Bilangan Real

No.
Date . .

Bilangan Real

Bilangan Rasional : - bil. asli
- bil. bulat
- pecahan
- tidak cukup untuk pengukuran

Bilangan Nol : $a \cdot 0 = 0 \cdot a = 0$

$\frac{0}{a} = 0$

$ab = 0$

$\frac{a}{b} = 0$

Selesaikan persamaan dengan sifat-sifat bilangan real

① $2x - 1 = 7$

$2x \cancel{-1} + \cancel{1} = 7 + 1$

$\frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 8$

$x = 4$

② $x^2 - 7x + 10 = 0$

Ingat : $(x+a)(x+b)$

$= x^2 + ax + bx + ab$

$= x^2 + (a+b)x + ab$

$\Leftrightarrow x^2 + (-5-2)x + (-5)(-2) = 0$

$\Leftrightarrow (x-5)(x-2) = 0$

$\frac{1}{x} = 1 \Leftrightarrow \cancel{x} \cdot \frac{1}{\cancel{x}} = x$

$\Leftrightarrow 1 = x$

Warning : $\frac{1}{x} < 1$

$\neq$ ~~$1 < x$~~

Date . .

Logika Matematika (Implikasi)

Jika P maka Q ($P \Rightarrow Q$)

- Jika $x + 3 < 8$, maka $x < 10$ ~~(benar)~~ (benar)
- Jika $x < 5$ maka $x^2 < 25$ (salah) ada x lain yg memenuhi $x^2 > 25$
- Jika $x^2 < 25$ maka $x < 5$ (benar)
- Jika $\frac{1}{x} < 1$ maka $1 < x$ (salah)

Logika Matematika (Biimplikasi)

P jika dan hanya jika Q ($P \Rightarrow Q$ dan $Q \Rightarrow P$)

- $x + 3 < 8$ jika dan hanya jika $x < 10$ (salah)
  ↳ $x + 3 < 8$ jika dan hanya jika $x < 5$
- $x^2 < 25$ jika dan hanya jika $x < 5$ (salah)
  ↳ $x^2 < 25$ jika dan hanya jika $-5 < x < 5$
- $\frac{5}{x} < 0$ jika dan hanya jika $x < 0$ (benar)

Logika Matematika (Quantifier)

-
- $\forall$ bil. real x berlaku $x^2 > 0$ (salah) $\rightarrow 0^2 = 0$
- $\exists x$ sehingga $\sqrt{x} = 100$ (betul) $\rightarrow x = 10^4$
- $\exists x$ sehingga $x > 10^{100}$ (betul) $\rightarrow x = 10^{101}$
- $\forall x$, $\exists y$ sehingga $y > x$. (betul)
- $\exists x$ sehingga $\forall y$ berlaku $y > x$ (salah) → tidak ada x semacam itu
  berapapun x ada y yg tdk memenuhi $y > x$
  Contoh : $y = x - 1$
- Terdapat x sehingga untuk setiap y berlaku $y^2 > x$ (betul)

No.
Date . .

Logika Matematika (Quantifier Lanjutan)

- $\exists \delta > 0$ sehingga berlaku implikasi :
  jika $x < \delta$ maka $2x < \frac{1}{100}$
  → Jika $x < \frac{1}{200}$ maka $2x < \frac{1}{100}$

- $\forall \ \varepsilon > 0$, terdapat $\delta > 0$ sehingga berlaku :
  jika $x < \delta$ maka $2x < \varepsilon$

$\varepsilon = 5$, pilih $\delta = \frac{5}{2}$

$\varepsilon = 2$, pilih $\delta = 1$

$\varepsilon = \frac{1}{10^{100}}$, pilih $\delta = \frac{1}{2.10^{100}}$

⇒ Diberikan berapapun $\varepsilon > 0$, pilih $\delta = \frac{\varepsilon}{2}$

SISTEM KOORDINAT CARTESIUS DAN GRAFIK PERSAMAAN

No.
Date . .

Sistem Koordinat Cartesius dan Grafik Persamaan

Jarak dua titik

$P(x_1, y_1)$ dan $Q(x_2, y_2)$

$$\boxed{d(P,Q) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$

Persamaan lingkaran di Bidang -xy

• Titik pusat $P(a,b)$ dan jari-jari r

$(x-a)^2 + (y-b)^2 = r^2$

• Bentuk persamaan lingkaran lainnya

$x^2 + y^2 + Ax + By + C = 0$

$P\left(-\frac{1}{2}A, -\frac{1}{2}B\right)$, $r = \sqrt{\left(-\frac{1}{2}A\right)^2 + \left(-\frac{1}{2}B\right)^2 - C}$

Tentukan pusat dan jari-jari lingkaran

$\Rightarrow x^2 + y^2 - 2x + 6y = 39$

$\Rightarrow P\left(-\frac{1}{2}(-2), -\frac{1}{2} \cdot 6\right) = P(1, -3)$

$\Rightarrow r = \sqrt{(1)^2 + (-3)^2 - 39}$

$= \sqrt{49} = 7$ //

Titik Tengah Segmen Garis

$P(x_1, y_1)$ dan $Q(x_2, y_2)$

$$\boxed{M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)}$$

Tentukan persamaan lingkaran dgn diameter AB, dimana:

$A(1,3)$, $B(3,-5)$

Jawab: (1) Cari titik pusat

$\Rightarrow P\left(\frac{1+3}{2}, \frac{3-5}{2}\right) = P(2, -1)$

(2) cari jari-jari

$\Rightarrow r = d(A,P)$

$= \sqrt{(1-2)^2 + (3+1)^2}$

$= \sqrt{1+16} = \sqrt{17}$ //

(3) Tulis persamaan lingkaran

$(x-2)^2 + (y+1)^2 = (\sqrt{17})^2$

$\therefore (x-2)^2 + (y+1)^2 = 17$ //

Kemiringan Garis di Bidang -xy

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

Persamaan Garis pada Bidang -xy

l suatu garis mell titik $(1,-2)$ dan $m = 3$

$\Rightarrow$ Apakah garis l mell $(3,2)$? $(5,7)$?

$\Rightarrow$ Gradien antara $(3,2)$ dan $(1,-2)$

$m_1 = \frac{-2-2}{1-3} = \frac{-4}{-2} = 2$ Tidak

$\Rightarrow$ Gradien antara $(5,7)$ dan $(1,-2)$

$m_2 = \frac{-2-7}{1-5} = \frac{-9}{-4}$ Tidak

$$\boxed{y - y_1 = m(x - x_1)}$$

Tentukan pers. garis yg mell titik $(2,7)$ dan $(6,3)$

Jawab: $m = \frac{3-7}{6-2} = -1$

$\rightarrow y - y_1 = m(x - x_1)$ atau $y - y_2 = m(x - x_2)$

$\therefore y - 7 = -(x-2) \lor y - 3 = -(x-6)$

$y = -x + 9$ // $\quad y = -x + 9$ //

• Bentuk lain persamaan garis

$ax + by = c \rightarrow m = \frac{-a}{b}$

$y = mx + c$

• Jarak antara $A(4,-1)$ dan $B(0,1)$

$\therefore d(A,B) = \sqrt{4^2 + 2^2}$

$= \sqrt{20}$

Hubungan dua garis di bidang - xy

• Kapan 2 garis sejajar?

Saat kedua bangun (segitiga) yg mell garis tsb sebangun. (m sama)

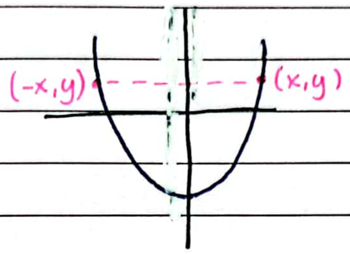
• Kapan 2 garis tegak lurus?

Saat kedua segitiga pada 2 garis juga sebangun dengan $m_1 = \frac{-1}{m_2}$

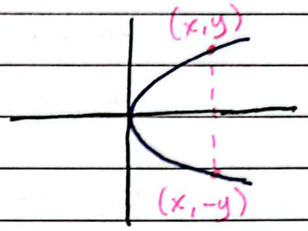
$\boxed{m_1 . m_2 = -1}$

syarat 2 garis ⊥

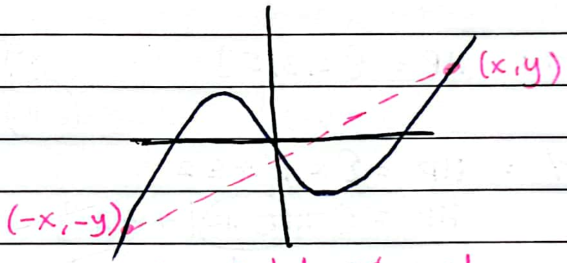
Kesimetrian Grafik



simetri thd sb-y



simetri thd sb-x



simetri thd titik asal

Tentukan jarak antara titik $A(4,-1)$ ke garis $y = 2x+1$

Jawab :

(1.) Cari gradien $y = 2x+1$

$y = 2x+1 \rightarrow m = 2$

(2.) Cari gradien garis yg mell $A(4,-1)$

$y - y_1 = m(x - x_1)$

$y + 1 = m(x-4)$

$y + 1 = \frac{-1}{2}(x-4)$

$y = -\frac{x}{2} + 1$

(3.) Cari titik potong antara $y = 2x+1$ dan $y = -\frac{x}{2} + 1$

$y - 2x = 1$

$y + \frac{x}{2} = 1$ −

$-\frac{5}{2}x = 0$

$x = 0$

$y = 2(0) + 1$

$y = 1$

$\Rightarrow B(0,1)$

Titik Potong dengan Sumbu Koordinat

• Tipot dgn sb-x

↳ $y = 0$

• Tipot dgn sb-y

↳ $x = 0$

Tentukan titik potong dgn sb-x dan sb-y

• $2x + 5y = 20$

Jawab : • Titik potong dgn sb-x

$y = 0$

$2x = 20$

$x = 10 \rightarrow (10,0)$

• Titik potong dgn sb-y

$x = 0$

$5y = 20 \Leftrightarrow y = 4 \rightarrow (0,4)$

Pertidaksamaan dan Nilai Mutlak

No.

Date . .

Pertidaksamaan

Himpunan Penyelesaian Pertidaksamaan adalah himpunan semua bil. real yang memenuhi pertidaksamaan tersebut.

c/ : Tentukan himp. penyelesaian

Jawab: $2x-1 \leq 3$

$\Leftrightarrow \quad 2x \leq 4$

$\Leftrightarrow \quad x \leq 2$

$HP = \{x \mid x \leq 2\}$ / $(-\infty, 2]$

Cara penulisan himpunan

① $HP = \{a, b, c\}$

c/ : $HP = \{2, 3, 5\}$

② $HP = \{x \mid P(x)\}$ / $HP = \{x \in \mathbb{R} \mid P(x)\}$ ↗ opsional

c/ : $HP = \{x \mid x \leq 2$

$HP = \{x \in \mathbb{R} \mid x \leq 2\}$

↳ opsional

Di kuliah ini cukup : $HP = \{x \mid P(x)\}$

Notasi Interval / Selang

- Interval tutup :

→ $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

- Interval buka :

→ $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

→ $(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$

→ $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$

- Interval setengah tutup / buka :

→ $[a, b)$ $(a, b] : \{x \mid a \leq x < b\}$ $\{x \mid a < x \leq b\}$

→ $[a, \infty)$, $(-\infty, b]$

$\{x \mid x \geq a\}$ $\{x \mid x \leq b\}$

* Tak hingga $(\infty, -\infty)$: bukan bil. real → ∞ selalu pake notasi ( )
Tdk boleh mengoperasikan ∞

Notasi Matematika

$\forall$: untuk setiap

$\in$: anggota

$\exists$: terdapat

$\subset$: himpunan bagian

$\mathbb{R}$: himpunan bil. real

$\mid$: sedemikian sehingga

$\emptyset$: himpunan kosong

Contoh soal!

1\. Tentukan himpunan penyelesaian dari $x^2 + 3x - 4 > 0$

Jawab : $x^2 + 3x - 4 > 0$

$(x+4)(x-1) > 0$

++ −− ++

−4 1

$HP = \{x \in \mathbb{R} \mid x < -4 \text{ atau } x > 1\}$

$HP = (-\infty, -4) \cup (1, \infty)$

2\. Tentukan himpunan penyelesaian dari pertidaksamaan

$$\frac{x+1}{x-2} \leq 0$$

Jawab : +++ −−− +++

−1 2

$HP = \{x \mid -1 \leq x < 2\}$

$HP : [-1, 2)$

No. ______

Date ______ . ______ . ______

3. Tentukan himpunan penyelesaian dari pertidaksamaan :

a.) $(x+1)^2(x-2) \geq 0$

b.) $(x+1)^2(x-2) < 0$

c.) $\dfrac{(x+1)^2}{x-2} \leq 0$

Jawab :

a.) $(x+1)^2(x-2) \geq 0$

---   --- [+++

-1 2

$HP = [2, \infty) \cup \{-1\}$

b.) $(x+1)^2(x-2) < 0$

---) (---) +++

-1 2

$HP = (-1, 2) \cup (-\infty, -1)$

c) $\dfrac{(x+1)^2}{x-2} \leq 0$

---   ---) +++

-1 2

$HP : (-\infty, 2)$

4. Tentukan himpunan penyelesaian dari pertidaksamaan :

$\dfrac{x-1}{x} > \dfrac{1}{2}$

Jawab : $\dfrac{x-1}{x} > \dfrac{1}{2}$

$\Leftrightarrow \dfrac{x-1}{x} - \dfrac{1}{2} > 0$

$\Leftrightarrow \dfrac{2x-2-x}{2x} > 0$

$\Leftrightarrow \dfrac{x-2}{2x} > 0$

+++) --- (+++

0 2

$HP = (-\infty, 0) \cup (2, \infty)$

Jika : $\dfrac{x}{x^2+1} \geq \dfrac{1}{2}$
$\Leftrightarrow 2x \geq x^2+1$ ✓ (kali silang)
karena $x^2 \geq 0 \Leftrightarrow x^2+1 \geq 1 > 0$

5. Selesaikan pertidaksamaan berikut :

$x^3 + x < 2x^2$

Jawab :

$x^3 + x < 2x^2$

$\Leftrightarrow x^3 + x - 2x^2 < 0$

$\Leftrightarrow x(x^2 - 2x + 1) < 0$

$\Leftrightarrow x(x-1)^2 < 0$

---) +++   +++

0 1

$HP : (-\infty, 0)$

Nilai Mutlak

- Untuk bilangan real x didefinisikan

$$|x| = \begin{cases} x &, x \geq 0 \\ -x &, x < 0 \end{cases}$$

Sifat - Sifat Nilai Mutlak

→ $|-a| = |a|$

→ $|ab| = |a||b|$

→ $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

→ $|a+b| \leq |a| + |b|$

→ Kaitan dgn akar kuadrat
$\sqrt{x^2} = |x|$

→ Kaitan dgn pertidaksamaan:
$|a| \leq |b| \Leftrightarrow a^2 \leq b^2$

Misalkan a, b bilangan real dengan $b > 0$.
Kapan berlaku:

→ $|a| \geq b$
$\Leftrightarrow a \geq b$ atau $a \leq -b$

→ $|a| > b$
$\Leftrightarrow a > b$ atau $a < -b$

→ $|a| \leq b$
$\Leftrightarrow -b \leq a \leq b$
$a \leq b$ dan $a \geq -b$

→ $|a| < b$
$\Leftrightarrow -b < a < b$
$a < b$ dan $a > -b$

Tentukan himp. penyelesaian dari pertidaksamaan nilai mutlak berikut ini:
(Ide : ubah pertidaksamaan di atas menjadi pertidaksamaan majemuk tanpa nilai mutlak)

a.) $|x-2| \geq 5$
b.) $|x+3| < 1$
c.) $|x^2-5| > 4$
d.) $\left|\frac{1}{x} - 3\right| \leq 1$

Jawab :

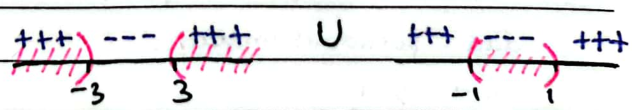
a.) $|x-2| \geq 5$
$\Leftrightarrow x-2 \geq 5$ atau $x-2 \leq -5$
$x \geq 7$ atau $x \leq -3$

HP : $[7, \infty) \cup (-\infty, -3]$

b.) $|x+3| < 1$
$\Leftrightarrow -1 < x+3 < 1$
$\Leftrightarrow -4 < x < -2$

HP : $(-4, -2)$

c.) i.) $|x^2-5| > 4$
$x^2 - 5 > 4$ atau $x^2 - 5 < -4$
$x^2 - 9 > 0$ atau $x^2 - 1 < 0$
$(x+3)(x-3) > 0 \cup (x+1)(x-1) < 0$



$HP_1 : (-\infty, -3) \cup (3, \infty) \cup HP_2 = (-1, 1)$

$HP = HP_1 \cup HP_2$
$HP = (3, \infty) \cup (-1, 1) \cup (-\infty, -3)$

c.2) $|x^2-5| \le 4$

$\Leftrightarrow \quad -4 \le x^2-5 \le 4$

$\Leftrightarrow \quad 1 \le x^2 \le 9$ ⇒ jangan lg sg (kurang tepat) di akar (akan salah!) $1 \le x \le 3$ → SALAH!

$\Leftrightarrow \quad x^2 \ge 1$ dan $x^2 \le 9$

$x^2-1 \ge 0$ | $x^2-9 \le 0$

$(x+1)(x-1) \ge 0$ | $(x+3)(x-3) \le 0$

+++ --- +++ (-1, 1) | +++ --- +++ (-3, 3)

digabung:

$-3 \quad -1 \quad 1 \quad 3$

HP = $[-3,-1] \cup [1,3]$

d.) $\left|\frac{1}{x}-3\right| \le 1$ Metode 1:

Jawab: $\Leftrightarrow \quad -1 \le \frac{1}{x}-3 \le 1$

$\frac{1}{x}-3 \ge -1$ dan $\frac{1}{x}-3 \le 1$

$\frac{1}{x}-2 \ge 0$ | $\frac{1}{x}-4 \le 0$

$\frac{1-2x}{x} \ge 0$ | $\frac{1-4x}{x} \le 0$

--- +++ --- (0, $1/2$) | +++ (0, $1/4$)

$0 \quad 1/4 \quad 1/2$

HP = $[1/4, 1/2]$

Metode 2:

$\boxed{|a| \le |b| \Leftrightarrow a^2 \le b^2}$

Cth: $|x-2| \le 3$

$\Leftrightarrow (x-2)^2 \le 3^2$

$\Leftrightarrow (x-2)^2 - 3^2 \le 0$

Tidak perlu dijabarkan

Gunakan: $a^2-b^2 = (a-b)(a+b)$

$\Leftrightarrow (x-2-3)(x-2+3) \le 0$

$\Leftrightarrow (x-5)(x+1) \le 0$

+++ --- +++ (-1, 5)

HP = $[-1, 5]$

Keuntungan metode 2: dapat dipakai jika kedua ruas memakai mutlak

Cth: $|x-5| \le |2x+3|$

$\Leftrightarrow (x-5)^2 \le (2x+3)^2$

$\Leftrightarrow 0 \le (2x+3)^2-(x-5)^2$

$\Leftrightarrow 0 \le (2x+3-(x-5))(2x+3+x-5)$

$\Leftrightarrow 0 \le (x+8)(3x-2)$

$\vdots$

(.) a.) Selesaikan $|2x+1| < -3$

b.) Selesaikan $|2x+1| > -3$

Jawab: Ingat: $|a| \ge 0$

a.) Tidak ada x shg $|2x+1| < -3$

∴ HP = { }

b.) $|2x+1| > -3$

Berapapun nilai x pasti ≥ 0

Maka, semua bil. real x pasti memenuhi: $|x-5| > -3$

HP = $(-\infty, \infty)$ = $\mathbb{R}$

Date . .

Pengayaan :

⇒ $|x+3| + |2x-4| < 5$

Jawab :

$|x+3| = \begin{cases} x+3, & \text{jika } x \geq -3 \\ -x-3, & \text{jika } x < -3 \end{cases}$

$|2x-4| = \begin{cases} 2x-4, & \text{jika } x \geq 2 \\ -2x+4, & \text{jika } x < 2 \end{cases}$

Dibagi Kasus

* Caranya akan panjang dan tidak akan digunakan di kalkulus

Soal Tutorial !

28. Jelaskan mengapa implikasi berikut benar :

satu arah

a.) $0 < c < 1 \Longrightarrow c^2(c+1) < 2$

Jawab :

Misalkan $0 < c < 1$

Maka $c^2 < 1$ dan $c+1 < 2$

Akibatnya $c^2(c+1) < 2$

c.) $2 < c < 3 \Longrightarrow \dfrac{c}{c^2-1} < 1$ //

Jawab :

$2 < c < 3 \Longrightarrow c < 3$

$c^2-1 > 3$

$\dfrac{1}{c^2-1} < \dfrac{1}{3}$

$c . \dfrac{1}{c^2-1} < 3 . \dfrac{1}{3}$

$\Longrightarrow \dfrac{c}{c^2-1} < 1$ //

Nama : Amelia Ufah Junadi
NIM : 19623229

No. Jum'at

Date 25 - 08 - 23

checkpoint

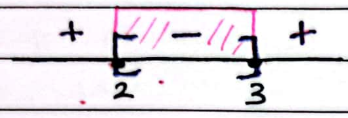
Latihan Soal !

1. Selesaikan pertidaksamaan ini :

$x^2 - 5x + 6 \leq 0$

Jawab : $x^2 - 5x + 6 \leq 0$

$\Leftrightarrow (x-3)(x-2) \leq 0$

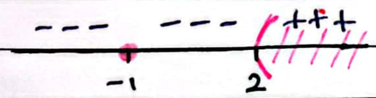


$\therefore$ HP = $[2, 3]$

2. Tentukan himpunan penyelesaian dari pertidaksamaan :

$\dfrac{(x+1)^2}{x-2} \geq 0$

Jawab : $\dfrac{(x+1)^2}{x-2} \geq 0$



$\therefore$ HP = $(2, \infty) \cup \{-1\}$

Nama : Anella Utari Gunadi
NIM : 19623229

No. Rabu
Date 30 - 08 - 23

1. Tentukan himpunan penyelesaian dari pertidaksamaan berikut :

a.) $|2x-3| \leq 1$

Jawab : $|2x-3| \leq 1$

$\Leftrightarrow \quad -1 \leq 2x-3 \leq 1$

$\Leftrightarrow \quad 2 \leq 2x \leq 4$

$\Leftrightarrow \quad 1 \leq x \leq 2$

HP = $[1, 2]$ //

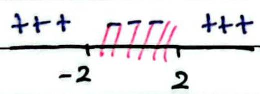
c.) $|2x^2-5| < 3$

Jawab :

$|2x^2-5| < 3$

$\Leftrightarrow \quad 2x^2-5 < 3$ dan $2x^2-5 > -3$

$2x^2-8 < 0$	$2x^2-2 > 0$
$2(x^2-4) < 0$	$2(x^2-1) > 0$
$2(x-2)(x+2) < 0$	$2(x-1)(x+1) > 0$



HP = $(-2, -1) \cup (1, 2)$ //

2. Jelaskan mengapa implikasi berikut benar :

b.) $1 < c < 2 \Rightarrow \dfrac{24}{(1+c)^5} < \dfrac{3}{4}$

Jawab :

Misalkan, $1 < c < 2$

Maka, $1+c > 2$

$\Leftrightarrow \quad (1+c)^5 > 32$

$\Leftrightarrow \quad \dfrac{1}{(1+c)^5} < \dfrac{1}{32}$

Akibatnya, $\dfrac{24}{(1+c)^5} < \dfrac{24}{32}$

$\Leftrightarrow \quad \dfrac{24}{(1+c)^5} < \dfrac{3}{4}$ //

No.

Date . .

Tutorial Bab 0 Bagian 1

1. Telaah konsep

a.) Jarak antara titik (x_1, y_1) dan (x_2, y_2) adalah

Jawab : $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

b.) Persamaan lingkaran yang berpusat di titik (a,b) dan berjari-jari r adalah ...

Jawab : $(x-a)^2 + (y-b)^2 = r^2$

c.) Titik tengah dari ruas garis yang menghubungkan (x_1, y_1) dan (x_2, y_2) adalah ...

Jawab : $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

d.) Misalkan $a \neq c$. Gradien garis yg melalui titik (a,b) dan (c,d) adalah...

Jawab : $m = \frac{d-b}{c-a}$

e.) Garis $y = mx + c$ adalah garis dengan gradien m dan memotong sumbu-y di titik c

f.) Penulisan $\{x \in \mathbb{R} : -3 \leq x \leq 5\}$ dalam notasi interval adalah ...

Jawab : $[-3, 5]$

g.) Jika $x < y$ dan $z < 0$, maka $xz > yz$

h.) Jika $x \geq 0$, maka $|x| = x$. Jika $x < 0$, maka $|x| = -x$

i.) Jarak dari x ke 6 pada garis bil. real adalah 5. Nilai x yang mungkin adalah ...

Jawab : $6 - x = 5$

$\Leftrightarrow x = 1$

2. Sederhanakan bentuk berikut :

a.) $3(-1(1+5-2)-4)-7$

$= 3(-1(4)-4)-7$

$= 3(-8)-7 = -24-7 = -31$

b.) $\frac{1}{2}\left(\frac{1}{4}\left(\frac{1}{5}+\frac{1}{6}\right)-\frac{1}{3}\right)$

$= \frac{1}{2}\left(\frac{1}{4}\left(\frac{11}{30}\right)-\frac{1}{3}\right)$

$= \frac{1}{2}\left(\frac{11}{120}-\frac{1}{3}\right)$

$= \frac{1}{2}\left(\frac{11-40}{120}\right) = -\frac{29}{240}$

c.) $\dfrac{\frac{2}{3}-\frac{1}{4}}{\frac{1}{2}+\frac{2}{5}}$

$= \dfrac{\frac{8-3}{12}}{\frac{5+4}{10}}$

$= \dfrac{\frac{5}{12}}{\frac{9}{10}} = \frac{5}{\cancel{12}_6} \cdot \frac{\cancel{10}^5}{9} = \frac{25}{54}$

d.) $3 - \dfrac{2}{1-\frac{3}{4}}$

$= 3 - 2 \cdot 4$

$= 3 - 8 = -5$

e.) $\dfrac{\frac{2}{5}-3}{5}+1$

$= \dfrac{\frac{2}{5}-\frac{15}{5}}{5}+1$

$= \dfrac{-\frac{13}{5}}{5}+1 = \frac{-13}{25}+1 = \frac{12}{25}$

No.

Date . .

f.) $(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})$

$= (\sqrt{7})^2 - (\sqrt{3})^2$

$= 7-3 = 4$

g.) $(\sqrt{7}-\sqrt{3})^2$

$= 7 - 2\sqrt{21} + 3$

$= 10 - 2\sqrt{21} = 2(5-\sqrt{21})$

3. Jabarkan atau sederhanakan bentuk berikut :

a.) $(4x-3)(x+1)$

Jawab : $4x^2 + 4x - 3x - 3$

$= 4x^2 + x - 3$

b.) $(3t-2)^2$

Jawab : $9t^2 - 12t + 4$

c.) $(2x^2 - x + 1)^2$

Jawab : $(2x^2 - x + 1)(2x^2 - x + 1)$

$= 4x^4 - 2x^3 + 2x^2 - 2x^3 + x^2 - x$

$+ 2x^2 - x + 1$

$= 4x^4 - 4x^3 + 5x^2 - 2x + 1$

d.) $(3y+2)^3$

Jawab : $(3y+2)(3y+2)(3y+2)$

$= (9y^2 + 12y + 4)(3y+2)$

$= 27y^3 + 18y^2 + 36y^2 + 24y + 12y + 8$

$= 27y^3 + 54y^2 + 36y + 8$

e.) $\frac{x^2-9}{x-3}$

Jawab : $\frac{(\cancel{x-3})(x+3)}{\cancel{x-3}} = x+3$

f.) $\frac{x^2-x-12}{x-4}$

Jawab : $\frac{(\cancel{x-4})(x+3)}{\cancel{x-4}} = x+3$

g.) $\frac{2}{3y-1} + \frac{y}{2y+1}$

Jawab : $\frac{4y+2+3y^2-y}{(3y-1)(2y+1)}$

$= \frac{3y^2+3y+2}{6y^2+y-1}$

4. Berikan contoh bilangan real diantara dua bilangan berikut :

a.) $\frac{1}{100}$ dan $\frac{1}{99}$

Jawab : $\left(\frac{1}{100} + \frac{1}{99}\right) \cdot \frac{1}{2}$

$= \frac{99+100}{9900} \cdot \frac{1}{2} = \frac{199}{19.800}$

c.) $-\frac{3}{4}$ dan $-\frac{4}{5}$

Jawab : $\left(\frac{-3}{4} - \frac{4}{5}\right) \frac{1}{2}$

$= \frac{-15-16}{20} \cdot \frac{1}{2} = \frac{-31}{40}$

5. Periksa apakah pernyataan-pernyataan berikut benar atau salah

a.) Jika $x<1$, maka $5x-1<10$

Jawab : Misalkan $x=0$

$\rightarrow 5x-1 = -1 < 10$

Benar

b.) Jika $x < 1$, maka $x^2 < 1$

Jawab : Misalkan $x = -5$

$\rightarrow x^2 = 25 > 1$

Salah

c.) Terdapat bil. real x shg $x^2 = 0{,}1$

Jawab : $\sqrt{0{,}1} = \sqrt{\frac{1}{10}} = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10} = x$

Benar

d.) Untuk setiap bil. real x, terdapat bilangan real y sehingga $x^2+1 < y$

Jawab :

6. Hitunglah jarak antara tiap dua titik berikut. Tentukan juga titik tengah dari ruas garis dengan titik ujung kedua titik berikut.

a.) $(7,2)$ dan $(3,2)$

Jawab : 1.) Jarak dua titik.

$d = \sqrt{(3-7)^2 + (2-2)^2}$

$= \sqrt{16} = 4$

2.) Titik tengah

$A\left(\frac{7+3}{2}, \frac{2+2}{2}\right)$

$= A(5, 2)$

b.) $(3,-5)$ dan $(-3,3)$

Jawab :

1.) Jarak dua titik

$d = \sqrt{(-3-3)^2 + (3-(-5))^2}$

$= \sqrt{36+64} = \sqrt{100} = 10$

2.) Titik Tengah

$B\left(\frac{3-3}{2}, \frac{-5+3}{2}\right)$

$= B(0, -1)$

7. Tentukan persamaan lingkaran yang memenuhi masing-masing kondisi yang diberikan berikut:

a.) Memiliki pusat $(-4,3)$ dan $r = 4$

Jawab : $(x+4)^2 + (y-3)^2 = 16$

b.) Memiliki pusat $(5,-4)$ dan melalui $(1,-1)$

Jawab :

$r = \sqrt{(5-1)^2 + (-4+1)^2}$

$= \sqrt{16+9} = \sqrt{25} = 5$

$\Rightarrow (x-5)^2 + (y+4)^2 = 5^2$

$\Leftrightarrow (x-5)^2 + (y+4)^2 = 25$

c.) Memiliki diameter AB dengan $A = (-13,-4)$ dan $B = (11,6)$

Jawab : $d(A,B) = \sqrt{(-13-11)^2 + (-4-6)^2}$

$= \sqrt{576+100}$

$= \sqrt{676} = 26$

$r = \frac{26}{2} = 13$

Pusat $\left(\frac{-13+11}{2}, \frac{-4+6}{2}\right)$

$= P(-1, 1)$

$\Rightarrow (x+1)^2 + (y-1)^2 = 13^2$

$\Leftrightarrow (x+1)^2 + (y-1)^2 = 169$

8. Tentukan pusat dan jari-jari lingkaran dari masing² persamaan lingkaran berikut:

a.) $x^2 + y^2 = 25$

Jawab : $P(0,0)$, $r = \sqrt{25} = 5$

b.) $(x+1)^2 + (y-4)^2 = 64$

Jawab : $P(-1,4)$

$r = \sqrt{64} = 8$

Date . .

c.) $x^2+y^2+10y=56$

<u>Jawab</u> : $x^2+y^2+10y-56=0$

$P\left(-\frac{1}{2}.0, -\frac{1}{2}.10\right) = P(0,-5)$

$r=\sqrt{0^2+(-5)^2-(-56)} = \sqrt{81} = 9$

d.) $x^2+6x+y^2-2y+1=0$

<u>Jawab</u>:

$x^2+y^2+6x-2y+1=0$

$P\left(-\frac{1}{2}.6, -\frac{1}{2}(-2)\right) = P(-3,1)$

$r=\sqrt{(-3)^2+1^2-1} = \sqrt{9} = 3$

9. Hitung gradien dari masing² garis yg mell kedua titik berikut dan tentukan juga persamaan garis tersebut.

a.) (0,0) dan (3,-3)

<u>Jawab</u> : $m = \frac{-3-0}{3-0} = -1$

$y-y_1 = m(x-x_1)$

$y+3 = -1(x-3)$

$y = -x+3-3$

$y+x=0$

b) (-5,1) dan (0,9)

<u>Jawab</u> : $m = \frac{9-1}{0+5} = \frac{8}{5}$

$y-y_1 = m(x-x_1)$

$y-9 = \frac{8}{5}(x-0)$

$y = \frac{8}{5}x+9$

10. Tentukan pers. garis yg memenuhi masing² kondisi berikut:

a.) Melalui (2,3) dan memiliki gradien -2.

<u>Jawab</u> :

$y-y_1 = m(x-x_1)$

$y-3 = -2(x-2)$

$y-3 = -2x+4$

$y = -2x+7$

b.) Mell (1,-2) dan sejajar dgn garis $2y = 3x+10$

<u>Jawab</u> : $2y = 3x+10$

$\Leftrightarrow y = \frac{3}{2}x+5$

$\rightarrow m_1 = \frac{3}{2} \longrightarrow m_2 = \frac{3}{2}$

$y-y_1 = m(x-x_1)$

$y+2 = \frac{3}{2}(x-1)$

$y = \frac{3}{2}x - \frac{7}{2}$

$2y = 3x-7$

11. Tentukan gradien dan perpotongan dgn sumbu-sumbu koordinat dari tiap garis berikut:

a.) $3y = 7x+21$

<u>Jawab</u> : $3y = 7x+21$

$\Leftrightarrow y = \frac{7}{3}x+7$

$\Rightarrow m = \frac{7}{3}$

$\Rightarrow$ Perpotongan sb-x : $\frac{7}{3}x+7=0$

$7/3\,x = -7$

$x = -7 \cdot \frac{3}{7} = -3$

$\downarrow$

$(-3,0)$

$\Rightarrow$ Perpotongan sb-y : $y = 0+7$

$= 7$

$\downarrow$

$(0,7)$

Nama : Anella Utari Gunadi

NIM : 19623229

No.

Date . .

Tutorial Bab 0 Bagian 1

10. Tentukan persamaan garis yg memenuhi masing² kondisi berikut :

c.) Melalui (2,1) dan tegak lurus dengan garis $2x + 3y = 5$

Jawab : $3y = -2x + 5$

$y = -\frac{2}{3}x + \frac{5}{3} \rightarrow m_1 = -\frac{2}{3}$

$m_2 \perp m_1$

$m_2 = \frac{3}{2}$

$\Rightarrow y - y_1 = m_2(x - x_1)$

$\Leftrightarrow y - 1 = \frac{3}{2}(x - 2)$

$\Leftrightarrow y - 1 = \frac{3}{2}x - 3$

$\Leftrightarrow y = \frac{3}{2}x - 2$ //

18. Tentukan titik potong dua kurva berikut. Kemudian, sketsa kedua grafik kurva pd bidang koordinat yang sama.

b.) garis $y = x + 5$ dan parabola $y = x^2 - 1$

Jawab : $y_1 = y_2$

$x^2 - 1 = x + 5$

$\Leftrightarrow x^2 - x - 6 = 0$

$\Leftrightarrow (x-3)(x+2) = 0$

$x_1 = 3$ atau $x_2 = -2$

$y_1 = 3+5 = 8$ $\quad y_2 = -2+5 = 3$

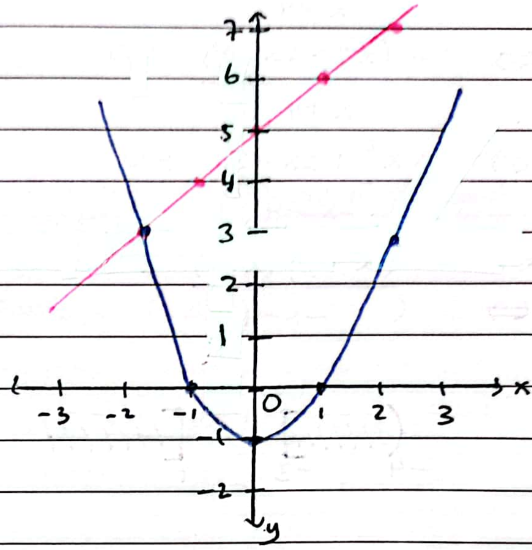
$A(3,8)$ $\quad B(-2,3)$

> $y = x + 5$

x	-2	-1	0	1	2
y	3	4	5	6	7

> $y = x^2 - 1$

x	-2	-1	0	1	2
y	3	0	-1	0	3



20. Tentukan himpunan penyelesaian pertaksamaan berikut.

e.) $x^3 - 3x^2 + 2x < 0$

$\Leftrightarrow x(x^2 - 3x + 2) < 0$

$\Leftrightarrow x(x-2)(x-1) < 0$

$- - - \quad + + + \quad - - - \quad + + +$ (titik 0, 1, 2)

HP = $(1, 2) \cup (-\infty, 0)$ //

21. Tentukan himpunan penyelesaian pertaksamaan berikut.

d.) $\frac{2x+3}{x+4} \leq x$

Jawab : $\Leftrightarrow \frac{2x+3}{x+4} - x \leq 0$

$\Leftrightarrow \frac{2x+3 - x(x+4)}{x+4} \leq 0$

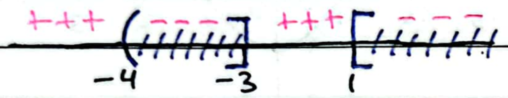
$$\Leftrightarrow \frac{2x+3-x^2-4x}{x+4} \le 0$$

$$\Leftrightarrow \frac{-x^2-2x+3}{x+4} \le 0$$

$$\Leftrightarrow \frac{-(x^2+2x-3)}{x+4} \le 0$$

$$\Leftrightarrow \frac{x^2+2x-3}{-x-4} \le 0$$

$$\Leftrightarrow \frac{(x+3)(x-1)}{-x-4} \le 0$$



HP $= (-4, -3] \cup [1, \infty)$ //

24. Tentukan himpunan penyelesaian dari pertidaksamaan berikut.

d.) $\left| \frac{x}{x-1} - \frac{3}{2} \right| \ge \frac{1}{2}$

Jawab:

$$\frac{x}{x-1} - \frac{3}{2} \ge \frac{1}{2} \quad \text{atau} \quad \frac{x}{x-1} - \frac{3}{2} \le -\frac{1}{2}$$

$$\Leftrightarrow \frac{x}{x-1} - 2 \ge 0 \quad \Big| \quad \Leftrightarrow \frac{x}{x-1} - 1 \le 0$$

$$\Leftrightarrow \frac{x-2x+2}{x-1} \ge 0 \quad \Big| \quad \Leftrightarrow \frac{x-x+1}{x-1} \le 0$$

$$\Leftrightarrow \frac{-x+2}{x-1} \ge 0 \quad \Big| \quad \Leftrightarrow \frac{1}{x-1} \le 0$$

HP $= (-\infty, 1) \cup (1, 2]$ //

Date . .

Fungsi dan Grafiknya

Fungsi → suatu aturan yg memetakan setiap unsur di suatu himpunan, disebut daerah asal / domain ke tepat satu unsur himpunan lain, disebut kodomain.

Daerah Asal Alami

Daerah asal alami dr fungsi f :

Himpunan semua bilangan real x sehingga f(x) terdefinisi.

