^{2n}\right)' =$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} \cdot x^{2n-1} \cdot 2n = 2 \sum_{n=0}^{+\infty} x^{2n-1} \cdot \frac{1}{(n-1)!} =$$

$$= 2 \sum_{n=0}^{+\infty} \frac{x^{2n-1} \cdot x}{(n-1)! \cdot x} = 2x \sum_{n=0}^{+\infty} \frac{x^{2(n-1)}}{(n-1)!} = 2x \sum_{n=0}^{+\infty} \frac{x^{2k}}{k!} =$$

$$= 2x e^{x^2}$$

11.

$f(x) = x e^{-x}, \quad x \in \mathbb{R}$

a) $e^{x} = \sum_{n=0}^{+\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$

$e^{-x} = \sum_{n=0}^{+\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{n!}$

$x e^{-x} = \sum_{n=0}^{+\infty} \frac{(-1)^n \cdot x^{n+1}}{n!}, \quad x \in \mathbb{R}$

b) Cálculo da soma da série $\sum_{n=1}^{+\infty} \frac{(-1)^n (n+1) 2^n}{n!}$

$(x e^{-x})' = \left(\sum_{n=0}^{+\infty} \frac{(-1)^n x^{n+1}}{n!} \right)' = \sum_{n=0}^{+\infty} \frac{(-1)^n (n+1) x^n}{n!} =$

$= e^{-x} - x e^{-x}$

$x = 2$

$\sum_{n=1}^{+\infty} \frac{(-1)^n (n+1) 2^n}{n!} = e^{-2} - 2e^{-2} = -e^{-2}$

13.

$f(x) = 2, \quad x \in [0, \pi]$

$\Rightarrow$ Série de Fourier dos senos

$a_n = \frac{2}{\pi} \int_0^{\pi} 2 \cos(nx)\, dx = \frac{4}{\pi} \int_0^{\pi} \left[\frac{\operatorname{sen}(nx)}{n} \right]_0^{\pi} = 0$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} 2\, dx = \frac{1}{\pi} \left[2x \Big|_{-\pi}^{\pi} \right] = \frac{2\pi - (-2\pi)}{\pi} =$

$= \frac{4\cancel{\pi}}{\cancel{\pi}} = 4$

$F_{par} = \frac{a}{2} + \sum_{n=0}^{+\infty} a_n \cos(nx) = \frac{4}{2} + \sum_{n=0}^{+\infty} 0 = \frac{4}{2} = 2$

⇒ Série de Fourier dos cossenos

$$b_n = \frac{2}{\pi}\int_0^{\pi} a_n \,\text{sen}(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} 2\,\text{sen}(nx)\,dx$$

$$= \frac{4}{\pi}\left[-\frac{\cos(nx)}{n}\Big|_0^{\pi}\right] = -\frac{4}{\pi}\left[\frac{\cos(n\pi)-\cos(0)}{n}\right] =$$

$$= -\frac{4}{\pi}\left(\frac{(-1)^n - 1}{n}\right) = \frac{4}{\pi}\left(\frac{-(-1)^n + 1}{n}\right) =$$

$$= \frac{4}{\pi}\left(\frac{(-1)^{n+1}+1}{n}\right) = \begin{cases} 0 & \text{, se } n\text{-par} \\ 2 & \text{, se } n\text{-ímpar} \end{cases}$$

14.

$$f(x) = \cos(x) \quad , \quad x \in [0, \pi]$$

função seno ⟶ função ímpar

série de Fourier dos senos: $\sum_{n=1}^{+\infty} b_n \,\text{sen}(nx)$

Extensão ímpar da função em $[0, \pi]$:

$$f_i(x) = \begin{cases} -\cos(-x) & , -\pi \leq x < 0 \\ 0 & , x = 0 \\ \cos(x) & , 0 < x \leq \pi \end{cases}$$

