6.

a) $x + y y' = 0$

$$x + y \frac{dy}{dx} = 0$$

$$x = -y \frac{dy}{dx}$$

$$\int x \, dx = -\int y \, dy$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + C \ , \ C \in \mathbb{R}$$

$$x^2 = -y^2 + C \ , \ C \in \mathbb{R}$$

$$\underline{\underline{x^2 + y^2 = C}} \ , \ C \in \mathbb{R}$$

c) $(t^2 - x t^2)\frac{dx}{dt} + x^2 = -t x^2$

$x(t)$: x dependente, t independente

$$(t^2 - xt^2)\frac{dx}{dt} = -t x^2 - x^2$$

$$\frac{dx}{dt} = \frac{-t x^2 - x^2}{t^2 - x t^2} = \frac{t x^2 + x^2}{x t^2 - t^2}$$

$$\frac{dx}{dt} = \frac{x^2 (t+1)}{(x-1)\, t^2}$$

$$\int \frac{x-1}{x^2} dx = \int \frac{t+1}{t^2} dt$$

- $\int \frac{x-1}{x^2} dx = \int \frac{1}{x} - \frac{1}{x^2} \, dx = \ln|x| + \frac{1}{x} + C_1, \ C_1 \in \mathbb{R}$
- $\int \frac{t+1}{t^2} dt = \int \frac{1}{t} + \frac{1}{t^2} \, dt = \ln|t| - \frac{1}{t} + C_2, \ C_2 \in \mathbb{R}$

$$\ln|x| + \frac{1}{x} = \ln|t| - \frac{1}{t} + C_1 \ , \ C_1 \in \mathbb{R}$$

$$\ln|x| - \ln|t| = -\frac{1}{x} - \frac{1}{t} + C_1, \ C_1 \in \mathbb{R}$$

$$\ln\left|\frac{x}{t}\right| = -\frac{1}{t} - \frac{1}{x} + C_1, \ C_1 \in \mathbb{R}$$

$$\frac{x}{t} = e^{-1/t - 1/x + C_1} = e^{-1/t} \cdot e^{-1/x}$$

$$\frac{x}{t} = e^{-1/t - 1/x} \cdot C, \ C \in \mathbb{R}$$

d) $(x^2-1)y' + 2xy^2 = 0$

$$(x^2-1)\frac{dy}{dx} + 2xy^2 = 0$$

$$\frac{dy}{dx} = \frac{-2xy^2}{x^2-1}$$

$$\frac{1}{y^2}\,dy = \frac{-2x}{x^2-1}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int \frac{-2x}{x^2-1}\,dx$$

$$-\frac{1}{y} = -\ln|x^2-1| + c$$

$$\cdot \int -\frac{2x}{x^2-1}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + c =$$

$u = x^2-1$ $\qquad = -\ln|x^2-1| + c$

$du = 2x\,dx$

$$\frac{1}{y} = \ln|x^2-1| - c$$

$$y = \frac{1}{\ln|x^2-1|-c}, \quad c \in \mathbb{R}$$

Resolver a seguinte EDO

$y' + p(x)\, y = q(x)$

1) $x y' - y = x - 1 \quad , \quad x > 0$

$p(x) = -\frac{1}{x} \quad , \quad q(x) = 1 - \frac{1}{x}$

$$y' - \frac{1}{x} y = 1 - \frac{1}{x}$$

Calcular fator integrante

$$\mu(x) = e^{\int p(x)\, dx} = e^{-\ln|x|} = \frac{1}{x} \quad , \quad x > 0$$

$$\underbrace{\frac{1}{x} y' - \frac{1}{x^2} y}_{} = \frac{1}{x} - \frac{1}{x^2}$$

$$\left(\frac{1}{x} y\right)' = \frac{1}{x} - \frac{1}{x^2}$$

$$\frac{1}{x} y = \int \frac{1}{x} - \frac{1}{x^2}\, dx$$

$$\frac{1}{x} y = \ln x + \frac{1}{x} + C \quad , \quad C \in \mathbb{R}$$

$$y = x \ln x + 1 + Cx \quad , \quad C \in \mathbb{R}$$

2) $xy' + y - e^{x} = 0 \quad , \quad x > 0$

$xy' + y = e^{x}$

$y' + \frac{1}{x} y = \frac{1}{x} e^{x}$

$p(x) = \frac{1}{x}$

$q(x) = \frac{e^{x}}{x}$

fator integrante:

$\mu(x) = e^{\int p(x)\, dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$xy' + y = e^{x}$