Contoh :

- $f(x) = x^2 \Rightarrow D_f = \{x \in \mathbb{R}\}$
- $f(x) = \sqrt{x} \Rightarrow D_f : \{x \mid x \geq 0\}$

$\sqrt{x}$ terdefinisi $\Leftrightarrow x \geq 0$ ↳ $[0, \infty]$

- $g(x) = \dfrac{1}{x-1} \Rightarrow D_g : \{x \mid x \neq 1\}$

↳ $(-\infty, 1) \cup (1, \infty)$

- $h(x) = \sqrt{x} + \dfrac{1}{x-1}$

$x \geq 0$; $x \neq 1$

$D_h : [0, 1) \cup (1, \infty)$

$= \{x \mid x \geq 0 \text{ dan } x \neq 1\}$

Contoh (2) :

Tentukan daerah asal alami dari fungsi - fungsi tersebut :

- $g(x) = \sqrt{1-x^2}$

Jawab : $D_g = -1 \leq x \leq 1$

$g(x)$ terdefinisi $\Leftrightarrow 1 - x^2 \geq 0$

$\Leftrightarrow x^2 - 1 \leq 0$

$\Leftrightarrow (x-1)(x+1) \leq 0$

+++ [- - -] +++
-1 1

$D_g = -1 \leq x \leq 1$

$= [-1, 1]$

- $h(x) = \dfrac{1}{\sqrt{x^2 - 3x + 2}}$

Jawab :

$h(x)$ terdefinisi $\Leftrightarrow x^2 - 3x + 2 \geq 0$ dan $\sqrt{x^2 - 3x + 2} \neq 0$

$x^2 - 3x + 2 > 0$

$(x-2)(x-1) > 0$

+++ - - - +++
1 2

$D_h : (-\infty, 1) \cup (2, \infty)$

Q: Tentukan daerah asal alami dari

$f(x) = \dfrac{1}{4\sqrt{1-x^2} - 1}$

Jawab :

$f(x)$ terdefinisi $\Rightarrow 1 - x^2 \geq 0$ dan $4\sqrt{1-x^2} - 1 \neq 0$

$\Rightarrow 1 - x^2 \geq 0$

$\Leftrightarrow (1-x)(1+x) \geq 0$

[/////]
-1 1

$\Rightarrow 4\sqrt{1-x^2} - 1 \neq 0$

$4\sqrt{1-x^2} \neq 1$

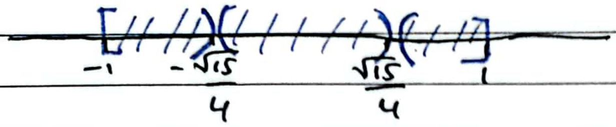
$4\sqrt{1-x^2} = 1$

$\sqrt{1-x^2} = \dfrac{1}{4}$

$1 - x^2 = \dfrac{1}{16}$

$x^2 = \dfrac{15}{16}$

$x = \pm \dfrac{\sqrt{15}}{4}$



$D_f : \left[-1, -\frac{\sqrt{15}}{4}\right) \cup \left(-\frac{\sqrt{15}}{4}, \frac{\sqrt{15}}{4}\right) \cup \left(\frac{\sqrt{15}}{4}, 1\right]$

Grafik Fungsi

Grafik fungsi $f \rightarrow$ grafik persamaan $y = f(x)$. Apa artinya?
Saat y adalah nilai dari fungsi $f(x)$ saat absisnya dimasukkan ke dalam x. $\rightarrow$ ada 1 output untuk 1 input.

$\rightarrow$ Daerah hasil (range) dari $f \rightarrow$ dinotasikan dengan R_f : himpunan semua $y = f(x)$ yang mungkin

$R_f = \{y \mid$ terdapat $x \in D_f$ sehingga $f(x) = y\}$

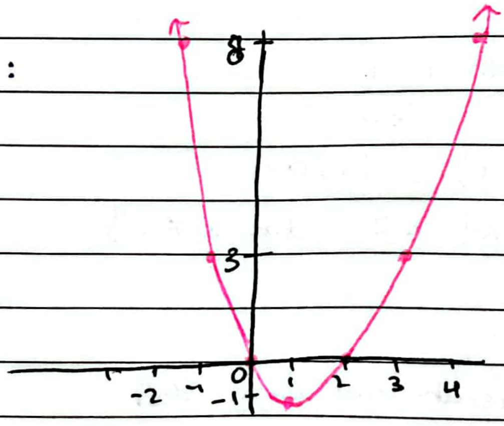
Contoh :

Diberikan fungsi $f(x) = x^2 - 2x$. Tentukan titik² pada grafik fungsi f dengan absis: $x = -2, -1, 0, 1, 2, 3$, lalu sketsalah grafik fungsi tersebut.

Jawab : $y = f(x) = x^2 - 2x$

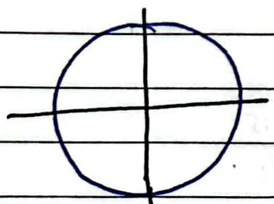
x	$f(x)$
-2	8
-1	3
0	0
1	-1
2	0
3	3

Grafiknya :

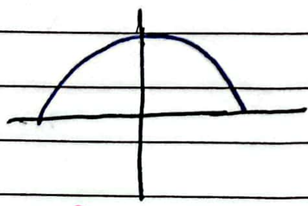


Diskusi

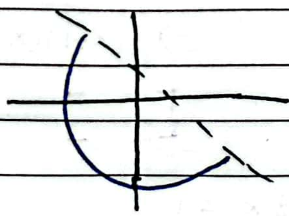
- Manakah di antara grafik berikut yang merupakan grafik suatu fungsi?



Bukan fungsi



Grafik fungsi



Bukan fungsi

CS Dipindai dengan CamScanner

No.

Date . .

→ Fungsi Ganjil

$$\boxed{f(-x) = -f(x)}$$

untuk setiap $x \in D_f$

→ fungsi ganjil dicerminkan thd $(0,0)$

→ Fungsi Genap

$$\boxed{f(-x) = f(x)}$$

untuk setiap $x \in D_f$

→ fungsi genap dicerminkan thd sumbu - y.

Contoh Soal !

Tentukan apakah fungsi - fungsi berikut merupakan fungsi ganjil, fungsi genap, atau tidak keduanya.

a.) $f(x) = x^4 - 2x^2 + 1$

Jawab: $f(-x) = (-x)^4 - 2(-x)^2 + 1$

$= x^4 - 2x^2 + 1 = f(x)$

∴ fungsi genap

b.) $f(x) = x^5 + 4x^3$

Jawab: $f(-x) = (-x)^5 + 4(-x)^3$

$= -x^5 - 4x^3$

$= -(x^5 + 4x^3) = -f(x)$

∴ fungsi ganjil

c.) $g(x) = \dfrac{x^4}{x^2+1}$

Jawab: $g(-x) = \dfrac{(-x)^4}{(-x)^2+1} = \dfrac{x^4}{x^2+1} = g(x)$

∴ fungsi genap

d.) $h(x) = \dfrac{x^7}{x^4+1}$

Jawab: $h(-x) = \dfrac{(-x)^7}{(-x)^4+1} = \dfrac{-x^7}{x^4+1} = -h(x)$

∴ fungsi ganjil

e.) $h(x) = x\sqrt{x^2+1}$

Jawab: $h(-x) = -x\sqrt{(-x)^2+1}$

$= -x\sqrt{x^2+1} = -h(x)$

∴ fungsi ganjil

Diskusi

- Apakah terdapat fungsi yang tidak ganjil dan tidak genap ?

  ⇒ ya, ada

- Apakah terdapat fungsi yang ganjil sekaligus genap ?

  ⇒ Ya, yaitu fungsi nol

- Apakah terdapat fungsi yang grafiknya simetri terhadap sumbu -x

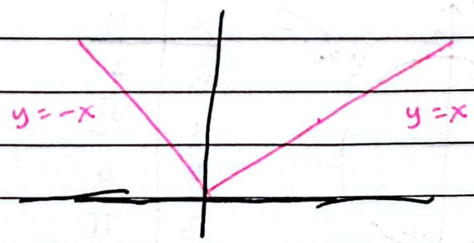
  ⇒ Ya, saat $f(x) = 0$

Fungsi Nilai Mutlak dan Grafiknya

Fungsi nilai mutlak dinotasikan dengan $f(x) = |x|$ didefinisikan dengan

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- $D_f = (-\infty, \infty)$
- $R_f = [0, \infty)$
- Grafik : $y = |x|$



Date . .

Fungsi Tangga (Floor)

- Fungsi tangga (floor), dinotasikan dengan $f(x) = [\![x]\!]$ didefinisikan sebagai bilangan bulat terbesar yang kurang dari atau sama dengan x.

Contoh :

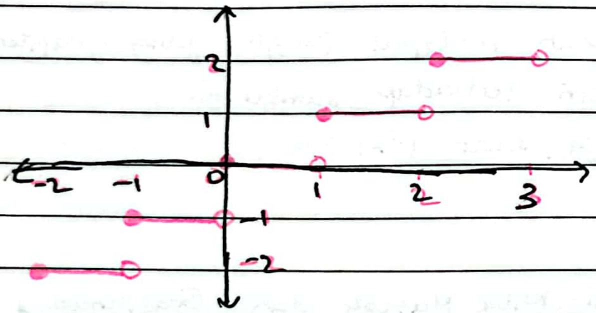
$f[\![0,?]\!] = 0$

$f[\![1,?]\!] = 1$

$f[\![-1,?]\!] = -2$

$f[\![-0,?]\!] = -1$

Grafik floor :



- $D_f = (-\infty, \infty)$
- $R_f = \mathbb{Z}$ (bilangan bulat)

Tambahan !

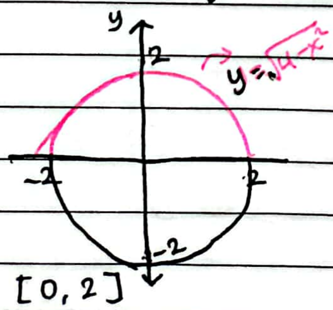
1. Sketsalah grafik fungsi $f(x) = \sqrt{4-x^2}$ kemudian tentukan R_f.

Jawab :

$y = \sqrt{4-x^2}$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$ < $P(0,0)$, $r = 2$



x	y
0	2
1	$\sqrt{3}$
2	0
3	TD
-1	$\sqrt{3}$
-2	0

→ $R_f = [0, 2]$

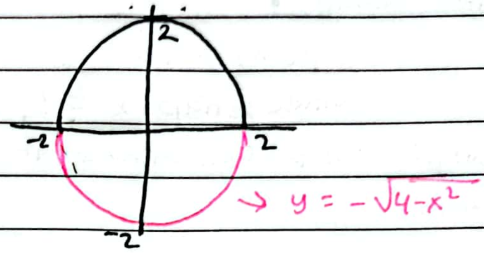
2. $g(x) = -\sqrt{4-x^2}$

Jawab :

$y = -\sqrt{4-x^2}$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$



→ $R_g = [-2, 0]$

Anella Utari Gunadi (19623229)

No.

Date . .

Tutorial Bab 0 Bagian 2

[2.] Misalkan $f(x) = 2x^2 - 3x$. Tentukan (dan sederhanakan) nilai atau ekspresi berikut:

d.) $\dfrac{f(2+h) - f(2)}{h}$

Jawab: $\dfrac{f(2+h) - f(2)}{h}$

$= \dfrac{2(2+h)^2 - 3(2+h) - (2(2)^2 - 3(2))}{h}$

$= \dfrac{2(h^2+4h+4) - 6 - 3h - 2}{h}$

$= \dfrac{2h^2 + 8h + 8 - 3h - 8}{h}$

$= \dfrac{2h^2 + 5h}{h}$

$= \dfrac{\cancel{h}(2h+5)}{\cancel{h}} = 2h+5$ //

[3.] Tentukan daerah asal fungsi² berikut.

e.) $f(x) = \sqrt{x} + \dfrac{1}{x-3}$

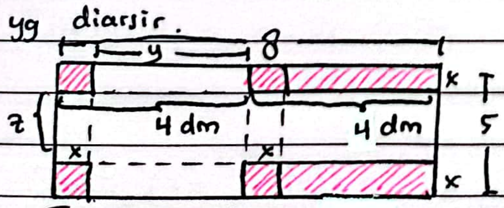
Jawab: $f(x)$ terdefinisi:

$x \geq 0$ dan $x - 3 \neq 0$

$x \neq 3$

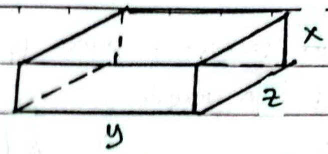
$\Rightarrow D_f = [0, 3) \cup (3, \infty)$ //

[16.] Suatu kotak tertutup dari karton ukuran 5 x 8 dm dgn memotong daerah yg diarsir.



Tentukan volume kotak yg dihasilkan sebagai fungsi dlm x dan tentukan daerah asal fungsi tsb.

Jawab:



$V = y \cdot z \cdot x$

$V = (4-x)(5-2x)x$

$V = x(2x^2 - 13x + 20)$

$V = 2x^3 - 13x^2 + 20x$ //

D_V =

$4 - x > 0$ dan $5 - 2x > 0$ dan $x > 0$

$-x > -4$ $\quad -2x > -5$

$x < 4$ $\quad x < \dfrac{5}{2}$

$\rightarrow D_V : \left(0, \dfrac{5}{2}\right)$ //

[12.] Tentukan daerah asal fungsi $f(x) = \sqrt{x} \cdot \sqrt{x-2}$ dan $g(x) = \sqrt{x(x-2)}$. Apakah keduanya sama? Jelaskan.

Jawab: $\Rightarrow f(x) = \sqrt{x} \cdot \sqrt{x-2}$

D_f :

$x \geq 0$ dan $x - 2 \geq 0$

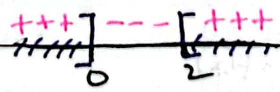
$x \geq 2$

$\rightarrow D_f = [2, \infty)$

$\Rightarrow g(x) = \sqrt{x(x-2)}$

D_g =

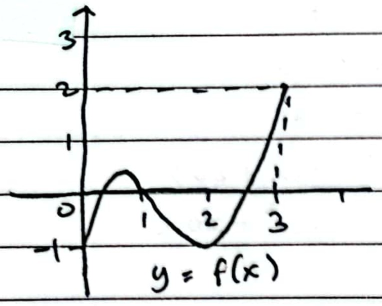
$x(x-2) \geq 0$



$\rightarrow D_g = (-\infty, 0] \cup [2, \infty)$

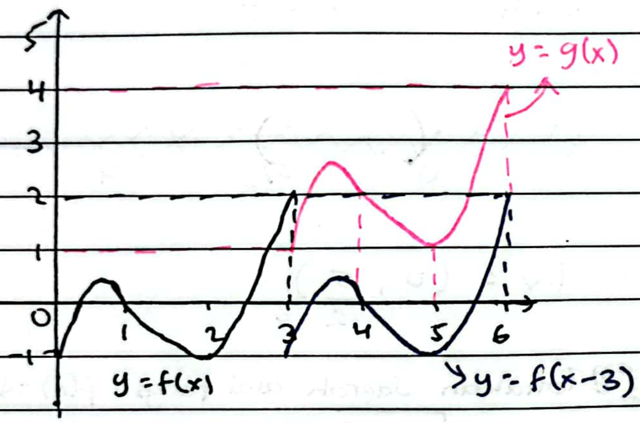
Kedua daerah asal tidak sama, karena jika $x < 0$ maka $f(x)$ tidak terdefinisi sedangkan $g(x)$ terdefinisi.

16. Diberikan fungsi f dgn $D_f = [0,3]$ dan $R_f = [-1,2]$ dgn grafik sebagai berikut.



Sketsa grafik fungsi berikut dan tentukan daerah asal dan daerah hasilnya.

e.) $g(x) = f(x-3) + 2$



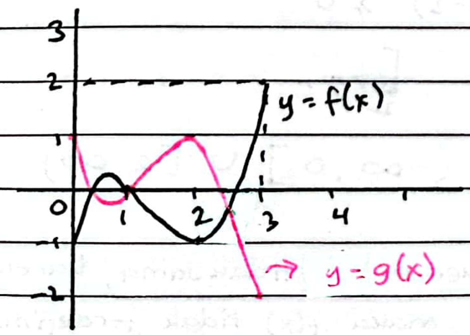
→ $D_g = [3, 6]$

→ $R_g = [1, 4]$

f.) $g(x) = -f(x)$

Jawab:

	$g(x) = -f(x)$
→ $f(0) = -1$	→ $g(0) = 1$
→ $f(1) = 0$	→ $g(1) = -0$
→ $f(2) = -1$	→ $g(2) = 1$
→ $f(3) = 2$	→ $g(3) = -2$



→ $D_g = [0, 3]$ → $R_g = [-2, 1]$

No.

Date . .

Operasi pada Fungsi

Misalkan, $f(x) = \sqrt{x}$ dan $g(x) = 2x - 5$.
Jika $h = f + g$ dan $k = \frac{f}{g}$, maka:

→ $h(x) = \sqrt{x} + 2x - 5$

→ $k(x) = \dfrac{\sqrt{x}}{2x-5}$

Daerah asal dan sketsa grafik

(1) Misalkan $f(x) = x + 2\sqrt{x}$ dan $g(x) = x - 2\sqrt{x}$

- Tentukan D_{f+g} dan $D_{f.g}$
- Sketsa grafiknya $f+g$ dan $f.g$

Jawab:

a.) $(f+g)(x) = f(x) + g(x)$

→ $D_f = [0, \infty)$

→ $D_g = [0, \infty)$

⇒ D_{f+g}:

$x \in D_f$ dan $x \in D_g$

$x = [0, \infty)$ dan $x = [0, \infty)$

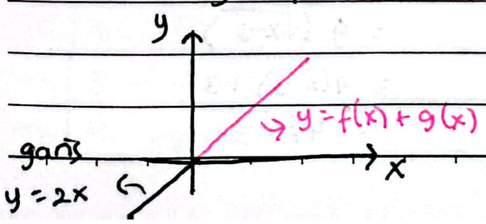
∴ $D_{f+g} = [0, \infty)$

b.) $D_{f.g}$

$(f.g)(x) = f(x).g(x)$
$= (x + 2\sqrt{x})(x - 2\sqrt{x})$
$= x^2 - 4x$

TAPI $D_{f.g} = D_f \cap D_g$
$= [0, \infty)$

Grafik $f+g$: $f(x) + g(x) = 2x$



2. Misalkan $f(x) = x^2 - 4$ dan $g(x) = x - 2$

- Tentukan $D_{f/g}$
- Sketsa grafik $\frac{f}{g}$

Jawab:

$\left(\frac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{x^2-4}{x-2} = x + 2$

→ $D_f = (-\infty, \infty)$

→ $D_g = (-\infty, \infty)$

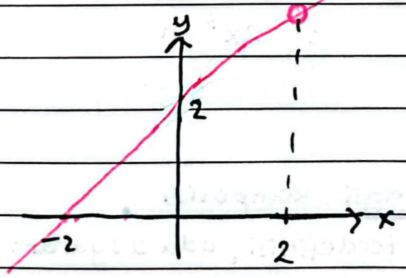
$D_{f/g}$: - $f(x)$ terdefinisi
- $g(x)$ terdefinisi
- $g(x) \neq 0$ → $x - 2 \neq 0$
$x \neq 2$

⇒ $D_{f/g} = (-\infty, 2] \cup [2, \infty)$

Sketsa grafik $D_{f/g}$:

$x + 2 = 0$, $x = -2$ $(-2, 0)$

$y = 0 + 2$, $y = 2$ $(0, 2)$



Daerah asal fungsi hasil operasi

Jika f, g fungsi, maka berlaku

- $D_{f+g} = D_f \cap D_g$
- $D_{f-g} = D_f \cap D_g$
- $D_{f.g} = D_f \cap D_g$
- $D_{\frac{f}{g}} = D_f \cap \{x \mid g(x)$ terdefinisi dan $g(x) \neq 0\}$

Date . .

Komposisi Fungsi

Fungsi f dan g dapat dikomposisikan dalam dua cara, menghasilkan fungsi baru $f \circ g$ dan $g \circ f$ dengan aturan:

→ $(f \circ g)(x) = f(g(x))$

→ $(g \circ f)(x) = g(f(x))$

Contoh soal :

11. Misal $f(x) = x-1$, $g(x) = 2x$, $h(x) = x^3$

Tentukan :

a.) $(f \circ g)(x)$

d.) $(f \circ g \circ h)(x)$

Jawab :

a.) $(f \circ g)(x) = f(g(x))$

$= f(2x)$

$= 2x - 1$

d.) $(f \circ g \circ h)(x) = f(g(h(x)))$

$= f(g(x^3))$

$= f(2x^3)$

$= 2x^3 - 1$ //

Daerah asal fungsi komposisi

- Agar $f(g(x))$ terdefinisi, ada 2 syarat:
  - → $g(x)$ terdefinisi, $x \in D_g$
  - → $f(g(x))$ terdefinisi, $g(x) \in D_f$

Contoh soal :

12. Misalkan $f(x) = \sqrt{x-3}$ dan $g(x) = 4x^2+3$.

a.) Tentukan D_f dan D_g

Jawab :

→ D_f : $x - 3 \geq 0$

$x \geq 3$

$D_f = [3, \infty)$ //

→ $D_g = (-\infty, \infty)$ //

b.) Tentukan $D_{f \circ g}$ dan $D_{g \circ f}$

Jawab : $D_{f \circ g}$:

$x \in D_g$ dan $g(x) \in D_f$

$x = (-\infty, \infty)$ $\quad 4x^2 + 3 \geq 3$

$4x^2 \geq 0$

$x^2 \geq 0$

++ ++

0

HP : $(-\infty, \infty)$

⇒ $D_{f \circ g} = (-\infty, \infty)$ //

$D_{g \circ f}$:

$x \in D_f$ dan $f(x) \in D_g$

$x \geq 3$ $\quad \sqrt{x-3} \in (-\infty, \infty)$

⇒ $D_{g \circ f} = [3, \infty)$ //

c.) Gambarkan grafik fungsi $f \circ g$ dan $g \circ f$

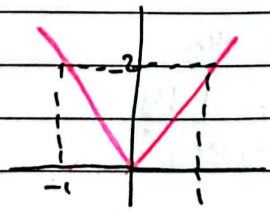
Jawab :

→ $(f \circ g)(x) = f(g(x))$

$= f(4x^2+3)$

$= \sqrt{4x^2+3-3}$

$= \sqrt{4x^2} = 2\sqrt{x^2} = 2|x|$



→ $(g \circ f)(x) = g(f(x))$

$= g(\sqrt{x-3})$

$= 4(x-3) + 3$

$= 4x - 9$

Anella Utari Gunadi (19623229)

No.

Date . .

Checkpoint 06 - 09 - 2023

1. Sketsa grafik $h(x) = \sqrt{6x - x^2}$ dan tentukan Rh.

Jawab: $y = \sqrt{6x - x^2}$

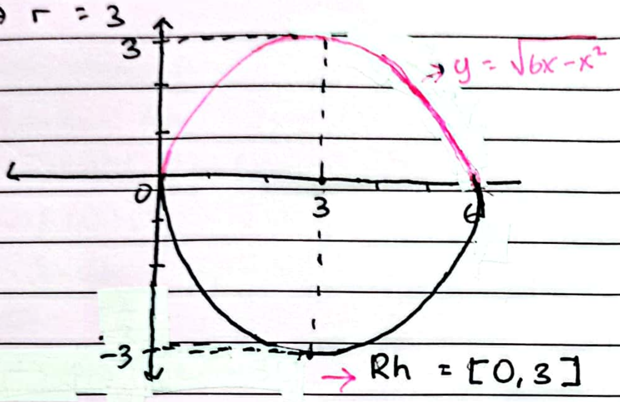
$$y^2 = 6x - x^2$$

$$x^2 + y^2 - 6x = 0$$

$$(x-3)^2 + (y-0)^2 = 3^2$$

→ P(3,0)

→ r = 3



→ $Rh = [0,3]$

2. $f(x) = x^2 + 1$

$g(x) = \sqrt{1-x}$

Tentukan Dfog dan sketsa grafiknya!

Jawab: $D_f = (-\infty, \infty)$, $D_g = (-\infty, 1]$

D_{fog} :

$x \in D_g$ dan $g(x) \in D_f$

$x \le 1$ $\quad \sqrt{1-x} \in (-\infty, \infty)$

$\Rightarrow D_{fog} = (-\infty, 1]$

Grafik fungsinya:

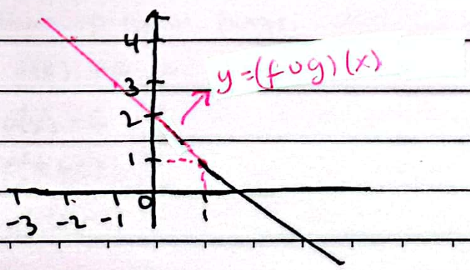
$$(f \circ g)(x) = f(g(x))$$

$$= f(\sqrt{1-x})$$

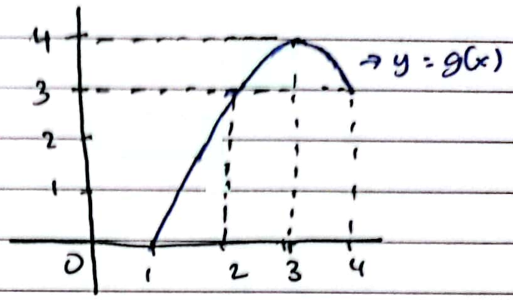
$$= (\sqrt{1-x})^2 + 1$$

$$= 1 - x + 1 = -x + 2$$

x	y
1	1
0	2
-1	3
-2	4



3.



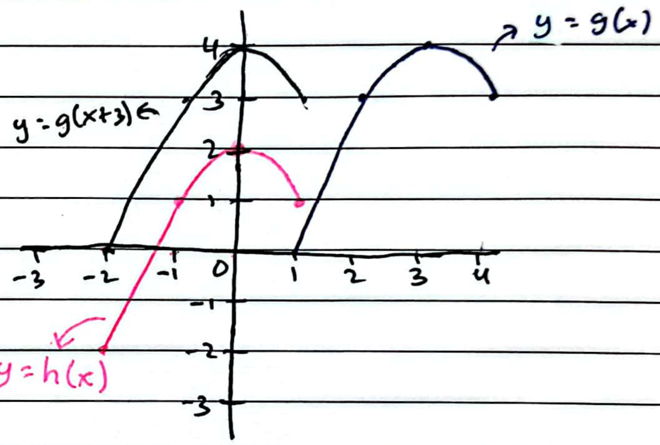
$D_g = [1,4]$

$R_g = [0,4]$

$h(x) = g(x+3) - 2$

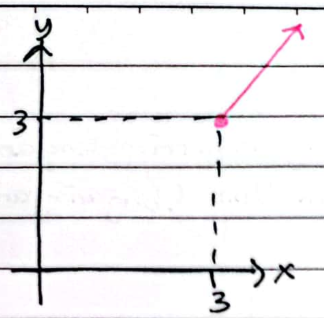
Sketsa $h(x)$, tentukan Dh dan Rh

Jawab:



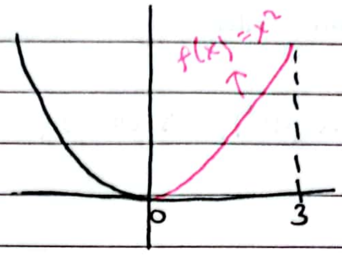
→ $D_h = [-2, 1]$

→ $R_h = [-2, 2]$



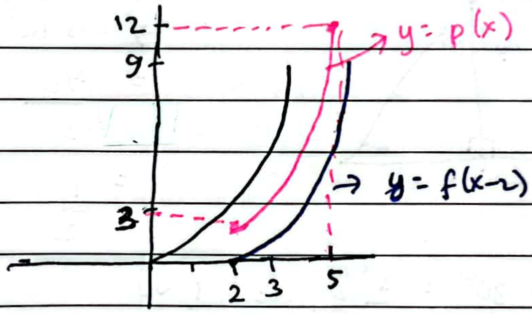
Contoh Soal!

$f(x) = x^2 \qquad D_f = [0,3]$



Tentukan grafik fungsi $p(x) = f(x-2)+3$ dan D_p & R_p !

Jawab :



→ $D_p = [2,5]$

→ $R_p = [3,12]$

Menuliskan fungsi sebagai komposisi

→ Tuliskan fungsi-fungsi berikut sebagai komposisi dua fungsi :

- $f(x) = (x+5)^7$

  Jawab : $f(x) = (p \circ q)(x) = p(q(x))$

  $q(x) = x + 5$

  $p(x) = x^7$ //

- $g(x) = \sqrt{x^2+1}$

  Jawab : $g(x) = (r \circ s)(x) = r(s(x))$

  $s(x) = x^2 + 1$

  $r(x) = \sqrt{x}$ //

→ Tuliskan fungsi $h(x) = \dfrac{1}{\sqrt{x+5}}$ sebagai komposisi tiga fungsi.

Jawab : $h(x) = (u \circ v \circ w)(x)$

$w(x) = x + 5$

$v(x) = \sqrt{x}$

$u(x) = \dfrac{1}{x}$ //

Translasi Grafik

- Misalkan f suatu fungsi dan $c > 0$

→ $y = f(x) + c$ digeser ke atas sebesar c

→ $y = f(x) - c$ digeser ke bawah sbsr c

→ $y = f(x+c)$ digeser ke kiri sebesar c

→ $y = f(x-c)$ digeser ke kanan sebesar c

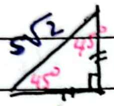
Date . .

Fungsi Trigonometri

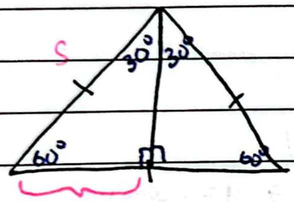
I. Fungsi Trigonometri dengan Segitiga Siku - Siku

→ Hitunglah $\sin(45^\circ)$, $\cos(60^\circ)$, dan $\tan(30^\circ)$

Jawab:



→ $\sin(45^\circ) = \dfrac{s}{s\sqrt{2}} = \dfrac{1}{2}\sqrt{2}$



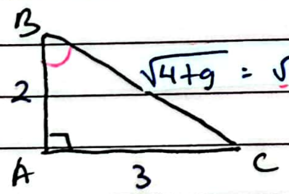
$\left.\right\} \dfrac{s}{2}\sqrt{3}$

→ $\cos 60^\circ = \dfrac{s/2}{s} = \dfrac{1}{2}$

→ $\tan 30^\circ = \dfrac{s/2}{s/2\sqrt{3}} = \dfrac{1}{3}\sqrt{3}$

→ Misal ABC segitiga siku - siku dengan $\angle BAC = 90^\circ$. $AB = 2$ satuan, $AC = 3$ satuan. Hitunglah $\sin B$, $\cos B$, dan $\tan C$.

Jawab :



→ $\sin B = \dfrac{3}{\sqrt{13}}$

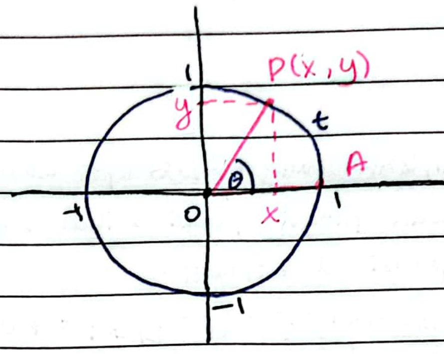
→ $\tan C = \dfrac{2}{3}$

→ $\cos B = \dfrac{2}{\sqrt{13}}$

<u>Kelemahan</u> : tidak bisa mendefinisikan sudut tumpul

$$\boxed{1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ} \quad \boxed{1^\circ = \left(\frac{\pi}{180}\right) \text{rad}}$$

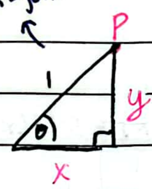
II. Fungsi Trigonometri dengan Lingkaran Unit (berjari-jari = 1)



→ $\dfrac{t}{2\pi} = \dfrac{\theta}{360^\circ} = \dfrac{\theta}{2\pi}$

→ $\boxed{\theta = t \text{ rad}}$ besar $\angle AOP$ dalam satuan rad

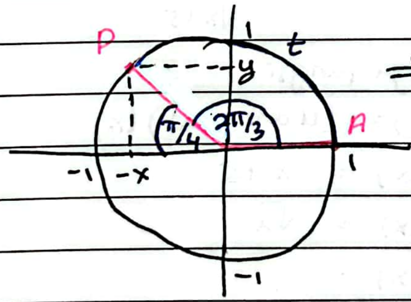
jari-jari



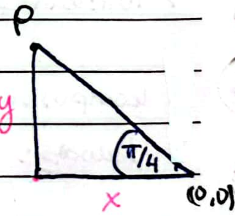
→ $\sin\theta = \dfrac{y}{1} = y \Rightarrow y = \sin\theta$

→ $\cos\theta = \dfrac{x}{1} = x \Rightarrow x = \cos\theta$

→ $P(x, y) = P(\cos\theta, \sin\theta)$
$= P(\cos t, \sin t)$



⇒



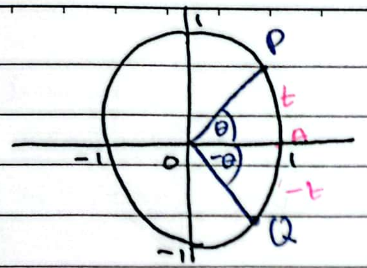
→ $P = (x, y) = P(\cos\theta, \sin\theta)$

→ $\cos\theta = \cos \pi/4 = \dfrac{1}{2}$

Karena titik P berada di x negatif maka $x = -\dfrac{1}{2} = \cos 2\pi/3$

→ $\sin \pi/4 = \dfrac{1}{2}\sqrt{3} = y = \sin 2\pi/3$

→ $P\left(-\dfrac{1}{2}, \dfrac{1}{2}\sqrt{3}\right)$



→ $\cos(t) = x$ → $\sin(t) = y$

→ $\cos(-t) = x$ → $\sin(-t) = -y$

→ $\sin(-t) = -\sin(t)$

↳ sinus fungsi ganjil

→ $\cos(-t) = \cos(t)$

↳ cosinus fungsi genap

III. Sifat - Sifat Dasar Fungsi Sinus dan Cosinus

Persamaan lingkaran dgn $r = 1$ dan $P(0,0)$:

$$x^2 + y^2 = 1$$

$$(\cos t)^2 + (\sin t)^2 = 1$$

$$\boxed{\cos^2 t + \sin^2 t = 1}$$

$$\sin(x-y) = \sin(x + (-y))$$
$$= \sin(x)\cos(-y) + \cos(x)\sin(-y)$$
$$= \sin(x)\cos(y) - \cos(x)\sin(y)$$

$$\cos 2x = \cos(x+x)$$
$$= \cos(x)\cos(x) - \sin(x)\sin(x)$$
$$= \cos^2 x - \sin^2 x$$
$$\sin^2 x + \cos^2 x = 1$$
$$= \cos^2 x + \cos^2 x - 1$$
$$= 2\cos^2 x - 1$$
$$= 1 - \sin^2 x - \sin^2 x$$
$$= 1 - 2\sin^2 x$$

Latsol !

Misalkan $0 < \theta < \pi/2$ dan $\pi/2 < \phi < \pi$ dgn $\cos(\theta) = 3/5$ dan $\sin(\phi) = 2/3$.

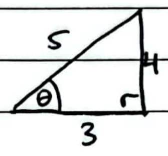
Hitunglah :

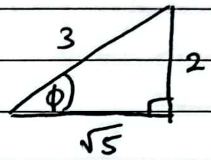
a.) $\sin(\theta)$

b.) $\cos(\phi)$

c.) $\sin(\phi - \theta)$

Jawab :





a.) $\sin\theta = \frac{4}{5}$

b.) $\cos\phi = \frac{\sqrt{5}}{3}$

c.) $\sin(\phi - \theta) = \sin\phi\cos\theta - \cos\phi\sin\theta$

$$= \frac{2}{\cancel{3}} \cdot \frac{\cancel{3}}{5} - \frac{\sqrt{5}}{3} \cdot \frac{4}{5}$$

$$= \frac{2}{5} - \frac{4\sqrt{15}}{15} = \frac{2}{5}\left(1 - \frac{2\sqrt{15}}{3}\right)$$

b.) atau

$$\sin^2\phi + \cos^2\phi = 1$$

$$\frac{4}{9} + \cos^2\phi = 1$$

$$\cos^2\phi = 1 - \frac{4}{9} = \frac{5}{9}$$

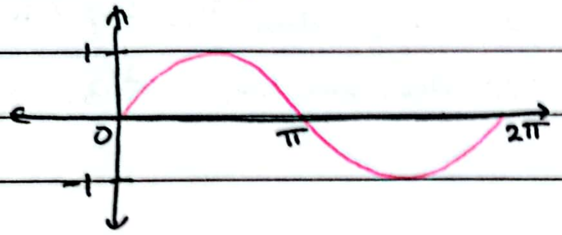
$$\cos\phi = -\frac{\sqrt{5}}{3}$$ "

IV. Grafik Fungsi Sinus dan Cosinus

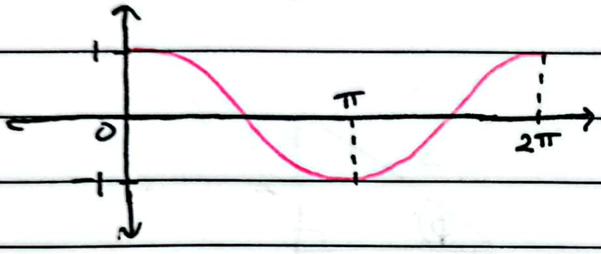
→ $D_{sin} = (-\infty, \infty)$	$D_{cos} = (-\infty, \infty)$
→ $R_{sin} = [-1, 1]$	$R_{cos} = [-1, 1]$
→ Sinus : ordinat dr titik P dan P ada di lingkaran berjari² 1	→ Cosinus : absis dr titik P dan P ada di lingkaran berjari-jari 1

Date

→ Grafik fungsi sinus :



→ Grafik fungsi cosinus :



No.

Date

Tutorial Bab 0 Bagian 2

1. a.) Jika grafik fungsi f melalui titik $(1,7)$, maka $f(1) = 7$

b.) Grafik fungsi ganjil yang memuat titik $(-1,9)$ juga akan memuat titik $(1,-9)$

c.) Jika fungsi genap yang memenuhi $f(x) = \sqrt{x}$ untuk $x \geq 0$, maka nilai $f(-1) = f(x) = \sqrt{1} = 1$

2. $f(x) = 2x^2 - 3x$. Tentukan:

a.) $f(3) = 2.9 - 3.3 = 9$

b.) $f(u-1) = 2(u-1)^2 - 3(u-1) = \ldots$

3. Tentukan daerah asal.

a.) $f(x) = x^2 + x - 1$

$\rightarrow D_f = (-\infty, \infty)$

b.) $f(x) = \sqrt{2x-6}$

Jawab: $2x - 6 \geq 0$

$2x \geq 6$

$x \geq 3$

$\rightarrow D_f = [3, \infty)$

c.) $f(x) = \dfrac{1}{x^2 + 3x + 2}$

Jawab: $x^2 + 3x + 2 \neq 0$

$(x+2)(x+1) \neq 0$

$x \neq -2 \cup x \neq -1$

$\rightarrow D_f = [-\infty, \infty] \setminus \{-2, -1\}$

d.) $f(x) = \dfrac{1}{\sqrt{x^2 - 4}}$

Jawab:

$x^2 - 4 > 0$

$(x+2)(x-2) > 0$

+++) - - - (+++

-2 $\quad$ 2

$\rightarrow D_f = (-\infty, -2] \cup [2, \infty)$

e.) $f(x) = \sqrt{x} + \dfrac{1}{x-3}$

Jawab:

$x \geq 0$ dan $x - 3 \neq 0$

$x \neq 3$

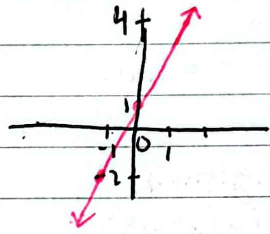
$D_f = [0, 3) \cup [3, 0]$

4. Sketsa dan tentukan daerah hasil

a.) $f(x) = 3x + 1$

$R_f = (-\infty, \infty)$

x	0	1	-1
y	1	4	-2



b.) $f(x) = x^2 - 4x$

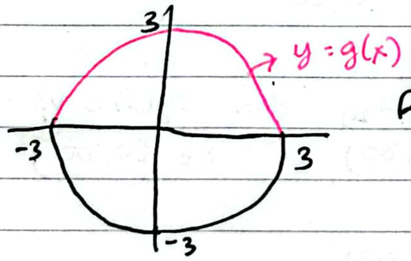
Jawab:

c.) $g(x) = \sqrt{9 - x^2}$

Jawab: $y = \sqrt{9 - x^2}$

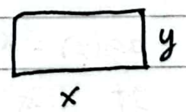
$y^2 = 9 - x^2$

$x^2 + y^2 = 9$



$R_f = [-3, 3]$

5. Keli $= 100$



$2(x+y) = 100$

$x + y = 50$

$y = 50 - x$

$L = x \cdot y = x(50 - x) = 50x - x^2$

$x > 0$ dan $50 - x > 0$

$x < 50$

$\rightarrow D_L = (0, 50)$

10. $f(x) = x+5$, $g(x) = 2x$, $h(x) = x^2 - x$

a.) $(f \circ g)(x) = f(g(x))$
$= f(2x)$
$= 2x+5$ //

b.) $(g \circ f)(x) = g(f(x))$
$= g(x+5)$
$= 2(x+5) = 2x+10$ //

c.) $(f \circ f)(x) = f(f(x))$
$= f(x+5)$
$= x+5+5 = x+10$ //

d.) $(f \circ g \circ h)(x) = f(g(h(x)))$
$= f(g(x^2-x))$
$= f(2x^2-2x)$
$= 2x^2-2x+5$ //

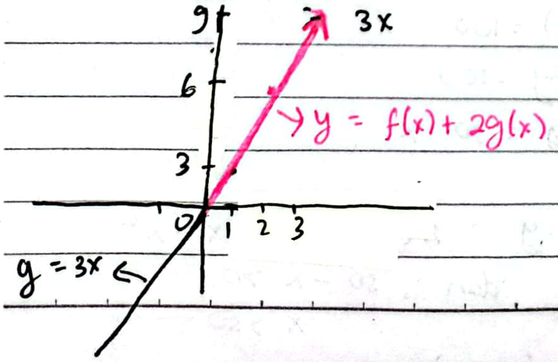
11. Misalkan $f(x) = x - 2\sqrt{x}$ dan $g(x) = x + \sqrt{x}$.
Tentukan D_{f+2g} dan sketsa grafiknya.