→ Série de Fourier dos senos

$$f_x \sim \sum_{n=1}^{+\infty} b_n \,\text{sen}(nx)\,dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos(x)\,\text{sen}(nx)\,dx =$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} \frac{1}{\cancel{2}}\left(\text{sen}(nx - x) + \text{sen}(nx + x)\right)dx =$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \text{sen}(nx + x)\,dx + \int_0^{\pi}\text{sen}(nx - x)\,dx\right] =$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} \text{sen}(x(n+1))\,dx + \int_0^{\pi}\text{sen}(x(n-1))\,dx\right] =$$

$$= \frac{1}{\pi}\left[-\left[\frac{\cos(x(n+1))}{n+1}\Big|_0^{\pi}\right] - \left[\frac{\cos(x(n-1))}{n-1}\Big|_0^{\pi}\right]\right]$$

$$= -\frac{1}{\pi}\left[\frac{\cos(\pi(n+1)) - \cos(0(n+1))}{n+1} + \frac{\cos(\pi(n-1)) - \cos(0(n-1))}{n-1}\right]$$

$$= -\frac{1}{\pi}\left[\frac{(-1)^{n+1} - 1}{n+1} - \frac{(-1)^{n-1} - 1}{n-1}\right]$$

$$\boxed{\begin{array}{c}(-1)^{n-1} = (-1)^{n+1} \\ n\text{-par} \\ n\text{-ímpar}\end{array}}$$

$$= -\frac{1}{\pi}\left[\frac{(n-1)\left((-1)^{n+1} - 1\right) - (n+1)\left((-1)^{n-1} - 1\right)}{(n+1)(n-1)}\right]$$

$$= -\frac{1}{\pi}\left[\frac{\left((-1)^{n+1} - 1\right)\left(n \cancel{-1} + n \cancel{+1}\right)}{n^2 - 1}\right]$$

$$= -\frac{1}{\pi}\left[\frac{\left((-1)^{n+1} - 1\right)(2n)}{n^2 - 1}\right] = \begin{cases} -\frac{1}{\pi}\left(\frac{(-2)(2n)}{n^2-1}\right) = \frac{4n}{\pi(n^2-1)} & , n\text{-par} \\ 0 & , n\text{-ímpar} \end{cases}$$

$$f(x) \sim \sum_{n=1}^{+\infty} \frac{4 \cdot (2k)}{\pi\left((2k)^2 - 1\right)} \operatorname{sen}(2k\pi) = \sum_{n=1}^{+\infty} \frac{8k}{\pi\left((2k)^2 - 1\right)} \operatorname{sen}(2k\pi)$$

→ Série de Fourier dos cossenos (função par)

$$f_p = \begin{cases} \cos(-x) & , x \in [-\pi, 0[\\ \cos(x) & , x \in [0, \pi] \end{cases}$$

$$= \begin{cases} \cos(x) & , x \in [-\pi, 0[\\ \cos(x) & , x \in [0, \pi] \end{cases}$$

$$= \cos(x) \ , \ x \in [-\pi, \pi]$$

$$f(x) \sim \cos(x)$$

15.

função f 2π - periódica

$f(x) = x^2, \quad -\pi \leq x \leq \pi$

a) Série de Fourier de f

f é uma função par, logo é uma série de Fourier dos cossenos

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos(nx) \, dx$$

$$\Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^{2+1}}{2+1}\right]_0^{\pi} =$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{2}{\cancel{\pi}} \cdot \frac{\pi^3}{3} =$$

$$= \frac{2\pi^2}{3}$$

$$\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)\,dx \quad ⊛$$

$$\cdot \int x^2 \cos(nx)\,dx = x^2\frac{\operatorname{sen}(nx)}{n} - \int 2x\,\frac{\operatorname{sen}(nx)}{n}\,dx =$$

$u = x^2 \qquad dv = \cos(nx)$

$du = 2x \qquad v = \dfrac{\operatorname{sen}(nx)}{n}$

$$= x^2\frac{\operatorname{sen}(nx)}{n} - \frac{2}{n}\int x \operatorname{sen}(nx)\,dx$$