$(x \cdot y)' = e^{x}$

$xy = \int e^{x}\, dx$

$xy = e^{x} + C \Rightarrow y = \frac{e^{x}}{x} + \frac{C}{x} \quad , \quad C \in \mathbb{R}$

3.

a) $y' + 2y = \cos x$ $\qquad p(x) = 2$

$q(x) = \cos x$

$$\mu(x) = e^{\int 2\,dx} = e^{2x}$$

$$e^{2x} y' + e^{2x} 2y = e^{2x} \cos x$$

$$(e^{2x} y)' = e^{2x} \cos x$$

$$e^{2x} y = \int e^{2x} \cos x \, dx$$

$$\cdot \int e^{2x} \cos x \, dx = \frac{1}{5}\left(e^{2x} \operatorname{sen} x + 2e^{2x} \cos x\right) + C$$

$u = e^{2x}$ $\qquad dv = \cos x$

$du = 2e^{2x}$ $\qquad v = \operatorname{sen} x$

$$= e^{2x} \operatorname{sen} x - \left[\int 2e^{2x} \cdot \operatorname{sen} x \, dx\right] =$$

$$= e^{2x} \operatorname{sen} x - 2\int e^{2x} \operatorname{sen} x \, dx$$

$u = e^{2x}$ $\qquad dv = \operatorname{sen} x$

$du = 2e^{2x}$ $\qquad v = -\cos x$

$$= e^{2x} \operatorname{sen} x - 2\left[-e^{2x} \cos x + 2\int e^{2x} \cos x \, dx\right]$$

$$\int e^{2x} \cos x \, dx = e^{2x} \operatorname{sen} x + 2e^{2x} \cos x - 4\int e^{2x} \cos x \, dx$$

$$5\int e^{2x} \cos x \, dx = e^{2x} \operatorname{sen} x + 2e^{2x} \cos x$$

$$\int e^{2x} \cos x \, dx = \frac{1}{5}\left(e^{2x} \operatorname{sen} x + 2e^{2x} \cos x\right) + C, C \in \mathbb{R}$$

$$e^{2x} y = \frac{1}{5}\left(e^{2x} \operatorname{sen} x + 2e^{2x} \cos x\right) + C$$

$$y = \frac{1}{5} \operatorname{sen} x + \frac{2}{5} \cos x + \frac{C}{e^{2x}}, \quad C \in \mathbb{R}$$

b) $x^3y' - y - 1 = 0$ $\quad y' + p(x)y = q(x)$

$x^3 y' - y = 1 \iff y' - \frac{1}{x^3}y = \frac{1}{x^3}$

$p(x) = -\frac{1}{x^3}$

$q(x) = \frac{1}{x^3}$

Cálculo do fator integrante:

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int -\frac{1}{x^3}\,dx} = e^{-\int x^{-3}\,dx} =$$

$$= e^{-\frac{x^{-3+1}}{-3+1}} = e^{-\frac{x^{-2}}{-2}} = e^{+\frac{1}{2x^2}}$$

$$e^{\frac{1}{2x^2}}y' - \frac{e^{\frac{1}{2x^2}}}{x^3}y = \frac{e^{\frac{1}{2x^2}}}{x^3} \iff$$

$$\left(e^{\frac{1}{2x^2}}y\right)' = \frac{e^{\frac{1}{2x^2}}}{x^3} \iff e^{\frac{1}{2x^2}}y = \int e^{\frac{1}{2x^2}} \cdot \frac{1}{x^3}\,dx$$

$$\iff e^{\frac{x^{-2}}{2}}y = \int \frac{e^{\frac{x^{-2}}{2}}}{x^3}\,dx \quad (*)$$

$$\int \frac{e^{\frac{x^{-2}}{2}}}{x^3}\,dx = \int -e^{u}\,du = -e^{u} = -e^{\frac{1}{2}x^{-2}}$$

$u = \frac{1}{2}x^{-2}$

$du = -\frac{\cancel{2}}{\cancel{2}}x^{-3} = -x^{-3} = -\frac{1}{x^3}\,dx$

$-du = \frac{1}{x^3}\,dx$

(*)

$$e^{\frac{1}{2}x^{-2}}y = -e^{\frac{1}{2}x^{-2}} + C, \quad C \in \mathbb{R}$$

$$y = -\frac{e^{\frac{1}{2}x^{-2}}}{e^{\frac{1}{2}x^{-2}}} + \frac{C}{e^{\frac{1}{2}x^{-2}}}, \quad C \in \mathbb{R}$$

$$y = -1 + C \cdot e^{-\frac{1}{2}x^{-2}}, \quad C \in \mathbb{R}$$

Folha 5 $y' + p(x)y = q(x)$

7.