Jawab: $D_f = [0, \infty)$
$2g = 2g(x) = 2(x+\sqrt{x}) = 2x + 2\sqrt{x}$
$D_{2g} = [0, \infty)$

$\Rightarrow D_{f+2g}$: $x \in D_f$ dan $x \in D_{2g}$
$x \in [0, \infty)$ $x \in [0, \infty)$

$\therefore D_{f+2g} = [0, \infty)$ //

$f(x) + 2g(x) = x - 2\sqrt{x} + 2x + 2\sqrt{x}$



13. Jika $f(x) = x^2 - 3x + 2$ dan $g(x) = x - 1$
Tentukan $D_{f/g}$ dan sketsa grafiknya!

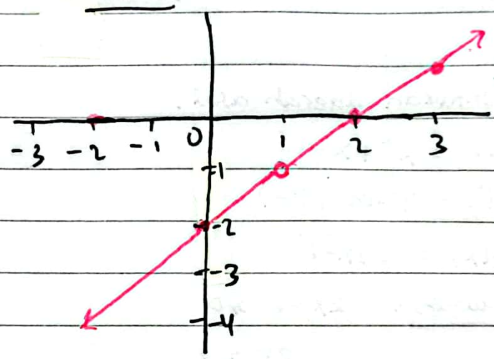
Jawab: $D_f = (-\infty, \infty)$
$D_g = (-\infty, \infty)$

$$\frac{f(x)}{g(x)} = \frac{x^2-3x+2}{x-1} = x-2$$
$\nearrow (x-2)(x-1)$

$D_{f/g}$: $x - 1 \neq 0$
$x \neq 1$

$\Rightarrow D_{f/g} = (-\infty, 1) \cup (1, \infty)$ //

Sketsa :



14. $f(x) = \sqrt{x-1}$ dan $g(x) = x^2 + 1$

a.) D_f dan D_g

Jawab: D_f : $x - 1 \geq 0$
$x \geq 1$

$\rightarrow D_f = [1, \infty)$
$\rightarrow D_g = (-\infty, \infty)$

b.) $D_{f \circ g}$ dan $D_{g \circ f}$:

Jawab:

$D_{f \circ g}$:
$x \in D_g$ dan $g(x) \in D_f$
$x \in (-\infty, \infty)$ $g(x) \in [1, \infty)$
$x^2 + 1 \geq 1$
$x^2 \geq 0$
$x \geq 0$

$\Rightarrow D_{f \circ g} = [0, \infty)$ //

$D_{g \circ f}$:

$x \in D_f$ dan $f(x) \in D_g$

$x \geq 1$ $\quad f(x) \in (-\infty, \infty)$

$\Rightarrow D_{g \circ f} = [1, \infty)$

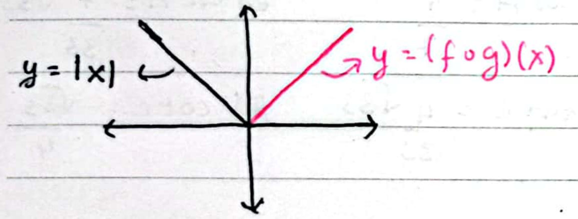
c.) Sketsa $f \circ g$ dan $g \circ f$

Jawab: $(f \circ g)(x) = f(g(x))$

$= f(x^2+1)$

$= \sqrt{x^2+1-1} = \sqrt{x^2}$

$(f \circ g)(x) = |x|$

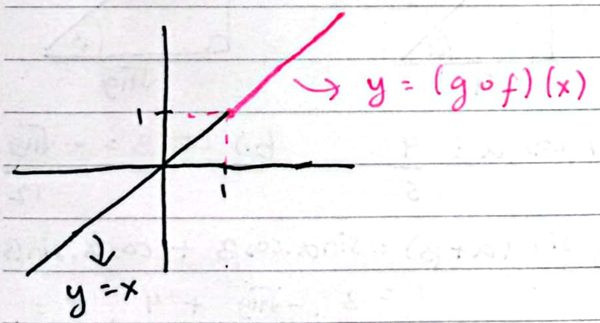


$\rightarrow (g \circ f)(x) = g(f(x))$

$= g(\sqrt{x-1})$

$= (\sqrt{x-1})^2 + 1$

$= x - 1 + 1 = x$



15. Tentukan suatu fungsi f dan g sehingga $h = f \circ g$ dengan h :

a.) $h(x) = \sqrt[3]{2x+7}$

Jawab: $f(x) = \sqrt[3]{x}$

$g(x) = 2x+7$

b.) $h(x) = (x^2+x-1)^5$

Jawab: $f(x) = x^5 \quad g(x) = x^2+x-1$

23. Periksa apakah fungsi-fungsi berikut merupakan fungsi ganjil, genap, atau bukan keduanya.

a.) $f(x) = x+2$

Jawab: $f(-x) = -x+2$

$\rightarrow$ bukan keduanya

b.) $f(x) = x^8 - 5x^4 + 1$

Jawab: $f(-x) = (-x)^8 - 5(-x)^4 + 1$

$= x^8 - 5x^4 + 1 = f(x)$

$\rightarrow$ fungsi genap

c.) $f(x) = x^3 + 2x$

Jawab: $f(-x) = (-x)^3 + 2(-x)$

$= -x^3 - 2x = -f(x)$

$\rightarrow$ fungsi ganjil

d.) $f(x) = \dfrac{2x}{x^4-1}$

Jawab: $f(-x) = \dfrac{2(-x)}{(-x)^4-1}$

$= \dfrac{-2x}{x^4-1} = -f(x)$

$\rightarrow$ fungsi ganjil

e.) $f(t) = 2t^2 - 3|t|$

Jawab: $f(-t) = 2(-t)^2 - 3|-t|$

$= 2t^2 - 3t = f(t)$

$\rightarrow$ fungsi genap

f.) $f(x) = x^2 - x + 1$

Jawab: $f(-x) = (-x)^2 - (-x) + 1$

$= x^2 + x + 1$

$\rightarrow$ bukan keduanya

g.) $f(x) = x \sin x + \cos x$

Jawab: $f(-x) = -x \cdot \sin(-x) + \cos(-x)$

$= x \sin x + \cos x = f(x)$

$\rightarrow$ fungsi genap

h.) $f(t) = \tan(t)/t^2$

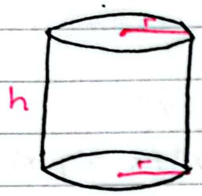
Jawab: $f(-t) = \tan(-t)/(-t)^2$

$= \dfrac{\sin(-t)}{\cos(-t) \cdot t^2} = -\dfrac{\tan t}{t^2} = -f(t)$

$\rightarrow$ fungsi ganjil

17. Kaleng tanpa tutup, $Lp = 100 \text{ cm}^2$. Nyatakan volume kaleng sebagai fungsi dari jari-jari alas dan tentukan daerah asal.

Jawab:



$Lp = 100 \text{ cm}^2$

$2\pi r^2 + 2\pi r h = 100$

$\pi r^2 + \pi r h = 50$

$h = \dfrac{50 - \pi r^2}{\pi r}$

Volume tabung $= \pi r^2 . h$

$V = \pi r^2 \left(\dfrac{50 - \pi r^2}{\pi r} \right)$

$V = 50r - \pi r^3$

D_v: $\pi r \neq 0$ $\pi r^2 < 50$

$r \neq 0$ $r^2 < \dfrac{50}{\pi}$

$r < \sqrt{\dfrac{50}{\pi}}$

$\therefore D_v = \left(0, \sqrt{\dfrac{50}{\pi}}\right)$ //

17. Ubah dari satuan derajat ke radian.

a.) $45^\circ = 45^\circ . \dfrac{\pi}{180} = \dfrac{1}{4}\pi$ rad //

b.) $60^\circ = 60 . \dfrac{\pi}{180}$ rad $= \dfrac{\pi}{3}$ rad //

c.) $150^\circ = 150 . \dfrac{\pi}{180} = \dfrac{5\pi}{6}$ rad //

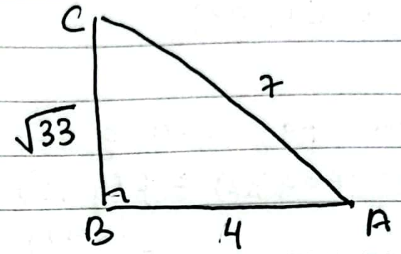
d.) $270^\circ = 270 . \dfrac{\pi}{180} = \dfrac{3\pi}{2}$ rad //

e.) $-30^\circ = 330^\circ . \dfrac{\pi}{180} = \dfrac{11\pi}{6}$ rad //

f.) $20^\circ = 20 . \dfrac{\pi}{180} = \dfrac{\pi}{9}$ rad //

18. $\triangle ABC$ dengan $\angle ABC = \dfrac{\pi}{2}$. Panjang $AB = 4$ dan panjang $AC = 7$. Tentukan nilai-nilai berikut.

Jawab:



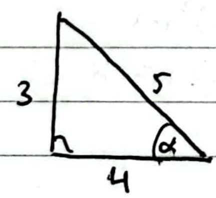
a.) $\sin A = \dfrac{\sqrt{33}}{7}$ d.) $\sec C = \dfrac{7\sqrt{33}}{33}$

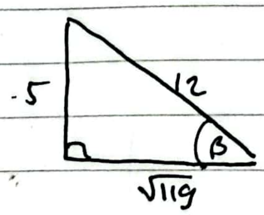
b.) $\cos A = \dfrac{4}{7}$ e.) $\csc A = \dfrac{7\sqrt{33}}{33}$

c.) $\tan C = \dfrac{4\sqrt{33}}{33}$ f.) $\cot C = \dfrac{\sqrt{33}}{4}$

20. $0 < \alpha < \dfrac{\pi}{2}$ dan $\dfrac{\pi}{2} < \beta < \pi$

$\sin\alpha = \dfrac{3}{5}$ dan $\sin\beta = \dfrac{5}{12}$





a.) $\cos\alpha = \dfrac{4}{5}$ b.) $\cos\beta = -\dfrac{\sqrt{119}}{12}$

c.) $\sin(\alpha+\beta) = \sin\alpha.\cos\beta + \cos\alpha.\sin\beta$

$= \dfrac{3}{5} . \dfrac{-\sqrt{119}}{2} + \dfrac{4}{5} . \dfrac{5}{12} = \ldots$

d.) $\cos(\alpha+\beta) = \cos\alpha.\cos\beta - \sin\alpha \sin\beta$

$= \dfrac{4}{5} . \dfrac{-\sqrt{119}}{2} - \dfrac{3}{5} . \dfrac{5}{12} = \ldots$

e.) $\sin(2\beta) = \sin(\beta+\beta)$

$= \sin\beta\cos\beta + \cos\beta\sin\beta$

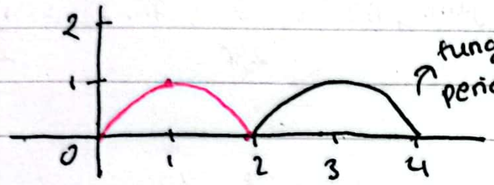
$= 2\sin\beta\cos\beta$

$= 2 . \dfrac{5}{12} . \dfrac{-\sqrt{119}}{2} = \dfrac{-5\sqrt{119}}{12}$

24. $D_f = (-\infty, \infty)$ → fungsi periodik 2 dan memenuhi $f(x) = 2x - x^2$ untuk $0 \le x \le 2$. Sketsa grafik fungsi f dan tentukan f(100) dan f(111).

Jawab: $f(x) = 2x - x^2$

x	0	1	2
y	0	1	0



Periode = 2

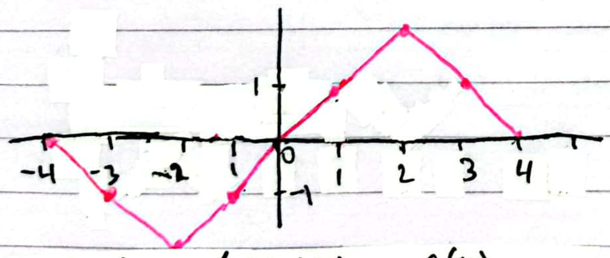
$f(100) = f(0) = 0$ → 100 mod 2 = 0

$f(111) = f(1) = 1$ → 111 mod 2 = 1

25. $D_g = (-\infty, \infty)$ → fungsi ganjil dan periode = 8. $g(x) = 2 - |x-2|$ untuk $0 \le x \le 4$. Sketsa grafik fungsi g dan tentukan g(100) dan g(103)

Jawab: $g(x) = 2 - |x-2|$

x	0	1	2	3	4
y	0	1	0	1	0



→ $g(100) = g(96+4) = g(4)$

→ $g(103) = g(104-1) = g(-1)$

$g(100) = g(4) = 0$

$g(103) = g(-1) = -1$

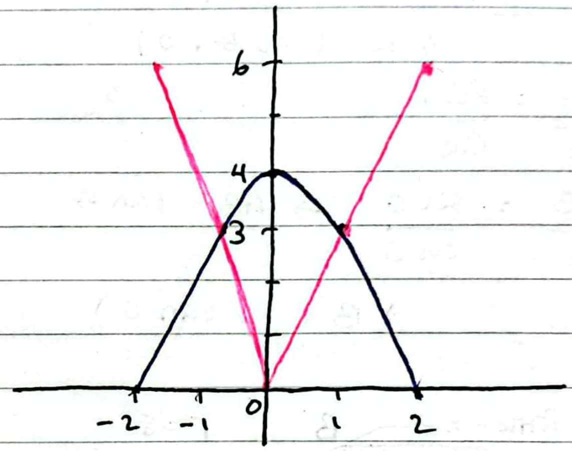
26. a.) Sketsa grafik fungsi $f(x) = 4 - x^2$ dan $g(x) = 3|x|$ pada satu bidang koordinat.

Jawab: $f(x) = 4 - x^2$

x	-2	-1	0	1	2
y	0	3	4	3	0

→ $g(x) = 3|x|$

x	-2	-1	0	1	2
y	6	3	0	3	6



b.) berdasarkan (a), tentukan HP pertidaksamaan $x^2 + 3|x| < 4$

Jawab: $x^2 + 3|x| < 4$

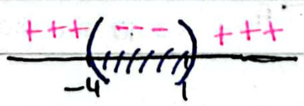
$3|x| < 4 - x^2$

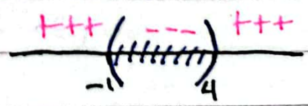
$3x < 4 - x^2$ dan $3x > -(4 - x^2)$

$x^2 + 3x - 4 < 0$ dan $3x > -4 + x^2$

$(x+4)(x-1) < 0$ dan $x^2 - 3x - 4 < 0$

$(x-4)(x+1) < 0$

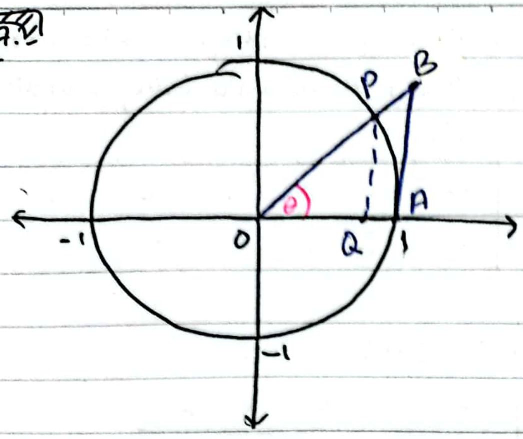




HP = $(-1, 1)$

Date

27.



a.) Tentukan koordinat titik B, P, Q (dalam θ)

Jawab: → $P(\cos\theta, \sin\theta)$ //

→ $Q(\cos\theta, 0)$ //

→ $\frac{AB}{OA} = \frac{PQ}{OQ}$;

$\frac{AB}{1} = \frac{\sin\theta}{\cos\theta}$ → $AB = \tan\theta$

→ $B(1, \tan\theta)$ //

27. b.) Luas ΔAOB, ΔAOP, juring AOP

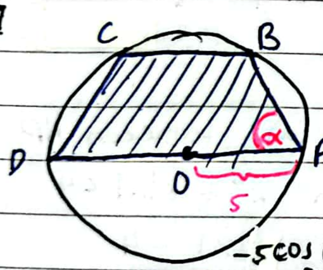
Jawab:

Luas ΔAOB : $\frac{1 \cdot \tan\theta}{2} = \frac{\tan\theta}{2}$

Luas ΔAOP : $\frac{1 \cdot \sin\theta}{2} = \frac{\sin\theta}{2}$

L juring AOP : $\frac{\theta}{2\pi} \cdot \pi \cdot 1^2 = \frac{1}{2}\theta$

28.



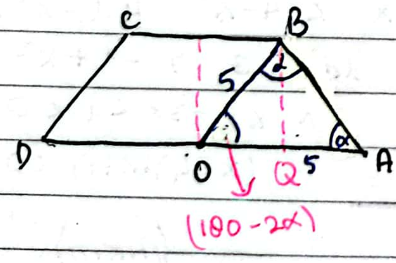
$r = 5$

Nyatakan luas trapesium sebagai fungsi dalam α.

$-5\cos(2\alpha)$ $\quad 5\sin(2\alpha)$

Jawab: $B(5\cos(180-2\alpha), 5\sin(180-2\alpha))$

$AD = 10$



$BQ = 5\sin(180-2\alpha)$

$BC = 2BQ$

$BC = 10\sin(180-2\alpha)$

$= 10\sin 2\alpha$

$L \text{ ⏢} : \frac{(a+b)t}{2} = \frac{(10 + 10\sin 2\alpha)\, 5\sin 2\alpha}{2}$

$= \frac{\overset{25}{\cancel{50}}(1 + 1\sin 2\alpha)\sin 2\alpha}{2}$

$= 25\sin 2\alpha + 25\sin^2 2\alpha$

$= 25\sin 2\alpha(1 + \sin 2\alpha)$

Bab 1.1

Date . .

Pengantar Limit

Kecepatan sesaat : "limit" dari v rata² saat interval waktu semakin pendek "menuju" nol.

misal : v rata² pada $t = [2,t]$

$$v(2) = \lim_{t \to 2} \frac{s(t) - s(2)}{t-2}$$

$$= \lim_{t \to 2} \frac{t^2 - 2^2}{t-2}$$

$$= \lim_{t \to 2} \frac{(\cancel{t-2})(t+2)}{\cancel{t-2}}$$

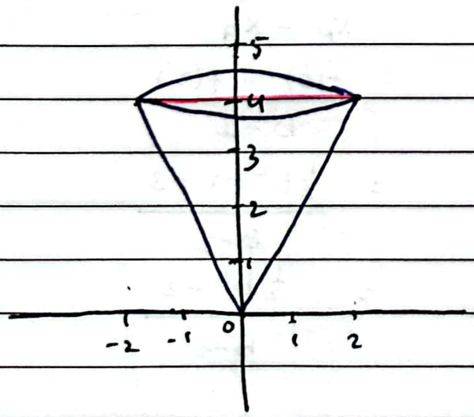
$$= \lim_{t \to 2} t+2$$

$$= 2+2 = 4$$

Kecepatan sesaat saat $t = 2$ adalah 4.

Menghitung volume menggunakan limit.

① Tangki berbentuk kerucut terbalik memiliki jari² 2 meter dan tinggi 4 meter. Tangki diisi air dgn debit konstan $\frac{1}{2}$ m^3 per menit.



a.) Hitung volume / kapasitas tangki.

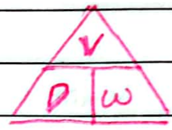
Jawab :

$$V = \frac{1}{3} \cdot \pi r^2 h \rightarrow V_{tangki}$$

$$= \frac{1}{3} \cdot \pi \cdot 4 \cdot 4 = \frac{16\pi}{3} \approx 16,\ldots$$

c.) Berapa volume air di dalam tangki setelah t menit?

Jawab :



$$V_{air}(t) = \text{debit} \times t$$

$$= \frac{1}{2} t$$

b.) Kapan tangki tersebut penuh? Notasikan sebagai t_1.

Jawab :

⇒ Penuh saat $V_{air} = V_{tangki}$

$$\frac{1}{2} t_1 = \frac{16\pi}{3}$$

$$t_1 = \frac{32\pi}{3}$$

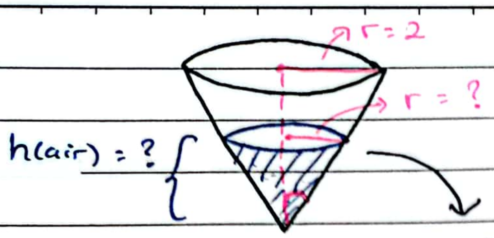
e.) Berapa laju perubahan tinggi air rata-rata tangki pada interval waktu $[0, t_1]$?

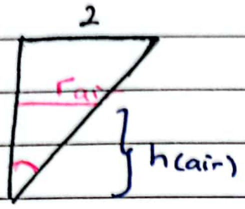
Jawab : Laju perubahan tinggi rata-rata

$$= \frac{h}{t_1} = \frac{4}{32/3\,\pi} = \frac{12}{32\pi}$$

d.) Berapa ketinggian air di dalam tangki setelah t menit?

Jawab :





$$\frac{4}{h(air)} = \frac{2}{r(air)}$$

$$\Rightarrow r_{air} = \frac{2}{4} h(air)$$

$$= \frac{1}{2} h(air)$$

$$\Rightarrow V_{air} = \frac{1}{3} \pi . r_{air}^2 . h_{(air)}$$

$$\frac{1}{2} t = \frac{1}{3} \pi . \left(\frac{1}{2} h(air)\right)^2 . h(air)$$

$$\frac{1}{2} t = \frac{1}{12} \pi . (h(air))^3$$

$$\therefore h(air) = \sqrt[3]{\frac{6t}{\pi}} \; //$$

misal: laju perubahan tinggi air rata² pada $[2, 3]$

$$= \frac{h(3) - h(2)}{3 - 2}$$

$$= \frac{\sqrt[3]{\frac{18}{\pi}} - \sqrt[3]{\frac{12}{\pi}}}{1} = \ldots \; //$$

f.) Berapa laju perubahan tinggi air rata-rata tangki pada interval waktu $[2, 2.01]$?

Jawab : laju perubahan

$$= \frac{h(2.01) - h(2)}{2.01 - 2}$$

$$= \frac{\sqrt[3]{\frac{12.06}{\pi}} - \sqrt[3]{\frac{12}{\pi}}}{0,01} \; //$$

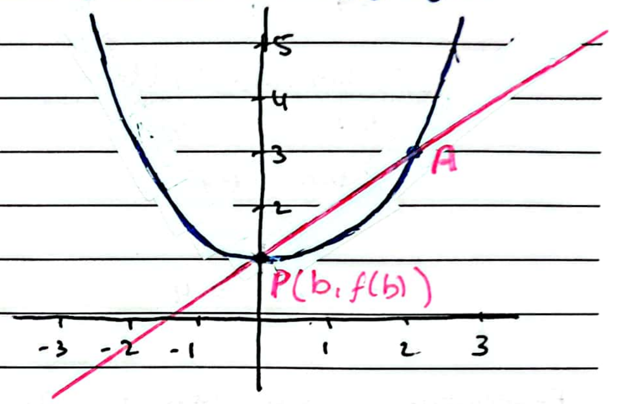
g.) Berapa laju perubahan tinggi air (sesaat) pada saat $t = 2$?

Jawab : laju perubahan tinggi sesaat

$$= \lim_{t \to 2} \frac{h(t) - h(2)}{t - 2}$$

$$= \lim_{t \to 2} \frac{\sqrt[3]{\frac{6t}{\pi}} - \sqrt[3]{\frac{12}{\pi}}}{t - 2} \; //$$

Kemiringan Garis Singgung



Misalkan P suatu titik pada grafik fungsi dengan koordinat $(b, f(b))$. Tentukan gradien garis AP sebagai fungsi dalam variabel b. Apakah percobaan ini bisa membantu menentukan gradien garis singgung ?

Jawab :

$$\Rightarrow (b, f(b)) = (b, \tfrac{1}{2} b^2 + 1)$$

$$\Rightarrow m_{AP} = \frac{y_P - y_A}{x_P - x_A}$$

$$= \frac{f(b) - f(2)}{b - 2}$$

$$= \frac{\frac{1}{2} b^2 + 1 - 3}{b - 2}$$

$$= \frac{\frac{1}{2} b^2 - 2}{b - 2}$$

$$\Rightarrow m = \lim_{b\to 2} m_{AP}$$

$$= \lim_{b\to 2} \frac{\frac{1}{2}b^2 - 2}{b-2}$$

$$= \lim_{b\to 2} \frac{\frac{1}{2}(b^2-4)}{b-2}$$

$$= \lim_{b\to 2} \frac{1}{2}(b+2)$$

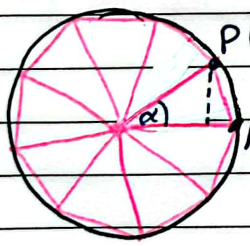
$$= \frac{1}{2}(2+2) = 2 \;//$$

→ Apakah percobaan ini bisa membantu menentukan gradien garis singgung?

→ Iya.

$\boxed{\text{Garis singgung di A = limit dari garis AP saat } P \to A}$

Luas segi-n dan luas lingkaran



- $\angle AOP = \frac{2\pi}{n}$
- $P\left(\cos\left(\frac{2\pi}{n}\right), \sin\left(\frac{2\pi}{n}\right)\right)$

$$\Rightarrow L_{AOP} = \frac{1}{2} \cdot 1 \cdot \sin\left(\frac{2\pi}{n}\right)$$

$$= \frac{1}{2} \sin\left(\frac{2\pi}{n}\right)$$

$$L \text{ segi-n} = n \cdot L_{AOP}$$

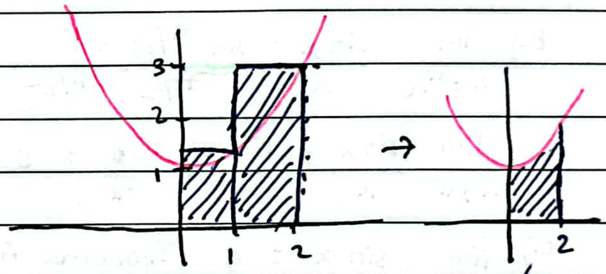
$$\boxed{L \text{ segi-n} = \frac{n}{2} \cdot \sin\left(\frac{2\pi}{n}\right) \;//}$$

Untuk $n \to \infty$, L segi-n → luas lingkaran

$$\lim_{n\to\infty} \frac{n}{2} \cdot \sin\left(\frac{2\pi}{n}\right) = \pi \cdot 1^2 = \pi$$

Luas daerah di bawah kurva

$$f(x) = \frac{1}{2}x^2 + 1$$



$$\int \left(\frac{1}{2}x^2 + 1\right) dx = \frac{1}{2} \cdot \frac{1}{3} x^3 + x + C$$

$$L = \int_0^2 \left(\frac{1}{2}x^2 + 1\right) dx$$

$$= \frac{1}{6}x^3 + x \Big|_0^2$$

$$= \frac{8}{6} + 2 = \frac{4}{3} + 2 = \frac{10}{3} \;//$$

Contoh Soal!

[2] $g(x) = \begin{cases} x^2 + 2x - 1, & x < 1 \\ 3x - 1, & x > 1 \\ 5, & x = 1 \end{cases}$

a.) Hitung $\lim_{x\to 1^-} g(x)$ dan $\lim_{x\to 1^+} g(x)$:

Jawab: $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-}(x^2+2x-1) = 1+2-1 = 2$ //

$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+}(3x-1) = 3.1 - 1 = 2$ //

b.) $\lim_{x \to 1} g(x)$ ada?

Jawab : Ada. Karena

$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x) = 2$

$\therefore \lim_{x \to 1} g(x) = 2$ //

3.2 Grafik fungsi $h(x) = \frac{\sin x}{x}$

a.) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (Teorema Apit) //

b.) $\lim_{x \to \pi/2} \frac{\sin x}{x} = \frac{\sin(\pi/2)}{\pi/2} = \frac{1}{\pi/2} = \frac{2}{\pi}$ //

$\lim_{x \to \pi} \frac{\sin x}{x} = \frac{\sin \pi}{\pi} = \frac{0}{\pi} = 0$ //

c.) $\lim_{x \to \infty} \frac{\sin x}{x} = 0$ (Teorema Apit) //

Conso !

3.2 Hitunglah $\lim_{x \to 3^-} [\![x]\!]$ dan $\lim_{x \to 3^+} [\![x]\!]$

Jawab: $\lim_{x \to 3^-} [\![x]\!] = 2$ //

$\lim_{x \to 3^+} [\![x]\!] = 3$ //

Apakah $\lim_{x \to 3} [\![x]\!]$ ada?

Jawab: Tidak ada, karena

$\lim_{x \to 3^-} [\![x]\!] \neq \lim_{x \to 3^+} [\![x]\!]$ //

Bagaimana dengan $\lim_{x \to 3/2} [\![x]\!]$?

Jawab : $\lim_{x \to 3/2} [\![x]\!] = 1$ //

Date . .

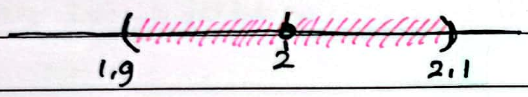
1.2 Definisi Limit Secara Formal / Rigor

→ Jelaskan makna dari :

$0 < |x-2| < \frac{1}{10}$

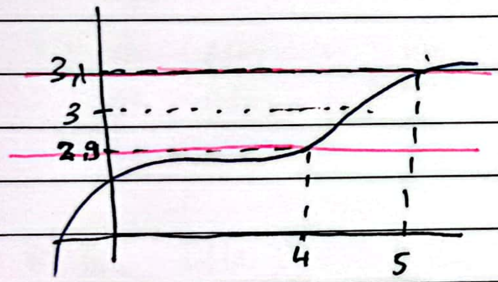
$|a-b| =$ jarak antara $a - b$

→ $x \neq 2$ dan jarak x dengan 2 kurang dari $\frac{1}{10}$



→ Jelaskan makna dari :

$|f(x) - 3| < \frac{1}{10}$



$4 < x < 5 \Rightarrow |f(x) - 3| < \frac{1}{10}$

Definisi Limit (Formal)

$\boxed{0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon}$

$\lim_{x \to c} f(x) = L$ apabila ;

Untuk setiap $\varepsilon > 0$, terdapat $\delta > 0$ sehingga berlaku :

Contoh Soal !

1.) Buktikan bahwa $\lim_{x \to 2} (2x+1) = 5$

Tentukan suatu $\delta > 0$ sehingga pernyataan berikut benar :

a.) Jika $0 < |x-2| < \delta$, maka $|(2x+1) - 5| < 1$

Jawab :

Perhatikan, $|(f(x)) - L| = |(2x+1) - 5|$

$= |2x - 4|$

$= |2(x-2)| < 1$

Pilih $\delta = \frac{1}{2}$

$0 < |x-2| < \frac{1}{2} \Rightarrow |2(x-2)| < 2.\frac{1}{2} = 1$

∴ Untuk $\varepsilon = 1$, terdapat δ yg memenuhi definisi limit.

b.) Jika $0 < |x-2| < \delta$, maka $|(2x+1) - 5| < \frac{1}{10}$

Jawab : $|(f(x)) - L| < \frac{1}{10}$

$|(2x+1) - 5| < \frac{1}{10}$

$|2x - 4| < \frac{1}{10}$

$|2(x-2)| < \frac{1}{10}$

Pilih $\delta = \frac{1}{20}$

$0 < |x-2| < \frac{1}{20} \Rightarrow |2(x-2)| < 2.\frac{1}{20} = \frac{1}{10}$

∴ Untuk $\varepsilon = \frac{1}{10}$, terdapat δ yang memenuhi definisi limit.

d.) Jika $0 < |x-2| < \delta$ maka $|(2x+1)-5| < \varepsilon$

Jawab : $|f(x) - L| < \varepsilon$

$|(2x+1)-5| < \varepsilon$

$|2x-4| < \varepsilon$

$|2(x-2)| < \varepsilon$

→ Pilih $\delta = \frac{\varepsilon}{2}$

∴ $0 < |x-2| < \frac{\varepsilon}{2} \Rightarrow |(2x+1)-5| < \varepsilon$

$\frac{|x-9|}{|\sqrt{x}+3|} < \varepsilon$

$\frac{|x-9|}{3} < \varepsilon$

Pilih $\delta = 3\varepsilon$

$0 < |x-9| < 3\varepsilon \Rightarrow \frac{|x-9|}{3} < \frac{3\varepsilon}{3}$

$\Rightarrow \frac{|x-9|}{3} < \varepsilon$ //

∴ $\lim_{x\to 9} \sqrt{x} = 3$ //

Buktikan bahwa $\lim_{x\to 3} (5x+7) = 22$

Jawab: $0 < |x-3| < \delta \Rightarrow |(5x+7)-22| < \varepsilon$

$|f(x) - L| < \varepsilon$

$|5x+7-22| < \varepsilon$

$|5x-15| < \varepsilon$

$|5(x-3)| < \varepsilon$

Pilih $\delta = \frac{\varepsilon}{5}$

Jika, $0 < |x-3| < \delta$

$\Rightarrow 0 < |x-3| < \frac{\varepsilon}{5}$

$\Rightarrow 0 < |5(x-3)| < 5 \cdot \frac{\varepsilon}{5}$

$\Rightarrow 0 < |5x-15| < \varepsilon$ //

[2] Buktikan bahwa $\lim_{x\to 9} \sqrt{x} = 3$

Jawab: $|\sqrt{x} - 3| < \varepsilon$

$\left|\frac{x-9}{\sqrt{x}+3}\right| < \varepsilon$

$\sqrt{x} \geq 0$

$\frac{1}{\sqrt{x}+3} = \frac{1}{0+3} = \frac{1}{3}$

$x-9 = (\sqrt{x}-3)(\sqrt{x}+3)$

$\sqrt{x}-3 = \frac{x-9}{\sqrt{x}+3}$

No.

Date . .

Bab 1.3

1.3 Teorema Limit

TEOREMA A : Teorema Limit Utama

1.) $\lim_{x \to c} k = k$

2.) $\lim_{x \to c} x = c$

3.) $\lim_{x \to c} k f(x) = k \lim_{x \to c} f(x)$

4.) $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

5.) $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$

6.) $\lim_{x \to c} [f(x) . g(x)] = \lim_{x \to c} f(x) . \lim_{x \to c} g(x)$

7.) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \rightarrow \lim_{x \to c} g(x) \neq 0$

8.) $\lim_{x \to c} [f(x)]^n = [\lim_{x \to c} f(x)]^n$

9.) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} \rightarrow \lim_{x \to c} f(x) > 0$ saat n genap

Teorema B : Teorema Substitusi

Jika f adalah fungsi polinomial atau fungsi rasional, maka

$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

Fungsi polinomial berbentuk:

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$a_0, \dots, a_n$ = konstanta real

Fungsi rasional berbentuk:

$\frac{p(x)}{g(x)}$

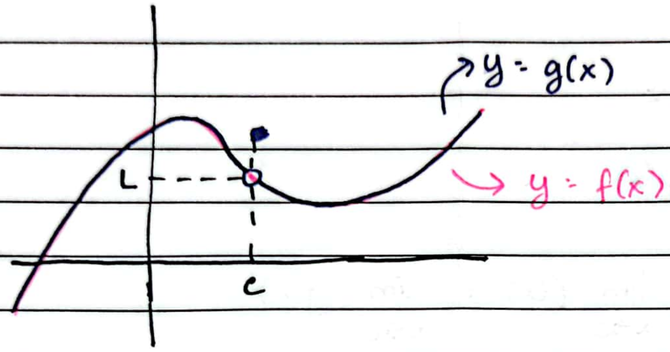
$p(x), g(x)$ = polinomial

Date . .

TEOREMA C

$f(x) = g(x)$ untuk x di sekitar c tapi mungkin $f(c)$ tidak sama dengan $g(c)$.

Ilustrasi :



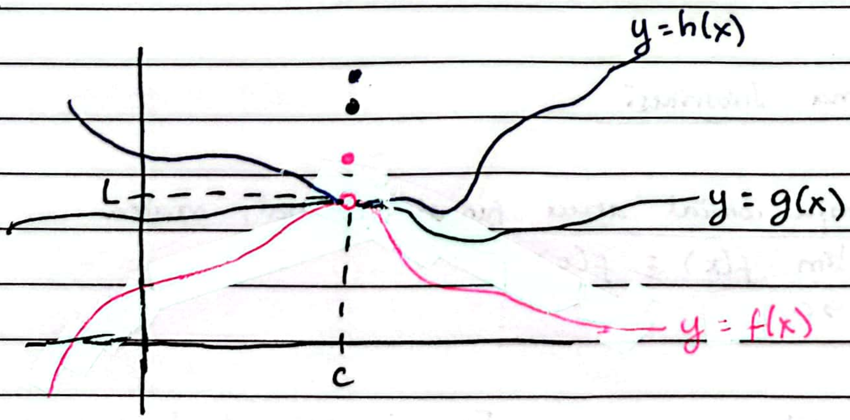
$$\boxed{\lim_{x \to c} f(x) = L = \lim_{x \to c} g(x)}$$

TEOREMA D : Teorema Apit

Diberikan f, g, h sehingga $f(x) \leq g(x) \leq h(x)$ untuk semua x mendekati c tetapi tidak terdefinisi saat c.

$$\boxed{\text{Jika } \lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L, \text{ maka } \lim_{x \to c} g(x) = L}$$

Ilustrasi :



$f(x) \leq g(x) \leq h(x)$

Jika, $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, maka $\lim_{x \to c} g(x) = L$

No.

Date . .