$u = x \qquad dv = \operatorname{sen}(nx)$

$du = 1 \qquad v = -\dfrac{\cos(nx)}{n}$

$$= x^2\frac{\operatorname{sen}(nx)}{n} - \frac{2}{n}\left[x\left(-\frac{\cos(nx)}{n}\right) - \int 1 \cdot \left(-\frac{\cos(nx)}{n}\right)dx\right]$$

$$= x^2\frac{\operatorname{sen}(nx)}{n} - \frac{2}{n}\left[-\frac{x\cos(nx)}{n} + \frac{1}{n}\int \cos(nx)\,dx\right]$$

$$= \frac{x^2\operatorname{sen}(nx)}{n} - \frac{2}{n}\left[-\frac{x\cos(nx)}{n} + \frac{1}{n}\left(\frac{\operatorname{sen}(nx)}{n}\right)\right]$$

$$= \frac{x^2\operatorname{sen}(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\operatorname{sen}(nx)}{n^3}$$

$sen(\pi) = sen(0) = 0$

(*)

$$a_n = \frac{2}{\pi}\left[\cancel{\frac{x^2 \, sen(nx)}{n}}^{0} + \frac{2x\cos(nx)}{n^2} - \cancel{\frac{2\,sen(nx)}{n^3}}^{0} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2x\cos(nx)}{n^2}\Big|_0^{\pi} \right] = \frac{2}{\cancel{\pi}}\left[\frac{2\cancel{\pi}\cos(n\pi)}{n^2} - 0\right] = \frac{4\cos(n\pi)}{n^2} =$$

$$= \frac{4(-1)^n}{n^2}$$

$$f(x) \sim \frac{\frac{2\pi^2}{3}}{\cancel{2}} + \sum_{n=1}^{+\infty} \frac{4(-1)^n}{n^2}\cos(nx) =$$

$$\sim \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4(-1)^n}{n^2}\cos(nx)$$

b) Mostre que

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{4}{n^2}\cos(nx), \quad \forall x \in [-\pi, \pi]$$

↓

uma vez que x^2 é contínua e diferenciável em $[-\pi, \pi]$

c) Usando a representação de f em série de Fourier, mostrar que

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

Se $x = 0$,

$$0^2 = \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{4}{n^2}\cos(n \cdot 0)$$

$$\Leftrightarrow -\frac{\pi^2}{3} = \sum_{n=1}^{+\infty} (-1)^n \frac{4}{n^2}$$

$$\Leftrightarrow -\frac{\pi^2}{3} = 4\sum_{n=1}^{+\infty} \frac{(-1)}{n^2} \Leftrightarrow -\frac{\pi^2}{12} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$$

d) Verificar que a série de Fourier de f é uniformemente convergente em $\mathbb{R}$

$$\frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4}{n^2} (-1)^n \cos(nx) \leq \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{4 \cdot (-1)^n}{n^2}$$

A série $\sum_{n=1}^{+\infty} \frac{4(-1)^n}{n^2}$ tem a mesma natureza que $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$

Estudando a série dos módulos de $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$

$\sum_{n=1}^{+\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{+\infty} \frac{1}{n^2}$, que é uma série de Dirichlet harmónica

com $r = 2 > 1$, logo é convergente

Pelo critério de Weierstrass, como $f(x) < a_n$, sendo a_n uma série convergente de termos positivos, então a série de Fourier de f é uniformemente convergente.