b) $xy + x + y'\sqrt{4+x^2} = 0 \quad , \; y(0) = 1$

$$y' + \underbrace{\frac{x}{\sqrt{4+x^2}}}_{p(x)} y = \underbrace{-\frac{x}{\sqrt{4+x^2}}}_{q(x)}$$

$$\int p(x)\,dx = \int \frac{x}{\sqrt{4+x^2}}\,dx = \frac{1}{2}\int \frac{1}{u^{1/2}}\,du =$$

$u = 4+x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\,\frac{u^{-1/2+1}}{-1/2+1} =$$

$$= \frac{1}{\cancel{2}}\,\frac{u^{1/2}}{\cancel{1/2}} = u^{1/2} = (4+x^2)^{1/2} = \sqrt{4+x^2}$$

$$\mu(x) = e^{\sqrt{4+x^2}}$$

$$\underbrace{e^{\sqrt{4+x^2}}\,y' + \frac{x\,e^{\sqrt{4+x^2}}}{\sqrt{4+x^2}}\,y}\; = -\frac{x\,e^{\sqrt{4+x^2}}}{\sqrt{4+x^2}}$$

$$\left(e^{\sqrt{4+x^2}}\cdot y\right)' = -\frac{x\,e^{\sqrt{4+x^2}}}{\sqrt{4+x^2}}$$

$$e^{\sqrt{4+x^2}}\,y = \int -\frac{x\,e^{\sqrt{4+x^2}}}{\sqrt{4+x^2}}\,dx \quad (*)$$

$$\int -\frac{x\,e^{\sqrt{4+x^2}}}{\sqrt{4+x^2}}\,dx = \; = -\int e^u\,du = -e^u =$$

$$= -e^{\sqrt{4+x^2}} + c, \; c \in \mathbb{R}$$

$u = \sqrt{4+x^2}$

$du = \frac{1}{\cancel{2}}(4+x^2)^{-1/2}\cdot \cancel{2}x$

$= \frac{x}{\sqrt{4+x^2}}\,dx$

(*) $e^{\sqrt{4+x^2}}\, y = \int -\frac{x\, e^{\sqrt{4+x^2}}}{\sqrt{4+x^2}}\, dx$

$$e^{\sqrt{4+x^2}}\, y = -\, e^{\sqrt{4+x^2}} + C, \; C \in \mathbb{R}$$

$$y = -1 + \frac{C}{e^{\sqrt{4+x^2}}}, \; C \in \mathbb{R}$$

$\boxed{y(0) = 1}$

x

$$1 = -1 + \frac{C}{e^{\sqrt{4+0^2}}}$$

$$1 = -1 + \frac{C}{e^2} \Leftrightarrow 2e^2 = C$$

Solução da equação do problema de valor inicial

$$y = -1 + \frac{2e^2}{e^{\sqrt{4+x^2}}}$$

Exemplo solução homogénea de grau 0

$$y' = \underbrace{\frac{y-x}{y+x}}_{f(x,y)}$$

$$f(\lambda x, \lambda y) = \frac{\lambda y - \lambda x}{\lambda y + \lambda x} = \frac{\cancel{\lambda}(y-x)}{\cancel{\lambda}(y+x)} = \frac{y-x}{y+x} =$$

$$= f(x,y), \lambda \neq 0$$

$\therefore$ f é uma função homogénea de grau 0 sempre que $\lambda \neq 0$

$$y' = \underbrace{\frac{x^2 + xy + y^2}{x^2}}_{f(x,y)}$$

f é homogénea de grau zero

$$z = \frac{y}{x}$$

Aplicar mudança de variável $zx = y \Rightarrow y' = z + xz'$

$$z + xz' = \frac{x^2 + x \cdot xz + (zx)^2}{x^2} =$$

$$= \frac{\cancel{x^2}(1 + z + z^2)}{\cancel{x^2}} = \underbrace{1 + z + z^2}_{f(1,z)}$$

$$\cancel{z} + xz' = 1 + \cancel{z} + z^2 \Leftrightarrow \underbrace{x\frac{dz}{dx} = 1 + z^2}_{\text{eq. diferencial de variáveis separáveis}}$$

$$\frac{1}{1+z^2}\, dz = \frac{1}{x}\, dx$$

$$\arctan z = \ln|x| + c \quad , c \in \mathbb{R}$$

$$\arctan\left(\frac{y}{x}\right) = \ln|x| + c$$

$$\frac{y}{x} = \tan(\ln|x| + c)$$

$$y = x\tan(\ln|x| + c), \quad c \in \mathbb{R}$$

Folha 5

$y' = f(x,y)$ → homogénea de grau 0

3. a) $(x^2+y^2)\,y' = xy$

$$y' = \underbrace{\frac{xy}{x^2+y^2}}_{f(x,y)}$$

f é homogénea de grau 0 → vamos provar isso

$$f(\lambda x, \lambda y) = \frac{\lambda x \lambda y}{(\lambda x)^2+(\lambda y)^2} = \frac{\lambda^2 xy}{\lambda^2 x^2 + \lambda^2 y^2} =$$

$$= \frac{\lambda^2 xy}{\lambda^2(x^2+y^2)} = \frac{xy}{x^2+y^2} = f(x,y), \quad \lambda \neq 0$$