Latihan Teorema Limit

Contoh 1: Hitung limit berikut dgn sifat.

a.) $\lim_{x\to 2} (3x^2 - 5x + 7)$

$= \lim_{x\to 2}(3x^2) - \lim_{x\to 2}(5x) + \lim_{x\to 2} 7$ sifat 4 & 5

$= 3 \lim_{x\to 2} x^2 - 5 \lim_{x\to 2} x + \lim_{x\to 2} 7$ sifat no. 3

$= 3\left(\lim_{x\to 2} x\right)^2 - 5 \lim_{x\to 2} x + \lim_{x\to 2} 7$ sifat no. 8

$= 3 \cdot 2^2 - 5 \cdot 2 + 7$ sifat 1 & 2

$= 9$ //

b.) $\lim_{x\to 2} \frac{2x}{x^2+7}$

$= \frac{\lim_{x\to 2}(2x)}{\lim_{x\to 2}(x^2+7)}$ sifat no. 7

$= \frac{2 \lim_{x\to 2} x}{\lim_{x\to 2} x^2 + \lim_{x\to 2} 7}$ sifat 3 & 4

$= \frac{2 \lim_{x\to 2} x}{\left(\lim_{x\to 2} x\right)^2 + \lim_{x\to 2} 7}$ sifat no. 8

$= \frac{2 \cdot 2}{2^2 + 7} = \frac{4}{11}$ // sifat no. 1 & 2

Contoh 2: $f(x)$ dan $g(x)$ dgn $\boxed{f(0) = 5 \text{ dan } g(0) = 2}$ tidak diperlukan

$\lim_{x\to 0} f(x) = 3$ dan $\lim_{x\to 0} g(x) = -1$

a.) $\lim_{x\to 0} (2f(x) + g(x))$

$= \lim_{x\to 0} 2f(x) + \lim_{x\to 0} g(x)$

$= 2 \lim_{x\to 0} f(x) + \lim_{x\to 0} g(x)$

$= 2 \cdot 3 + (-1) = 5$ //

c.) $\lim_{x\to 0} \sqrt{1 - f(x) \cdot g(x)}$

$= \sqrt{\lim_{x\to 0} (1 - f(x) \cdot g(x))}$

$= \sqrt{\lim_{x\to 0} 1 - \lim_{x\to 0} f(x) \cdot g(x)}$

$= \sqrt{\lim_{x\to 0} 1 - \lim_{x\to 0} f(x) \cdot \lim_{x\to 0} g(x)}$

$= \sqrt{1 - 3 \cdot (-1)}$

$= \sqrt{4} = 2$ //

Polinom → fungsi berbentuk

$a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 x^0$

⇒ $a_0, a_1, \dots$ → konstanta real

Fungsi Rasional → fungsi berbentuk $\dfrac{p(x)}{q(x)}$

dengan $p(x), q(x)$ → polinom

c/ : $f(x) = \dfrac{x^2 + 7x - 5}{x^7 - 2x + 1}$

Contoh : Hitung dgn Teorema Substitusi

a.) $\lim\limits_{x \to 1} (x^{10} - 5x^4 - 3x^2 + 5x + 7)$

$= 1^{10} - 5.1^4 - 3.1^2 + 5.1 + 7 = 5$ //

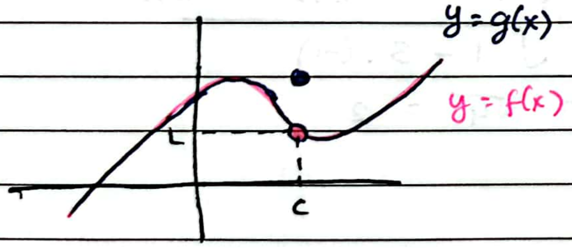
b.) $\lim\limits_{x \to 2} \dfrac{2x^3 - 3x^2 + 1}{x^2 + x}$

$= \dfrac{2.8 - 3.4 + 1}{4 + 2} = \dfrac{5}{6}$ //

Teorema C :

$f(x) = g(x)$ untuk x di sekitar c

tapi mungkin $f(c)$ tidak sama dgn $g(c)$



$\lim\limits_{x \to c} f(x) = L = \lim\limits_{x \to c} g(x)$

c/ : $\lim\limits_{x \to 2} \dfrac{x^2 - 3x + 2}{x^2 - 4} = \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}(x+2)}$

Teorema C → $= \lim\limits_{x \to 2} \dfrac{x-1}{x+2}$

Teorema Substitusi → $= \dfrac{2-1}{2+2} = \dfrac{1}{4}$ //

2. $\lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

$= \lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9} \cdot \dfrac{\sqrt{x} + 3}{\sqrt{x} + 3}$

$= \lim\limits_{x \to 9} \dfrac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x} + 3)}$

$= \lim\limits_{x \to 9} \dfrac{1}{\sqrt{x} + 3}$ ← Teorema C (fungsinya berubah)

$= \dfrac{1}{\sqrt{9} + 3} = \dfrac{1}{6}$ //

Teorema Apit

1. Hitunglah $\lim\limits_{x \to 0} x^2 \cos\left(\dfrac{1}{x}\right)$

Jawab :

Perhatikan : $-1 \le \cos\left(\frac{1}{x}\right) \le 1$

$\underbrace{-x^2}_{f(x)} \le x^2 \cos\left(\frac{1}{x}\right) \le \underbrace{x^2}_{h(x)}$

$\lim\limits_{x \to 0} -x^2 = 0$

$\lim\limits_{x \to 0} x^2 = 0$

Berdasarkan T. Apit : $\lim\limits_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$

Alternatif Penyelesaian :

$-1 \le \cos\left(\frac{1}{x}\right) \le 1$

$-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$

Anella Utari Gunadi (19623229)

No.

Date . .

Checkpoint 20-09-2023

1. Diketahui: $f(1) = 0$ dan $g(1) = -5$

$\lim_{x\to 1} f(x) = 2$, $\lim_{x\to 1} g(x) = 7$

Hitunglah $\lim_{x\to 1} \frac{g(x) - 3f(x)}{\sqrt{x^2+x+2}}$

Jawab:

$$\lim_{x\to 1}\left(\frac{g(x)-3f(x)}{\sqrt{x^2+x+2}}\right)$$

$$= \frac{\lim_{x\to 1}(g(x) - 3f(x))}{\lim_{x\to 1}\sqrt{x^2+x+2}}$$

$$= \frac{\lim_{x\to 1} g(x) - \lim_{x\to 1} 3f(x)}{\sqrt{\lim_{x\to 1}(x^2+x+2)}}$$

$$= \frac{\lim_{x\to 1} g(x) - 3\lim_{x\to 1} f(x)}{\sqrt{\lim_{x\to 1} x^2 + \lim_{x\to 1} x + \lim_{x\to 1} 2}}$$

$$= \frac{\lim_{x\to 1} g(x) - 3\lim_{x\to 1} f(x)}{\sqrt{(\lim_{x\to 1} x)^2 + \lim_{x\to 1} x + \lim_{x\to 1} 2}}$$

$$= \frac{7 - 3.2}{\sqrt{1^2+1+2}} = \frac{1}{\sqrt{4}} = \frac{1}{2} //$$

2. Hitunglah $\lim_{x\to 3} \frac{x^2-9}{x^3-5x^2+6x}$

Jawab:

$$\lim_{x\to 3} \frac{x^2-9}{x^3-5x^2+6x}$$

$$= \lim_{x\to 3} \frac{(x-3)(x+3)}{x(x^2-5x+6)}$$

$$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{x\cancel{(x-3)}(x-2)}$$

$$= \lim_{x\to 3} \frac{x+3}{x(x-2)}$$

$$= \frac{3+3}{3(3-2)} = \frac{6}{3} = 2 //$$

3. Hitunglah $\lim_{x\to 0^-} x \sin\left(\frac{1}{x^2}\right)$

Jawab:

$$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1$$

$$f(x) \leftarrow -x \geq x \cdot \sin\left(\frac{1}{x^2}\right) \geq x \rightarrow h(x)$$

$$\lim_{x\to 0^-}(-x) \geq \lim_{x\to 0^-} x \sin\left(\frac{1}{x^2}\right) \geq \lim_{x\to 0^-} x$$

$$0 \geq \lim_{x\to 0^-} x \sin\left(\frac{1}{x^2}\right) \geq 0$$

$\therefore$ Berdasarkan Teorema Apit

$$\lim_{x\to 0^-} x \sin\left(\frac{1}{x^2}\right) = 0 //$$

1.4 Limit Fungsi Trigonometri

Teorema A : Limit Fungsi Trigonometri Dasar

1.) $\lim_{t \to c} \sin t = \sin c$

2.) $\lim_{t \to c} \cos t = \cos c$

3.) $\lim_{t \to c} \tan t = \tan c$

Teorema B : Limit Fungsi Trigonometri Khusus

1.) $\lim_{t \to 0} \frac{\sin t}{t} = 1$

2.) $\lim_{t \to 0} \frac{1-\cos t}{t} = 0$

Latihan soal!

6. $\lim_{\theta \to 0} \frac{\sin 3\theta}{2\theta} = \lim_{\theta \to 0} \frac{\sin(3\theta) \cdot \frac{3}{2}}{2\theta \cdot \frac{3}{2}}$

$= \lim_{\theta \to 0} \frac{3}{2} \cdot \frac{\sin 3\theta}{3\theta}$

$= \frac{3}{2} \cdot \lim_{\theta \to 0} \frac{\sin 3\theta}{3\theta}$

$= \frac{3}{2} \cdot 1 = \frac{3}{2}$ //

atau

Misal, $t = 3\theta \Rightarrow \theta = \frac{t}{3}$

$\lim_{t \to 0} \frac{\sin t}{2 \cdot \frac{t}{3}} = \lim_{t \to 0} \frac{3}{2} \cdot \frac{\sin t}{t}$

$= \frac{3}{2} \cdot 1 = \frac{3}{2}$ //

10. $\lim_{\theta \to 0} \frac{\tan 5\theta}{\sin 2\theta} = \lim_{\theta \to 0} \frac{\sin 5\theta}{\cos 5\theta} \cdot \frac{1}{\sin 2\theta}$

$= \lim_{\theta \to 0} \frac{1}{\cos 5\theta} \left(\frac{\sin 5\theta \cdot 5\theta}{5\theta}\right)\left(\frac{2\theta}{\sin 2\theta \, 2\theta}\right)$

$= \lim_{\theta \to 0} \frac{1}{\cos 5\theta} \cdot \frac{\sin 5\theta}{5\theta} \cdot \frac{2\theta}{\sin 2\theta} \cdot \frac{5}{2}$

$= \frac{5}{2} \cdot \lim_{\theta \to 0} \frac{1}{\cos 5\theta} \cdot \frac{\sin 5\theta}{5\theta} \cdot \frac{2\theta}{\sin 2\theta}$

$= \frac{5}{2} \cdot \frac{1}{1} \cdot 1 \cdot 1 = \frac{5}{2}$ //

14. $\lim_{\theta \to 0} \frac{\sin^2\theta}{\theta^2} = \lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta}\right)^2$

$= \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right)^2$

$= 1^2 = 1$ //

Identitas : $1-\cos(2\theta) = 2\sin^2\theta$

$(1-\cos\theta)(1+\cos\theta) = \sin^2\theta$

11. $\lim_{x \to 0} \frac{1-\cos(10x)}{x^2}$

$= \lim_{x \to 0} \frac{2\sin^2(5x)}{x^2}$

$= 2\left(\lim_{x \to 0} \frac{\sin(5x)}{x}\right)^2$

$= 2\left(\lim_{x \to 0} \frac{\sin 5x}{5x} \cdot 5\right)^2$

$= 2(1 \cdot 5)^2 = 50$ //

12. $\lim_{x \to 0} \frac{1-\cos(12x)}{x \sin(3x)}$

$= \lim_{x \to 0} \frac{2\sin^2(6x)}{x \sin 3x}$

$= 2 \lim_{x \to 0} \frac{\sin^2(6x)}{x \sin(3x)}$

Date . .

$$= 2 \lim_{x\to 0} \frac{\sin 6x . \sin 6x}{x . \sin 3x}$$

$$= 2 \lim_{x\to 0} \frac{\sin 6x}{6.x} . 6 . \frac{\sin 6x}{6x} . \cancel{6x} \frac{3x}{\sin 3x} . \frac{1}{\cancel{3x}}$$

$$= 2.6.6.\frac{1}{3} \lim_{x\to 0} \frac{\sin 6x}{6x} . \frac{\sin 6x}{6x} . \frac{3x}{\sin 3x}$$

$$= 2.6.6.\frac{1}{3} \; 1.1.1 = 24$$

Limit dengan pemisalan

c.) $\lim_{x\to \pi/4} \frac{\cos x - \sin x}{x - \pi/4}$

$\sin x - \cos x = \sqrt{2} \sin(x - \pi/4)$

$$= \lim_{x\to \pi/4} \frac{-\sqrt{2}\sin(x-\pi/4)}{x - \pi/4}$$

$$= -\sqrt{2} \lim_{x\to \pi/4} \frac{\sin(x-\pi/4)}{x-\pi/4} = -\sqrt{2}$$

b.) $\lim_{x\to -1} \frac{\sin(x^2-1)}{x+1}$

$$= \lim_{x\to -1} \frac{\sin(x^2-1)}{(x^2-1)/(x-1)}$$

$$= \lim_{x\to -1} \frac{\sin(x^2-1)}{x^2-1} . (x-1)$$

$$= 1.(-1-1) = -2$$

Dipindai dengan CamScanner

11. Limit Fungsi Trigonometri Khusus

Theorem B

1. $\lim_{t \to 0} \frac{\sin t}{t} = 1$

2. $\lim_{t \to 0} \frac{1 - \cos t}{t} = 0$

Latsol 1

16.

$\lim_{\theta \to 0} \frac{\sin 3\theta}{2\theta}$

$= \lim_{\theta \to 0} \frac{\sin 3\theta \cdot \frac{3}{2}}{2\theta \cdot \frac{3}{2}}$

$= \lim_{\theta \to 0} \frac{3}{2} \cdot \frac{\sin 3\theta}{3\theta} = \frac{3}{2} \cdot 1 = \frac{3}{2}$ //

cara lain :

misal $t = 3\theta \Leftrightarrow \theta = \frac{t}{3}$

$\lim_{t \to 0} \frac{\sin t}{2 \cdot \frac{t}{3}} = \lim_{t \to 0} \frac{3}{2} \cdot \frac{\sin t}{t}$

$= \frac{3}{2} \lim_{t \to 0} \frac{\sin t}{t}$

$= \frac{3}{2}$ //

18.

$\lim_{\theta \to 0} \frac{\tan (5\theta)}{\sin (2\theta)}$

$= \lim_{\theta \to 0} \frac{\sin (5\theta)}{\cos (5\theta)} \cdot \frac{1}{\sin (2\theta)}$

$= \lim_{\theta \to 0} \frac{1}{\cos (5\theta)} \left(\frac{\sin (5\theta)}{5\theta} \cdot \cancel{5\theta} \right) \left(\frac{2\theta}{\sin (2\theta)} \cdot \frac{1}{\cancel{2\theta}} \right)$

$= \lim_{\theta \to 0} \frac{5}{2} \cdot \frac{1}{\cos 5\theta} \cdot \frac{\sin 5\theta}{5\theta} \cdot \frac{2\theta}{\sin 2\theta}$

$= \frac{5}{2} \cdot \lim_{\theta \to 0} \frac{1}{\cos 5\theta} \cdot \lim_{\theta \to 0} \frac{\sin 5\theta}{5\theta} \cdot \lim_{\theta \to 0} \frac{2\theta}{\sin 2\theta}$

$= \frac{5}{2} \cdot \frac{1}{\cos (0)} \cdot 1 \cdot 1 = \frac{5}{2}$ //

← Theorem B

14.

$\lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta^2}$

$= \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \right)^2 = \left(\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \right)^2$

$= 1^2 = 1$ //

TAMBAHAN !

limit terkait $1 - \cos (nx)$

Gunakan identitas

$$\boxed{\begin{array}{c} 1 - \cos (2\theta) = 2 \sin^2 (\theta) \\ \text{atau} \\ (1 - \cos \theta)(1 + \cos \theta) = \sin^2 \theta \end{array}}$$

1.

$\lim_{x \to 0} \frac{1 - \cos (10x)}{x^2}$

$= \lim_{x \to 0} \frac{2 \sin^2 (5x)}{x^2}$

$= 2 \left(\lim_{x \to 0} \left(\frac{\sin 5x}{x} \right) \right)^2$

$= 2 \left(\lim_{x \to 0} \frac{\sin 5x}{5x} \cdot 5 \right)^2$

$= 2 \cdot 1 \cdot 5^2 = 50$ //

2.

$\lim_{x \to 0} \frac{1 - \cos (12x)}{x \sin 3x}$

$= \lim_{x \to 0} \frac{1 - \cos (12x)}{x \sin 3x} \cdot \frac{1 + \cos (12x)}{1 + \cos (12x)}$

$= \lim_{x \to 0} \frac{1 - \cos^2 (12x)}{x \sin (3x) (1 + \cos (12x))}$

$= \lim_{x \to 0} \frac{\sin^2 (12x)}{x \sin (3x) (1 + \cos (12x))}$

$= \lim_{x \to 0} \frac{1}{1 + \cos (12x)} \cdot \frac{\sin (12x)}{x} \cdot \frac{\sin (12x)}{\sin (3x)}$

$= \lim_{x \to 0} \frac{1}{1 + \cos 12x} \cdot \frac{\sin 12x}{12x} \cdot \frac{\sin 12x}{12x} \cdot \frac{3x}{\sin 3x} \cdot \frac{12 \cdot 12}{3}$

No.

Date . .

$= \frac{1}{2} \cdot 1 \cdot 1 \cdot 1 \cdot \frac{12 \cdot 12}{3} = 24$

c. $\lim_{x \to \pi/4} \frac{\cos x - \sin x}{x - \pi/4}$

$\boxed{\sin x - \cos x = \sqrt{2} \sin (x - \pi/4)}$

$= \lim_{x \to \pi/4} \frac{-\sqrt{2} \sin (x - \pi/4)}{x - \pi/4}$

$= \lim_{x \to \pi/4} -\sqrt{2} \cdot \frac{\sin (x - \pi/4)}{x - \pi/4} = -\sqrt{2}$

No.

Date . .

Bab 1.5

1.5 Limit Tak Hingga dan Limit Bernilai Tak Hingga

Teorema Dasar

- $\lim_{x\to\infty} \frac{1}{x} = 0$
- $\lim_{x\to-\infty} \frac{1}{x} = 0$

Latsol !

1. $\lim_{x\to\infty} \frac{5}{x^3}$

$= 5 \lim_{x\to\infty} \frac{1}{x^3}$

$= 5 \left(\lim_{x\to\infty} \frac{1}{x}\right)^3 = 5.0^3 = 0$ //

2. $\lim_{x\to\infty} \frac{2x+1}{3x-2}$

$= \lim_{x\to\infty} \frac{(2x+1)/x}{(3x-2)/x}$

$= \lim_{x\to\infty} \frac{2+\frac{1}{x}}{3-\frac{2}{x}}$

$= \frac{\lim_{x\to\infty}(2+\frac{1}{x})}{\lim_{x\to\infty}(3-\frac{2}{x})}$

$= \frac{\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty} 3 - \lim_{x\to\infty}\frac{2}{x}}$

$= \frac{2+0}{3-0} = \frac{2}{3}$ //

3. $\lim_{x\to\infty} \frac{2x^2-x}{x^2+5x-7}$ pembilang dan penyebut dibagi x^2

$= \lim_{x\to\infty} \frac{2-\frac{1}{x}}{1+\frac{5}{x}-\frac{7}{x^2}}$

$= \cancel{\frac{2-0}{1+0-0^2}} = \frac{2-0}{1+5.0-7.0^2} = 2$ //

4. $\lim_{x\to-\infty} \frac{\sqrt{x^2+7}}{2x+3}$

$x<0 \Rightarrow \sqrt{x^2} = -x$

Hati-hati : $x\to-\infty$ maka $x<0$

Untuk $x<0$

$\frac{\sqrt{x^2+7}}{x} = (-)\sqrt{\frac{x^2+7}{x^2}}$

$= \lim_{x\to-\infty} \frac{\sqrt{x^2+7}\,/x}{2x+3\,/x}$

$= \lim_{x\to-\infty} \frac{-\sqrt{(x^2+7)/x^2}}{(2x+3)/x}$

$= \lim_{x\to-\infty} \frac{-\sqrt{1+7/x^2}}{2+3/x}$

$= \frac{-\sqrt{1+0}}{2+0}$

$= -\frac{1}{2}$ //

Teorema Apit untuk Limit Tak Hingga

1. $\lim_{x \to \infty} \frac{\sin x}{x}$ → untuk $x \to \infty$, $\sin x$ berosilasi

Tambahan

$\lim_{x \to \infty} \sin x$ → tidak ada

$-1 \le \sin x \le 1$

$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

$\lim_{x \to \infty} \frac{-1}{x} \le \lim_{x \to \infty} \frac{\sin x}{x} \le \lim_{x \to \infty} \frac{1}{x}$

$0 \le \lim_{x \to \infty} \frac{\sin x}{x} \le 0$

$\therefore \lim_{x \to \infty} \frac{\sin x}{x} = 0$

2. $\lim_{x \to \infty} \frac{7x + 2\cos x}{3x-1}$

$-1 \le \cos x \le 1$

$-2 \le 2\cos x \le 2$

$7x - 2 \le 7x + 2\cos x \le 7x + 2$

$\frac{7x-2}{3x-1} \le \frac{7x + 2\cos x}{3x-1} \le \frac{7x+2}{3x-1}$

$\lim_{x \to \infty} \frac{7x-2}{3x-1} \le \lim_{x \to \infty} \frac{7x+2\cos x}{3x-1} \le \lim_{x \to \infty} \frac{7x+2}{3x-1}$

$\frac{7}{3} \le \lim_{x \to \infty} \frac{7x+2\cos x}{3x-1} \le \frac{7}{3}$

$\therefore \lim_{x \to \infty} \frac{7x+2\cos x}{3x-1} = \frac{7}{3}$

Asimtot Datar

$\boxed{\lim_{x \to \infty} f(x) = b \text{ atau } \lim_{x \to -\infty} f(x) = b}$

Asimtot datar dari : $f(x) = \frac{7x + 2\cos x}{3x-1}$

$\lim_{x \to \infty} \frac{7x + 2\cos x}{3x-1} = \frac{7}{3}$

$\therefore$ Asimtot datar : $\boxed{y = \frac{7}{3}}$ //

garis

Limit Bernilai Tak Hingga (∞ atau $-\infty$)

1. $\lim_{x \to 1^+} \frac{2x-1}{x-1}$

Jawab: $x \to 1^+$, $2x - 1 \to 1$

$x - 1 \to 0^+$

$\Rightarrow \frac{2x-1}{x-1} > 0$

$\therefore \lim_{x \to 1^+} \frac{2x-1}{x-1} = +\infty$ //

$\lim_{x \to 1^-} \frac{2x-1}{x-1} = -\infty$

$x < 1$, $x - 1 < 0$

sementara $2x - 1 > 0$

$\therefore \lim_{x \to 1^-} \frac{2x-1}{x-1} = -\infty$ //

2. $\lim_{x \to 2^-} \frac{2x-7}{3x-6}$

Jawab: $x \to 2^-$, $2x - 7 < 0 = -3$

$3x - 6 < 0$

$\therefore \frac{2x-7}{3x-6} > 0 \Rightarrow \lim_{x \to 2^-} \frac{2x-7}{3x-6} = +\infty$ //

· Tentukan asimtot datar & tegak

a.) $f(x) = \dfrac{2x-1}{x-1}$

› As. datar : garis $y = 2$

› As. tegak : garis $x = 1$

b.) $f(x) = \dfrac{x^2-1}{x^2-3x+2}$

› As. datar : garis $y = 1$

› As. tegak : $f(x) = \dfrac{(x+1)\cancel{(x-1)}}{(x-2)\cancel{(x-1)}}$

$f(x) = \dfrac{x+1}{x-2}$

As. tegak : garis $x = 2$

c.) $f(x) = \dfrac{3x+1}{\sqrt{x^2+7}}$

$\lim\limits_{x \to \infty} \dfrac{3x+1}{\sqrt{x^2+7}} = \lim\limits_{x \to \infty} \dfrac{(3x+1)/x}{\sqrt{x^2+7}/x}$

$= \lim\limits_{x \to \infty} \dfrac{3+\frac{1}{x}}{\sqrt{1+\frac{7}{x^2}}}$

$= \dfrac{3}{1} = 3$

$\lim\limits_{x \to -\infty} \dfrac{3x+1}{\sqrt{x^2+7}} = \lim\limits_{x \to -\infty} \dfrac{(3x+1)/x}{\sqrt{x^2+7}/x}$

$= \lim\limits_{x \to -\infty} \dfrac{3+\frac{1}{x}}{-\sqrt{1+\frac{7}{x^2}}}$

$= \dfrac{3}{-1} = -3$

› As. datar : garis $y = 3$ dan garis $y = -3$

› As. tegak : tidak ada

d.) $f(x) = \dfrac{x|x|-4}{x^2-5x+6}$

Jawab :

$|x| = x$ dan $-x$

› As. datar : garis $x = 1$ dan $x = -1$

Kandidat as. tegak

$x^2-5x+6 = 0$

$(x-3)(x-2) = 0$

$\lim\limits_{x \to 2} \dfrac{x|x|-4}{x^2-5x+6} = \lim\limits_{x \to 2} \dfrac{x^2-4}{x^2-5x+6}$

$= \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}(x+2)}{(x-3)\cancel{(x-2)}}$

$= \dfrac{2+2}{2-3} = \dfrac{4}{-1} = -4$ bukan $\pm\infty$

→ $x = 2$ bukan as. tegak

$\lim\limits_{x \to 3^+} \dfrac{x|x|-4}{x^2-5x+6} = \lim\limits_{x \to 3^+} \dfrac{x^2-4}{x^2-5x+6}$

$= \lim\limits_{x \to 3^+} \dfrac{\cancel{(x-2)}(x+2)}{(x-3)\cancel{(x-2)}}$

$= \lim\limits_{x \to 3^+} \dfrac{x+2}{x-3} = +\infty$

› As. tegak : garis $x = 3$

Bab 1.6

1.6 Kekontinuan dan TNA

Kekontinuan di satu titik

Fungsi f dikatakan kontinu di titik c, jika :

$$\boxed{\lim_{x\to c} f(x) \text{ ada dan } \lim_{x\to c} f(x) = f(c)}$$

Kekontinuan fungsi hasil operasi

• Teorema 1

Jika f, g kontinu di $x = c$, maka

$\to f \pm g,\ f \cdot g,\ \frac{f}{g},\ f^n,\ f^{\frac{1}{n}}$

juga kontinu di $x = c$

• Teorema 2

Jika g kontinu di $x = c$ dan f kontinu di $x = g(c)$, maka $f \circ g$ kontinu di $x = c$

Contoh !

[17] Periksa apakah fungsi berikut kontinu di $x = 2$

a) $f(x) = \dfrac{x^3 - \cos x}{x - 1}$

Jawab :

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3 - \cos x}{x-1}$$

$$= \frac{\lim_{x\to 2}(x^3 - \cos x)}{\lim_{x\to 2}(x-1)}$$

$$= \frac{\lim_{x\to 2} x^3 - \lim_{x\to 2} \cos x}{\lim_{x\to 2}(x-1)}$$

$$= \frac{2^3 - \cos(2)}{2-1}$$

$$= 8 - \cos(2)$$

$$f(2) = \frac{2^3 - \cos(2)}{2-1} = 8 - \cos(2)$$

$\therefore f$ kontinu di $x = 2$

b) $h(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 5, & x = 2 \end{cases}$

Jawab :

$\to h(2) = 5$

$\to \lim_{x\to 2} h(x) = \lim_{x\to 2} \dfrac{x^2-4}{x-2} = 4$

$\therefore h$ tidak kontinu di $x = 2$

[19] Tentukan konstanta a sehingga fungsi

$f(x) = \begin{cases} 2x^2 + a, & x < 3 \\ ax + 4, & x \geq 3 \end{cases}$

kontinu di setiap bilangan real.

Jawab:

• Untuk $x > 3$, $f(x) = ax + 4$ polinom sehingga f kontinu di x tersebut.

• Untuk $x < 3$, $f(x) = 2x^2 + a$ polinom shg f kontinu di x tsb

• Untuk $x = 3$ (?)

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (2x^2 + a)$$

$$= 2 \cdot 3^2 + a = 18 + a$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (ax+4) = 3a + 4$$

Agar kontinu : $\lim_{x\to 3} f(x)$ ada dan

$$\lim_{x\to 3} f(x) = f(3)$$

Date . .

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x)$

$\Leftrightarrow \quad 3a + 4 = 18 + a$

$\Leftrightarrow \quad 2a = 14$

$\Leftrightarrow \quad a = 7$

→ Untuk $a = 7$, $\lim_{x \to 3} f(x) = 18 + 7 = 25$

dan $f(3) = 3a + 4 = 3 \cdot 7 + 4 = 25$

∴ Fungsi tsb kontinu di $x = 3$ untuk $a = 7$.

Removable Discontinuity

Tinjau $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 1 \text{ dan } x \neq 0 \\ 0, & x = 1 \text{ atau } x = 0 \end{cases}$

- Tentukan titik-titik di mana f tidak kontinu
- Tentukan apakah ketidakkontinuan tsb dapat dihapuskan atau tidak.

Jawab: $f(0) = 0$ dan $f(1) = 0$

f tidak kontinu di $x = 0$ dan $x = 1$

Cek di $x = 0$:

$\lim_{x \to 0} \frac{\sin x}{x} = 1$, $f(0) = 0$

∴ f tidak kontinu di $x = 0$ tetapi ketidakkontinuannya dapat dihapuskan dgn cara mendefinisikan $f(0) = 1$

Cek di $x = 1$

$\lim_{x \to 1} \frac{\sin x}{x} = \sin 1$, $f(1) = 0$

∴ ketidakkontinuan dapat dihapuskan dengan cara mendefinisikan $f(1) = \sin 1$

→ ketidakkontinuan di $x = 1$ dapat dihapuskan.

Teorema Nilai Antara

Misalkan f sebuah fungsi yg terdefinisi pada interval $[a,b]$ dan w suatu bil. yg terletak di antara $f(a)$ dan $f(b)$

Dua kemungkinan: $f(a) \leq w \leq f(b)$

$f(a) \geq w \geq f(b)$

Contoh!

Tunjukkan pers. $x^3 - x^2 = 1$ memiliki setidaknya satu akar pada int $[1,2]$.

Jawab: c adalah akar dari $x^3 - x^2 = 1$

$\Rightarrow c^3 - c^2 = 1$

→ Pilih $f(x) = x^3 - x^2$, $w = 1$

interval $= [1, 2]$

→ Periksa syarat² TNA:

1.) f kontinu pada $[1,2]$ karena f polinom ✓

2.) $f(a) = f(1) = 1^3 - 1^2 = 0$

$f(b) = f(2) = 2^3 - 2^2 = 4$

Betul bahwa $f(a) \leq w = 1 \leq f(b)$

→ Berdasarkan TNA: terdapat $1 \leq c < 2$ sehingga $f(c) = w$

$c^3 - c^2 = 1$

Dengan kata lain, c adalah akar dari persamaan $x^3 - x^2 = 1$ pada interval $[1, 2]$.

Date . .

Catatan : TNA tdk memberikan kesimpulan apa-apa saat :

1.) f kontinu di $[a,b]$

2.) w tidak diantara $f(a)$ dan $f(b)$

Tetapi tidak berarti :

Tidak ada $a \leq c \leq b$ shg $f(c) = w$

Kesimpulan : Jadi berdasarkan TNA terdapat $c \in [0, \pi/2]$ sehingga

$f(c) = w$

$\Leftrightarrow \cos c - 2c = -1$

$\Leftrightarrow \cos c = 2c - 1$

Contoh :

Tunjukkan bahwa terdapat suatu bilangan real c sehingga

$$\cos c = 2c - 1$$

Jawab : $\cos c = 2c - 1$

$\Leftrightarrow \cos c - 2c = -1$

Pilih $f(x) = \cos x - 2x$, $w = -1$

Cari a dan b shg $f(a) \leq w \leq f(b)$

	x	$f(x)$
$a \leftarrow$	0	1
$b \leftarrow$	$\pi/2$	$-\pi$

$-\pi \leq w \leq 1$

Cek 2 syarat TNA :

1.) f kontinu di $[a,b]$?

$\Rightarrow$ Ya, karena fungsi cos dan $2x$ keduanya kontinu sehingga $\cos(x) - 2x$ juga kontinu pada interval $[0, \pi/2]$

2.) $f(a) \leq w \leq f(b)$

$-\pi \leq w = -1 \leq 1$

Benar

No.

Date . .

Bab 2.1 - 2.2

2.1 Turunan

Definisi

$m_{tan} = \lim_{h \to 0} m_{sec} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

Misal, $c+h = x \Rightarrow h = x-c$

saat $h \to 0$, maka $x \to c$

$= \boxed{\lim_{x \to c} \frac{f(x) - f(c)}{x-c}}$ ← Definisi alternatif

Laju perubahan tinggi sesaat

$= \lim_{t \to 2} \frac{h(t) - h(2)}{t-2} = \lim_{t \to 2} \frac{\sqrt[3]{\frac{6t}{\pi}} - \sqrt[3]{\frac{12}{\pi}}}{t-2}$

$\boxed{\lim_{t \to c} \frac{f(t) - f(c)}{t-c}}$ Laju sesaat

Notasi :

$y = f(x)$

$\Rightarrow f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x) = D_x(f(x))$

Contoh !

2. $f(x) = 13x - 7$, hitunglah $f'(4)$

Jawab :

$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$

$= \lim_{h \to 0} \frac{(13(4+h) - 7) - (13(4) - 7)}{h}$

$= \lim_{h \to 0} \frac{52 + 13h - 7 - 52 + 7}{h}$

$= \lim_{h \to 0} \frac{13\cancel{h}}{\cancel{h}} = 13$

2.2

Buktikan bahwa :

$g(x) = |x|$ tidak memiliki turunan di $x = 0$

$g(0) = 0$

→ $\lim_{h \to 0^+} \frac{g(h) - g(0)}{h-0} = \lim_{h \to 0^+} \frac{|h| - 0}{h}$

$= \lim_{h \to 0^+} \frac{h}{h} = 1$

→ $\lim_{h \to 0^-} \frac{g(h) - g(0)}{h-0} = \lim_{h \to 0^-} \frac{|h| - 0}{h}$

$= \lim_{h \to 0^-} \frac{-h}{h} = -1$

$\lim_{h \to 0^+} \frac{g(h) - g(0)}{h-0} \neq \lim_{h \to 0^-} \frac{g(h) - g(0)}{h-0}$

Bab 2.3 – 2.4

Date . .

Aturan - Aturan Penentuan Turunan

Teorema A : Aturan Fungsi Konstanta

Jika $f(x) = k$, maka $f'(x) = 0$

$$\boxed{D_x(k) = 0}$$

Teorema B : Aturan Fungsi Identitas

Jika $f(x) = x$, maka $f'(x) = 1$

$$\boxed{D_x(x) = 1}$$

Teorema C : Aturan Pangkat

Jika $f(x) = x^n$, maka $f'(x) = n x^{n-1}$

$$\boxed{D_x(x^n) = n x^{n-1}}$$

Contoh !

1. $f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$ //

2. $f(x) = \frac{1}{x^2} = x^{-2}$

$f'(x) = -2x^{-3} = \frac{-2}{x^3}$ //

Teorema D : Aturan Kelipatan Konstanta

$(kf)'(x) = k.\, f'(x)$

$$\boxed{D_x(k.f(x)) = k.\, D_x(f(x))}$$

Teorema E : Aturan Penjumlahan

$(f+g)'(x) = f'(x) + g'(x)$

$$\boxed{D_x(f(x) + g(x)) = D_x(f(x)) + D_x(g(x))}$$

Teorema F : Aturan Pengurangan

$(f-g)'(x) = f'(x) - g'(x)$

$$\boxed{D_x(f(x) - g(x)) = D_x(f(x)) - D_x(g(x))}$$

Contoh :

Tentukan $D_x(x^7 - 7x^4 + 5x + 10)$

Jawab :

$D_x(x^7) - D_x(7x^4) + D_x(5x) + D_x(10)$

$= 7x^6 - 28x^3 + 5$ //

Teorema G : Aturan Perkalian

Jika $u = f(x)$ dan $v = g(x)$

$$\boxed{(f(x).\,g(x))' = (uv)' = u'v + uv'}$$

Teorema H : Aturan Pembagian

Jika $u = f(x)$ dan $v = g(x)$

$$\boxed{\left(\frac{f(x)}{g(x)}\right)' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

Contoh !

1. $f(x) = (2x^2 + 7x + 3)\sin x$, $f'(0) = ?$

Jawab: $f'(x) = (4x+7)\sin x + (2x^2+7x+3)\cos x$

$f'(0) = (0+7).\sin 0 + (0+0+3)\cos 0$

$= 0 + 3.1 = 3$ //

2. Tentukan turunan dari $g(x) = \frac{\cos x}{2x+1}$

Jawab: $g'(x) = \frac{-\sin x(2x+1) - (\cos x)(2)}{(2x+1)^2}$

$= \frac{-(2x+1)(\sin x) - 2\cos x}{(2x+1)^2}$ //

2.5 Aturan Rantai

Aturan Rantai

$$\boxed{D_x(f(g(x))) = f'(g(x)).g'(x) \quad \text{atau} \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}}$$

Contoh Soal!

1. $y = (2x^2-4x+1)^{60}$

$\frac{dy}{dx} = 60.(2x^2-4x+1)^{59}.(4x-4)$

2. Tentukan turunan dari:

a.) $g(x) = f(x^3)$

Jawab: $g'(x) = f'(x^3).3x^2$

b.) $h(x) = (f(x))^3$

Jawab: $\underset{h'(x)=}{\downarrow}\ 3(f(x))^2.f'(x)$

3. a.) Tentukan turunan dari:

(i) $h(t) = t\cos(t^2)$

Jawab:

$h'(t) = 1.\cos(t^2) + t.(-\sin(t^2)).2t$

$= \cos(t^2) - 2t^2.\sin(t^2)$

b.) Misalkan y suatu fungsi dalam x

Jika $w = 5y^3 + 3x^2\sin y$

Hitunglah $\frac{dw}{dx}$!

Jawab: y bukan konstanta

$\frac{d(5y^3)}{dx} = 5.3y^2.\underset{\downarrow\, y'}{\frac{dy}{dx}}$

Jika $v = \sqrt{y+x} \Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{y+x}}.\left(\frac{dy}{dx}+1\right)$

$\frac{dw}{dx} = 15y^2.\frac{dy}{dx} + \left(6x\sin y + 3x^2.\cos y.\frac{dy}{dx}\right)$

Aturan Rantai Komposisi Tiga Fungsi

$D_x(f(g(h(x)))) = f'(g(h(x))).g'(h(x)).h'(x)$

$\frac{dy}{dx} = \frac{dy}{dw}.\frac{dw}{du}.\frac{du}{dx}$

Contoh Soal!

1. Tentukan turunan dari $F(x) = \tan^5(x^3)$ dan $g(x) = \frac{1}{\sqrt{x^4+\sin x}}$

Jawab: $F'(x) = 5\tan^4(x^3).\sec^2(x^3).3x^2$

atau

$u = x^3 \quad w = \tan(u) \quad y = w^5$

$\frac{dy}{dx} = \frac{dy}{dw}.\frac{dw}{du}.\frac{du}{dx}$

$= 5w^4.\sec^2(u).3x^2$

$= 5\tan^4(x^3).\sec^2(x^3).3x^2$

boleh pake cara langsung

6. Diberikan $f(1)=2$, $f'(1)=-1$, $g(1)=\pi/6$, $g'(1)=1$. Tentukan $F'(1)$ dimana $F(x) = f(x).\cos g(x)$

Jawab: $F(x) = f(x).\cos(g(x))$

$F'(x) = f'(x).\cos(g(x)) + f(x).(-\sin(g(x))).g'(x)$

$F'(1) = f'(1).\cos(g(1)) + f(1).(-\sin(g(1))).g'(1)$

$= -1.\cos(\pi/6) - 2.\sin(\pi/6).1$

$= -\frac{1}{2}\sqrt{3} - 2.\frac{1}{2}.1$

$= -\frac{1}{2}\sqrt{3} - 1$

Date . .

70 Perangkat roda piston mempunyai jari-jari 1 kaki dan berputar berlawanan arah jarum jam sebesar 2 rad/s. Panjang batang penghubungnya adalah 5 kaki. Titik P berada di (1,0) saat $t=0$.

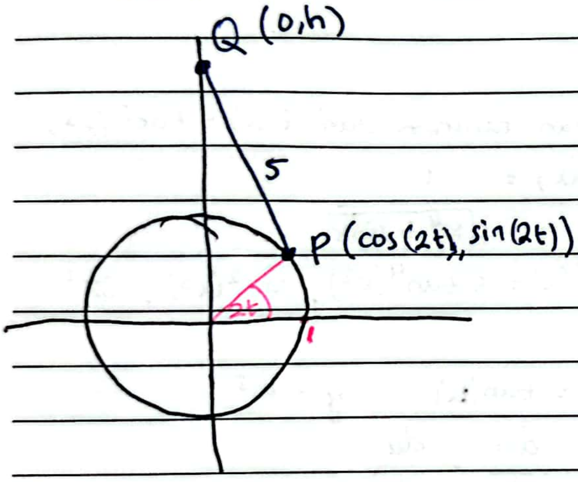
a.) Cari koordinat P saat waktu t.

b.) Cari koordinat-y Q saat waktu t (koordinat-x nya selalu 0)

c.) Cari kecepatan Q terhadap waktu.

$\left(D_u(\sqrt{u}) = \dfrac{1}{2\sqrt{u}} \right)$

Jawab:



a.) $P(\cos(2t), \sin(2t))$

b.) $d(P,Q) = \sqrt{(\sin(2t)-h)^2 + (\cos(2t)-0)^2}$

$5 = \sqrt{\sin^2(2t) - 2h.\sin(2t) + h^2 + \cos^2(2t)}$

$5 = \sqrt{1 - 2h\sin(2t) + h^2}$

$h^2 - 2h\sin(2t) + 1 = 25$

$h^2 - 2h\sin(2t) = 24$

$h(h - 2\sin(2t)) = 24$

$h = \dfrac{24}{h - 2\sin(2t)}$

⋮

c.) $v = \dfrac{dh}{dt} = \ldots$

Date . .