e) justificar que

$$\frac{x^3 - \pi^2 x}{3} = \sum_{n=1}^{+\infty} (-1)^n \frac{4}{n^3} \operatorname{sen}(nx), \forall x \in [-\pi, \pi]$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{4}{n^2} \cos(nx)$$

$$x^3 - \frac{\pi^2}{3} = \sum_{n=1}^{+\infty} (-1)^n \frac{4}{n^2} \cos(nx)$$

$$\int_0^x \left(t^2 - \frac{\pi^2}{3} \right) dt = \int_0^x \sum_{n=1}^{+\infty} (-1)^n \frac{4}{n^2} \cos(nt)\, dt$$

$$\left[\frac{t^3 - \pi^2 t}{3} \right]_0^x = \sum_{n=1}^{+\infty} (-1)^n \frac{4}{n^2} \left[\frac{\operatorname{sen}(tn)}{n} \right]_0^x$$

$$\frac{x^3 - \pi^2 x}{3} = \sum_{n=1}^{+\infty} (-1)^n \frac{4}{n^2} \operatorname{sen}(nx)$$

Folha de exercícios 2

2.

Sabe-se que:

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n \, , \quad x \in \,]-1,1[$$

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \, , \quad x \in \mathbb{R}$$

$$\operatorname{sen} x = \sum_{n=0}^{+\infty} \frac{(-1)^n \, x^{2n+1}}{(2n+1)!} \, , \quad x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{+\infty} \frac{(-1)^n \, x^{2n}}{(2n)!} \, , \quad x \in \mathbb{R}$$

2.

a) e^{-x^2}

$$e^{-x^2} \underset{\substack{\downarrow \\ \text{substituição} \\ -x^2 = y}}{=} e^{y} = \sum_{n=0}^{+\infty} \frac{y^n}{n!}, \quad y \in \mathbb{R}$$

$$= \sum_{n=0}^{+\infty} \frac{(-x^2)^n}{n!}, \quad (-x^2) \in \mathbb{R}$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n \cdot x^{2n}}{n!}, \quad x \in \mathbb{R}$$

b) $$\operatorname{senh}(3x) = \frac{e^{3x} - e^{-3x}}{2} = \frac{e^{3x}}{2} - \frac{e^{-3x}}{2} =$$

↳ substituição
$y = 3x$
$z = -3x$

$$= \frac{e^y}{2} - \frac{e^z}{2} = \sum_{n=0}^{+\infty} \frac{y^n}{2n!} - \sum_{n=0}^{+\infty} \frac{z^n}{2n!} =$$

$$= \frac{1}{2} \sum_{n=0}^{+\infty} \frac{y^n}{n!} - \frac{1}{2} \sum_{n=0}^{+\infty} \frac{z^n}{n!} =$$

$$= \frac{1}{2} \sum_{n=0}^{+\infty} \frac{y^n - z^n}{n!} = \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(3x)^n - (-3x)^n}{n!} =$$

$$= \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(3x)^n - (-1)^n (3x)^n}{n!} = \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(3x)^n}{n!} \underbrace{\left(1 - (-1)^n\right)}$$

$$= \begin{cases} 0, & n\text{-par} \\ 2, & n\text{-ímpar} \end{cases}$$

$$= \frac{1}{\cancel{2}} \sum_{n=0}^{+\infty} \frac{(3x)^{2K+1}}{(2K+1)!} \times \cancel{2} =$$

$$= \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(3x)^{2K+1}}{(2K+1)!}$$

$$\boxed{\cos^2 x = \frac{1+\cos(2x)}{2}}$$

c) $2\cos^2 x = \cancel{2}\left(\frac{1+\cos(2x)}{\cancel{2}}\right) = 1 + \cos(2x) \underset{\downarrow}{=} 1+\cos(y) =$

substituição
$y = 2x$

$$= 1 + \sum_{n=0}^{+\infty} \frac{(-1)^n\, y^{2n}}{(2n)!}, \quad y \in \mathbb{R}$$

$$= 1 + \sum_{n=0}^{+\infty} \frac{(-1)^n \cdot (2x)^{2n}}{(2n)!}, \quad x \in \mathbb{R}$$

