Persiapan KBF Matematika
BAB 3: Turunan &
BAB 4: Integral

Kamis/16 NOV 2023

Definisi Integral Tentu
(Limit Jumlah Riemann)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\bar{x_i})\Delta x$$

= luas di atas sb x − luas di bawah sb. x

$\Delta x = \frac{b-a}{n}$ $\quad x_i = a + i\Delta x$

Riemann Kiri : $\bar{x_i} = x_{i-1}$
Riemann Kanan : $\bar{x_i} = x_i$

TDK 1

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

TDK 2

$$\int f(x)\,dx = F(x) \Rightarrow \int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

Contoh:

$$\int_1^4 x^2\,dx = \frac{x^3}{3}\Big|_1^4 = \frac{4^3-1^3}{3} = 21$$

$$\int_{x=0}^{x=9} \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx = \int_0^9 \sqrt{1+\sqrt{x}}\left(\frac{1}{\sqrt{x}}dx\right)$$

$u(x) = 1+\sqrt{x}$
$du = \frac{1}{2\sqrt{x}}\,dx$
$u(0) = 1$
$u(9) = 4$

$$= \int_{u=1}^{u=4} \sqrt{u}\; 2\,du = 2\cdot\frac{u^{3/2}}{3/2}\Big|_1^4 = \frac{4}{3}\left(4^{3/2} - 1^{3/2}\right) = \frac{28}{3}$$

Masalah:

$$\int_0^2 \frac{1}{1+x^2}\,dx \qquad \int_0^2 \frac{x}{1+x^2}\,dx = \int_1^5 \frac{1}{u}\,\frac{du}{2}$$

$u(x) = 1+x^2 \longrightarrow du = 2x\,dx$
$u(0) = 1$
$u(2) = 5$

Integral Numerik

1. Jumlah Riemann Kiri/Kanan/Titik Tengah
   ↳ mengestimasi luas dgn persegi panjang
2. Aturan Trapesium → luas dgn trapesium
3. Aturan Simpson → luas dgn daerah yang dibatasi parabola

$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$ Riemann Kiri

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$ Riemann Kanan

$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$ Riemann Tengah

$M_n = \sum_{i=1}^{n} f\left(\frac{2x_i-1}{2}\right)\Delta x$

- Jumlah Riemann

> Riemann Kiri

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$L_n = \Delta x\left[f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1})\right]$$

> Riemann Kanan

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \Delta x\left[f(x_1) + f(x_2) + \dots + f(x_n)\right]$$

> Riemann Tengah

$$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

- Aturan Trapesium

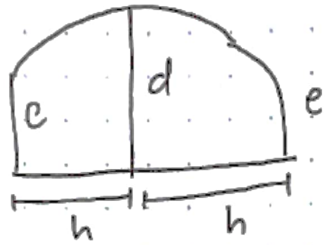
$$T_n = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}\Delta x$$

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + f(x_1) + f(x_2) + \dots + f(x_i) + \dots + \right]$$

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + \underbrace{2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1})}_{\text{yang ditengah aja yang dua}} + f(x_n)\right]$$

u/ ujung "cuma satu" aja

Aturan Simpson



$$A = \frac{h}{3}(c + 4d + e)$$

$$P_n = \sum_{i=1}^{k} \frac{\Delta x}{3} [f(x_{(2i-2)}) + 4f(x_{2i-1}) + f(x_{2i})]$$

$$= \sum_{i=1}^{k} \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

polanya: 1-4-2-4-2-4-2 ...-2-4-2-4-2-4-1

Contoh Soal:

1) Aprosimasi $\int_1^3 \frac{1}{1+x^2}\,dx$ dgn jumlah Riemann kiri, aturan trapesium, dan simpson dgn $n=4$. Kemudian tentukan galat mutlak maksimum.

Pembahasan:

$a = 1$, $b = 3$, $f(x) = \frac{1}{1+x^2}$, $n = 4$

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$$

$$x_i = a + i\Delta x = 1 + \frac{1}{2}i$$

$$L_4 = \Delta x\,[f(x_0) + f(x_1) + f(x_2) + f(x_3)]$$

$$L_4 = \frac{1}{2}[f(1) + f(1{,}5) + f(2) + f(2{,}5)] \approx 0{,}5728$$

$$R_4 = \frac{1}{2}[f(1{,}5) + f(2) + f(2{,}5) + f(3)] \approx 0{,}3728$$

$$M_4 = \frac{1}{2}[f(1{,}25) + f(1{,}75) + f(2{,}25) + f(2{,}75)] \approx \cancel{0{,}3728}\ 0{,}4591$$

$$T_4 = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)]$$

$$T_4 = \frac{\frac{1}{2}}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)]$$

$$T_4 = \frac{1}{4}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$T_4 = \frac{1}{4}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)] \approx 0{,}4728$$

$$P_4 = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$P_4 = \frac{\frac{1}{2}}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$P_4 = \frac{1}{6}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$P_4 \approx 0{,}4637$$

Nilai Sesungguhnya:

$$\int_1^3 \frac{1}{1+x^2}\,dx = \tan^{-1} 3 - \frac{\pi}{4}$$

$$\int_1^3 \frac{1}{1+x^2} \approx 0{,}46365$$

2.) Tentukan n sehingga aturan trapesium akan mengaproksimasi $\int_1^3 \frac{1}{x}\,dx$ dgn galat E_n yg memenuhi $|E_n| \le 0{,}01$

$$E_n = \frac{-(b-a)^3}{12n^2} f''(c), \text{ for } a < c \le b$$

Pembahasan:

$f''(x)$ punya nilai max pd $[a,b]$

$$|E_n| \le \left|\frac{(b-a)^3}{12n^2}\right| \cdot \max\{|f''(x)|\}$$

$$|f''(c)| \le \max_{[a,b]}\{|f''(x)|\}$$

$a = 1 \quad b = 3 \quad f(x) = \frac{1}{x}$

$$f'(x) = -\frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}, \ [1,3]$$

$$\max_{[1,3]}\left\{\left|\frac{2}{x^3}\right|\right\} = \left|\frac{2}{1^3}\right| = |2| = 2$$

Agar max maka pembaginya sekecil mungkin, $x = 1$

$\boxed{\text{Notes :}}$ $f(x) = |x|, \ [-5,1]$ max = 5

$$|E_n| \le \frac{8}{12n^2} \cdot 2 = \frac{4}{3n^2} \le 0{,}01$$

$$\frac{3n^2}{4} \ge 100$$

$$n^2 \ge \frac{400}{3}$$

$$n \ge \frac{20}{\sqrt{3}}$$

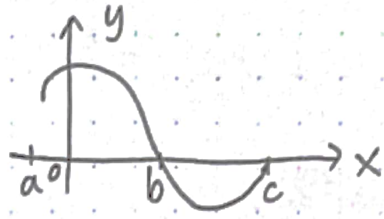
tanda pembulatan keatas

$$n \ge \left\lceil \frac{20}{\sqrt{3}} \right\rceil = \left\lceil \frac{20}{\sqrt{3}} \right\rceil = 12$$

Jadi, agar $|E_n| \le 0{,}01$, pilih $n = 12$

Pembahasan Soal KBF FTI

1.) $h(x)$



Misalkan $f(x) = \int_a^x h(t)\,dt$

Grafik yg merepresentasikan grafik $f(x)$

Jawab:

$f(x) = \int_a^x h(t)\,dt \Leftrightarrow f'(x) = h(x)$ gradien g

$f'(x) = h(x)$
$\hookrightarrow$ gradien h $\rightarrow$ gambarnya adalah grafik $f'(x)$

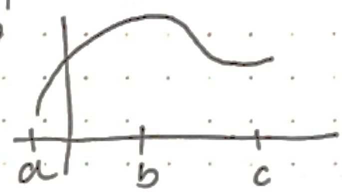
$f'(x)$ menyatakan kemonotonan

f naik pd $(a,b) \rightarrow \oplus$ positif

f turun pd $(b,c) \rightarrow \ominus$ negatif

dan tidak ada singular

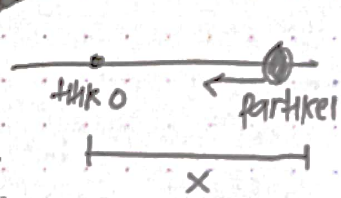
Maka grafik yg merepresentasikan grafik $f(x)$ adalah



2.) Sebuah partikel bergerak pd sumbu x positif mendekat ke titik asal O laju partikel pd setiap saat berbanding lurus dgn kuadrat jaraknya ke titik O. Pd saat $t = 0$, partikel berada 5 cm dari titik O dan bergerak dgn laju 25 cm/detik

Jawab:

(A) Benar : partikel tidak pernah mencapai titik O



Misal $x(t)$ jarak partikel ke titik O

$v(t) = x'(t)$

Laju partikel $\sim kx^2$

$V(t) \approx \frac{dx}{dt} = kx^2$

$t = 0 \rightarrow x = 5$ cm
$\rightarrow V = 25$ cm/s

$x(0) = 5$, $v(0) = -25$
$\hookrightarrow$ mendekati

$\Rightarrow V(0) = k \cdot x(0)^2$
$-25 = k \cdot 25$
$k = 1$ //

$\therefore v(t) = \frac{dx}{dt} = -x^2$

$\Rightarrow \frac{dx}{x^2} = k \, dt$

$\frac{x^{-1}}{-1} = kt + c$
$\hookrightarrow k = 1$

$\frac{x^{-1}}{-1} = -t + c$

$\frac{1}{x} = t + C$

$x(0) = 5 \longrightarrow \frac{1}{x} = t + c$
$\frac{1}{5} = 0 + c$
$c = \frac{1}{5}$ //

$\frac{1}{x} = t + \frac{1}{5} = \frac{5t+1}{5}$

$X(t) = \frac{5}{5t+1}$, $t \geq 0$

Karena $t \geq 0$, maka $x(t) > 0$ sehingga partikel tidak pernah mencapai titik 0.

(B) Benar : persamaan posisi partikel adalah $\frac{dx}{dt} = -x^2$

$X(0,1) = \frac{5}{0,5+1} = \frac{5}{1,5} = \frac{10}{3}$

$V(0,1) = -\left(\frac{10}{3}\right)^2 = -\frac{100}{9}$

(C) salah posisi saat $t = 0,1$ s, partikel berada pd 10 cm dari titik 0

(D) salah posisi saat $t = 0,1$ s, kecepatan partikel berada pada 100 cm/s.
$\hookrightarrow$ karena $v(0,1) = -\frac{100}{9}$ cm/s bukan 100