2.6 Turunan Tingkat Tinggi

Turunan	Notasi Leibniz
$f'(x)$	$\frac{dy}{dx}$
$f''(x)$	$\frac{d^2y}{dx^2}$
$f'''(x)$	$\frac{d^3y}{dx^3}$
$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$
$\vdots$	$\vdots$
$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$

Contoh!

$f(x) = 2x^3 - 4x^2 + 7x - 8$

$\rightarrow f'(x) = 6x^2 - 8x + 7$

$\rightarrow f''(x) = 12x - 8$

$\rightarrow f'''(x) = 12$

$\rightarrow f^{(4)}(x) = 0$

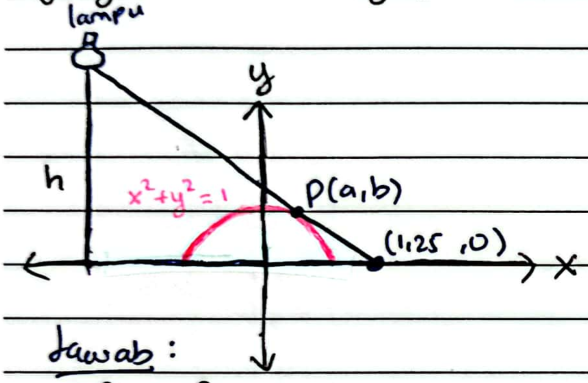
2.7 Turunan Implisit

Turunan Implisit

↳ Suatu metode untuk menentukan $\frac{dy}{dx}$ dari dalam x dan y tanpa menentukan y sebagai fungsi dalam x (secara eksplisit)

Contoh :

149 Tentukan tinggi h pada lampu jika titik (1,25 ; 0) berada di ujung daerah yang terkena cahaya?



Jawab :

$$x^2 + y^2 = 1$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$ → a (sebagai konstanta x), → b (sbg konstanta y)

→ m garis singgung : $m = \frac{-a}{b}$

⇒ Pers. garis singgung :

$$y - b = m(x-a)$$

$$y - b = \frac{-a}{b}(x-a)$$

→ Garis tsb memotong di titik $\left(\frac{5}{4}, 0\right)$

↳ $0 - b = \frac{-a}{b}\left(\frac{5}{4} - a\right)$

$$b = \frac{a}{b}\left(\frac{5}{4} - a\right)$$

→ Titik P ada di setengah lingkaran

⇒ $a^2 + b^2 = 1$

→ $b = \frac{a}{b}\left(\frac{5}{4} - a\right)$

$$b^2 = \frac{5}{4}a - a^2$$

$$a^2 + b^2 = \frac{5}{4}a$$

$$1 = \frac{5}{4}a$$

$$a = \frac{4}{5}$$

⇒ $b = \frac{4/5}{b}\left(\frac{5}{4}\right) - \frac{(4/5)^2}{b}$

$$b = \frac{1}{b} - \frac{16}{25b}$$

$$b = \frac{25-16}{25b} = \frac{9}{25b}$$

$$b^2 = \frac{9}{25} \Rightarrow b = \frac{3}{5}$$

- Tinjau grafik pers. $y^2 = 4 - x^2$
- Grafik tsb mell titik $A(1, \sqrt{3})$
- Tentukan pers. garis singgung di titik A

Jawab :

Cari gradien = turunan pertama = $\frac{dy}{dx}$

$$y^2 = 4 - x^2$$

$$y = \sqrt{4-x^2} \rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{4-x^2}} \cdot (-2x)$$

$$= \frac{-x}{\sqrt{4-x^2}}$$

No.

Date . .

saat $x = 1 \rightarrow m = \frac{-1}{\sqrt{4-1^2}} = \frac{-1}{\sqrt{3}}$

PGS : $y - \sqrt{3} = \frac{-1}{\sqrt{3}}(x-1)$

Diberikan pers. $y^3 - y = x^2 + 4x + 3$
Cari kemiringan garis singgung di titik $A(-1,1)$

Jawab : $y^3 - y = x^2 + 4x + 3$
Turunkan secara implisit

$$3y^2 \frac{dy}{dx} - \frac{dy}{dx} = 2x + 4$$

$$\frac{dy}{dx}(3y^2 - 1) = 2x + 4$$

$$\frac{dy}{dx} = \frac{2x+4}{3y^2-1}$$

Kemiringan garis singgung:

$$m = \frac{2(-1)+4}{3(1)^2-1} = \frac{2}{2} = 1$$

PGS : $y - 1 = 1(x-(-1))$

$y = x + 2$

17. $4x^3 + 7xy^2 = 2\tan(y)$

$$12x^2 + 7y^2 + 7x \cdot 2y \cdot \frac{dy}{dx} = 2\sec^2(y)\frac{dy}{dx}$$

$$14xy\frac{dy}{dx} - 2\sec^2(y)\frac{dy}{dx} = -12x^2 - 7y^2$$

$$\frac{dy}{dx}(14xy - 2\sec^2(y)) = -12x^2 - 7y^2$$

$$\frac{dy}{dx} = \frac{-12x^2 - 7y^2}{14xy - 2\sec^2(y)}$$

No. ______

Date ___ . ___ . ___

Checkpoint Bab 2.5 – 2.7

1. $G(x) = \dfrac{1}{\sqrt{x^4+\sin x}} = (x^4+\sin x)^{-1/2}$

Jawab:

$u = x^4 + \sin x \qquad y = u^{-1/2}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -\frac{1}{2} \cdot u^{-3/2} \cdot (4x^3 + \cos x)$$

$$= \frac{-1}{2 \cdot u^{3/2}} (4x^3 + \cos x)$$

$$= \frac{-1}{2\sqrt{(x^4+\sin x)^3}} \cdot (4x^3 + \cos x)$$

$$G'(x) = \frac{-4x^3 - \cos x}{2\sqrt{(x^4+\sin x)^3}}$$

2. Tentukan PGS:

$\sqrt{y} + xy^2 = 5$ di titik $(4,1)$

Jawab:

$$\frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} + 1 \cdot y^2 + x \cdot 2y \cdot \frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} + 2xy \frac{dy}{dx} = -y^2$$

$$\left(\frac{1}{2\sqrt{y}} + 2xy\right)\frac{dy}{dx} = -y^2$$

$$\frac{dy}{dx} = \frac{-y^2}{\frac{1}{2\sqrt{y}} + 2xy}$$

$\Rightarrow$ m di titik $(4,1) = \dfrac{-1^2}{\frac{1}{2.1} + 2.4.1} = \dfrac{-1}{17/2}$

$$m = -\frac{2}{17}$$

Persamaan garis singgung:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{-2}{17}(x - 4)$$

$$y = -\frac{2}{17}x + \frac{25}{17}$$

Date . .

2.8 Laju Berkaitan

Perubahan rata-rata dari f pada interval waktu [a,b] :

$$\frac{f(b)-f(a)}{b-a}$$

Laju perubahan sesaat dari f pada saat $t=a$:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a) = \frac{df}{dt}$$

→ Dlm bbrp situasi, bbrp kuantitas yg berubah-ubah thd waktu, bisa memiliki hubungan / kaitan dlm bentuk persamaan.

Contoh:

$$u^2+2w^3 = v^2w+7$$

Laju perubahan :

$$\frac{d(u^2+2w^3)}{dt} = \frac{d}{dt}(v^2w+7)$$

$$\boxed{2u\frac{du}{dt} + 6w\frac{dw}{dt} = 2v\frac{dv}{dt}w + v^2\frac{dw}{dt}}$$

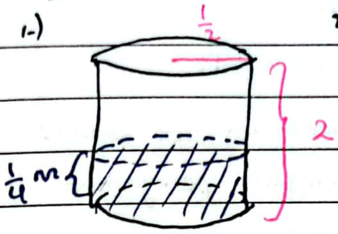
Persamaan yg menggambarkan kaitan antara laju perubahan u, v, w thd t

Contoh soal !

- Suatu tangki berbentuk tabung $d=1\,m$, $h=2\,m$
- Berisi air penuh. Laju air keluar dr tabung $= \frac{1}{2}\ m^3/menit$
- Tentukan laju perubahan tinggi air pd saat tinggi air $= \frac{1}{4}$ meter.

Jawab:

1.)



2.) kuantitas² yg berubah seiring waktu berjalan

V = volume air

h = tinggi air

3.) $V = \pi r^2 h$

$= \pi \left(\frac{1}{2}\right)^2 h$

4.) Turunkan secara implisit thd t

$$\frac{dV}{dt} = \frac{1}{4}\pi\frac{dh}{dt}$$

5.) Masukkan informasi yg diketahui

Debit air $= \frac{dV}{dt} = -\frac{1}{2}$ → (−) karena V air berkurang

$$\Rightarrow -\frac{1}{2} = \frac{1}{4}\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{2}{\pi}\ m/menit \ //$$

h=10 m

- Balon dilepaskan dari satu titik ~~dari~~ sejauh 10 m dari pengamat yg diam. Saat $h=5m$, balon bergerak ke atas dgn laju 1m/detik. Tentukan:

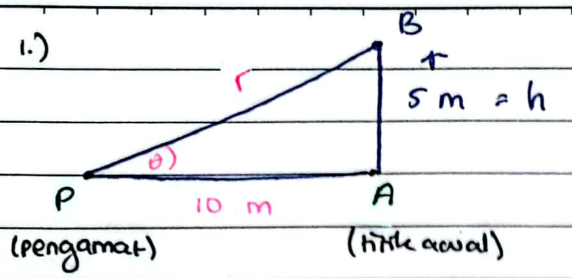
a.) Laju perubahan jarak antara balon & pengamat

b.) Laju perubahan luas segitiga yg dibentuk balon, pengamat, dan titik awal balon.

c.) Laju perubahan sudut yg dibentuk oleh balon, pengamat, dan titik awal balon.

Jawab:

1.)



2.) Kuantitas yang berubah thd waktu

s = jarak balon dengan pengamat

θ : sudut antara balon, pengamat, dan titik awal

L : luas antara balon, pengamat, dan titik awal

h : tinggi balon

a.) 3.) $s^2 = 100 + h^2$ ← kaitan s dan h

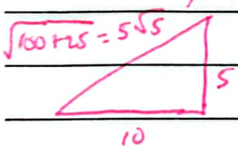
4.) Turunkan implisit thd t

$$2s\frac{ds}{dt} = 0 + 2h\frac{dh}{dt}$$

5.) Dik : Pada saat $h = 5$ m, $\frac{dh}{dt} = 1$ m/s

Dit : $\frac{ds}{dt}$ pada saat yg sama ?

$\to s = 5\sqrt{5}$



$$s\frac{ds}{dt} = h\frac{dh}{dt}$$

$$5\sqrt{5}\frac{ds}{dt} = 5\frac{dh}{dt}$$

$$5\sqrt{5}\frac{ds}{dt} = 5 \cdot 1$$

$$\boxed{\frac{ds}{dt} = \frac{5}{5\sqrt{5}} = \frac{1}{\sqrt{5}} \text{ m/s}}$$

c.) θ = sudut $\angle BPA$

$$\tan\theta = \frac{h}{10}$$

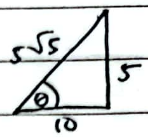
Turunkan thd t :

$$\sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot \frac{dh}{dt}$$

Dik : $h = 5 \to \frac{dh}{dt} = 1$

Dit : $\frac{d\theta}{dt}$

$\sec^2\theta \to$



$$= \frac{1}{\cos^2\theta}$$

$$= \frac{1}{(10/5\sqrt{5})^2} = \left(\frac{5\sqrt{5}}{10}\right)^2 = \frac{5}{4}$$

$$\Rightarrow \sec^2(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot \frac{dh}{dt}$$

$$\frac{5}{4} \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot 1$$

$$\boxed{\frac{d\theta}{dt} = \frac{1}{10} \cdot \frac{4}{5} = \frac{2}{25} \text{ rad/s}}$$

b.) $L = \frac{10 \cdot h}{2} = \frac{5 \cdot h}{2}$

turunkan thd t

$$\frac{dL}{dt} = \frac{5}{2}\frac{dh}{dt}$$

$$\boxed{\frac{dL}{dt} = \frac{5}{2} \text{ m}^2/\text{s}}$$

• Bagaimana jika pengamat berjalan mendekati titik awal balon dengan laju 0,5 m/s ? Dik : saat $h = 5$ m, pengamat ada di jarak 12 m dr titik awal balon, balon bergerak ke atas dgn laju 1 m/s.

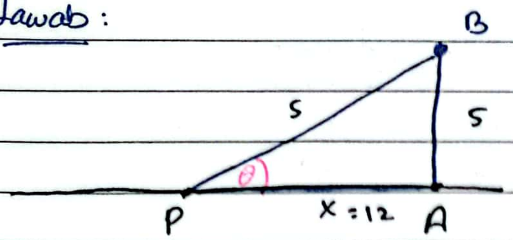
a.) laju perubahan jarak

b.) laju perubahan luas

c.) laju perubahan sudut

Date . .

Jawab :



B:

a.) $s^2 = x^2 + h^2$

$2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2h \frac{dh}{dt}$

$\cancel{2}s \frac{ds}{dt} = x \frac{dx}{dt} + h \frac{dh}{dt}$

$13 \frac{ds}{dt} = 12.(-0{,}5) + 5.(1)$

$\boxed{\frac{ds}{dt} = \frac{-6+5}{13} = \frac{-1}{13} \text{ m/s}}$ //

b.) $L = \frac{x.h}{2} = \frac{x}{2} . \frac{h}{2}$

$\frac{dL}{dt} = \frac{1}{2}\frac{dx}{dt} . \frac{h}{2} + \frac{x}{2} . \frac{1}{2} . \frac{dh}{dt}$

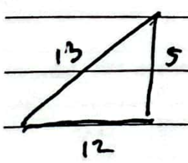
$\frac{dL}{dt} = \frac{1}{2}(-0{,}5) . \frac{5}{2} + \frac{12}{2} . \frac{1}{2} . 1$

$\boxed{\frac{dL}{dt} = -\frac{5}{8} + 3 = \frac{19}{8} \text{ m}^2/\text{s}}$

c) θ : sudut ∠BPA

$\tan\theta = \frac{h}{x}$

$\sec^2(\theta) . \frac{d\theta}{dt} = \frac{\frac{dh}{dt} x - h . \frac{dx}{dt}}{x^2}$



$\sec^2\theta = \frac{1}{\cos^2\theta}$

$= \frac{1}{(\frac{12}{13})^2} = \left(\frac{13}{12}\right)^2 = \frac{169}{144}$

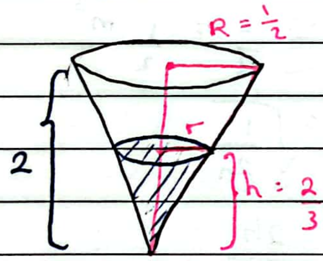
$\frac{169}{144} . \frac{d\theta}{dt} = \frac{1.12 - 5.(-0{,}5)}{12^2}$

$\frac{169}{144} \frac{d\theta}{dt} = \frac{14{,}5}{144}$

$\boxed{\frac{d\theta}{dt} = \frac{14{,}5}{169} \text{ rad/s}}$ //

- Air dialirkan ke dalam wadah berbentuk kerucut terbalik dengan panjang diameter 1 meter dan tinggi 2 meter dengan laju ½ m³/menit. Tentukan laju perubahan tinggi air dalam wadah tsb saat air di wadah sudah terisi setinggi 2/3 meter.

Jawab :

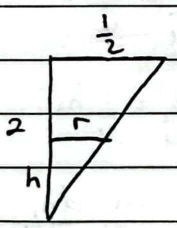


V = volume air dlm wadah

r = jari² permukaan air

h = tinggi air

$V = \frac{1}{3}\pi r^2 h$



$\frac{r}{h} = \frac{1/2}{2}$

$r = \frac{1}{4} h$

$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h = \frac{1}{3}\pi . \frac{1}{16} h^2 = \frac{1}{48}\pi h^2$

$V = \frac{1}{48} h^2$

$\frac{dV}{dt} = \frac{1}{48} . 2h . \frac{dh}{dt}$

$\frac{1}{2} = \frac{1}{486} . \cancel{2} . \frac{\cancel{2}}{3} . \frac{dh}{dt}$

$\frac{dh}{dt} = 18$ m/menit

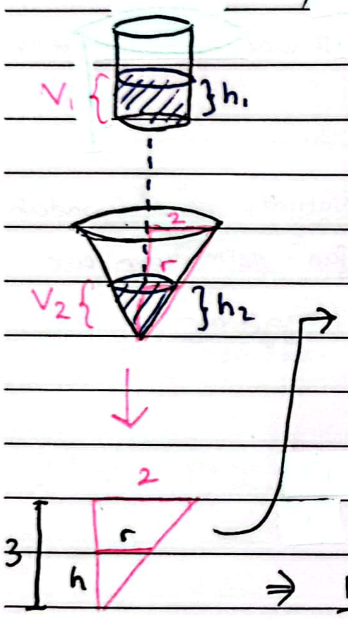
//

- Air mengalir dari tangki 1 bentuk silinder ke dalam tangki 2 bentuk kerucut terbalik
- Tangki 1 : $d = 1\,m$ $h = 3\,m$
  Tangki 2 : $d = 2\,m$ $h = 3\,m$
- Mula-mula tangki 1 terisi air penuh. Tb-tb air keluar (bocor) dan masuk ke tangki 2. Tinggi air di tangki 1 sisa 1 meter.
- Tinggi air di tangki 1 berkurang dengan laju 1/3 m/menit. tentukan laju perubahan tinggi air di tangki 2.

Jawab :

$$V_1 + V_2 = V_0$$

$$\pi r_1^2 h_1 + \frac{1}{3}\pi r_2^2 h_2 = \pi r_1^2 h$$



$$\frac{1}{4}h_1 + \frac{1}{3}r_2^2 h_2 = \frac{3}{4}$$

$$\frac{1}{3}r_2^2 h_2 = \frac{3}{4} - \frac{1}{4}h_1$$

$$\frac{r}{h} = \frac{2}{3}$$

$$r = \frac{2h}{3}$$

$$\Rightarrow \frac{1}{3}\cdot\frac{4}{9}h^2 \cdot h = \frac{3}{4} - \frac{1}{4}h_1$$

$$\frac{4}{27}h_2^3 = \frac{3}{4} - \frac{1}{4}h_1$$

$$h_2^3 = \frac{4}{27}\left(\frac{3}{4} - \frac{1}{4}h_1\right)$$

$$h_2^3 = \frac{1}{9} - \frac{1}{27}h_1$$

$$h_2 = \left(\frac{1}{9} - \frac{1}{27}h_1\right)^{1/3}$$

$$\frac{dh_2}{dt} = \frac{1}{3}\left(\frac{1}{9} - \frac{1}{27}h_1\right)^{-2/3} \cdot \left(\frac{-1}{27}\right)\frac{dh_1}{dt}$$

$$\frac{dh_2}{dt} = \frac{1}{3\sqrt[3]{\left(\frac{1}{9} - \frac{1}{27}\cdot 1\right)^2}} \cdot \left(\frac{-1}{27}\right)\cdot\left(\frac{-1}{3}\right)$$

$$\frac{dh_2}{dt} = \frac{1}{3^5\sqrt[3]{1/729}}$$

$$\frac{dh_2}{dt} = \frac{1}{27}\ m/s$$ //

2.9 Diferensial dan Hampiran

Definisi Diferensial

Misal. $y = f(x)$ suatu fungsi

Diferensial variabel x

↳ dx, perubahan pada variabel x

$dx = \Delta x$

Diferensial variabel y

↳ dy, didefinisikan sebagai:

$dy = f'(x)\, dx$

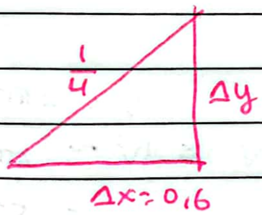
Misal : $y = f(x) = \sqrt{x}$

$x_0 = 4 \rightarrow y_0 = f(4) = \sqrt{4} = 2$

bagaimana jika $x_1 = 4{,}6 \rightarrow y_1 = ?$

$y = x^{1/2}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$



$m = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

$m = \dfrac{\Delta y}{0{,}6}$

$\dfrac{1}{4} = \dfrac{\Delta y}{0{,}6}$

$\Delta y = 0{,}15$ //

$\Delta y \approx 0{,}15$

$\sqrt{4{,}6} \approx 2 + 0{,}15 = 2{,}15$ //

Catatan: $\sqrt{4{,}6} = 2{,}144761\ldots$

$\Delta y \approx f'(x_0) \cdot \Delta x$

$f(x_1) - f(x_0) \approx f'(x_0) \cdot \Delta x$

$$\boxed{f(x_1) \approx f(x_0) + f'(x_0) \cdot \Delta x}$$

diferensial dr variabel y ↵

Contoh !

c.) $y = \sin(x^4 - 3x^2 + 4)$

Jawab :

$dy = f'(x)\, dx$ ⟵ aturan rantai

$= \cos(x^4 - 3x^2 + 4) \cdot (4x^3 - 6x)\, dx$ //

12. $y = \dfrac{1}{x}$. Temukan nilai dy.

a.) $x = 1$, $dx = 0{,}5$

Jawab : $y = \dfrac{1}{x} = x^{-1}$

$dy = -x^{-2}\, dx$

$dy = \dfrac{-1}{x^2}\, dx$

$dy = \dfrac{-1}{1^2} \cdot 0{,}5 = -0{,}5$ //

b.) $x = -2$, $dx = 0{,}75$

Jawab :

$dy = \dfrac{-1}{x^2}\, dx$

$dy = \dfrac{-1}{(-2)^2} \cdot 0{,}75 = -\dfrac{0{,}75}{4}$ //

16. $y = x^2 - 3$. Tentukan Δy dan dy

a.) $x = 2$ dan $dx = \Delta x = 0{,}5$

Jawab :

$\Delta y = f(x_1) - f(x_0)$

$= f(2{,}5) - f(2)$

$= (2{,}5^2 - 3) - (2^2 - 3)$

$= (2{,}5^2 - 2^2)$

$= (2{,}5 + 2)(2{,}5 - 2)$

$= (4{,}5)(0{,}5) = 2{,}25$ //

Date . .

$dy = f'(x)\,dx$

$= f'(2)\,0{,}5 = 2.2.0{,}5 = 2$ //

1. Taksir nilai berikut:

a.) $\sqrt[3]{7{,}85}$

Jawab: $f(x) = \sqrt[3]{x} = x^{1/3}$

Pilih, $x_0 = 8$

$f'(x_0) = \frac{1}{3} x_0^{-2/3} = \frac{1}{3\sqrt[3]{x_0^2}}$

$f'(8) = \frac{1}{3\sqrt[3]{8^2}} = \frac{1}{12}$

$\Delta x = dx = 7{,}85 - 8 = -0{,}15$

$\sqrt[3]{7{,}85} = f(x_0 + \Delta x)$

$= f(x_0) + f'(x_0)\,dx$

$= \sqrt[3]{8} + \frac{1}{12}(-0{,}15)$

$\sqrt[3]{7{,}85} = 2 - \frac{1}{80} = \frac{159}{80}$ //

1. Gunakan diferensial untuk menaksir pertambahan/peningkatan luas permukaan suatu kubus jika panjang rusuknya bertambah dari 3 cm menjadi 3,2 cm.

Jawab:

Misal, x = panjang rusuk kubus

$L = 6x^2$ → luas permukaan kubus

$\Delta L \approx dL = L'(x)\,dx = 12x\,dx$

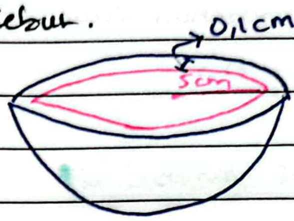
$x = 3 \quad \Delta x = 0{,}2$

$\Delta L \approx dL = 12(3).0{,}2 = 7{,}2$

∴ luas permukaan kubus tersebut bertambah sebesar $7{,}2\ cm^2$.

2. Suatu wadah dengan bahan besi berbentuk setengah bola (tanpa tutup) memiliki jari-jari bagian dalam 5 cm dan ketebalan 0,1 cm. Dengan menggunakan diferensial, taksir volume besi yang diperlukan untuk membuat wadah tersebut.

Jawab:



Volume besi = $V_{luar} - V_{dalam}$

$= V(5{,}1) - V(5) = \Delta V$

$V(r) = \frac{1}{\cancel{2}} . \frac{\cancel{4}^{2}}{3} \pi r^3$

$\Delta V \approx dV = V'(r)\,dr$

$= \frac{6\pi r^2}{3} dr = 2\pi r^2\,dr$

$\Delta V \approx dV \approx 2\pi . 5^2 . 0{,}1$

$= 5\pi\ cm^3$

∴ Volume besi yang diperlukan sekitar $5\pi\ cm^3$.

Anella Utari Gunadi (19623229)

No.

Date

Checkpoint Bab 2.8 - 2.9

1.

$r_2 = 12{,}25$

$r_1 = 12{,}5$

$\Delta r = 0{,}25$

$L = 4\pi r^2$

$\Delta L \approx dL$

$= 4\pi \cdot 2r\, dr$

$= 8\pi r\, dr$

$r = 12{,}5 \quad dr = -0{,}25$

$\Delta L \approx 8\pi \cdot 12{,}5 \cdot (-0{,}25) = \boxed{-25\pi \text{ cm}^2}$ //

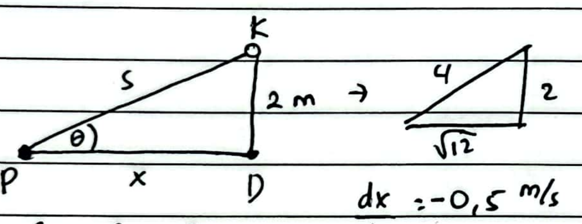
$\frac{1}{(\sqrt{12}/4)^2} \frac{d\theta}{dt} = \frac{-2}{(\sqrt{12})^2} \cdot (-0{,}5)$

$\frac{16}{12} \cdot \frac{d\theta}{dt} = \frac{-2}{12} (-0{,}5)$

$\frac{d\theta}{dt} = \frac{1}{12} \cdot \frac{12}{16}$

$\frac{d\theta}{dt} = \boxed{\frac{1}{16} \text{ rad/s}}$ //

3.



$\frac{dx}{dt} = -0{,}5 \text{ m/s}$

a) $s^2 = 2^2 + x^2$

$2s \frac{ds}{dt} = 0 + 2x \frac{dx}{dt}$

$2 \cdot 4 \frac{ds}{dt} = 0 + 2 \cdot \sqrt{12}(-0{,}5)$

$8 \frac{ds}{dt} = -\sqrt{12}$

$\frac{ds}{dt} = \frac{-2\sqrt{3}}{8} = \boxed{\frac{-\sqrt{3}}{4} \text{ m/s}}$ //

b.) $\tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{3}\sqrt{3}$

$\theta = 30^\circ = \boxed{\frac{\pi}{6} \text{ rad}}$ //

c.) $\tan\theta = \frac{2}{x} = 2x^{-1}$

$\sec^2(\theta) \cdot \frac{d\theta}{dt} = -2x^{-2} \frac{dx}{dt}$

$\frac{1}{\cos^2(\theta)} \frac{d\theta}{dt} = \frac{-2}{x^2} \frac{dx}{dt}$

3.1 Maksimum dan Minimum

Titik Ekstrem

→ Definisi : Misalkan f suatu fungsi yg terdefinisi di S dan c suatu titik di S. Kita katakan :

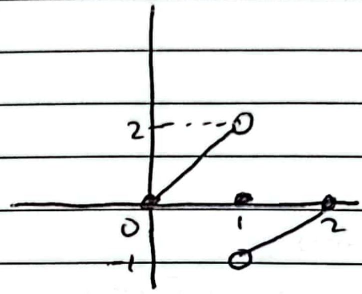
⇒ $f(c)$ = nilai maksimum dari f di S jika $f(c) \geq f(x) \quad \forall x \in S$

⇒ $f(c)$ = nilai minimum dari f di S jika $f(c) \leq f(x) \quad \forall x \in S$

⇒ $f(c)$ = nilai ekstrem dari f di S jika $f(c)$: nilai max dan min

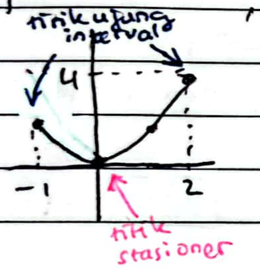
Teorema I

Jika f kontinu pada interval tutup $[a, b]$, maka fungsi f pasti mencapai nilai maks dan min pada $[a, b]$

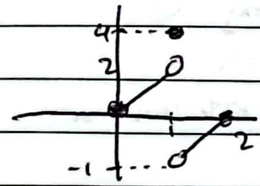


nilai max : tidak ada
nilai min : tidak ada

$f(x) = x^2$, pada $S = [-1, 2]$

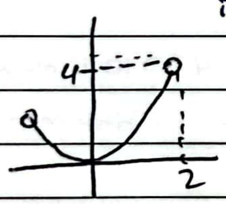


nilai maks : $f(2) = 4$
nilai min : $f(0) = 0$
titik maks = 2
-"- min = 0



nilai max = $f(1) = 4$
nilai min = tidak ada

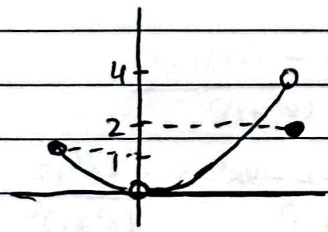
jika $f(x) = x^2$, pada $S = (-1, 2)$
interval buka



nilai maks = tdk ada
nilai min = $f(0) = 0$

$$f(x) = \begin{cases} x^2, & x \neq 0, x \neq 2 \\ 1, & x = 0 \\ 2, & x = 2 \end{cases}$$

pada $[-1, 2]$



nilai maks: tdk ada
nilai min: tidak ada

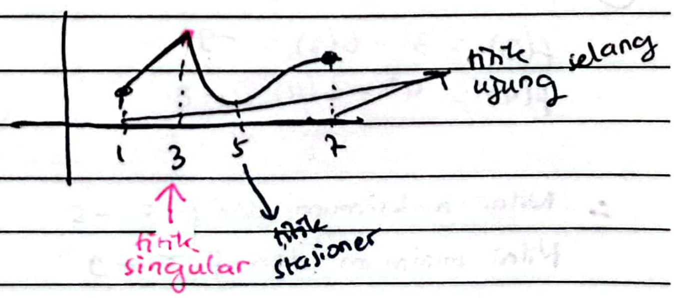
↑ f tidak kontinu

Teorema II

Jika $f(c)$ adalah nilai max dan min fungsi f pd interval tutup $[a, b]$, maka c adalah titik kritis, yaitu 1 dr 3 kemungkinan :

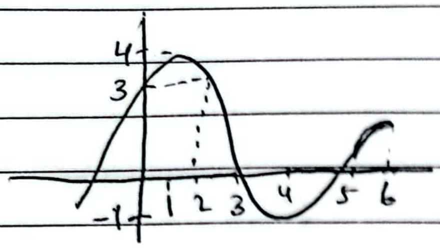
- titik stasioner, $f'(c) = 0$
- titik batas / ujung interval, yaitu $c = a$ atau $c = b$
- titik singular, $f'(c)$ tidak ada

Titik singular



titik = x nilai = y

Menentukan Nilai Ekstrim



Tentukan nilai max dan min dr f pada $S = [2,6]$

nilai max : 3

nilai min : -1

(hanya lihat pd interval yg diketahui)

Latihan!

Tentukan titik - titik kritis

b.) $f(x) = x^2 - 6x$ pd interval $[1,4]$

Jawab : → titik ujung selang (1.)

↳ $x = 1$ dan $x = 4$

→ titik singular & titik stasioner

$f(x) = x^2 - 6x$

$f'(x) = 2x - 6$

→ $f'(x)$ = selalu terdefinisi

∴ Titik singular tidak ada

$f'(x) = 0$

$2x - 6 = 0$

$2x = 6$

$x = 3$

→ Titik stasioner = 3 → ada di $S[1,4]$

(jangan lupa cek interval)

∴ Ada tiga titik kritis :

$x = 1$, $x = 3$, $x = 4$

(2.) $f(1) = 1^2 - 6 = -5$

$f(3) = 3^2 - 6(3) = -9$

$f(4) = 4^2 - 6(4) = -8$

∴ Nilai maksimum dari f = -5

Nilai minimum dari f = -9

d. $f(x) = x^{2/3}$ pada interval $[-1,8]$

Jawab : → Titik ujung selang

$x = -1$ dan $x = 8$

→ Titik singular dan stasioner

$f(x) = x^{2/3}$

$f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

Jika $x = 0$ maka $f'(x)$ = tidak ada

Titik singular = $x = 0$

$f'(x) = 0$

$\frac{2}{3\sqrt[3]{x}} = 0$ → tidak bisa

Titik stasioner : tidak ada

∴ Ada tiga titik kritis

$x = -1$, $x = 0$, dan $x = 8$

(2) $f(-1) = (-1)^{2/3} = \sqrt[3]{(-1)^2} = 1$

$f(0) = 0^{2/3} = 0$

$f(8) = 8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

∴ Nilai maks = 4 , titik max = $x = 8$

Nilai min : 0 , titik min : $x = 0$

e.) $f(x) = \frac{2x}{x^2+1}$ pada interval $[-1,2]$

Jawab :

→ Titik ujung selang

$x = -1$, dan $x = 2$

$f'(x) = \frac{2(x^2+1) - (2x)(2x)}{(x^2+1)^2}$

$= \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2}$

Date . .

$$f'(x) = \frac{2(1-x^2)}{(x^2+1)^2}$$

→ Titik singular : tidak ada
karena $f'(x)$ selalu terdefinisi

$f'(x) = 0$

$\frac{2(1-x^2)}{(x^2+1)^2} = 0$

$2(1-x^2) = 0$

$2 - 2x^2 = 0$

$2x^2 = 2$

$x^2 = 1$

$x = 1$ dan $x = -1$ keduanya masuk di interval $[-1, 2]$

→ Titik stasioner : $x = 1$ dan $x = -1$

∴ Ada 3 titik kritis
$x = -1$, $x = 1$, $x = 2$

(2) $f(-1) = \frac{2(-1)}{(-1)^2+1} = -1$

$f(1) = \frac{2(1)}{1^2+1} = 1$

$f(2) = \frac{2(2)}{2^2+1} = \frac{4}{5}$

∴ Nilai maks : 1 , titik maks : $x = 1$
Nilai min : -1 , titik min : $x = -1$

Date . .

3.2 Kemonotonan dan Kecekungan

Arti Kemonotonan

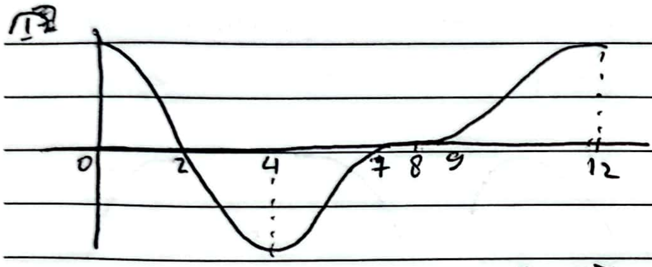
↳ Dikatakan monoton naik pd interval jika grafik f naik dari kiri ke kanan pd interval tsb.

→ f naik pd interval I jika $\forall\, x_1, x_2 \in I$ dgn $x_1 < x_2$ berlaku:

$$f(x_1) < f(x_2)$$

Contoh 1:

1.

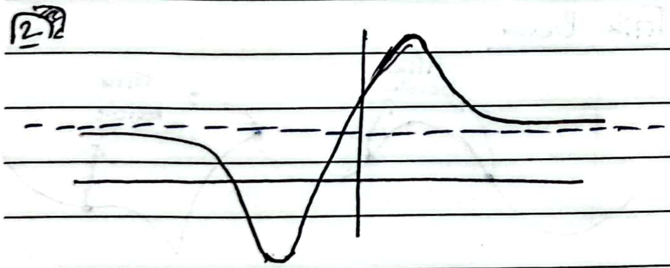


→ f monoton turun pd interval $[0,4]$

f monoton naik pd interval $[4,12]$

atau $[4,8)$ dan $(8,12]$

2.



⇒ f monoton turun pd interval $(-\infty,-1]$ (atau $(-\infty,-1)$)

dan pada $[1,\infty)$ (atau $(1,\infty)$)

⇒ f naik pada $[-1,1]$ atau $(-1,1)$.

3.

⇒ f turun pada interval $(-\infty,-1]$ dan pada $[-1,3]$ dan pada $[3,\infty)$

//

Teorema

Jika f dapat diturunkan pada (a,b) dan $f'(x) > 0\ \forall x \in (a,b)$, maka f monoton naik pada interval ~~tersebut~~

Jika f dapat diturunkan pada (a,b) dan $f'(x) < 0\ \forall x \in (a,b)$ maka f monoton turun pd interval tsb.

Latihan 1

b.) $f(x) = x(x-3)^2$

Jawab:

$$f'(x) = (x-3)^2 + x \cdot 2(x-3) \cdot 1$$
$$= (x-3)^2 + 2x(x-3)$$
$$= (x-3)(x-3+2x)$$
$$= (x-3)(3x-3)$$

$f'(x) > 0$ dan $f'(x) < 0$

$(x-3)(3x-3) > 0$ $(x-3)(3x-3) < 0$

+ − +
1 3

$f'(x) < 0$ pd $(1,3)$

$f'(x) > 0$ pd $(-\infty,1) \cup (3,\infty)$

⇒ f monoton turun pada interval $(1,3)$

⇒ f monoton naik pd interval $(-\infty,1)$ dan $(3,\infty)$

//

a.) $f(x) = \dfrac{2x}{x^2+1}$

Jawaban: $f'(x) = \dfrac{2(1-x^2)}{(x^2+1)^2} = \dfrac{2(1-x)(1+x)}{(x^2+1)(x^2+1)}$

− + −
−1 1

⇒ $f'(x) < 0$ pada interval $(-\infty, -1) \cup (1, \infty)$

⇒ $f'(x) > 0$ pada interval $(-1, 1)$

⇒ f monoton turun pada interval $(-\infty, -1)$ dan pada $(1, \infty)$ //

⇒ f monoton naik pada interval $(-1, 1)$

//

Variasi Soal !

a.) $g'(x) = x^2 - 6x$

Jawab : $g'(x) = x^2 - 6x = x(x-6)$

$+ \quad - \quad +$

$0 \quad 6$

f turun pada $(0, 6)$

f naik pada $(-\infty, 0)$ dan pada $(6, \infty)$

b.) $g'(x) = \dfrac{x^2}{x-1}$

Jawab :

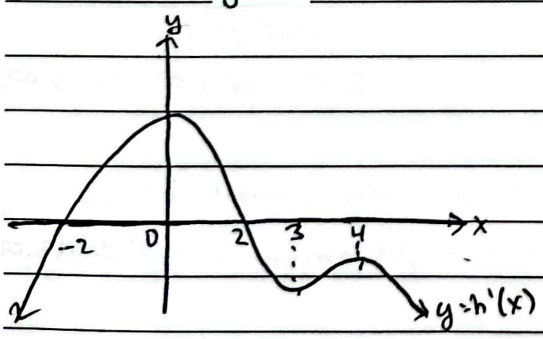
$- \quad - \quad +$

$0 \quad 1$

g turun pada $(-\infty, 0)$ dan pada $(0, 1)$

g naik pada $(1, \infty)$

c.) Grafik $y = h'(x)$:



⇒ h turun pada $(-\infty, -2)$ dan pada $(2, \infty)$ //

⇒ h naik pada $(-2, 2)$ //

Kecekungan

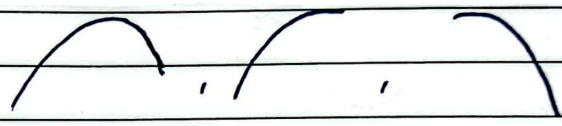
⇒ f cekung ke atas pd interval (a, b) jika f' monoton naik pd interval tsb.

Teorema

Jika f dpt diturunkan dua kali pd (a, b)

$f''(x) > 0$ → cekung ke atas

f cekung ke bawah pd interval (a, b) jika f' monoton turun pd (a, b)

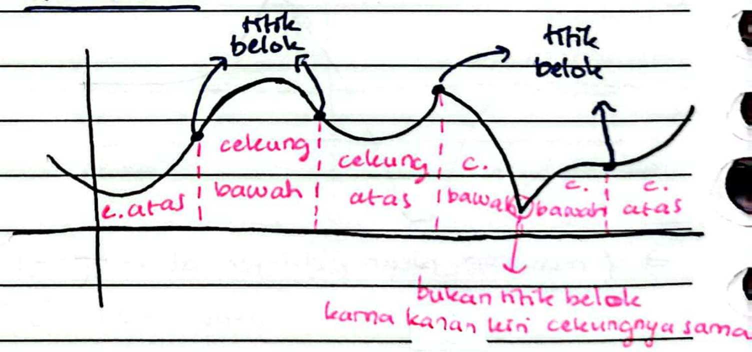


Teorema

⇒ Jika f dpt diturunkan dua kali pd (a, b)

$f''(x) < 0$ → cekung ke bawah

Titik Belok



⇒ Titik $x = c$ dikatakan titik belok dari fungsi f jika terjadi perubahan kecekungan pada fungsi f di sebelah kiri dan kanan titik c.