$\boxed{zx = y}$ → substituição $\quad y' = z + xz' \quad z = \frac{y}{x}$

$$z + xz' = \frac{x^2 z}{x^2 + x^2 z^2} = \frac{x^2 z}{x^2(1+z^2)} = \underbrace{\frac{z}{1+z^2}}_{f(1,z)}$$

$$z + xz' = \frac{z}{1+z^2} \Leftrightarrow x\frac{dz}{dx} = \frac{z}{1+z^2} - z$$

$$\Leftrightarrow x\frac{dz}{dx} = \frac{z - z(1+z^2)}{1+z^2} \Leftrightarrow x\frac{dz}{dx} = \frac{-z^3}{1+z^2}$$

$$\Leftrightarrow -\frac{1+z^2}{z^3}\,dz = \frac{1}{x}\,dx$$

$$\int \frac{1+z^2}{z^3}\,dz = \int\left(\frac{1}{z^3} + \frac{1}{z}\right)dz = \int\left(z^{-3} + \frac{1}{z}\right)dz =$$

$$= \frac{z^{-2}}{-2} + \ln|z| + c, \quad c \in \mathbb{R}$$

$$= -\frac{1}{2z^2} + \ln|z| + c$$

$$\frac{1}{2z^2} - \ln|z| = \ln|x| + c, \quad c \in \mathbb{R}$$

$$\frac{1}{2z^2} = \ln|xz| + c \Leftrightarrow \frac{x^2}{2y^2} = \ln|y| + c, \quad c \in \mathbb{R}, \ y \neq 0$$

$y = 0$ é uma solução singular da equação inicial

EDO de Bernoulli (exemplo)

$$y' + y = e^{x} y^{2} \longrightarrow \alpha = 2$$

$$y^{-2} y' + y^{-2} y = e^{x}$$

$$y^{-2} y' + y^{-1} = e^{x}$$

Substituição : $z = y^{1-\alpha} = y^{1-2} = y^{-1}$

$$z' = -y^{-2} \cdot y'$$

$$-z' = y^{-2} \cdot y'$$

$$-z' + z = e^{x} \iff z' - z = -e^{x}$$

$p(x) = -1$
$q(x) = -e^{x}$

$$z' + (1-\alpha)\, a(x)\, z = (1-\alpha)\, b(x)$$

$$\int p(x)\,dx = \int -1\,dx = -x$$

$$\mu(x) = e^{-x}$$

$$\underbrace{e^{-x} z' - z e^{-x}}_{} = -e^{x} \cdot e^{-x}$$

$$(e^{-x} z)' = -1$$

$$e^{-x} z = \int -1\,dx$$

$$e^{-x} z = -x + c, \; C \in \mathbb{R}$$

$$z = \frac{(c-x)}{e^{-x}}, \; C \in \mathbb{R}$$

$$z = (c-x)\, e^{x}, \; C \in \mathbb{R}$$

$$\frac{1}{y} = (c-x)\, e^{x}$$

$$y = \frac{1}{(c-x)\, e^{x}}, \; C \in \mathbb{R}$$

18/05/2020

Folha 5.

$y' = f(x,y)$

10. b) $y'\left(1 - \ln\frac{y}{x}\right) = \frac{y}{x}, \quad x > 0$

Provar:

$$\Leftrightarrow y' = \underbrace{\frac{y}{x} \cdot \frac{1}{1 - \ln\left(\frac{y}{x}\right)}}_{f(x,y)}$$

$\begin{cases} f(x,y) \text{ é homogénea de grau } 0 \\ f(\lambda x, \lambda y) = \ldots = f(x,y) \end{cases}$

