Universidade de Aveiro
Departamento de Matemática

Cálculo II — Agrupamento IV 2019/2020

Ficha de Exercícios 6
Transformadas de Laplace (e aplicação às EDO lineares)

1. Para cada uma das funções seguintes, determine $F(s) = \mathcal{L}\{f(t)\}$:

   (a) $f(t) = 2\,\text{sen}\,(3t) + t - 5e^{-t}$; (b) $f(t) = e^{2t}\cos(5t)$; (c) $f(t) = te^{3t}$;
   (d) $f(t) = \pi - 5e^{-t}t^{10}$; (e) $f(t) = (3t-1)\,\text{sen}\,t$;
   (f) $f(t) = (1 - H_\pi(t))\,\text{sen}\,t$; (g) $f(t) = (t-2)^2 e^{2(t-2)} H_2(t)$.

2. Para cada uma das funções seguintes, determine $\mathcal{L}^{-1}\{F(s)\}$:

   (a) $F(s) = \dfrac{2s}{s^2-9}$; (b) $F(s) = \dfrac{4}{s^7}$; (c) $F(s) = \dfrac{1}{s^2+6s+9}$;
   (d) $F(s) = \dfrac{1}{s^2+s-2}$; (e) $F(s) = \dfrac{1}{s^2+4s+6}$; (f) $F(s) = \dfrac{3s-1}{s^2-4s+13}$;
   (g) $F(s) = \dfrac{4s+e^{-s}}{s^2+s-2}$; (h) $F(s) = \dfrac{s}{(s^2+4)^2}$.

3. Calcule o valor dos seguintes integrais impróprios, usando transformadas de Laplace:
   (a) $\displaystyle\int_0^{+\infty} t^{10}\, e^{-2t} dt$; (b) $\displaystyle\int_0^{+\infty} e^{-3t} t\,\text{sen}\, t\, dt$.

4. Seja $f : \mathbb{R} \to \mathbb{R}$ uma função diferenciável. Sabendo que $f'(t) + 2f(t) = e^t$ e que $f(0) = 2$, determine a expressão de $f(t)$.

5. Calcule:
   (a) $\mathcal{L}\{(t - 2 + e^{-2t})\cos(4t)\}$; (b) $\mathcal{L}^{-1}\left\{\dfrac{2s-1}{s^2-4s+6}\right\}$; (c) $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s-1)(s^2+2s+5)}\right\}$.

6. Usando transformadas de Laplace mostre que
$$t^m * t^n = \frac{m!\,n!}{(m+n+1)!}\, t^{m+n+1} \qquad (m, n \in \mathbb{N}_0).$$

7. Determine a solução da equação
$$y'(t) = 1 - \text{sen}\, t - \int_0^t y(\tau)\, d\tau$$
   que satisfaz a condição $y(0) = 0$.

8. Resolva cada um dos seguintes problemas de Cauchy usando transformadas de Laplace.

   (a) $3x' - x = \cos t\,, \quad x(0) = -1$;
   (b) $\dfrac{d^2y}{dt^2} + 36y = 0\,, \quad y(0) = -1\,, \quad \dfrac{dy}{dt}(0) = 2$;
   (c) $y'' + 2y' + 3y = 3t\,, \quad y(0) = 0\,, \quad y'(0) = 1$;
   (d) $y''' + 2y'' + y' = x\,, \quad y(0) = y'(0) = y''(0) - 1 = 0$;
   (e) $y'' + y' = \dfrac{e^{-t}}{2}\,, \quad y(0) = 0 = y'(0)$.

9. Resolva o seguinte problema de valores iniciais recorrendo às transformadas de Laplace:
$$y'' + y = t^2 + 1\,, \quad y(\pi) = \pi^2\,, \quad y'(\pi) = 2\pi.$$
   (Sugestão: Efetuar a substituição definida por $x = t - \pi$).

Soluções

1.