Contoh :

• Jika $f''(x) < 0$ pada interval $(-1, 2)$ dan $f''(x) > 0$ pada interval $(2, 3)$, maka $x = 2$ adalah titik belok.

Latihan!

• Tentukan interval kecekungan dan titik (jika ada) dari fungsi f berikut.

a. $f(x) = x^2 - 6x$

Jawab: $f(x) = x^2 - 6x$

$f'(x) = 2x - 6$

$f''(x) = 2 \quad > 0$

∴ Grafik fungsi f cekung ke atas pada interval $(-\infty, \infty)$.

b.) $f(x) = x(x-3)^2$

Jawab: $f(x) = x(x-3)^2$

$= x(x^2 - 6x + 9)$

$= x^3 - 6x^2 + 9x$

→ $f'(x) = 3x^2 - 12x + 9$

→ $f''(x) = 6x - 12$

--- +++

2

⇒ Grafik f cekung ke atas pada interval $(2, \infty)$

⇒ Grafik f cekung ke bawah pada interval $(-\infty, 2)$

⇒ $x = 2$ merupakan titik belok

c) $f(x) = \frac{2x}{x^2+1}$

Jawab: $f'(x) = \frac{2(x^2+1) \quad 2x(2x)}{(x^2+1)^2}$

$= \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2}$

→ $f''(x) = \frac{-4x(x^2+1)^2 - (-2x^2+2)\,4x(x^2+1)}{(x^2+1)^4}$

$= \frac{-4x(x^2+1)^2 + (8x^3 - 8x)(x^2+1)}{(x^2+1)^4}$

$= \frac{-4x^3 - 4x + 8x^3 - 8x}{(x^2+1)^3}$

⇒ $f''(x) = \frac{4x^3 - 12x}{(x^2+1)^3}$

$= \frac{4x(x^2-3)}{(x^2+1)^3}$

--- --- --- +++

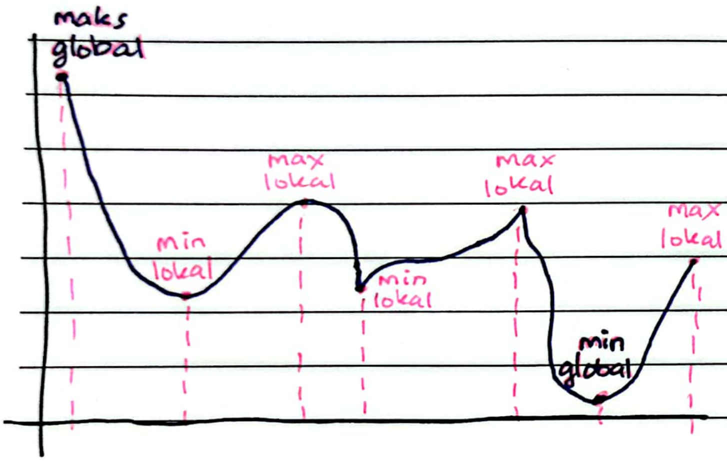
$-1 \quad 0 \quad \sqrt{3}$

3.3 Ekstrem Lokal pada Interval Buka

Titik Ekstrem Lokal

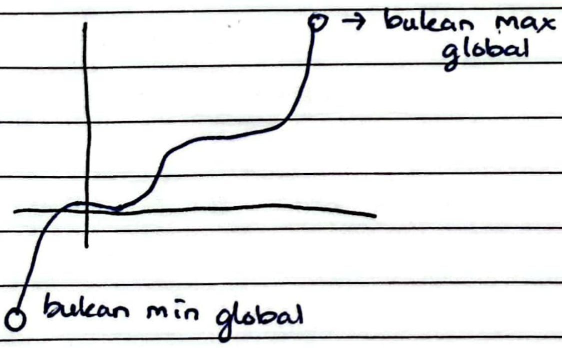
- $f(c)$ disebut nilai maksimum lokal jika terdapat interval $(a,b) \subset S$ yg memuat c sehingga $x = c$ merupakan titik maksimum f pada (a,b). Begitupun sebaliknya.

- $f(c)$ disebut nilai ekstrem lokal jika $f(c)$ adalah nilai maks / min lokal.



Maks-Min pada Interval Buka

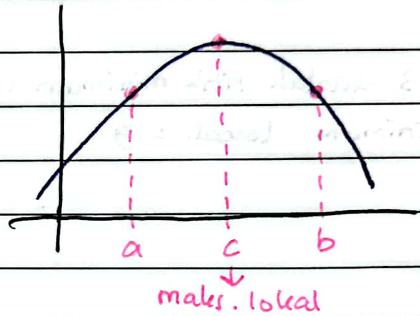
→ Jika f pada interval buka, f tidak selalu memiliki nilai maks/min global.



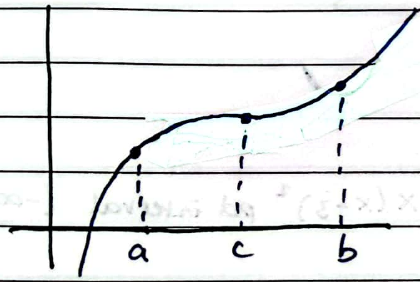
Uji Turunan Pertama

Misal $x = c$ adalah titik kritis dari f

- Jika $f'(x) > 0$ pada (a,c) dan $f'(x) < 0$ pada (c,b), maka $f(c)$ adalah nilai maksimum lokal. (Begitupun sebaliknya)



- Jika $f'(x) > 0$ pada (a,c) dan $f'(x) > 0$ pada (c,b) maka $x = c$ bukan titik ekstrem lokal.



tidak ada nilai ekstrem lokal

Uji Turunan Kedua (khusus titik stasioner)

Misal $x = c$ adalah titik stasioner dan f dapat diturunkan dua kali.

- Jika $f''(c) > 0$ maka $f(c)$: nilai min lokal
- Jika $f''(c) < 0$ maka $f(c)$: nilai max lokal

→ x : titik max / min

→ $f(x)$: nilai max / min

Latihan!

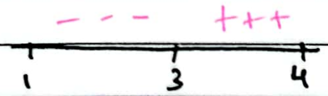
Tentukan titik dan nilai ekstrem lokal!

a.) $f(x) = x^2 - 6x$ pd interval $(1,4)$

Jawab :

$f'(x) = 2x - 6 = 2(x-3)$

- Titik stasioner : $x = 3$
- Singular : tidak ada



$f(3) = -9$

$\Rightarrow$ Titik $x = 3$ adalah titik minimum lokal

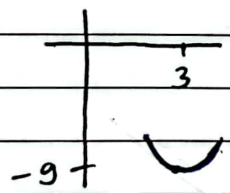
$\Rightarrow$ Nilai minimum lokal $= 9$

atau

$\rightarrow f''(x) = 2$

$f''(3) = 2 > 0$ (cekung atas)

$\therefore$ Titik $x = 3$ adalah titik min lokal dgn nilai min lokal $= -9$



c.) $f(x) = x(x-3)^2$ pd interval $(-\infty, \infty)$

Jawab :

$f'(x) = (x-3)^2 + 2x(x-3) \cdot 1$

$= (x-3)(x-3+2x)$

$= (x-3)(3x-3) = 3(x-3)(x-1)$

$f(1) = 4$

$f(3) = 0$

- Titik stasioner : $x = 1$ dan $x = 3$

$\Rightarrow$ Titik max lokal : $x = 1$, Nilai : 4

$\Rightarrow$ Titik min lokal : $x = 3$, Nilai : 0

atau

$f'(x) = 3(x^2 - 4x + 3)$

$f''(x) = 3(2x - 4)$

$f''(x) = 6(x-2)$

Masukkan titik stasioner

$f''(1) = 6(1-2) = -6 < 0$

$f''(3) = 6(3-2) = 6 > 0$

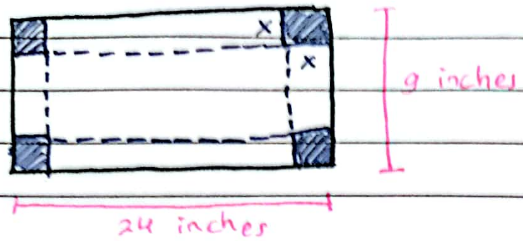
$\Rightarrow$ Titik $x = 1$ adalah titik max lokal

$\Rightarrow$ Titik $x = 3$ adalah titik min lokal

3.4 Pemodelan Matematika

Contoh !

1.



Berapakah volume maksimum dari balok yang dapat dibuat ?

Jawab :

→ Fungsi objektif :

$V = p \times l \times t$

$= (24-2x)(9-2x)(x)$

Dv :

syarat , $x > 0$ dan $x < \frac{9}{2}$

→ $Dv : (0, \frac{9}{2})$

$V = (24-2x)(9-2x)x$

$= 2((12-x)(9-2x)x)$

$= 2(108x - 33x^2 + 2x^3)$

$V' = 2(6x^2 - 66x + 108)$

$= 12(x^2 - 11x + 18)$

$= 12(x-9)(x-2)$

Titik kritis : $x = 9$ atau $\boxed{x = 2}$

TM

$$\underset{0}{(}\ \overset{+++}{\ldots}\ \underset{2}{|}\ \overset{---}{\ldots}\ \underset{9/2}{)}$$

$x = 2$: titik maksimum global

$\Rightarrow V_{max} = (24-2(2))(9-2(2)) \cdot 2$

$= 20 \cdot 5 \cdot 2$

$= 200 \text{ inches}^3$

3.5 Grafik Fungsi dengan Kalkulus

Contoh!

Sketsa grafik fungsi $f(x) = x(x-3)^2$

- $D_f : (-\infty, \infty)$
- Tipot sb-x : $(0,0)$ dan $(3,0)$
- Tipot sb-y : $(0,0)$
- Kesimetrian & keperiodikan : tdk ada
- Asimtot : tidak ada (bkn f rasional)
- Titik kritis, kemonotonan, kecekungan, titik belok :

$\rightarrow f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$

Titik stasioner : $f'(x) = 0$

$\Rightarrow x = 1$ atau $x = 3$

$y = 4$ $\quad y = 0$

$(1,4)$ $\quad (3,0)$

+++ $\quad$ --- $\quad$ +++

1 $\quad$ 3

$\therefore$ • f monoton naik pada interval $(-\infty, 1)$ dan pada int $(3, \infty)$

• f monoton turun pada int $(1,3)$

$\Rightarrow f''(x) = 6x - 12 = 6(x-2)$

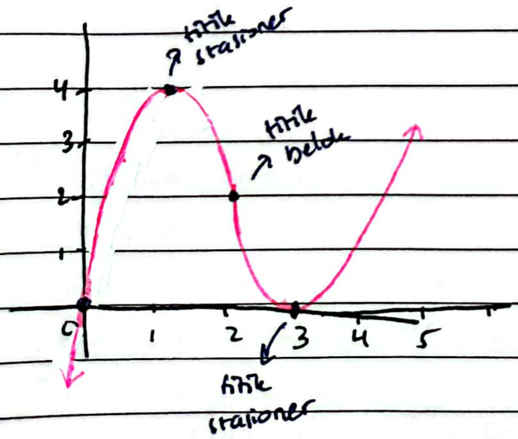
--- $\quad$ +++

2

$\Rightarrow$ f cekung ke bawah pada int $(-\infty, 2)$

$\Rightarrow$ f cekung ke atas pada int $(2, \infty)$

$\Rightarrow$ titik belok : $x = 2 \rightarrow y = 2$ $\quad (2,2)$



Latihan!

• Sketsa grafik fungsi

$g(x) = \frac{2}{27}(x+1)^3(x-3)$

Jawab : • $D_g : (-\infty, \infty)$

- Tipot sb-x : $(-1,0)$ dan $(3,0)$
- Tipot sb-y : $(0, -\frac{2}{9})$
- Titik singular : tdk ada (polinom)
- Titik kritis, kemonotonan, kecekungan titik belok :

$\rightarrow g'(x) = \frac{8}{27}(x+1)^2(x-2)$

Titik stasioner : $g'(x) = 0$

$\Rightarrow x = -1$ dan $x = 2$

$y = 0$ $\quad y = -2$

$(-1,0)$ $\quad (2,-2)$

--- $\quad$ --- $\quad$ +++

-1 $\quad$ 2

$\Rightarrow$ naik pada interval $(2, \infty)$

$\Rightarrow$ turun pada int $(-\infty, -1)$ dan pada int $(-1, 2)$

$\rightarrow g''(x) = \frac{8}{9}(x+1)(x-1)$

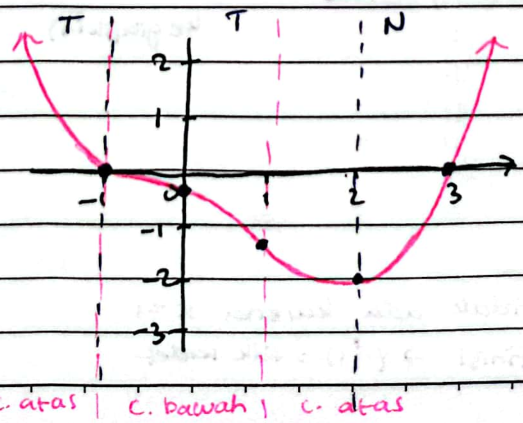
+++ $\quad$ --- $\quad$ +++

-1 $\quad$ 1

$\Rightarrow$ cekung bawah pada interval $(-1,1)$

$\Rightarrow$ cekung atas pada int $(-\infty,-1)$ dan pada int $(1,\infty)$

titik belok : $(-1, 0)$ dan $(1, -\frac{32}{27})$



Date . .

Fungsi Rasional

- Sketsa grafik fungsi $f(x) = \dfrac{2x}{x-1}$

Jawab :

- $D_f = (-\infty, 1) \cup (1, \infty)$
- Tipot sb-x : (0,0)
- Tipot sb-y : (0,0)
- Asimtot datar : $y = 2$
- Asimtot tegak : $x = 1$

$\rightarrow f'(x) = \dfrac{2(x-1) - 2x}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$

Titik stasioner : $f'(x) = 0 \rightarrow$ tidak ada

Singular : $f'(x)$ tidak ada tapi $f'(1)$ tidak terdefinisi

(x = 1 bukan singular, karena f'(1) tdk terdef)

$- - - \quad\quad - - -$

$\overline{\quad\quad\quad\quad 1 \quad\quad\quad\quad}$

monoton turun pada int $(-\infty, 1)$ dan int$(1, \infty)$

$\rightarrow f''(x) = (-2)(-2)(x-1)^3 \cdot 1$

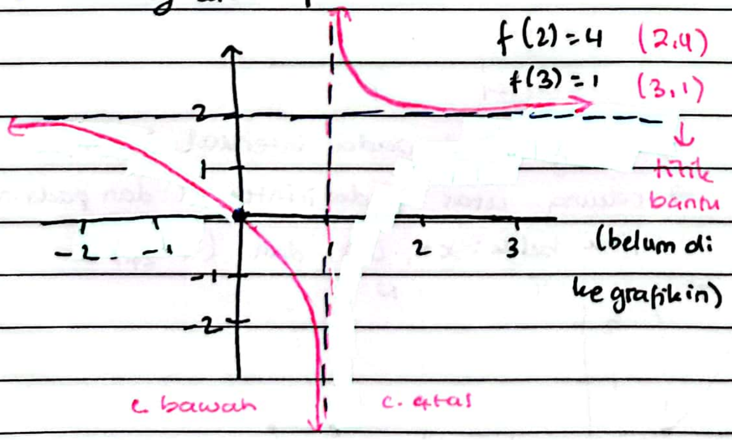
$= 4(x-1)^3$

$- - - \quad\quad + + +$

$\overline{\quad\quad\quad\quad 1 \quad\quad\quad\quad}$

$\Rightarrow$ cekung bawah pd int $(-\infty, 1)$

$\Rightarrow$ cekung atas pd int $(1, \infty)$



$\Rightarrow$ titik belok tidak ada karena $x = 1$ tidak terdefinisi $\rightarrow f(1) =$ tdk terdef

Dipindai dengan CamScanner

3.6 Teorema Nilai Rata-Rata (untuk Turunan)

Teorema A : TNR untuk Turunan

Jika f kontinu di [a,b] dan mempunyai turunan di (a,b), maka setidaknya terdapat satu bilangan c di (a,b), dimana:

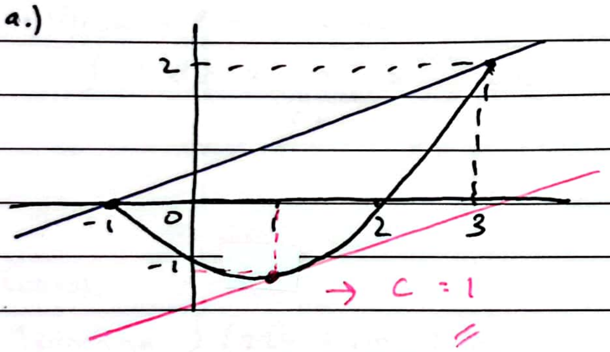
$$\boxed{\frac{f(b) - f(a)}{b-a} = f'(c)}$$

atau sering ditulis :

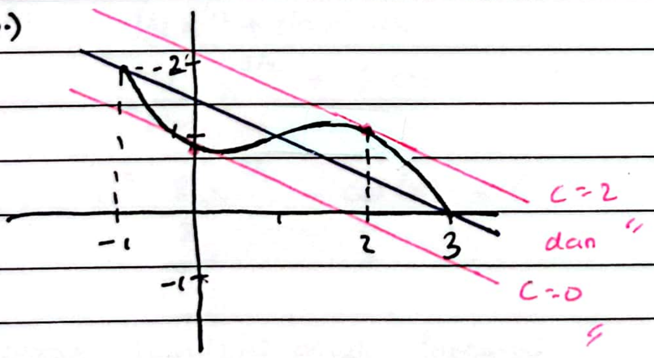
$$\boxed{f(b) - f(a) = f'(c)\,(b-a)}$$

21. Tentukan semua nilai c yg memenuhi TNR di int [-1,3]

a.)



b.)



22. Tentukan apakah TNR untuk turunan berlaku pada f berikut.

a.) $f(x) = x^2 - x + 1 \quad [-1,3]$

Jawab : f kontinu dan $f'(x)$ terdef di $(-1,3)$,

TNR : Ada c sehingga :

$$\frac{f(3) - f(-1)}{b-a} = f'(c)$$

$$\Leftrightarrow \frac{7-3}{4} = 2c-1$$

$$1 = 2c-1$$

$$c = 1$$

b.) $h(x) = x^{2/3} \quad [-8, 27]$

Jawab : $h'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

$h'(0) \Rightarrow$ tidak ada

∴ Syarat TNR tidak terpenuhi

46. Di sebuah pertandingan, kuda A dan kuda B mulai di titik yg sama. Buktikan kecepatan mereka sama di rentang waktu tertentu.

Jawab : Misal : S_A : jarak yg ditempuh A

S_B : —〃— B

Buktikan ada $c \in [t_0, t_1]$ sehingga $S_A'(c) = S_B'(c)$

Definisikan : $f(t) = S_A(t) - S_B(t)$

Berdasarkan TNR ada $c \in [t_0, t_1]$ shg

$$\frac{f(t_1) - f(t_0)}{t_1 - t_0} = f'(c)$$

$$0 = S_A'(c) - S_B'(c)$$

$$\therefore S_A'(c) = S_B'(c)$$

TNR pada Masalah Gerak

$s(t)$ = fungsi posisi suatu objek saat t

Pada int waktu $[a,b]$, akan ada suatu waktu (misal $t = c$) dgn $c \in (a,b)$, shg :

$$\boxed{\frac{s(b) - s(a)}{b-a} = s'(c)}$$

Contoh :

(no. 46 di hal sebelumnya)

Date . .

3.8 Anti_Turunan

Definisi Anti-Turunan

Jika f adalah turunan F, maka F disebut sebuah anti-turunan dari f.

Sifat-Sifat Integral Tak Tentu

Teorema C :

1.) $\int k f(x)\, dx = k \int f(x)\, dx$

2.) $\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$

3.) $\int (f(x) - g(x))\, dx = \int f(x)\, dx - \int g(x)\, dx$

Integral-Integral Dasar

- Untuk $r \neq -1$, berlaku :

$$\boxed{\int x^r\, dx = \frac{1}{r+1} x^{r+1} + C}$$

Contoh

Tentukan $\int (2\sqrt{x} + \sin x)\, dx$

Jawab: $\int (2\sqrt{x} + \sin x)\, dx$

$= \int (2x^{1/2} + \sin x)\, dx$

$= \frac{2}{3/2} x^{3/2} + (-\cos x) + C$

$= \frac{4}{3} x^{3/2} - \cos(x) + C$ //

Metode Substitusi untuk Integral Tak Tentu

- Teorema

Jika F suatu anti-turunan dari f, maka :

$$\boxed{\int f(g(x)) . g'(x)\, dx = F(g(x)) + C}$$

Contoh :

1. $\int x \sin(x^2)\, dx$

Jawab:

Pilih $u = x^2$

$du = 2x\, dx$

$\int \sin(x^2) . x\, dx = \int \sin(u) . \frac{1}{2}\, du$

$= \int \frac{1}{2} \sin(u)\, du$

$= -\frac{1}{2} \cos(u) + C$

$= -\frac{1}{2} \cos(x^2) + C$ //

2. $\int \frac{x^2}{\sqrt{x^3+1}}\, dx$

Jawab :

Pilih $w = x^3 + 1$

$dw = 3x^2\, dx$

$\frac{dw}{3} = x^2\, dx$

$\Rightarrow \int \frac{1}{\sqrt{w}} . \frac{1}{3}\, dw$

$= \int \frac{w^{-1/2}}{3}\, dw$

$= \frac{1}{3} . 2 . w^{1/2} + C = \frac{2}{3} \sqrt{x^3+1} + C$ //

3.9 Pengantar Persamaan Diferensial

Contoh

$y = \sqrt{x^2+1}$ merupakan solusi PD $\frac{dy}{dx} = \frac{x}{y}$?

Jawab :

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}} = \frac{x}{y}$$

(Benar) //

∴ $y = \sqrt{x^2+1}$ merupakan sebuah solusi.

1. Cari persamaan xy dari suatu kurva melalui $(-1,2)$ dan kemiringan garis singgung = 2 kali koordinat x.

Jawab: $y = f(x)$

melalui $(-1,2) \rightarrow f(-1) = 2$

$$\rightarrow \frac{dy}{dx} = 2x$$

$$\Rightarrow y = \int 2x\, dx = x^2 + C$$

∴ Solusi umum PD di atas adalah $x^2 + C$.

$\rightarrow f(-1) = 2$

$$y = x^2 + c$$
$$2 = (-1)^2 + c$$
$$c = 1 \rightarrow y = x^2 + 1$$

//

∴ Persamaan kurva yg memenuhi syarat pada soal adalah $y = x^2 + 1$ //

3. Percepatan gravitasi = 32 kaki/s. Objek dilemparkan dari ketinggian 1000 kaki dengan kecepatan 50 kaki/s. Tentukan kecepatan dan tinggi benda setelah 4 s!

Jawab :

$$\frac{dv}{dt} = 32 \Rightarrow v = 32t + C$$

saat $t = 0$, $v = 50$

$$50 = 0 + C$$
$$C = 50$$

$$\frac{ds}{dt} = 32t + 50$$

$$s = 16t^2 + 50t + c$$

saat $t = 0 \Rightarrow s = 1000$

$$1000 = 0 + 0 + c$$
$$c = 1000$$

$$\rightarrow s = 16t^2 + 50t + 1000 \quad //$$

saat $t = 4$

∴ $s = 16(4)^2 + 50(4) + 1000$ //

Metode Pemisah Variabel

2. Selesaikan persamaan diferensial:

$$\frac{dy}{dx} = \frac{x + 3x^2}{y^2}$$

lalu cari solusi untuk $y = 6$ saat $x = 0$.

Jawab :

$$y^2 \cdot dy = (x + 3x^2)\, dx$$
$$\int y^2\, dy = \int (x + 3x^2)\, dx$$
$$\frac{1}{3}y^3 + C_1 = \frac{1}{2}x^2 + x^3 + C_2$$

$$\frac{1}{3}y^3 = \frac{1}{2}x^2 + x^3 + C$$

Date . .

$$y^3 = \frac{3}{2}x^2 + 3x^3 + C$$

$$y = \sqrt[3]{\frac{3x^2}{2} + 3x^3 + C}$$ ⇒ solusi umum

⇒ $x = 0 \rightarrow y = 6$

$$6 = \sqrt[3]{0 + 0 + C}$$

$$6 = \sqrt[3]{C}$$

$$C = 216$$

∴ Solusinya adalah $y = \sqrt[3]{\frac{3x^2}{2} + 3x^3 + 216}$ //

12. $\frac{du}{dt} = u^3(t^3 - t)$; $u = 4$ saat $t = 0$

Jawab :

$$\frac{du}{u^3} = (t^3 - t)\,dt$$

$$u^{-3}\,du = (t^3 - t)\,dt$$

$$\int u^{-3}\,du = \int (t^3 - t)\,dt$$

$$-\frac{1}{2}u^{-2} + C_1 = \frac{1}{4}t^4 - \frac{1}{2}t^2 + C_2$$

$$-\frac{1}{2u^2} = \frac{1}{4}t^4 - \frac{1}{2}t^2 + C$$

$$\frac{1}{u^2} = -\frac{1}{2}t^4 + t^2 + C$$

$$\frac{1}{u} = \pm\sqrt{-\frac{1}{2}t^4 + t^2 + C}$$

∴ $u = \pm\frac{1}{\sqrt{-\frac{1}{2}t^4 + t^2 + C}}$ //

$u = 4$ saat $t = 0$

$$4 = \frac{1}{\sqrt{0 + 0 + C}}$$

$$4 = \frac{1}{\sqrt{C}}$$

$C = \frac{1}{16}$ ⇒ ∴ $u = \frac{1}{\sqrt{-\frac{1}{2}t^4 + t^2 + \frac{1}{16}}}$ //

Tutorial Bab 3 Bag 1

Anella Utari Gunadi (19623229)

NO

DATE

3. Tentukan titik kritis, nilai max, dan nilai min.

a) $f(x) = x^3 - 6x^2 + 10 \quad [-2,1]$

Jawab :

→ Titik ujung selang

$x = -2$ dan $x = 1$

$f'(x) = 3x^2 - 12x$

→ Titik singular : tidak ada

→ Titik stasioner :

$f'(x) = 0$

$3x^2 - 12x = 0$

$3x(x-4) = 0$

$x = 0$ dan $x = 4$ → TM karena tidak pd int $[-2,1]$

→ Titik Kritis :

$x = -2$, $x = 0$, dan $x = 1$

$f(-2) = -8 - 24 + 10 = -22$

$f(0) = 10$

$f(1) = 1 - 6 + 10 = 5$

→ Nilai maks : 10

→ Nilai min : -22

16. a.) $f(x) = x^3 - 6x^2 + 9x - 1$

Jawab :

$f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x-3)(x-1)$

→ Titik singular : tidak ada

→ Titik stasioner : $x = 3$ dan $x = 1$

$f''(x) = 6x - 12$

$f''(1) = 6 - 12 = -6 < 0$

$f''(3) = 18 - 12 = 6 > 0$

⇒ $x = 1$ adalah titik maksimum lokal

$f(1) = 1 - 6 + 9 - 1 = 3$

⇒ $x = 3$ adalah titik minimum lokal

$f(3) = 27 - 54 + 27 - 1 = -1$

⇒ Nilai maksimum lokal : 3

⇒ Nilai minimum lokal : -1

19. a.) $y = 6x^5 - 10x^3 + 2$

Jawab : $f(x) = 6x^5 - 10x^3 + 2$

• $D_f = (-\infty, \infty)$

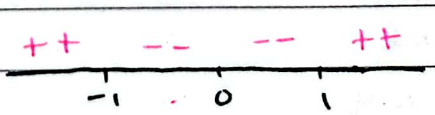
• Titik potong sumbu $y = (0,2)$

$f'(x) = 30x^4 - 30x^2$

$= 30x^2(x^2 - 1) = 30x^2(x-1)(x+1)$

→ Titik stasioner : $x = 0$, $x = 1$, dan $x = -1$

$(0,2), (1,-2), (-1,6)$ ← $y = 2$ $\quad y = -2$ $\quad y = 6$



⇒ f monoton naik pada interval $(-\infty, -1)$ dan pada interval $(1, \infty)$

⇒ f monoton turun pada interval $(-1, 0)$ dan pada interval $(0, 1)$

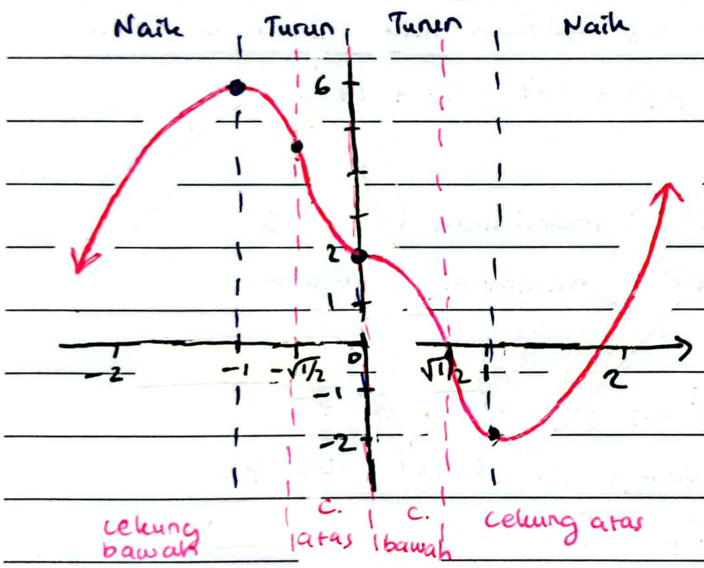
$f''(x) = 30(4x^3 - 2x)$
$= 60x(2x^2 - 1)$

$---$ $+++$ $---$ $+++$
$-\sqrt{1/2}$ 0 $\sqrt{1/2}$

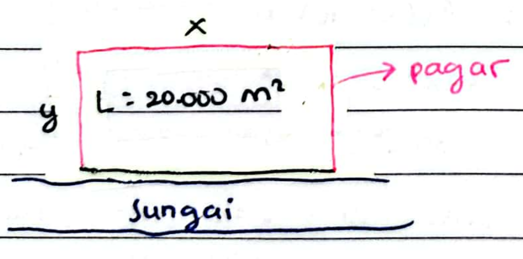
⇒ f celung ke bawah pada interval $(-\infty, -\sqrt{\frac{1}{2}})$ dan interval $(0, \sqrt{\frac{1}{2}})$

⇒ f celung ke atas pada interval $(-\sqrt{\frac{1}{2}}, 0)$ dan interval $(\sqrt{\frac{1}{2}}, \infty)$

⇒ titik belok : $x = -\sqrt{\frac{1}{2}}$ dan $x = \sqrt{\frac{1}{2}}$



15.



$L = p \times l$

$20.000 = x \cdot y$

$y = \frac{20.000}{x}$

→ Keliling pagar $= x + 2y$

$= x + 2\left(\frac{20000}{x}\right)$

$= x + \frac{40.000}{x}$

$= \frac{x^2 + 40.000}{x}$

$D_x = (0, \infty)$

Turunkan keliling pagar :

$K' = 1 - \frac{40.000}{x^2} = \frac{x^2 - 40.000}{x^2}$

$= \frac{(x+200)(x-200)}{x^2}$

Titik kritis : $x = 200$ atau $x = -200$

TM

$---$ $+++$
0 200

⇒ $x = 200$ adalah titik minimum global

Maka, ambil $x = 200$ m

$y = \frac{20.000}{200} = 100$ m

Keli pagar : $200 + 2(100) = 400$ m

∴ Agar kawat pagar yang dipergunakan paling kecil, padang gembala harus memiliki ukuran panjang 200 meter dan lebar 100 meter.

NO

DATE

22. Teorema Nilai Rata-rata

c) $f(t) = t + \sin(2t)$, $[0, \pi]$

Jawab: $a = 0$, $b = \pi$

$f'(t) = 1 + 2\cos(2t)$

Syarat TNR :

→ f kontinu dan $f'(x)$ terdefinisi pada $(0, \pi)$

Terdapat c sehingga :

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{f(\pi) - f(0)}{\pi - 0} = 1 + 2\cos(2c)$$

$$\frac{\pi + 0 - 0}{\pi} = 1 + 2\cos(2c)$$

$$1 = 1 + 2\cos(2c)$$

$$2\cos(2c) = 0$$

$$\cos(2c) = 0$$

$$c = \frac{\pi}{4} \text{ //}$$

∴ Nilai c yang memenuhi TNR pada fungsi $f(t) = t + \sin(2t)$ pada interval $[0, \pi]$ adalah $\frac{\pi}{4}$. //

Anella Utari Gunadi - 19623229

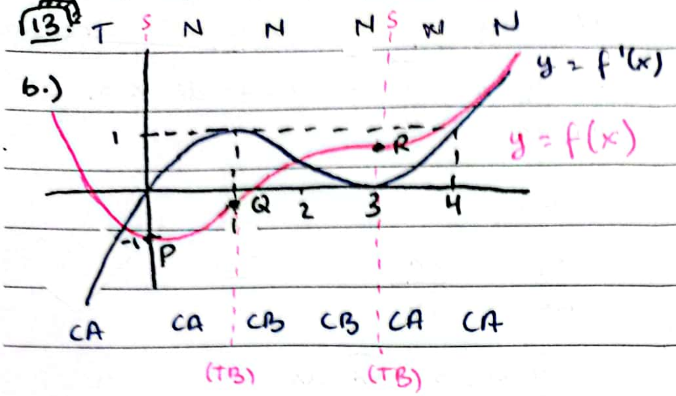
Tutorial MTK Bab 3

NO

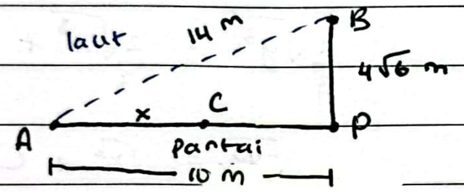
DATE

Bagian 1

13\. b.)



20\.



$AP = \sqrt{14^2 - (4\sqrt{6})^2}$

$= \sqrt{196 - 96} = \sqrt{100} = 10 \text{ m}$

$CP = 10 - x$

$CB^2 = CP^2 + PB^2$

$= (10-x)^2 + 96$

waktu

$W = W_{lari} + W_{renang}$

$= \frac{\text{jarak lari}}{v \text{ lari}} + \frac{\text{jarak renang}}{v \text{ renang}}$

$= \frac{x}{5} + \frac{\sqrt{(10-x)^2+96}}{1}$ → minimum

Titik kritis :

→ Titik ujung : $x = 0$ dan $x = 10$

→ Titik stasioner

$W'(x) = 0$

$\Rightarrow \frac{1}{5} + \frac{1}{2}\left((10-x)^2+96\right)^{-\frac{1}{2}}\left(-2(10-x)\right) = 0$

$\Leftrightarrow \frac{1}{5} + \frac{x-10}{\sqrt{(10-x)^2+96}} = 0$

$\Leftrightarrow \frac{\sqrt{(10-x^2)+96} + 5x - 50}{5\sqrt{(10-x)^2+96}} = 0$

$\Leftrightarrow \left(\sqrt{(10-x)^2+96}\right)^2 = (50-5x)^2$

$\Leftrightarrow (10-x)^2 + 96 = 2500 - 500x + 25x^2$

$\Leftrightarrow 100 - 20x + x^2 + 96 = 2500 - 500x + 25x^2$

$\Leftrightarrow 0 = 24x^2 - 480x + 2304$

$\Leftrightarrow 0 = x^2 - 20x + 96$

$\Leftrightarrow 0 = (x-8)(x-12)$

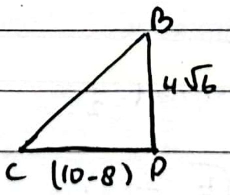
$\Leftrightarrow x = 8 \cup x = 12$
(TM)

→ Titik singular : tidak ada

→ $W(0) = 0 + \sqrt{196} = 14$

→ $W(8) = \frac{8}{5} + \sqrt{100} = \frac{58}{5}$ → minimum

→ $W(10) = \frac{10}{5} + \sqrt{96} = 2 + 4\sqrt{6}$



$CB = \sqrt{2^2 + (4\sqrt{6})^2}$

$CB = 10 \text{ m}$

∴ Jarak terkecil adalah jalan 8 meter lalu berenang 10 m.