Aplicando o método: $zx = y \Rightarrow z = \frac{y}{x}$

$$z + xz' = z \cdot \frac{1}{1 - \ln z}$$

$$\Leftrightarrow z + x\frac{dz}{dx} = \frac{z}{1 - \ln z} \Leftrightarrow x\frac{dz}{dx} = \frac{z}{1-\ln z} - z$$

$$\Leftrightarrow x\frac{dz}{dx} = \frac{z\ln z \cancel{-z} \cancel{+z}}{1 - \ln z} \Leftrightarrow \frac{1 - \ln z}{z \ln z}\, dz = \frac{1}{x}\, dx \quad ⊛$$

$$\int \frac{1 - \ln z}{z\ln z}\, dz = \int \frac{1}{z\ln z}\, dz - \int \frac{\cancel{\ln z}}{z\cancel{\ln z}}\, dz =$$

$u = \ln z$

$du = \frac{1}{z}\, dz$

$$= \int \frac{1}{u} - \ln|z| = \ln|u| - \ln|z| = \ln\left|\ln|u|\right| - \ln|z|$$

integrando...

$$⊛ \Leftrightarrow \ln\left|\ln|z|\right| - \ln|z| = \ln|x| + c_1$$

$$\Leftrightarrow \ln\left|\ln(z)\right| = \ln|xz| + c_1$$

$$\Leftrightarrow \ln\left|\ln\frac{y}{x}\right| = \ln|y| + c_1 \Leftrightarrow \ln\left|\frac{y}{x}\right| = y \cdot e^{c_1}$$

$$\Leftrightarrow \ln\left|\frac{y}{x}\right| = y \cdot C \Leftrightarrow \frac{y}{x} = e^{yC} \Leftrightarrow y = xe^{yC}, C \in \mathbb{R}$$

Exercício:
$$\begin{cases} x^2 y' - 2xy = 3y^4 \\ y(1) = 1/2 \end{cases}$$

$$x^2 y' - 2xy = 3y^4 \quad (4) \longrightarrow \alpha = 4$$

$$y' - \frac{2}{x} y = \frac{3y^4}{x^2}$$

$$y^{-4} \cdot y' - \frac{2}{x} y^{-3} = \frac{3}{x^2}$$

H.V.

$$z = y^{1-4} = y^{-3}$$

$$z = y^{-3}$$

$$z' = -3 y^{-4} \cdot y' \iff \frac{z'}{-3} = y^{-4} \cdot y'$$

$$\frac{z'}{-3} - \frac{2}{x} z = \frac{3}{x^2} \iff \boxed{z' + \frac{6}{x} z = -\frac{9}{x^2}} \quad (*)$$

Usando fator integrante $\quad p(x) = \frac{6}{x}, \quad g(x) = -\frac{9}{x^2}$

$$\mu(x) = e^{\int \frac{6}{x} dx} = e^{6 \cdot \ln|x|} = e^{\ln|x|^6} = x^6$$

$(*)$

$$\underbrace{z' \cdot x^6 + 6x^5 z} = -9x^4$$

$$(x^6 \cdot z)' = -9x^4 \iff x^6 \cdot z = -\frac{9}{5} x^5 + C, \ C \in \mathbb{R}$$

$$\iff z = -\frac{9}{5} \cdot \frac{1}{x} \cdot \frac{C}{x^6}, \ C \in \mathbb{R}$$

$$z = y^{-3}$$

$$z = \frac{1}{y^3}$$

$$\iff \frac{1}{y^3} = -\frac{9}{5} \cdot \frac{1}{x} + \frac{C}{x^6}$$

$$y = y(x)$$

$$y(1) = 1/2$$

$x = 1$

$$\frac{1}{(y(1))^3} = -\frac{9}{5} \cdot \frac{1}{1} + \frac{C}{1^6}$$

$$\frac{1}{(1/2)^3} = -\frac{9}{5} + C \iff 8 + \frac{9}{5} = C \iff C = \frac{49}{5}$$

Solução do p.v.i: $\quad \frac{1}{y^3} = -\frac{9}{5} \frac{1}{x} + \frac{49}{5x^6}$

Slide 28 (exemplo)

$$y' - 2y = \underbrace{e^{5x}}_{b(x)}$$

Equação homogénea associada é $y' - 2y = 0$ **

Solução geral de ** :

$$\frac{dy}{dx} - 2y = 0 \Leftrightarrow \frac{dy}{dx} = 2y \Leftrightarrow \frac{1}{2y}\,dy = dx$$

$$\Leftrightarrow \frac{1}{2}\ln y = x + C_1 \Leftrightarrow \ln y = 2x + C_2$$

$$\Leftrightarrow y = e^{2x} \cdot e^{C_2} \Leftrightarrow y = e^{2x} \cdot C, \ C \in \mathbb{R}$$