(a) $\dfrac{6}{s^2+9}+\dfrac{1}{s^2}-\dfrac{5}{s+1}\,,\quad s>0;$ (b) $\dfrac{s-2}{(s-2)^2+25}\,,\quad s>2;$ (c) $\dfrac{1}{(s-3)^2}\,,\quad s>3;$

(d) $\dfrac{\pi}{s}-\dfrac{5\cdot 10!}{(s+1)^{11}}\,,\quad s>0;$ (e) $\dfrac{6s}{(s^2+1)^2}-\dfrac{1}{s^2+1}\,,\quad s>0;$

(f) $\dfrac{1}{s^2+1}+\dfrac{e^{-\pi s}}{s^2+1}\,,\quad s>0;$ (g) $e^{-2s}\dfrac{2!}{(s-2)^3}\,,\quad s>2.$

2.

(a) $2\cosh(3t)=e^{3t}+e^{-3t}\,,\quad t\geq 0;$ (b) $\dfrac{t^6}{180}\,,\quad t\geq 0;$

(c) $t\,e^{-3t}\,,\quad t\geq 0;$ (d) $\dfrac{1}{3}e^t-\dfrac{1}{3}e^{-2t}\,,\quad t\geq 0;$

(e) $\dfrac{e^{-2t}}{\sqrt{2}}\,\mathrm{sen}\,(\sqrt{2}t)\,,\quad t\geq 0;$ (f) $e^{2t}\Big(3\cos(3t)+\dfrac{5}{3}\mathrm{sen}\,(3t)\Big)\,,\quad t\geq 0.$

(g) $\dfrac{4}{3}\,e^t+\dfrac{8}{3}\,e^{-2t}+\dfrac{1}{3}\,H_1(t)e^{t-1}-\dfrac{1}{3}\,H_1(t)e^{-2t+2};$ (h) $\dfrac{1}{4}\,t\,\mathrm{sen}\,(2t).$

3. (a) $\dfrac{10!}{2^{11}};$ (b) $\dfrac{3}{50}.$

4. $f(t)=\dfrac{1}{3}e^t+\dfrac{5}{3}e^{-2t}.$

5.

(a) $\dfrac{s^2-16}{(s^2+16)^2}-\dfrac{2s}{s^2+16}+\dfrac{s+2}{(s+2)^2+16}\,,\quad s>0;$

(b) $e^{2t}\left(2\cos(\sqrt{2}\,t)+\dfrac{3}{\sqrt{2}}\,\mathrm{sen}\,(\sqrt{2}\,t)\right)\,,\quad t\geq 0.$ (c) $\dfrac{1}{4}\,e^t-\dfrac{1}{4}\,e^{-t}\,\cos(2t)+\dfrac{3}{4}\,e^{-t}\,\mathrm{sen}\,(2t)\,,\quad t\geq 0.$

6. –

7. $\left(1-\dfrac{t}{2}\right)\mathrm{sen}\,t.$

8.

(a) $x(t)=\dfrac{3}{10}\,\mathrm{sen}\,t-\dfrac{1}{10}\,\cos t-\dfrac{9}{10}\,\mathrm{e}^{\frac{t}{3}};$ (b) $y(t)=\dfrac{1}{3}\,\mathrm{sen}\,(6t)-\cos(6t);$

(c) $y(t)=t-\dfrac{2}{3}+\dfrac{2}{3\sqrt{2}}\,\mathrm{e}^{-t}\mathrm{sen}\,(\sqrt{2}\,t)+\dfrac{2}{3}\,\mathrm{e}^{-t}\cos(\sqrt{2}\,t);$

(d) $y(x)=\dfrac{1}{2}\,(x^2-4x+8)-2e^{-x}(x+2);$ (e) $y(t)=\dfrac{e^{-t}}{2}\,\big(e^t-t-1\big).$

9. $y(t)=(t-\pi)^2+2\pi(t-\pi)+\pi^2-1+\cos(t-\pi)=t^2-1-\cos t.$