Bantex

Dipindai dengan CamScanner

NO

DATE

Bagian 2

26. f.) $\int \frac{x\cos x - \sin x}{x^2}\,dx = \frac{\sin x}{x} + C$

Jawab :

Misal, $f(x) = \frac{\sin x}{x}$

$f'(x) = \frac{\cos x \cdot x - \sin x \cdot 1}{x^2}$

$= \frac{x \cdot \cos x - \sin x}{x^2}$

$\therefore \int \frac{x\cos x - \sin x}{x^2}\,dx = \frac{\sin x}{x} + C$ (Benar) //

27. Tentukan integral tak tentu berikut.

b.) $\int \frac{(t+1)\sqrt{t}}{t^2}\,dt$

Jawab :

$\int \frac{(t+1)\sqrt{t}}{t^2}\,dt = \int \frac{(t+1)\,t^{\frac{1}{2}}}{t^2}\,dt$

$= \int (t+1)\,t^{-3/2}\,dt$

$= \int (t^{-1/2} + t^{-3/2})\,dt$

$= 2t^{\frac{1}{2}} - 2t^{-\frac{1}{2}}$

$= 2\sqrt{t} - \frac{2}{\sqrt{t}}$ //

28. Selesaikan persamaan diferensial

d.) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$; $y = 4$ di $x = 1$

Jawab :

$\frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{y}}$

$\sqrt{y}\,dy = \sqrt{x}\,dx$

$\int \sqrt{y}\,dy = \int \sqrt{x}\,dx$

$\frac{2}{3}y^{3/2} + C_1 = \frac{2}{3}x^{3/2} + C_2$

$\frac{2}{3}\sqrt{y^3} = \frac{2}{3}\sqrt{x^3} + C$

$\sqrt{y^3} = \sqrt{x^3} + C$

Saat $y = 4$ di $x = 1$

$\sqrt{4^3} = \sqrt{1^3} + C$

$8 = 1 + C$

$C = 7$

$\therefore$ Solusi khusus nya adalah :

$\sqrt{y^3} = \sqrt{x^3} + 7$ //

Bab 4

4.1 Luas

Sifat-Sifat Sigma

1.) $\sum_{i=m}^{n} (a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$

2.) $\sum_{i=m}^{n} k a_i = k \sum_{i=m}^{n} a_i$

Rumus Penting

1.) $\sum_{i=1}^{n} k = kn$

2.) $\sum_{i=1}^{n} i = \frac{1}{2} n(n+1)$

3.) $\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$

4.) $\sum_{i=1}^{n} i^3 = \frac{1}{4} n^2(n+1)^2$

Contoh!

b.) $\frac{1}{n} \sum_{i=1}^{n} \left(\frac{5i}{n} + 7 \right)$

$= \frac{1}{n} \left(\sum_{i=1}^{n} \frac{5i}{n} + \sum_{i=1}^{n} 7 \right)$

$= \frac{1}{n} \left(\frac{5}{n} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 7 \right)$

$= \frac{1}{n} \left(\frac{5}{\cancel{n}} \left(\frac{1}{2} \cancel{n}(n+1) \right) + 7n \right)$

$= \frac{1}{n} \left(\frac{5}{2} (n+1) + 7n \right)$

$= \frac{1}{n} \left(\frac{5n}{2} + 7n + \frac{5}{2} \right)$

$= \frac{1}{n} \left(\frac{19n}{2} + \frac{5}{2} \right.$

$= \frac{19}{2} + \frac{5}{2n}$ //

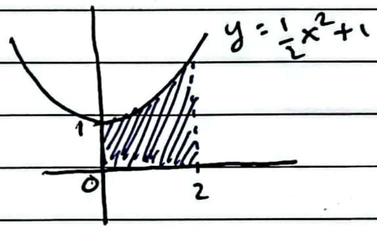
c.) $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{5i}{n} + 7 \right)$

$= \lim_{n \to \infty} \left(\frac{19}{2} + \frac{5}{2n} \right) = \frac{19}{2}$ //

Menghitung Luas

Contoh :

$f(x) = \frac{1}{2} x^2 + 1$ dari $x = 0$ sampai $x = 2$



$L = \int_0^2 \left(\frac{1}{2} x^2 + 1 \right) dx$

$= \frac{1}{6} x^3 + x \Big|_0^2$

$= \frac{1}{6} \cdot 8 + 2 = \frac{4}{3} + 2 = \frac{10}{3}$ //

4.2. Integral Tentu

Definisi

Limit dari jumlah Riemann kanan

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \left(\frac{b-a}{n}\right)i\right) \cdot \frac{b-a}{n}$$

Limit dari jumlah Riemann kiri

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}) \cdot \Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + \frac{b-a}{n}(i-1)\right) \cdot \frac{b-a}{n}$$

Contoh !

c.) $\int_0^5 (3 - 8x^3)\,dx$

Jawab :

$$\int_0^5 (3-8x^3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

$$x_i = a + \frac{b-a}{n} i = \frac{5i}{n}$$

$$f(x_i) = 3 - 8\left(\frac{5i}{n}\right)^3$$

$$= 3 - 8 \cdot 125 \cdot \frac{i^3}{n^3} = 3 - \frac{1000}{n^3} \cdot i^3$$

$$\Delta x = \frac{b-a}{n} = \frac{5}{n}$$

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(3 - \frac{1000}{n^3} \cdot i^3\right) \cdot \frac{5}{n}$$

$$= \frac{5}{n}\left(\sum_{i=1}^{n} 3 - \sum_{i=1}^{n} \frac{1000}{n^3} \cdot i^3\right)$$

$$= \frac{5}{n}\left(\sum_{i=1}^{n} 3 - \frac{1000}{n^3} \sum_{i=1}^{n} i^3\right)$$

$$= \frac{5}{n}\left(3n - \frac{1000}{n^3}\left(\frac{1}{2}n(n+1)\right)^2\right)$$

$$= \frac{5}{n}\left(3n - \frac{1000}{n^3} \cdot \frac{1}{4} n^2 (n+1)^2\right)$$

$$= \frac{5}{n} \cdot \left(3n - \frac{250}{n}(n+1)^2\right)$$

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = 15 - 1250 \cdot \frac{(n+1)^2}{n^2}$$

$$\therefore \int_0^5 (3-8x)\,dx = \lim_{n\to\infty} \left(15 - 1250 \cdot \frac{(n+1)^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \left(15 - 1250\left(\frac{n^2+2n+1}{n^2}\right)\right)$$

$$= 15 - 1250(1+0+0)$$

$$= -1235$$

Nyatakan limit dari jumlah Riemann sebagai integral tentu

a.) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(1+\frac{2i}{n}\right)^3 - 2\left(1+\frac{2i}{n}\right) + 9\right)\frac{2}{n}$

Jawab :

$$\frac{b-a}{n} = \frac{2}{n}$$

$$b - a = 2$$

$$a = 1 \rightarrow b = 2+1 = 3$$

$$x_i = 1 + \frac{2i}{n}$$

$$f(x_i) = x_i^3 - 2x_i + 9$$

$$f(x) = x^3 - 2x + 9$$

$\therefore$ Integral tentu =

$$\int_1^3 (x^3 - 2x + 9)\,dx$$

Date . .

b) $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 + \frac{21 i}{n}} \cdot \frac{21}{n}$

Jawab :

$b - a = 21$

$a = 4 \rightarrow b = 21 + 4 = 25$

$\therefore \int_4^{25} \sqrt{x}\, dx$

Contoh jika f(x) negatif

$\rightarrow f(x) = -2x$

$\int_0^1 f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) . \Delta x$

Jawab :

$\lim_{n\to\infty} \sum_{i=1}^{n} \left(-\frac{2i}{n}\right) \cdot \frac{1}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-\frac{2i}{n^2}\right)$

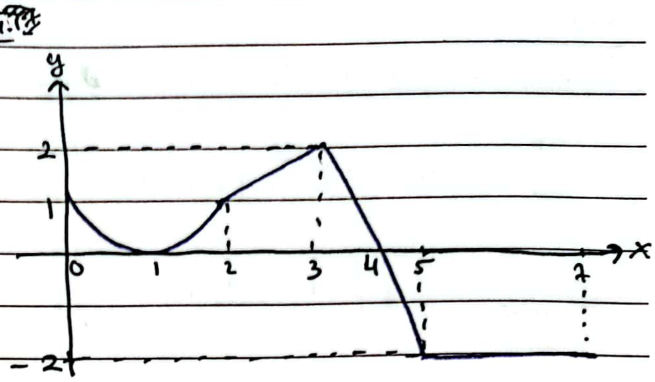
$= \lim_{n\to\infty} \left(-\frac{2}{n^2}\right) \sum_{i=1}^{n} i$

$= \lim_{n\to\infty} \left(-\frac{2}{n^2}\right) \cdot \frac{1}{2} n(n+1)$

Untuk f(x) negatif :

$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\, \Delta x = -$ luas daerah

Contoh :



Hitunglah $\int_a^b f(x)\, dx$ untuk a, b :

a.) $a = 0$, $b = 4$

Jawab :

$\int_0^4 f(x)\, dx$

$= \left(1.2 - \frac{1}{2}\pi.1\right) + \frac{(2+1).1}{2} + \frac{1.2}{2}$

$= 2 - \frac{\pi}{2} + \frac{3}{2} + 1$

$= \frac{9}{2} - \frac{\pi}{2}$ //

b.) $a = 4$ $b = 7$

Jawab :

$\int_4^7 f(x)\, dx = \frac{1}{2}.1(-2) + 2.(-2) = -5$ //

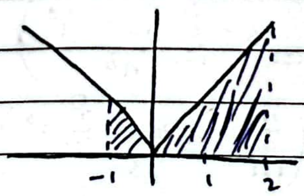
c.) $a = 0$ $b = 7$

Jawab :

$\int_0^7 f(x)\, dx = \frac{9}{2} - \frac{\pi}{2} - 5 = -\frac{1}{2} - \frac{\pi}{2}$ //

Hitung $\int_{-1}^{2} f(x)\, dx$ dgn terlebih dahulu membuat sketsa grafik fungsi f.

a.) $f(x) = |x|$

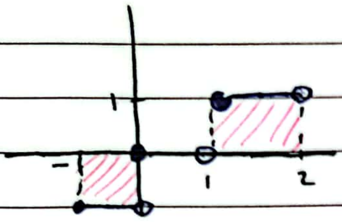


$\int_{-1}^{2} f(x)\, dx$

$= \frac{1.1}{2} + \frac{2.2}{2}$

$= \frac{5}{2}$ //

b.) $f(x) = [\![x]\!]$

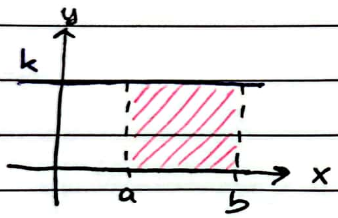


$\rightarrow \int_{-1}^{2} [\![x]\!]\, dx$

$= -1 + 0 + 1 = 0$

Sifat-sifat (dan Konsensus)

- $\int_a^b k\, dx = k(b-a)$
- $\int_a^a f(x)\, dx = 0$



- $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$
- $\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$

Contoh :

$\int_0^7 f(x)\, dx = 8$, $\int_7^5 f(x)\, dx = 3$

$\int_0^5 f(x)\, dx = \ldots$

Jawab :

$\int_0^7 f(x)\, dx = \int_0^5 f(x)\, dx + \int_5^7 f(x)\, dx$

$\int_0^5 f(x)\, dx = \int_0^7 f(x)\, dx - \int_5^7 f(x)\, dx$

$= \int_0^7 f(x)\, dx + \int_7^5 f(x)\, dx$

$= 8 + 3$

$= 11$

Misalkan f dan g dua fungsi yang kontinu pada $[0,3]$ dan diketahui

$\int_0^1 f(x)\, dx = -2$, $\int_1^3 f(x)\, dx = 5$,

$\int_1^3 g(x)\, dx = 1$, $\int_0^3 g(x)\, dx = 4$

Hitung integral berikut:

a.) $\int_0^3 f(x)\, dx$

$= \int_0^1 f(x)\, dx + \int_1^3 f(x)\, dx$

$= -2 + 5 = 3$

b.) $\int_1^3 5g(x)\, dx$

$= 5 \int_1^3 g(x)\, dx = 5 \cdot 1 = 5$

c.) $\int_0^3 (f(x) + g(x))\, dx$

$= \int_0^3 f(x)\, dx + \int_0^3 g(x)\, dx$

$= \int_0^1 f(x)\, dx + \int_1^3 f(x)\, dx + \int_0^3 g(x)\, dx$

$= -2 + 5 + 4$

$= 7$

d.) $\int_1^3 (2f(t) - 3g(t))\, dt$

$= \int_1^3 2f(t)\, dt - \int_1^3 3g(t)\, dt$

$= 2\int_1^3 f(t)\, dt - 3\int_1^3 g(t)\, dt$

$= 2 \cdot 5 - 3 \cdot 1$

$= 7$

No.

Date . .

4.4 Teorema Dasar Kalkulus II

Definisi

$\int_a^b f(x)\, dx = F(b) - F(a)$

Contoh :

2. Hitung integral tentu dgn menggunakan TDK II.

a.) $\int_0^1 \sqrt{x}\, dx$

Jawab : $\int_0^1 \sqrt{x}\, dx = \int_0^1 x^{1/2}\, dx$

$= \frac{2}{3} x^{3/2} \Big|_0^1$

$= \frac{2}{3} - 0 = \frac{2}{3}$ //

b.) $\int_0^{\pi/2} (2\cos(x) + 1)\, dx$

Jawab : $(2 \sin x + x) \Big|_0^{\pi/2}$

$= 2 + \frac{\pi}{2} - 0 = 2 + \frac{\pi}{2}$ //

Metode Substitusi untuk Integral Tentu

1. $\int_0^1 (2x+1)^6\,dx$

Jawab: misal : $u = 2x+1$

① cara 1 :

Metode subs. untk integral tak tentu

$\frac{du}{dx} = 2$

$dx = \frac{1}{2}du$

$\rightarrow \int u^6 \cdot \frac{1}{2}du$

$= \frac{1}{2}\cdot\frac{1}{7}\cdot u^7 + C$

$= \frac{1}{14}(2x+1)^7 + C$

∴ Berdasarkan TDK II :

$\int_0^1 (2x+1)^6\,dx$

$= \frac{1}{14}(2x+1)^7 \Big|_0^1$

$= \frac{3^7}{14} - \frac{1^7}{14} = \frac{3^7-1}{14}$ //

② Cara 2 :

$\int_0^1 (2x+1)^6 dx \qquad u = 2x+1$

$\begin{cases} x=0 \rightarrow u=1 \\ x=1 \rightarrow u=3 \end{cases}$

$\int_1^3 u^6 \cdot \frac{1}{2}du$

$= \frac{1}{2}\cdot\frac{1}{7}\,u^7 \Big|_1^3$

$= \frac{1}{14}\cdot 3^7 - \frac{1}{14}\cdot 1^7 = \frac{3^7-1}{14}$ //

3. $\int_0^{\sqrt{\pi/2}} t\cos(t^2)\,dt$

Jawab: misal : $u = t^2$

$du = 2t\,dx$

$dx = \frac{1}{2t}du$

$\int \cancel{t}\cos(u)\cdot\frac{1}{2\cancel{t}}du \qquad t=0 \rightarrow u=0$

$t = \sqrt{\pi/2} \rightarrow u = \pi/2$

$= \int_0^{\pi/2} \frac{1}{2}\cos(u)\,du$

$= \frac{1}{2}\sin(u)\Big|_0^{\pi/2}$

$= \frac{1}{2} - 0 = \frac{1}{2}$ //

Menghitung limit jumlah Riemann dgn TDK II

1. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(1+\frac{2i}{n}\right)^3 - 2\left(1+\frac{2i}{n}\right)+1\right)\frac{2}{n}$

Jawab: $a = 1 \qquad b = 2+1 = 3$

$f(x_i) = x_i^3 - 2x_i + 1$

$\Rightarrow \int_1^3 (x^3 - 2x + 1)\,dx$

$= \frac{1}{4}x^4 - x^2 + x \Big|_1^3$

$= \frac{3^4}{4} - 3^2 + 3 - \left(\frac{1}{4} - 1 + 1\right)$

$= \frac{3^4}{4} - \frac{1}{4} - 6 = 14$ //

2. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\sec^2\left(\frac{\pi i}{4n}\right)\right)\frac{\pi}{4n}$

Jawab: $a = 0 \qquad b = \pi/4$

$f(x_i) = \sec^2(x_i)$

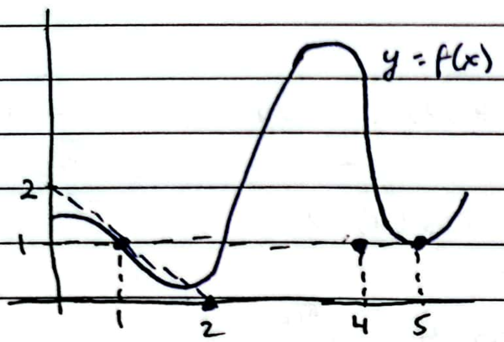
$\rightarrow \int_0^{\pi/4} \sec^2(x)\,dx$

$= \tan(x)\Big|_0^{\pi/4} = 1 - 0 = 1$ //

Date . .

Menghitung Integral Tentu dari Grafik

Example 15 :



$y = f(x)$ di $(1,1)$ dan $(5,1)$

Berdasarkan grafik, apakah integral di bawah ini bernilai (+), (−), nol ?

b.) $\int_1^5 f'(x)\,dx$

Jawab: $\int_1^5 f'(x)\,dx$

$= f(x) \Big|_1^5$

$= f(5) - f(1) = 1 - 1 = 0$ //

c.) $\int_1^5 f''(x)\,dx$

Jawab:

$\int_1^5 f''(x)\,dx$

$= f'(x) \Big|_1^5$

$= f'(5) - f'(1)$ → kemiringan GS

$= 0 - (-1) = 1$ //

d.) $\int_1^5 f'''(x)\,dx$

Jawab:

$\int_1^5 f'''(x)\,dx$

$= f''(x) \Big|_1^5$

$= f''(5) - f''(1)$ → kecekungan

$= (+) - (-)$

$= +$ //

a.) $\int_1^5 f(x)\,dx$

Jawab : $\int_1^5 f(x)\,dx = +$ //

Karena grafik berada di atas sumbu-x maka luasnya +.

• **Masalah Gerak**

Contoh :

$v(t) = 2t - 3\sqrt{t} + 5$

Hitung jarak ditempuh pd int t [1, 9]

Jawab :

$s'(t) = v(t)$

$\Rightarrow s(9) - s(1)$

$= \int_1^9 v(t)\,dt$ → Brds. TDK II

$= \int_1^9 (2t - 3\sqrt{t} + 5)\,dt$

$= t^2 - \cancel{3} . \frac{2}{\cancel{3}} t^{3/2} + 5t \Big|_1^9$

$= (81 - 54 + 45) - (1 - 2 + 5)$

$= 72 - 4 = 68$ satuan //

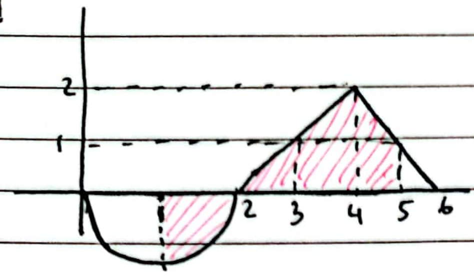
Anella Utari Gunadi (19623229)

Checkpoint Bab 4.2

No.

Date . .

1.



Hitunglah $\int_1^5 f(x)\,dx$

Jawab :

$\int_1^5 f(x)\,dx = -\int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^5 f(x)\,dx$

$= -\left(\frac{1}{4}\cdot\pi\cdot 1\right) + \frac{2\cdot 2}{2} + \frac{(1+2)\cdot 1}{2}$

$= -\frac{1}{4}\pi + 2 + \frac{3}{2} = -\frac{1}{4}\pi + \frac{7}{2}$ //

2. Diketahui :

$\int_0^5 f(x)\,dx = 7$, $\int_5^3 f(x)\,dx = 3$

$\int_0^3 g(x)\,dx = 5$

Hitunglah $\int_0^3 (2f(x) - g(x))\,dx$

Jawab :

$\int_0^3 (2f(x)) - g(x))\,dx$

$= \int_0^3 2f(x)\,dx - \int_0^3 g(x)\,dx$

$= 2\int_0^3 f(x)\,dx - \int_0^3 g(x)\,dx$

$= 2\left(\int_0^5 f(x)\,dx + \int_5^3 f(x)\,dx\right) - \int_0^3 g(x)\,dx$

$= 2(7 + 3) - 5$

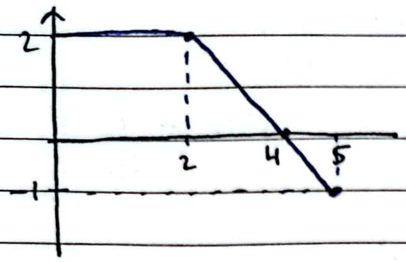
$= 20 - 5$

$= 15$

//

4.3 Teorema Dasar Kalkulus I

Misal



Definisikan $G(x) = \int_0^x f(t)\,dt$ utk $0 \le x \le 5$

Jawab:

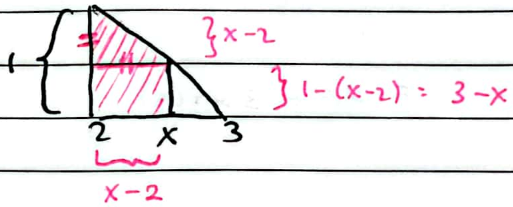
Cari formula umum:

$$G(x) = \begin{cases} x, & 0 \le x \le 2 \\ 2 - \frac{1}{2}x^2 - 3x + 4, & 0 \le x \le 3 \\ \frac{5}{2} - \frac{1}{2}(x-3)^2, & 0 \le x \le 5 \end{cases}$$

$\int_0^5 f(x)\,dx =$ luas pd int $[1,5]$

$= 6 - \frac{1}{2} = \frac{11}{2}$

untuk $2 \le x \le 3$



$\int_0^4 f(x)\,dx = 6$

$\int_0^2 f(x)\,dx = 4$

$\int_0^1 f(x)\,dx = 2$

$= \frac{1}{2}(1+3-x)(x-2)$

$= \frac{1}{2}(4-x)(x-2)$

$= \frac{1}{2}x^2 + 3x - 4$

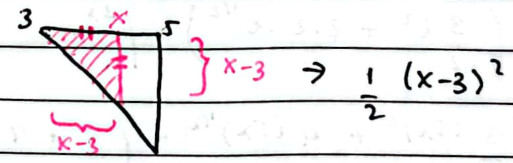
Definisi

$F(x) = \int_0^x f(s)\,ds$

→ Fungsi Akumulasi

: menyatakan luas "bertanda" dari 0 sampai x.

untuk $3 \le x \le 5$



$\rightarrow \frac{1}{2}(x-3)^2$

Contoh:

1. $f(x) = 2x + 3\sin(x)$

$F(x) = \int_0^x f(t)\,dt$

$= \int_0^x (2t + 3\sin(t))\,dt$

$= t^2 - 3\cos t \Big|_0^x$

$= x^2 - 3\cos x + 3$

∴ Formula fungsi akumulasi:

$F(x) = x^2 - 3\cos x + 3$

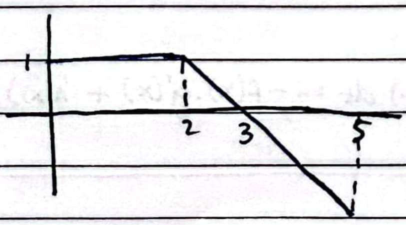
TDK I: Definisi

Jika $F(x) = \int_a^x f(t)\,dt$, maka berlaku

$\boxed{F'(x) = f(x)}$

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

2.



Date . .

Contoh : Hitunglah $F'(x)$

a.) $F(x) = \int_0^x (2t + 3\sin(t))\, dt$

Jawab : $F(x) : x^2 - 3\cos(x) + 3$

↑ TDK II (1.)

$\frac{d}{dx} F(x) : \frac{d}{dx}(x^2 - 3\cos(x) + 3)$

$: 2x - 3(-\sin x) + 0$

$= 2x + 3\sin(x)$

(2) Dengan TDK I

$F'(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \int_0^x \underbrace{(2t + 3\sin t)}_{f(t)}\, dt$

$= 2x + 3\sin(x)$

CATATAN :

Hati-hati jika batas bawah integral bukan konstanta atau batas atas bukan x

b.) $F(x) = \int_1^{5x} \left(3t + \frac{2}{\sqrt{t}}\right) dt$

Jawab : (1.) Cara TDK II

$F(x) : \left(\frac{3}{2}t^2 + 2.2.t^{1/2}\right) \Big|_1^{5x}$

$= \frac{3}{2}(5x)^2 + 4(5x)^{1/2} - \left(\frac{3}{2} + 4\right)$

$F'(x) = \frac{3}{2}.2(5x).5 + 4.\frac{1}{2}(5x)^{-1/2}.5 - 0$

$= 3.(5x).5 + \frac{2}{\sqrt{5x}}.5$

(2.) Cara TDK I

misal $u = 5x$

$\frac{d}{dx} \underbrace{\int_1^u \left(3t + \frac{2}{\sqrt{t}}\right) dt}$

$w = \int_1^u \left(3t + \frac{2}{\sqrt{t}}\right) dt$

$\Rightarrow \frac{dw}{dx} = \frac{dw}{du} \cdot \frac{du}{dx}$

$= \left(3u + \frac{2}{\sqrt{u}}\right) 5$

$= 5\left(3(5x) + \frac{2}{\sqrt{5x}}\right)$ //

→ $F(x) = \int_1^{x^2} \left(3t + \frac{2}{\sqrt{t}}\right) dt$

Jawab : misal $u = x^2$

$w = \int_1^u \left(3t + \frac{2}{\sqrt{t}}\right) dt$

$\frac{dw}{dx} = \frac{dw}{du} \cdot \frac{du}{dx} = \left(3u + \frac{2}{\sqrt{u}}\right) . 2x$

$= 2x\left(3x^2 + \frac{2}{\sqrt{x^2}}\right)$ //

b.) $\int_x^4 \sqrt{t^6 + 2}\, dt$

Jawab : $\frac{d}{dx}\left(-\int_4^x \sqrt{t^6+2}\right) dt$

$= -\sqrt{x^6+2}$ //

d.) $\int_x^{2x+1} \cos(\theta^2)\, d\theta$

Jawab :

$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

$\int_x^{2x+1} \cos(\theta^2)\, d\theta = \int_x^0 \cos\theta^2 d\theta + \int_0^{2x+1} \cos\theta^2 d\theta$

$: -\int_0^x \cos\theta^2 d\theta + \int_0^{2x+1} \cos\theta^2 d\theta$

$F'(x) : \frac{d}{dx}\left(\int_x^0 \cos(\theta^2)\,d\theta + \int_0^{2x+1} \cos(\theta^2)\,d\theta\right)$

$: \frac{d}{dx}\left(-\int_0^x \cos(\theta^2)\,d\theta + \int_0^{2x+1} \cos(\theta^2)\,d\theta\right)$

$: -\cos(x^2) + \cos((2x+1)^2) . 2$ //

Formula :

$$\boxed{\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt : -f(g(x)).g'(x) + f(h(x)).h'(x)}$$

4.5 Nilai Rata-rata (Integral)

Sebuah benda brgrk lurus dgn kecepatan sesaat saat t diberikan oleh fungsi $v(t)$ pada int $a \le t \le b$

Hitung kecepatan rata-rata ! $[a,b]$

Jawab :

kecepatan rata² = $\dfrac{\text{jarak yang ditempuh}}{\text{waktu tempuh}}$

$$= \frac{s(b) - s(a)}{b-a}$$

$$= \frac{1}{b-a}\left(s(t) \Big|_a^b \right)$$

$$= \frac{1}{b-a} \int_a^b v(t)\, dt$$

Definisi :

$$\boxed{v\ \text{rata}^2 = \frac{1}{b-a} \int_a^b v(t)\, dt}$$

Example 1

Cari v rata² pada $f(x) = x \sin x^2$ pada interval $[0, \sqrt{\pi}]$

Jawab :

$$\bar{v} = \frac{1}{\sqrt{\pi} - 0} \int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$$

$u = x^2$

$du = 2x\,dx$

$dx = \dfrac{du}{2x}$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\pi} \cancel{x} \sin(u)\, \frac{du}{2\cancel{x}}$$

$$= \frac{1}{\sqrt{\pi}} \left(\frac{1}{2} (-\cos u) \Big|_0^{\pi} \right)$$

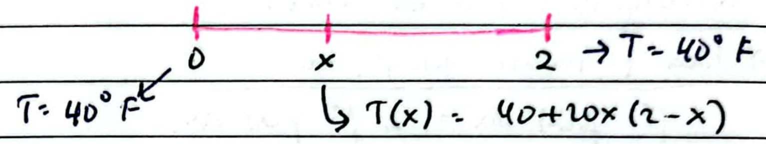
$$= \frac{1}{\sqrt{\pi}} \left(\frac{1}{2} (-\cos \pi - (-\cos 0)) \right)$$

$$= \frac{1}{\sqrt{\pi}} \left(\frac{1}{2} (1+1) \right) = \frac{1}{\sqrt{\pi}} \text{ //}$$

Example 2 (TNR)

Hitunglah suhu rata² dr sebuah besi dgn panjang 2 kaki, suhu pada x $T(x) = 40 + 20x(2-x)$. Apakah ada suatu titik dimana suhu asli = suhu rata² ?

Jawab :



↳ $T(x) = 40 + 20x(2-x)$

$= 40 + 40x - 20x^2$

$$T\ \text{rata}^2 = \frac{1}{2-0} \int_0^2 T(x)\, dx$$

$$= \frac{1}{2} \int_0^2 (40 + 40x - 20x^2)\, dx$$

$$= \frac{1}{2} \left(\left(40x + 20x^2 - \frac{20}{3} x^3 \Big|_0^2 \right) \right)$$

$$= \frac{1}{2} \left(80 + 80 - \frac{160}{3} \right)$$

$$= \frac{160}{3}\ ^\circ F \approx 53.33\ ^\circ F \text{ //}$$

⇒ suhu di suatu titik = suhu rata² ?

ada, karena F(x) kontinu

$$40 + 40x - 20x^2 = \frac{160}{3}$$

$$2 + 2x - x^2 = \frac{8}{3}$$

$$-3x^2 + 6x + 6 = 8$$

$$-3x^2 + 6x - 2 = 0$$

$$3x^2 - 6x + 2 = 0$$

$$x = \frac{6 \pm \sqrt{36 - 24}}{6}$$

$$= \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{\sqrt{3}}{3} \text{ //}$$

Teorema A : TNR untuk Integral

Jika f kontinu di [a,b] dan ada c di antara a dan b sehingga:

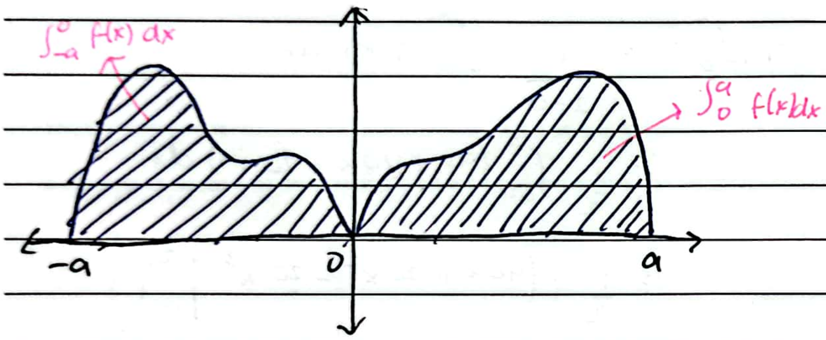
$$\boxed{f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt}$$

(contoh : example 2)

Teorema B : Teorema Simetri

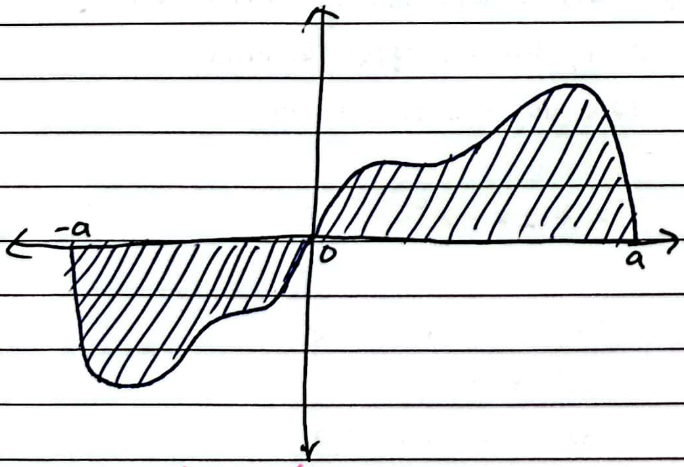
→ Jika f fungsi genap, maka:

$$\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx}$$



→ Jika f fungsi ganjil, maka:

$$\boxed{\int_{-a}^{a} f(x)\,dx = 0}$$



$\int$up $- \int$down $= 0$

(saling menghilangkan)

Hitung integral berikut:

a.) $\int_{-1}^{1} (x^4 + 2x^2 + 1)\,dx$

Jawab: f fungsi genap

$\int_{-1}^{1} (x^4 + 2x^2 + 1)\,dx$

$= 2\int_0^1 (x^4 + 2x^2 + 1)\,dx$

$= 2\left(\left(\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right)\Big|_0^1\right)$

$= 2\left(\frac{1}{5} + \frac{2}{3} + 1\right)$ //

b.) $\int_{-1}^{1} \left(\sin x + 7x^3 + \frac{3x^7}{x^2 + \cos x}\right)dx$

Jawab:

$f(-x) = \sin(-x) + 7(-x)^3 + \frac{3(-x)^7}{(-x)^2 + \cos(-x)}$

$= -\sin x - 7x^3 - \frac{3x^7}{x^2 + \cos x}$

$= -\left(\sin x + 7x^3 + \frac{3x^7}{x^2 + \cos x}\right)$

$f(-x) = -f(x)$

→ f fungsi ganjil

$\therefore \int_{-1}^{1} \left(\sin x + 7x^3 + \frac{3x^7}{x^2 + \cos x}\right)dx = 0$ //

c.) $\int_{-2}^{2} (x^3 \sin^4(x) - x\cos(2x) + x^4 + 5x^3 - 1)\,dx$

Jawab:

$f(x) = x^3 \sin^4(x) \rightarrow$ f ganjil

$x\cos(2x) \rightarrow$ f ganjil

$x^4 \rightarrow$ f genap

$5x^3 \rightarrow$ f ganjil

$1 \rightarrow$ f genap

(verifikasi f ganjil)

$\rightarrow \int_{-2}^{2} (x^4 - 1)\,dx = 2\int_0^2 (x^4 - 1)\,dx$

$= 2\left(\left(\frac{1}{5}x^5 - x\right)\Big|_0^2\right)$

$= 2\left(\frac{32}{5} - 2\right)$ //

Keperiodikan

Teorema C : Keperiodikan

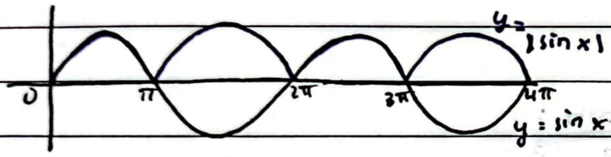
Jika f periodik dgn periode p, maka

$$\boxed{\int_{a+p}^{b+p} f(x)\,dx = \int_a^b f(x)\,dx}$$

Soal 1

$\int_0^{100\pi} |\sin x|\,dx$

Jawab :



$\Rightarrow f(x) = |\sin x|$ memiliki periode π

$\therefore = \int_0^{100\pi} |\sin x|\,dx = 100 \int_0^{\pi} |\sin x|\,dx$

$= 100 \int_0^{\pi} \sin x\,dx$

$= 100 \left(-\cos x \Big|_0^{\pi}\right)$

$= 100 \left(-(-1) - (-1)\right)$

$= 100(2) = 200$ //

$\int_0^{\pi} |\cos x|\,dx$:

$= \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx$

$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx = 2$ //

karna $\cos x \geq 0$ pd $[0, \pi/2]$

$\cos x \leq 0$ pd $[\pi/2, \pi]$

$\therefore \int_0^{\pi} |\cos x|\,dx = 2$ //

$\int_0^{\pi} |\cos x|\,dx = \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx$

$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx$

$\to \cos x \geq 0\ [0, \frac{\pi}{2}]$ $= 2$ //

$\cos x \leq 0\ [\frac{\pi}{2}, \pi]$

Fungsi f kontinu pada $(-\infty, \infty)$ dgn periode 10.

Dik: $\int_0^4 f(x)\,dx = 7$ dan $\int_{-5}^{5} f(x)\,dx = 11$

Hitunglah :

a.) $\int_0^{10} f(x)\,dx$

Jawab : $\int_0^5 f(x)\,dx + \int_{5-10}^{10-10} f(x)\,dx$

$= \int_0^5 f(x)\,dx + \int_{-5}^{0} f(x)\,dx$

$= \int_{-5}^{5} f(x)\,dx$

$= 11$

b.) $\int_{44}^{50} f(x)\,dx$

$= \int_{34}^{40} = \int_{24}^{30} = \int_{14}^{20} = \int_4^{10} = \int_0^{10} - \int_0^4$

$= 11 - 7$

$= 4$ //

c.) $\int_{33}^{83} f(x)\,dx$

Jawab : $\int_{33}^{83} f(x)\,dx = 55$

$\int_{33}^{83} f(x)\,dx = \int_3^{13} f(x)\,dx$

4.6 Integrasi Numerik

Metode 1-3 : Jumlah Riemann

• **Jumlah Riemann Kiri**

$$\boxed{\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_{i-1})\,\Delta x}$$

dengan : $x_{i-1} = a + \frac{(b-a)}{n}(i-1)$

$\Delta x = \frac{b-a}{n}$

• **Jumlah Riemann Kanan**

$$\boxed{\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i)\,\Delta x}$$

dengan $x_i = a + \frac{(b-a)}{n}\,i$

$\Delta x = \frac{b-a}{n}$

• **Jumlah Riemann Tengah**

$$\boxed{\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x}$$

Metode 4 : Trapesium

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \frac{1}{2}\left(f(x_{i-1}) + f(x_i)\right)\Delta x$$

$$= \frac{1}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\right).\Delta x$$

Metode 5 : Parabola (Aturan Simpson)

$$\int_a^b f(x)\,dx \approx \frac{1}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right).\Delta x$$

Catatan!
Metode ini hanya bisa untuk
n genap saja.

Contoh :

$\int_1^3 \frac{1}{1+x^2}\,dx$, $n = 4$

Jawab :

$x_0 = 1 \quad 3/2 \quad 2 \quad 5/2 \quad 3 = x_n = x_4$

→ Riemann Kiri

$\approx \left(f(x_0) + f(x_1) + f(x_2) + f(x_3)\right)\Delta x$

$= \left(\frac{1}{2} + \frac{4}{13} + \frac{1}{5} + \frac{4}{29}\right)\frac{1}{2}$ //

→ Trapesium :

$\approx \frac{1}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)\Delta x$

$= \frac{1}{2}\left(\frac{1}{2} + 2.\frac{4}{13} + 2.\frac{1}{5} + 2.\frac{4}{29} + \frac{1}{10}\right)\frac{1}{2}$ //

→ Parabolic :

$\approx \frac{1}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$

$= \frac{1}{3}\left(\frac{1}{2} + 4.\frac{4}{13} + 2.\frac{1}{5} + 4.\frac{4}{29} + \frac{1}{10}\right).\frac{1}{2}$ //

Date . .

Galat Taksiran / Hampiran

Taksiran selalu memiliki galat :

$\int_a^b f(x)\,dx$ = aproksimasi + E_n

galat : nilai integral sebenarnya − nilai taksiran

Teorema A

Untuk c di [a,b]

1.) Left Riemann Sum

$E_n = \frac{(b-a)^2}{2n} f'(c)$

2.) Right Riemann Sum

$E_n = -\frac{(b-a)^2}{2n} f'(c)$

3.) Midpoint Riemann Sum

$E_n = \frac{(b-a)^3}{24n^2} f''(c)$

4.) Metode Trapesium

$E_n = -\frac{(b-a)^3}{12n^2} f''(c)$

5.) Metode Parabola

$E_n = -\frac{(b-a)^5}{180n^4} \cdot f^{(4)}(c)$

Example 5 : $\int_1^4 \frac{1}{1+x}\,dx$

Tentukan suatu batas dgn galat mutlak

a.) Riemann Kiri , n = 100

Jawab:

$|E_{100}| = \left|\frac{(b-a)^2}{2n} \cdot f'(c)\right|$

$= \left|\frac{(4-1)^2}{2.100} \cdot f'(c)\right|$

$= \frac{9}{200} \cdot f'(c)$

$f(x) = \frac{1}{1+x} = 1(1+x)^{-1}$

$f'(x) = \frac{-1}{(1+x)^2}$

$|E_{100}| = \frac{9}{200}\left|-\frac{1}{(1+c)^2}\right|$

$= \frac{9}{200} \cdot \frac{1}{(1+c)^2}$ dengan $1 \le c \le 4$

$\le \frac{9}{200} \cdot \frac{1}{(1+1)^2} = \frac{9}{800}$ //

b.) Riemann Tengah, n = 100

Jawab:

$|E_{100}| = \left|\frac{(b-a)^3}{24n^2} \cdot f''(c)\right|$

$= \left|\frac{27}{240.000} \cdot f''(c)\right|$

→ $f'(x) = -(1+x)^{-2}$

$f''(x) = 2(1+x)^{-3} = \frac{2}{(1+x)^3}$

$|E_{100}| = \left|\frac{27}{240.000} \cdot \frac{2}{(1+c)^3}\right|$

$\le \frac{27}{240.000} \cdot \frac{2}{8} = \frac{27}{960.000}$ //

c.) Metode Parabola, n = 100

Jawab :

$|E_{100}| = \left|\frac{-(b-a)^5}{180n^4} f^{(4)}(c)\right|$

$= \left|\frac{-243}{180.100^4} \cdot f^{(4)}(c)\right|$

$f'''(x) = -6(1+x)^{-4}$

$f^{(4)}(x) = 24(1+x)^{-5} = \frac{24}{(1+x)^5}$

$$|E_{100}| \leq \frac{243}{180 \cdot 100^4} \cdot \frac{24}{32} = \frac{243}{180 \cdot 100^4} \cdot \frac{3}{4}$$

Dik: $f''(x) = \dfrac{x}{1+x^2}$

Jika $\int_1^4 f(x)\,dx$ ditaksir dgn trapesium, $n = 1000$, tentukan batas atas utk galat mutlak!