d) $\frac{1}{4+x^2} = \frac{1}{4\left(1+\frac{x^2}{4}\right)} \underset{\downarrow}{=} \frac{1}{4} \cdot \frac{1}{1+y} = \frac{1}{4} \cdot \frac{1}{1-(-y)} \underset{\downarrow}{=}$

substituição
$y = \left(\frac{x}{2}\right)^2$

substituição
$z = -y$

$$= \frac{1}{4} \cdot \frac{1}{1-z} = \frac{1}{4} \cdot \sum_{n=0}^{+\infty} z^n, \quad |z| < 1$$

$$= \frac{1}{4} \cdot \sum_{n=0}^{+\infty} (-y)^n, \quad |-y| < 1$$

$$= \frac{1}{4} \cdot \sum_{n=0}^{+\infty} (-1)^n y^n, \quad |y| < 1$$

$$= \frac{1}{4} \cdot \sum_{n=0}^{+\infty} (-1)^n \left(\frac{x}{2}\right)^{2n}, \quad \left|\frac{x}{2}\right| < 1$$

$$= \frac{1}{4} \cdot \sum_{n=0}^{+\infty} (-1)^n \left(\frac{x}{2}\right)^{2n}, \quad |x| < 2$$

3.

a) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^n}{n!} + \dots$

$$\sum_{n=0}^{+\infty} (-1)^n \frac{x^n}{n!} = \sum_{n=0}^{+\infty} \frac{(-x)^n}{n!} \underset{\downarrow}{=} \sum_{n=0}^{+\infty} \frac{y^n}{n!} = e^y = e^{-x}$$

substituição
$y = -x$

b) $1 - x^3 + x^6 - x^9 + \dots + (-1)^n x^{3n} + \dots$

$$\sum_{n=0}^{+\infty} (-1)^n x^{3n} = \sum_{n=0}^{+\infty} 1^n \cdot (-1)^n \cdot x^{3n} =$$

$$= \sum_{n=0}^{+\infty} (-1)^{2n} \cdot (-1)^n \cdot x^{3n} = \sum_{n=0}^{+\infty} (-1)^{3n} \cdot x^{3n} =$$

$$= \sum_{n=0}^{+\infty} (-x)^{3n} = \sum_{n=0}^{+\infty} \left((-x)^3\right)^n = \sum_{n=0}^{+\infty} y^n =$$

substituição
$y = (-x)^3$

$$= \frac{1}{1-y} = \frac{1}{1-((-x)^3)} = \frac{1}{1+x^3}$$

5. Cálculo da soma das séries indicadas

a) $$\sum_{n=0}^{+\infty} \frac{(-1)^n 2^{2n} \pi^{2n}}{(2n)!} = \sum_{n=0}^{+\infty} \frac{(-1)^n (2\pi)^{2n}}{(2n)!} =$$

$$= \cos(2\pi) = 1$$

c) $$\sum_{n=0}^{+\infty} \frac{2n+1}{2^n n!} = \sum_{n=0}^{+\infty} \frac{2n}{2^n n!} + \sum_{n=0}^{+\infty} \frac{1}{2^n n!} =$$

$$= \sum_{n=0}^{+\infty} \frac{1}{2^{n-1}(n-1)!} + \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^n \cdot \frac{1}{n!} =$$

$\longrightarrow k = n - 1$

$$= \sum_{n=0}^{+\infty} \frac{1}{2^k k!} + e^{1/2} = 2\sqrt{e}$$

b) $$\sum_{n=0}^{+\infty} \frac{1}{3^n (n+1)} = \sum_{n=0}^{+\infty} \frac{3}{3 \times 3^n (n+1)} = 3 \sum_{n=0}^{+\infty} \frac{1}{3^{n+1}(n+1)} =$$

$$= 3 \sum_{n=0}^{+\infty} \frac{\left(\frac{1}{3}\right)^{n+1}}{n+1} = 3 \ln\left(1 - \frac{1}{3}\right) = 3 \ln\left(\frac{2}{3}\right)$$