$y_p = \frac{1}{3} e^{5x}$ é solução particular de $y' - 2y = e^{5x}$

$$y_p' - 2y_p = \frac{5}{3}e^{5x} - \frac{2}{3}e^{5x} = \frac{3}{3}e^{5x} = e^{5x}$$ ✓

Pelo Teorema do slide 28, uma solução geral da equação completa é

$$y = \underbrace{\frac{1}{3}e^{5x}}_{\substack{\text{particular} \\ y_p}} + \underbrace{e^{2x} \cdot C}_{\substack{\text{homogénea} \\ y_h}}, \ C \in \mathbb{R}$$

Folha 5

13. Considere a EDO linear homogénea (de coeficientes não constantes)

$(1-x)y'' + xy' - y = 0$, com $x \in]1, +\infty[$

a) Mostrar que $\{e^x, x\}$ é um S.F.S da equação.

$$W(x, e^x) = \begin{bmatrix} x & e^x \\ 1 & e^x \end{bmatrix}$$

$$\det(W) = xe^x - e^x = e^x(x-1) \neq 0$$

Logo $\{e^x, x\}$ é um SFS da equação

b) Solução geral da edo

$\{x, e^x\}$

$$y = C_1 x + C_2 e^x \quad , C_1, C_2 \in \mathbb{R}$$

Digitalizada com CamScanner

Slide 36 (exemplo)

$$y^{(5)} + 2y^{(4)} + 4y^{(3)} + 8y^{(2)} + 4y' + 8y = 0$$

Polinómio característico:

$p(x) = x^5 + 2x^4 + 4x^3 + 8x^2 + 4x + 8$

$p(x) = 0$

	1	2	4	8	4	8
-2		-2	0	-8	0	-8
	1	0	4	0	4	$\boxed{0}$ → resto

$$(x+2)\underbrace{(x^4 + 4x^2 + 4)}$$

$\Leftrightarrow (x^2 + 2)^2 = 0$

$\Leftrightarrow x^2 + 2 = 0$

$\Leftrightarrow x^2 = -2$

$\Leftrightarrow x = \pm i\sqrt{2}$

Exemplo: 20/05/2020

1) $2y^{(5)} - 8y^{(4)} + 8y''' = 0$ equação linear homogénea de ordem 5

polinómio característico

$$p(x) = 2x^5 - 8x^4 + 8x^3 = 0$$

$$2x^3(x^2 - 4x + 4) = 0$$

$$x^3 = 0 \quad \vee \quad x^2 - 4x + 4 = 0$$

$$x = 0 \quad \vee \quad (x-2)^2 = 0$$

$$\underline{x = 0} \quad \vee \quad \underline{x = 2}$$

$$SFS = \{ e^{0x}, x e^{0x}, x^2 e^{0x}, e^{2x}, x e^{2x} \}$$

$$= \{ 1, x, x^2, e^{2x}, x e^{2x} \}$$

Solução geral da EDO:

Solução geral da equação homogénea

$$\boxed{C_1 + C_2 x + C_3 x^2 + C_4 e^{2x} + C_5 x e^{2x}}, \; C_1, C_2, C_3, C_4, C_5 \in \mathbb{R}$$

2) $y'' + 2y' + 5y = 0$

$$p(x) = x^2 + 2x + 5 = 0$$

$$x_1 = -1 + 2i, \quad x_2 = -1 - 2i$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1} = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$SFS = \{ e^{-x} \cos(2x), e^{-x} \operatorname{sen}(2x) \}$$

Solução geral da EDO:

$$C_1 e^{-x} \cos(2x) + C_2 e^{-x} \operatorname{sen}(2x), \; C_1, C_2 \in \mathbb{R}$$

Folha 5

14.

e) $y'' + 4y = 0$

$$p(x) = x^2 + 4 = 0$$
$$x^2 = -4$$
$$x = \pm 2i$$
$$x_1 = 2i, \quad x_2 = -2i$$

$$SFS = \{ e^{\alpha x} \cos(\beta x) \; ; \; e^{\alpha x} sen(\beta x) \}$$

Sistema fundamental de soluções

$$= \{ e^{0x} \cos(2x) \; ; \; e^{0x} sen(2x) \}$$

$$= \{ \cos(2x) \; , \; sen(2x) \}$$

Solução geral:

$$C_1 \cos(2x) + C_2 \, sen(2x), \; C_1, C_2 \in \mathbb{R}$$

xemplo slide 40

$$1y'' + y = \operatorname{cosec} x \quad , x \in]0, \pi[$$

1) Solucionar a homogénea

$$y'' + y = 0$$
$$p(x) = x^2 + 1 = 0$$
$$x^2 = -1$$
$$x \pm i$$

$$SFS = \{ e^{0x} \cos(x) , e^{0x} \operatorname{sen}(x) \}$$

Solução geral homogénea:

$$y = C_1 \cos(x) + C_2 \operatorname{sen}(x) \quad , C_1, C_2 \in \mathbb{R}$$

2) Solução particular

$$y_p = C_1(x) \cos x + C_2(x) \operatorname{sen} x$$

onde

$$\begin{bmatrix} \cos x & \operatorname{sen} x \\ -\operatorname{sen} x & \cos x \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \operatorname{cosec}(x) \end{bmatrix} \leftarrow \frac{b(x)}{a_0}$$

$$C_1'(x) = \frac{\begin{vmatrix} 0 & \operatorname{sen} x \\ \operatorname{cosec} x & \cos x \end{vmatrix}}{\underbrace{\begin{vmatrix} \cos x & \operatorname{sen} x \\ -\operatorname{sen} x & \cos x \end{vmatrix}}} = \frac{-\operatorname{cosec} x \cdot \operatorname{sen} x}{1} =$$

$$= \frac{1}{\operatorname{sen} x}$$

$$\det = \cos^2 x + \operatorname{sen}^2 x$$
$$= 1$$

$$= -\frac{1}{\cancel{\operatorname{sen} x}} \cdot \cancel{\operatorname{sen} x} = -1$$

$$\boxed{C_1'(x) = -1}$$

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6.

c) $y'' + y = 2y + 3 - 6x$

$\Leftrightarrow y'' - y' - 2y = \underbrace{3 - 6x}_{b(x)}$

· Solução da homogénea

$y'' - y' - 2y = 0$

$p(x) = x^2 + x - 2 = 0 \Leftrightarrow (x+2)(x-1) = 0$ → polinómio caracteristico

$\Leftrightarrow x = -2 \lor x = 1$

$SFS = \{ e^{-2x}, e^{x} \}$

Solução geral da homogénea:

$y_h = C_1 e^{2x} + C_2 e^{x} \quad , C_1, C_2 \in \mathbb{R}$

· Solução particular

$y_p = C_1(x) e^{-2x} + C_2(x) e^{x}$ onde

$$\begin{bmatrix} e^{-2x} & e^{x} \\ -2e^{-2x} & e^{x} \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 3-6x \end{bmatrix} \leftarrow \frac{b(x)}{a_0}$$

derivadas

$$C_1' = \frac{\begin{bmatrix} 0 & e^{x} \\ 3-6x & e^{x} \end{bmatrix}}{\begin{bmatrix} e^{-2x} & e^{x} \\ -2e^{-2x} & e^{x} \end{bmatrix}} = \frac{-e^{x}(3-6x)}{e^{-x} + 2e^{-x}} =$$

$$= \frac{-\cancel{3}e^{x}(1-2x)}{+\cancel{3}e^{-x}} = -e^{-2x}(1-2x)$$

$$c_2'(x) = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & 3-6x \end{vmatrix}}{3e^{-x}} = \frac{e^{-2x}(3-6x)}{3e^{-x}} =$$

$$= \frac{\cancel{3}e^{-2x}(1-2x)}{\cancel{3}e^{-x}} = e^{-x}(1-2x)$$

$$\begin{cases} c_1'(x) = -e^{2x}(1-2x) \\ c_2'(x) = e^{-x}(1-2x) \end{cases}$$

$$c_1(x) = \int -e^{2x}(1-2x)\,dx = \int -e^{2x} + 2e^{2x}x\,dx$$

$$\cdot \int 2e^{2x}\cdot x\,dx = xe^{2x} - \int e^{2x}\,dx =$$

$u = 2x \quad du = 2$

$dv = e^{2x} \quad v = \frac{e^{2x}}{2}$

$$= -\frac{e^{2x}}{2} + xe^{2x} - \frac{e^{2x}}{2} = -e^{2x} + xe^{2x}$$

$$c_2(x) = \int e^{-x}(1-2x)\,dx = \int e^{-x} - 2e^{-x}\cdot x\,dx =$$

$$\cdot \int -2x\,e^{-x}\,dx = 2xe^{-x} + 2e^{-x}$$

$$= -e^{-x} + 2xe^{-x} + 2e^{-x} = e^{-x} + 2xe^{-x}$$

$$y_p = \left(-e^{2x} + x e^{2x}\right) e^{-2x} + \left(e^{-x} + 2x e^{-x}\right) e^{x}$$

$$= \cancel{-1} + x \cancel{+1} + 2x = \boxed{3x}$$

Solução geral da 1ª equação

$$y = y_h + y_p$$

$$= c_1 e^{-2x} + c_2 e^{x} + 3x, \quad c_1, c_2 \in \mathbb{R}$$

Exemplo slide 43 (parecido)

Encontrar equação particular de:

$$y' - 3y = \underbrace{2x}_{b(x)} \quad ⊛$$

$$2x = \underbrace{p_m(x)}\, e^{\alpha x} \cos(\beta x)$$

$e^{\alpha x} = 1$, logo $\alpha = 0$

$\cos(\beta x) = 1$, logo $\beta = 0$

zero do polinômio

Como $\underset{\alpha}{0} + i\underset{\beta}{0} = 0$ não é raiz de $p(x)$ então $K = 0$

$$\underbrace{p_m(x) = 2x}_{\text{grau } 1}$$

$$\boxed{m = 1}$$

mesmo grau

• Solução particular

$$y_p = x^K e^{\alpha x}\left[A(x)\cos(\beta x) + B(x)\,\text{sen}(\beta x)\right]$$

$$= x^K e^{0x}\left[A(x)\cos(0) + B(x)\,\text{sen}(0)\right]$$

$$= x^K \cdot A(x)$$

$$= x^0 A(x)$$

$$= A(x)$$

$A(x)$ é de grau 1 então $A(x) = a_0 x + a_1$

$y_p = a_0 x + a_1$

Como y_p deve ser solução particular de ⊛,

$y_p' - 3y_p = 2x \Leftrightarrow a_0 - 3(a_0 x + a_1) = 2x$

derivada de y_p

$\underbrace{-3a_0 x}_{} + \underbrace{a_0 - 3a_1}_{} = 2x + 0$

$-3a_0 = 2$ $\quad = 0$

igualdade de polinómios

$\Leftrightarrow a_0 = -\frac{2}{3}$

$a_0 - 3a_1 = 0$

$\Leftrightarrow -\frac{2}{3} - 3a_1 = 0$

$\Leftrightarrow a_1 = -\frac{2}{9}$

Logo, $y_p = -\frac{2}{3}x - \frac{2}{9}$

Exercício:

$$5y + 2y' + y'' = 2e^{-2x} \quad (*)$$

Calcular a solução geral

· Solução da homogénea

$$5y + 2y' + y'' = 0$$

$$p(x) = x^2 + 2x + 5 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1} = \frac{-2 \pm \sqrt{-16}}{2} =$$

$$= \frac{-2 + 4i}{2} \vee \frac{-2 - 4i}{2}$$

$$= -1 + 2i \quad \vee \quad -1 - 2i$$

$$= -1 \pm 2i \longrightarrow \alpha = -1$$

$$\beta = 2$$

$$SFS = \{ e^{-x} \cos(2x) \, , \, e^{-x} \operatorname{sen}(2x) \}$$

$$y_h = A e^{-x} \cos(2x) + B e^{-x} \operatorname{sen}(2x)$$

Solução particular:

$$2e^{-2x} \cdot 1 = P_m(x) \, e^{\alpha x} \cos(\beta x) \, , \quad \beta = 0 \, , \; \alpha = -2$$

$\cos(0)$

$$P_m(x) = 2 \longrightarrow \text{grau zero}, \; m = 0$$

$\alpha + \beta i = -2 + 0i = -2$ é raiz do polinómio característico? $\boxed{\text{Não}}$

Logo, $K = 0$

Solução particular:

$$y_p = x^0 e^{-2x}\left[Q(x)\cos(\beta x) + R(x)\,\text{sen}(\beta x)\right]$$

$Q(x)$ e $R(x)$ têm grau zero, logo são constantes

$$y_p = x^0 e^{-2x}\left[C\cos(0x) + D\,\text{sen}(0x)\right]$$

$$= e^{-2x}\cdot C \quad , \quad C \in \mathbb{R} \text{ a determinar}$$

Como y_p deve ser solução particular de ⊛

$$5y_p + 2y_p' + y_p'' = 2e^{-2x}$$

$$5Ce^{-2x} + 2\cdot(-2Ce^{-2x}) + 4Ce^{-2x} = 2e^{-2x}$$

$$5Ce^{-2x} \cancel{-4Ce^{-2x}} + \cancel{4Ce^{-2x}} = 2e^{-2x}$$

$$5Ce^{-2x} = 2e^{-2x}$$

$$C = 2/5$$

Logo,

$$y_p = 2/5\, e^{-2x}$$

Solução geral de ⊛:

$$y = y_h + y_p = Ae^{-x}\cos(2x) + Be^{-x}\,\text{sen}(2x) + \frac{2}{5}e^{-2x}$$

$$A, B \in \mathbb{R}$$