Jawab:

$$|E_{100}| = \left|\frac{(b-a)^3}{12n^2} \cdot f''(c)\right|$$

$$= \frac{8}{12.000.000} \cdot \frac{c}{1+c^2}, \quad 1 \leq c \leq 3$$

$$\leq \frac{8}{12.000.000} \cdot \frac{3}{2} = \frac{1}{1.000.000}$$

Jika $f''(x) = \dfrac{\sin x}{x^2+1}$, $2 \leq c \leq 5$

$$|f''(c)| = \left|\frac{\sin c}{c^2+1}\right| \leq \frac{1}{2^2+1} = \frac{1}{5}$$

Example 6

Berapa besar n jika galat $\leq 0,00001$ pada $\int_1^4 \frac{1}{1+x}\,dx$.

a.) J. Riemann Kanan

Jawab:

$$|E_n| = \left|\frac{(b-a)^2}{2n} \cdot f'(c)\right|, \quad 1 \leq c \leq 4$$

$$= \frac{9}{2n} \cdot \frac{1}{(1+c)^2}$$

$$\leq \frac{9}{2n} \cdot \frac{1}{4}$$

$$\leq \frac{9}{8n} \leq \frac{1}{100000}$$

$$\Rightarrow n \geq \frac{9 \times 100.000}{8} = 9 \times 12.500$$

b.) Metode Trapesium

$$|E_n| = \left|\frac{(b-a)^3}{12n^2} \cdot f''(c)\right|$$

$$= \frac{27}{12n^2} \cdot \frac{2}{(1+c)^3}, \quad 1 \leq c \leq 4$$

$$\leq \frac{27}{12n^2} \cdot \frac{2}{8} \leq \frac{1}{100.000}$$

$$\Rightarrow n^2 \geq \frac{27}{48} \cdot 100.000$$

$$n \geq \sqrt{\frac{27}{48} \cdot 100000}$$

Anella Utari Gunadi (14623229)

PR Bab 4.3 , 4.5 , 4.6

NO

DATE

4.3. Misalkan $f(x) = \int_{3x}^{0} \sqrt{t^2+3\sin(t)+4}\; dt$

Hitunglah $f(0)$ dan $f'(0)$

Jawab :

$$f(x) = \int_{3x}^{0} \sqrt{t^2+3\sin(t)+4}\; dt$$

$$= -\int_{0}^{3x} \sqrt{t^2+3\sin(t)+4}\; dt$$

misal : $u = 3x$

$$w = \int_0^{3x} \sqrt{t^2+3\sin(t)+4}\; dt$$

$$f'(x) = \frac{d}{dx} - \int_0^{u} \sqrt{t^2+3\sin(t)+4}\; dt$$

$$= -\frac{dw}{dx} = -\frac{dw}{du}\cdot\frac{du}{dx} = -\sqrt{u^2+3\sin(u)+4}\cdot 3$$

$$f'(x) = -3\sqrt{x^2+3\sin(x)+4}$$

$\Rightarrow$ $f(0) = \int_{3(0)}^{0} \sqrt{t^2+3\sin(t)+4}\; dt$

$= \int_0^0 \sqrt{t^2+3\sin(t)+4}\; dt$

$= 0$ //

$\Rightarrow$ $f'(0) = -3\sqrt{0^2+3\sin(0)+4}$

$= -3 \cdot 2 = -6$ //

4.4. Tentukan nilai rata-rata fungsi $f(x) = 3x + \sin^7(2x+1)$ pada selang $[-2,1]$.

Jawab :

Nilai rata-rata : $\frac{1}{1-(-2)} \int_{-2}^{1} (3x+\sin^7(2x+1))\; dx$

$= \frac{1}{3}\left(\int_{-2}^{1} 3x\, dx + \int_{-2}^{1} \sin^7(2x+1)\, dx\right)$ $\Rightarrow$ misal $u = 2x+1$

$du = 2\,dx$

$= \frac{1}{3}\left(\int_{-2}^{1} 3x\, dx + \int_{-3}^{3} \sin^7 u\; \frac{1}{2}\, du\right)$

NO

DATE

$$= \frac{1}{3}\left(\int_{-2}^{1} 3x\,dx + \frac{1}{2}\int_{-3}^{3} \sin^7(u)\,du\right)$$

$$= \frac{1}{3}\left(\frac{3x^2}{2}\Big|_{-2}^{1} + 0\right)$$

$\Rightarrow f(x) = \sin^7(u)$

$f(-x) = -\sin^7(u) \Rightarrow$ fungsi ganjil

$$= \frac{1}{3}\left(\frac{3}{2} - 6\right)$$

$$= \frac{1}{3}\left(-\frac{9}{2}\right) = \frac{-3}{2} \,//$$

4.6. Diberikan integral tentu $\int_0^1 \frac{2x}{x+1}\,dx$. Taksir nilai integral tsb dengan:

a.) Metode trapesium, $n = 5$

Jawab :

$x_0 = 0 \quad 1/5 \quad 2/5 \quad 3/5 \quad 4/5 \quad 1 = x_n = x_5$

$\Delta x = \frac{1}{5}$

$$\Rightarrow \int_0^1 \frac{2x}{x+1}\,dx \approx \sum_{i=1}^{5} \frac{1}{2}\left(f(x_{i-1}) + f(x_i)\right).\Delta x$$

$$= \frac{1}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right).\Delta x$$

$$= \frac{1}{2}\left(0 + 2.\frac{1}{3} + 2.\frac{4}{7} + 2.\frac{3}{4} + 2.\frac{8}{9} + 1\right)\frac{1}{5}$$

$$= \frac{1}{10}\left(\frac{2}{3} + \frac{8}{7} + \frac{3}{2} + \frac{16}{9} + 1\right)$$

$$= \frac{1}{10}\left(\frac{84 + 144 + 189 + 224 + 126}{126}\right)$$

$$= \frac{1}{10}.\frac{767}{126}$$

$$= \frac{767}{1260} \,//$$

Bantex

Anetta Ufan Bunadi ([illegible])

Tutorial Bab 4

NO

DATE

2.2 b.) $\int_{-2}^{4} (2x-1)\, dx$

Jawab :

$a = -2$, $b = 4$

$\frac{b-a}{n} = \frac{6}{n}$, $x_i = -2 + \frac{6i}{n}$

$\Rightarrow \int_{-2}^{4} (2x-1)\, dx$

$= \lim_{n \to \infty} \sum_{i=1}^{n} \left(2\left(-2+\frac{6i}{n}\right)-1\right) \frac{6}{n}$

$\sum_{i=1}^{n} \left(2\left(-2+\frac{6i}{n}\right)-1\right) \frac{6}{n}$

$= \frac{6}{n}\left(\sum_{i=1}^{n} -4 + \frac{12i}{n} - 1\right)$

$= \frac{6}{n}\left(\sum_{i=1}^{n} -5 + \sum_{i=1}^{n} \frac{12i}{n}\right)$

$= \frac{6}{n}\left(\sum_{i=1}^{n} -5 + \frac{12}{n} \sum_{i=1}^{n} i\right)$

$= \frac{6}{n}\left(-5n + \frac{12}{n}\left(\frac{1}{2}n(n+1)\right)\right)$

$= \frac{6}{n}\left(-5n + 6n + 6\right)$

$= \frac{6}{n}(n+6)$

$= 6 + \frac{36}{n}$

$\Rightarrow \int_{-2}^{4} (2x-1)\, dx$

$= \lim_{n \to \infty} \left(6 + \frac{36}{n}\right)$

$= 6 + 0 = 6$ //

3.1 d.) $\int_{0}^{6} f(x)\, dx$

$= \int_0^1 f(x)\, dx + \int_1^3 f(x)\, dx + \int_3^6 f(x)\, dx$

$= \frac{1.2}{2} - \frac{1}{2}\pi . 1^2 + \frac{(3+2)1}{2}$

$= 1 - \frac{\pi}{2} + \frac{5}{2} = \frac{7-\pi}{2}$ //

9.1 b.) $\int_0^{2x} f(t)\, dt = 8x^2 + \sin(6x)$

Jawab : TDK I :

$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$

$\frac{d}{dx} \int_0^{2x} f(t)\, dt = \frac{d}{dx}\left(8x^2 + \sin(6x)\right)$

$f(2x) . 2 = 16x + 6\cos(6x)$

$f(2x) = 8x + 3\cos(6x)$

$\therefore f(x) = 4x + 3\cos(3x)$ //

11.1 b.) $\int_1^3 \frac{2x+1}{(x^2+x)^4}\, dx$

Jawab : $u = x^2 + x$

$du = (2x+1)\, dx$

$dx = \frac{du}{2x+1}$

$\rightarrow \int \frac{2x+1}{u^4} . \frac{du}{2x+1}$

$= \int_2^{12} u^{-4}\, du$

$= \frac{1}{-3} u^{-3} \Big|_2^{12} = \frac{1}{-3.12^3} + \frac{1}{3.2^3}$

$= -\frac{1}{5184} + \frac{1}{24} = \frac{215}{5184}$ //

NO
DATE

18. Aturan trapesium

$L = \int_0^{80} f(x)\, dx$

$\approx \sum_{i=1}^{n} \frac{1}{2} \left(f(x_{i-1}) + f(x_i) \right) . \Delta x$

$= \frac{1}{2} (f(0) + 2f(10) + 2f(20) + 2f(30)$

$+ 2f(40) + 2f(50) + 2f(60) +$

$2f(70) + f(80)) . 10$

$= \frac{1}{2} (75 + 2.71 + 2.60 + 2.45 + 2.45 +$

$2.52 + 2.57 + 2.60 + 59) \; 10$

$= \frac{1}{\cancel{2}} (914) . \cancel{10}^{5}$

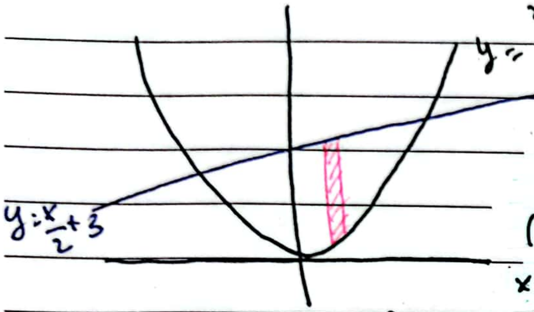
$= 4570$

//

5.1 Luas

Contoh :

- Hitunglah luas daerah yg dibatasi kurva $y = \frac{x^2}{2}$ dan garis $y = \frac{x}{2} + 3$



$\frac{x^2}{2} = \frac{x}{2}+3$

$x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$x = -2 \vee x = 3$

$\Delta A_i \approx \left(\frac{x_i}{2}+3 - \frac{x^2}{2}\right)\Delta x$

$A \approx \sum_{i=1}^{n}\left(\frac{x_i}{2}+3-\frac{x_i^2}{2}\right)\Delta x$

$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{x_i}{2}+3-\frac{x_i^2}{2}\right)\Delta x$

$= \int_{-2}^{3}\left(\frac{1}{2}x+3-\frac{x^2}{2}\right)dx$

$= \left(\frac{1}{4}x^2 + 3x - \frac{1}{6}x^3\right)\Big|_{-2}^{3}$

$= \left(\frac{9}{4}+9-\frac{27}{6}\right) - \left(1-6+\frac{8}{6}\right)$

$= \ldots$

$\Delta A \approx (2x - x^2 - x^3)\,\Delta x$

$A = \int_0^1 (2x - x^2 - x^3)\,dx$

$= \left(x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1$

$= 1 - \frac{1}{3} - \frac{1}{4} = \frac{12-4-3}{12} = \frac{5}{12}$ //

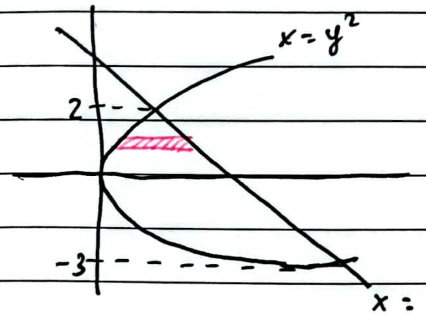
- Hitunglah luas daerah yang dibatasi oleh parabola $y^2 = x$ dan garis $x + y = 6$

$y^2 = 6 - y$

$y^2 + y - 6 = 0$

$(y-2)(y+3) = 0$

$y = 2 \vee y = -3$



$\Delta A \approx (6 - y - y^2)\,\Delta y$

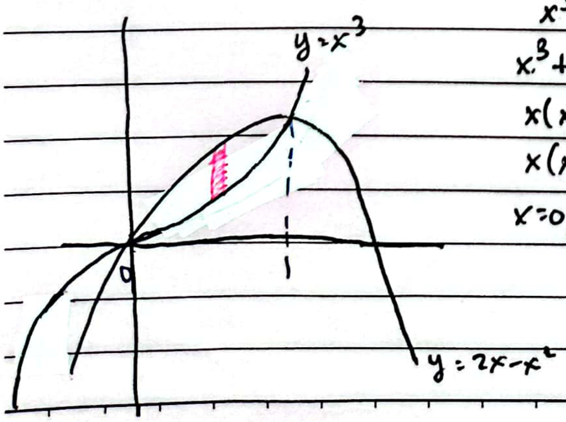
$A = \int_{-3}^{2}(6-y-y^2)\,dy$

$= \left(6y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_{-3}^{2}$

$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right)$

$=$

- Hitunglah luas daerah di Kuadran I yang dibatasi oleh kurva $y = x^3$ dan parabola $y = 2x - x^2$



$x^3 = 2x - x^2$

$x^3 + x^2 - 2x = 0$

$x(x^2 + x - 2) = 0$

$x(x+2)(x-1) = 0$

$x = 0, x = -2, x = 1$

- Hitung luas daerah yang dibatasi oleh grafik $y = \sqrt{5-x}$, garis $y = 2x$ & $y = 1$

$2x = \sqrt{5-x}$

$4x^2 = 5 - x$

$4x^2 + x - 5 = 0$

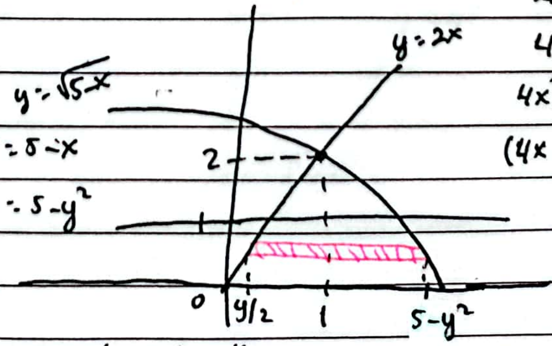
$(4x+5)(x-1) = 0$

$x = 1$

$y = \sqrt{5-x}$

$y^2 = 5 - x$

$x = 5 - y^2$



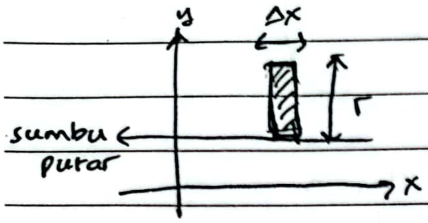
$\Delta A \approx \left(5 - y^2 - \frac{y}{2}\right)\Delta y$

$A = \int_0^1 \left(5 - y^2 - \frac{y}{2}\right)dy$

Metode Cakram

Cakram = Tabung

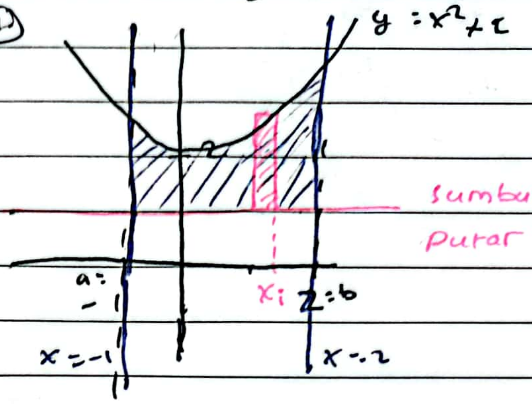
$V_{cakram} = \pi r^2 h$



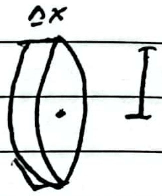
$y = x^2 + 2$, garis $x = -1$, $x = 2$ (Latihan)

sumbu putar → $y = 1$

(1)



(3)



$r = x_i^2 + 2 - 1 = x_i^2 + 1$

$= x^2 + 1$

$\Delta V \approx \pi (x^2+1)^2 . \Delta x$

(5) $V = \int_{-1}^{2} \pi (x^2+1)^2 \, dx$

$= \pi \int_{-1}^{2} (x^4 + 2x^2 + 1) \, dx$

$= \pi \left(\left(\frac{1}{5} x^5 + \frac{2}{3} x^3 + x \right) \Big|_{-1}^{2} \right)$

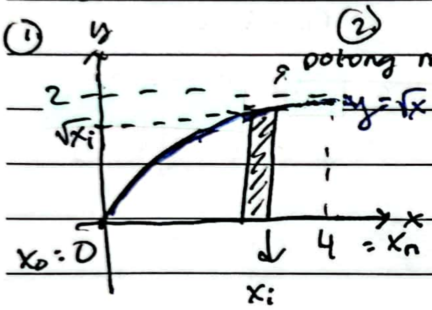
$= \pi \left(\left(\frac{32}{5} + \frac{16}{3} + 2 \right) - \left(\frac{-1}{5} - \frac{2}{3} - 1 \right) \right)$

$=$

Example 1

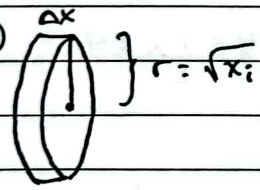
$y = \sqrt{x}$, $x = 4$

(1)



(2) potong menjadi sub interval (berbentuk seperti cakram)

(3.)



$r = \sqrt{x_i}$

Volume potongan

$= \Delta V_i \approx$ volume cakram

$= \pi . r^2 . h$

$= \pi (\sqrt{x_i})^2 . \Delta x$

$= \pi . x_i . \Delta x$

(4.) $V \approx \sum_{i=1}^{n} \pi . x_i . \Delta x$

(5.) $V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi x_i . \Delta x$

$= \int_0^4 \pi x \, dx$

TDK II

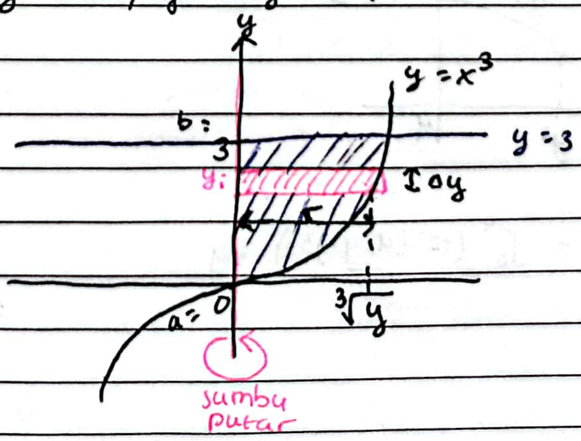
$= \frac{\pi}{2} x^2 \Big|_0^4$

$= 8\pi$

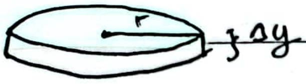
//

Example 2

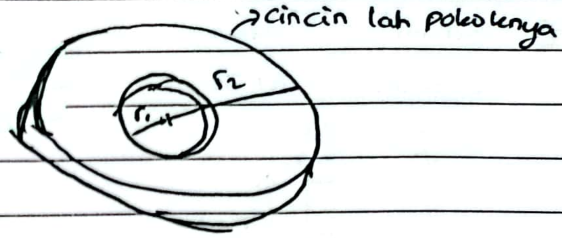
$y = x^3$, garis $y = 3$, sumbu putar: sb-y



$r = \sqrt[3]{y}$ $\quad \Delta V \approx \pi (\sqrt[3]{y})^2 . \Delta y$ $\quad$ <u>Metode Cincin</u>



$= \pi y^{2/3} . \Delta y$



$\therefore V = \int_0^3 \pi y^{2/3} . dy$

$= \pi . \frac{3}{5} y^{5/3} \Big|_0^3$

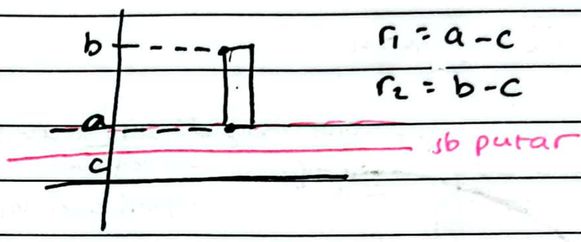
$V = \pi . \frac{3}{5} \; 3^{5/3}$

$V = \pi r_2^2 h - \pi r_1^2 h$

$= \pi (r_2^2 - r_1^2) h$

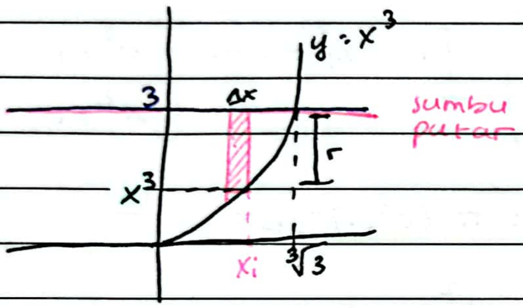
<u>Variasi</u>

$y = x^3$, $y = 3$, sumbu putar → $y = 3$



$r_1 = a - c$

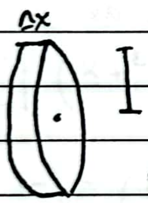
$r_2 = b - c$



<u>Contoh!</u>

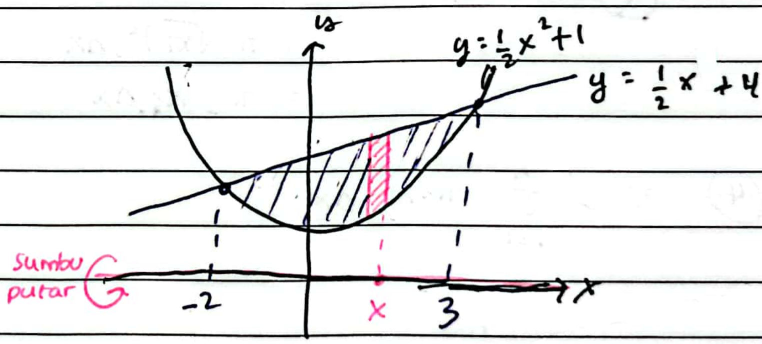
$y = \frac{1}{2}x^2 + 1$ dan $y = \frac{1}{2}x + 4$

sb-putar : sb-x



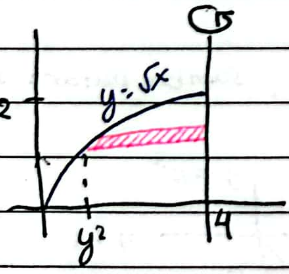
$r = 3 - x^3$

$\Delta V \approx \pi (3 - x^3)^2 . \Delta x$

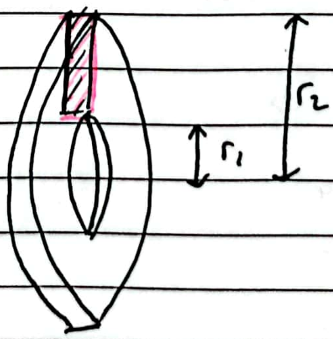


$V = \int_0^{\sqrt[3]{3}} \pi (3 - x^3)^2 . dx$

$y = \sqrt{x}$, sumbu putar : $x = 4$



$\Delta V = \pi (4 - y^2)^2 \Delta y$



$r_1 = \frac{1}{2}x^2 + 1$

$r_2 = \frac{1}{2}x + 4$

$V = \int_0^4 (\pi (4 - y^2)^2) \; dy$

$\Delta V \approx \pi ((\frac{1}{2}x + 4)^2 - (\frac{1}{2}x^2 + 1)) \; \Delta x$

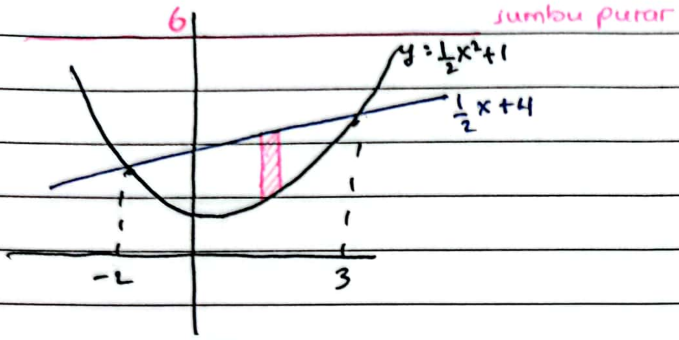
$V = \int_{-2}^{3} \pi ((\frac{1}{2}x + 4)^2 - (\frac{1}{2}x^2 + 1)) \; dx$

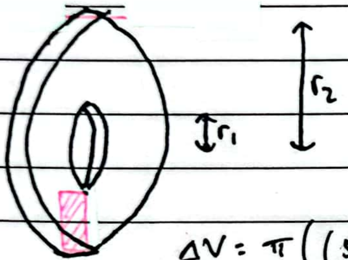
Date

Variasi

$y = \frac{1}{2}x^2 + 1$ dan $y = \frac{1}{2}x + 4$

sb putar $\rightarrow$ $y = 6$





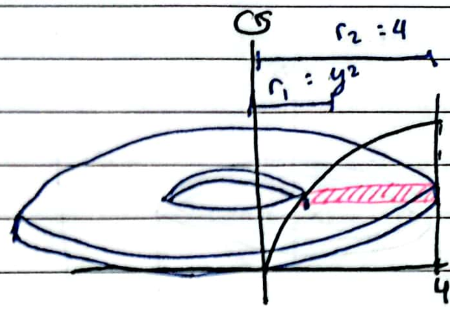
$r_1 = 6 - (\frac{1}{2}x + 4)$

$r_2 = 6 - (\frac{1}{2}x^2 + 1)$

$\Delta V = \pi\left((5 - \frac{1}{2}x^2) - (2 - \frac{1}{2}x)\right)\Delta x$

$V = \int_{-2}^{3} \pi\left((5-\frac{1}{2}x^2) - (2 - \frac{1}{2}x)\right) dx$

b.) Jika sb-putar $\rightarrow$ $x = 0$

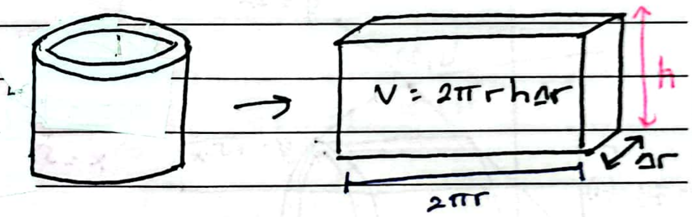


$\Delta V = \pi\left((4)^2 - (y^2)^2\right) \Delta y$

$V = \int_0^2 (16 - y^4)\, dy$

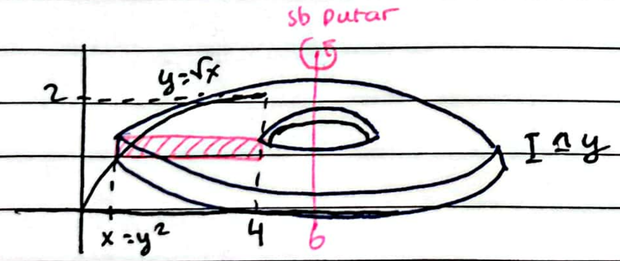
Metode Kulit Tabung

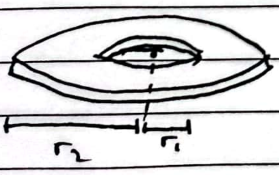
Jika kulit tabung dibuka, akan menjadi seperti balok



$y = \sqrt{x}$, $x = 4$, $x = 0$

a.) Jika sb-putar $\rightarrow$ $x = 6$





$r_1 = 6 - 4 = 2$

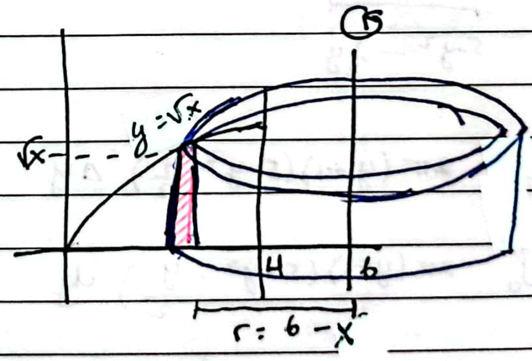
$r_2 = 6 - y^2$

$\Delta V = \pi\left((6-y)^2 - 4\right) \Delta y$

$V = \int_0^2 \left((6-y)^2 - 4\right) dy$

$y = \sqrt{x}$, $x = 4$, $x = 0$

a.) sumbu putar $\rightarrow$ $x = 6$



$\Delta V \approx 2\pi r h . \Delta r$

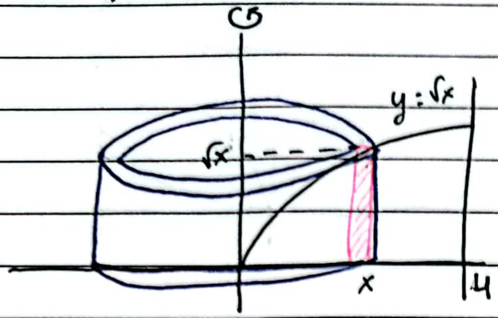
$= 2\pi(6-x)\sqrt{x}\, \Delta x$

$V = \int_0^4 2\pi(6-x)\sqrt{x}\, dx$

Date . .

b.) $y = \sqrt{x}$, $x = 4$, $x = 0$

Sumbu putar : $x = 0$



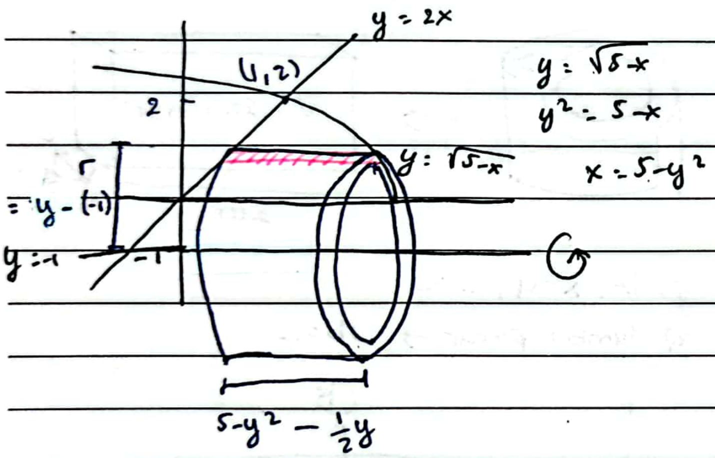
$\Delta V \approx 2\pi x \cdot \sqrt{x}\, \Delta x$

$$V = \int_0^4 2\pi x \sqrt{x}\, dx$$

Variasi

$y = 2x$, $y = \sqrt{5-x}$

c.) Sumbu putar : $y = -1$



$y = \sqrt{5-x}$

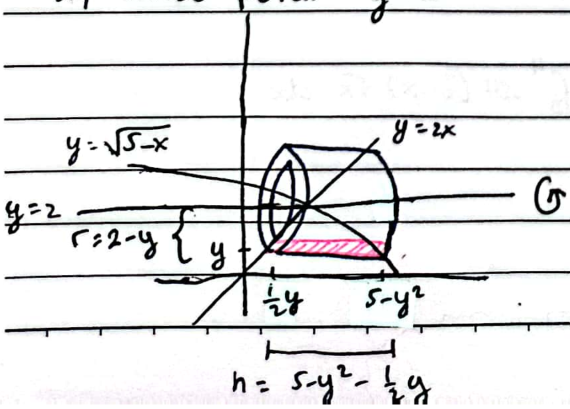
$y^2 = 5 - x$

$x = 5 - y^2$

$\Delta V \approx 2\pi (y+1)(5 - y^2 - \frac{y}{2})\, \Delta y$

$$V = \int_0^2 2\pi (y+1)(5 - y^2 - \frac{y}{2})\, dy$$

d.) Sumbu putar $y = 2$



$\Delta V \approx 2\pi (2-y)(5 - y^2 - \frac{1}{2}y)\, \Delta y$

$$V = \int_0^2 2\pi (2-y)\left(5 - y^2 - \frac{1}{2}y\right) dy$$

Metode Irisan Sejajar

→ bukan benda putar!

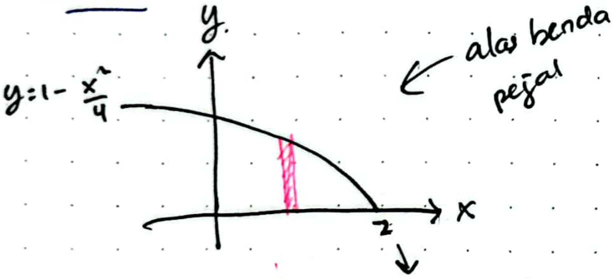
Contoh!

Example 5

Alas sebuah benda pejal berada di kuadran I yg dibatasi oleh $y = 1 - \frac{x^2}{4}$, sb-x, sb-y. Penampang benda pejal jika dipotong tegak lurus sb-x berbentuk persegi.

Tentukan volumenya!

Jawab:



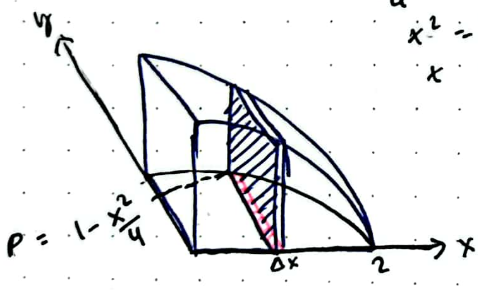
$y = 1 - \frac{x^2}{4}$

$0 = 1 - \frac{x^2}{4}$

$\frac{x^2}{4} = 1$

$x^2 = 4$

$x = \pm 2$



$\Delta V = p \times t \times \text{tebal}$

$= p^2 \times \text{tebal}$

$= \left(1-\frac{x^2}{4}\right)^2 \Delta x$

$V = \int_0^2 \left(1-\frac{x^2}{4}\right)^2 dx$

$= \int_0^2 \left(1 - \frac{1}{2}x^2 + \frac{x^4}{16}\right) dx$

$= \left(x - \frac{1}{6}x^3 + \frac{1}{80}x^5\right) \Big|_0^2$

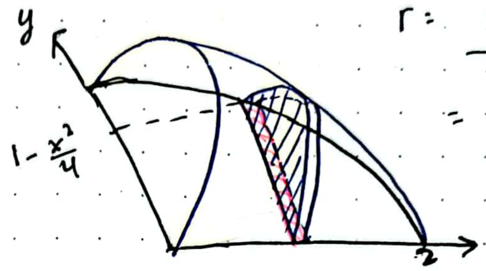
$= 2 - \frac{8}{6} + \frac{32}{80}$ //

Example 5

(sama seperti sebelumnya)

Penampang benda pejal selalu setengah lingkaran.

Jawab:



$r = \frac{1-\frac{x^2}{4}}{2}$

$= \frac{1}{2} - \frac{x^2}{8}$

$\Delta V \approx$ volume setengah cakram

$= \frac{1}{2} \pi r^2 \Delta x$

$= \frac{1}{2} \pi \left(\frac{1}{2} - \frac{x^2}{8}\right)^2 \Delta x$

$V = \int_0^2 \left(\frac{1}{2}\pi \left(\frac{1}{2} - \frac{x^2}{8}\right)^2\right) dx$

$= \int_0^2 \frac{1}{2}\pi\left(\frac{1}{2}\left(1 - \frac{x^2}{4}\right)\right)^2 dx$

$= \int_0^2 \frac{1}{8}\pi \left(1-\frac{x^2}{4}\right)^2 dx$

$= \frac{1}{8}\pi \int_0^2 \left(1-\frac{x^2}{4}\right)^2 dx$

$= \frac{1}{8}\pi \left(2 - \frac{8}{6} + \frac{32}{80}\right)$ //

Variasi

Contoh 3:

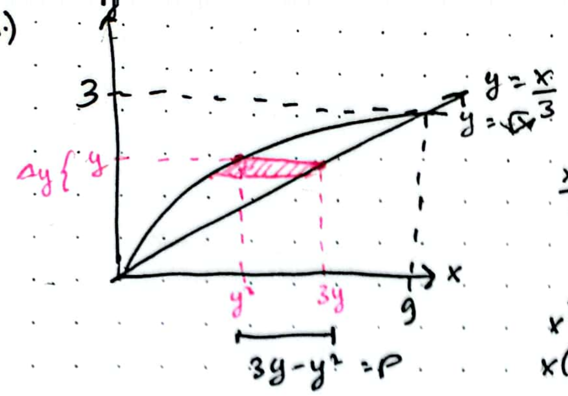
Alas sebuah benda pejal ada di kudran I yg dibatasi oleh $y = \sqrt{x}$ dan $y = \frac{x}{3}$.

Penampang sejajar sumbu x (tegak lurus sb-y) selalu berbentuk:

a.) persegi panjang dgn tinggi 2

b.) Segitiga sama sisi.

a.)



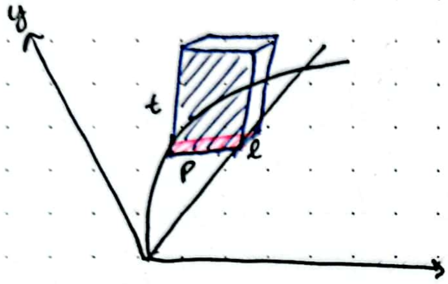
$\frac{x}{3} = \sqrt{x}$

$\frac{x^2}{9} = x$

$x^2 - 9x = 0$

$x(x-9) = 0$

$x = 9 \vee x = 0$



$\Delta V \approx p \cdot l \cdot t$

$= (3y - y^2) \cdot 2 \cdot \Delta y$

$V = \int_0^3 2(3y - y^2)\, dy$

$= 2 \int_0^3 (3y - y^2)\, dy$

$= 2\left(\left(\frac{3}{2}y^2 - \frac{1}{3}y^3\right)\Big|_0^3\right)$

$= 2\left(\frac{27}{2} - 9\right)$

$= 27 - 18$

$= 9$ //

$\Delta V \approx \frac{1}{2} \cdot a \cdot t \cdot \text{tebal}$

$= \frac{1}{2}(3y - y^2) \cdot \frac{1}{2}\sqrt{3}(3y - y^2) \cdot \Delta y$

$= \frac{1}{4}\sqrt{3}(3y - y^2)^2 \Delta y$

$V = \int_0^3 \frac{1}{4}\sqrt{3}(3y - y^2)^2\, dy$

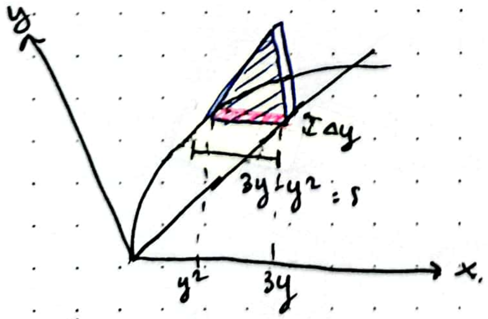
$= \frac{1}{4}\sqrt{3} \int_0^3 (3y - y^2)^2\, dy$

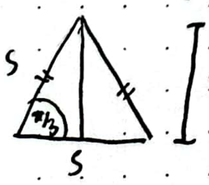
$= \frac{1}{4}\sqrt{3} \int_0^3 (9y^2 - 6y^3 + y^4)\, dy$

$= \frac{1}{4}\sqrt{3}\left(\left(3y^3 - \frac{6}{4}y^4 + \frac{1}{5}y^5\right)\Big|_0^3\right)$

$= \frac{1}{4}\sqrt{3}\left(3^4 - \frac{6}{4}(3)^4 + \frac{3^5}{5}\right)$ //

b.)





$t = s \cdot \sin\frac{\pi}{3} = s \cdot \frac{1}{2}\sqrt{3}$

$= (3y - y^2)\frac{1}{2}\sqrt{3}$

5.5 Kerja / Usaha

1) Andi mendorong suatu benda dgn gaya konstan 15 N sehingga benda tsb bergeser sejauh 2 meter. Hitunglah kerja yg dilakukan Andi

Jawab :

$W = 15 \cdot 2 = 30 \text{ J}$

2) Massa 5 kg diangkat ke ketinggian 0,5 m hingga 1 m. Hitung kerja!

Jawab :

$W = m \cdot g \cdot \Delta x$

$= 5 \cdot 10 \ (1 - 0{,}5)$

$= 25 \text{ J}$

3) 5 kubus dgn $s = 25$ cm disusun vertikal. Massa 1 kubus = 1 kg. Jika semua kubus diangkat sampai ketinggian 2 m, Hitung kerja!

Jawab :

$W = m \cdot g \cdot \Delta x$

$= (1 \cdot 2 + 1 \cdot 1{,}75 + 1 \cdot 1{,}5 + 1 \cdot 1{,}25 + 1 \cdot 1) 10$

$= (2 + 1{,}75 + 1{,}5 + 1{,}25 + 1) \ 10$

$= 75 \text{ J}$

Contoh 1

Gaya F berpindah dari $x = 5$ sampai $x = 10$
$F(x) = 2x + 1$. Hitung kerja!

Jawab :

$W = \int_5^{10} F(x) \, dx$

$= \int_5^{10} (2x + 1) \, dx$

$= (x^2 + x) \Big|_5^{10}$

$= (100 + 10) - (25 + 5)$

$= 110 - 20 = 90 \text{ J}$

Example 1

Pegas panjangnya 0,2 m ditarik oleh gaya 12 N dan bertambah panjang 0,04 m. Hitung kerja jika ditarik dari panjang awal sampai 0,3 m.

Jawab :

$F = kx$

$12 = k \cdot 0{,}04$

$k = \frac{12}{0{,}04} = 300 \text{ N/m}$

$\Delta W = F(x_i) \cdot \Delta x$

$= k \, x_i \cdot \Delta x$

$= 300 \, x \cdot \Delta x$

$W = \int_0^{0,1} kx \, dx$

$= \int_0^{0,1} 300 \, x \, dx$

$= 150 x^2 \Big|_0^{0,1} = 150 (0{,}01) = 1{,}5 \text{ J}$

Variasi

Kerja/usaha yang dilakukan utk menarik pegas dari 0,25 menjadi 0,3 m

Jawab: $0{,}3 - 0{,}25 = 0{,}05$

$W = \int_0^{0,05} 300 \, x \, dx$

$= 150 x^2 \Big|_0^{0,05}$

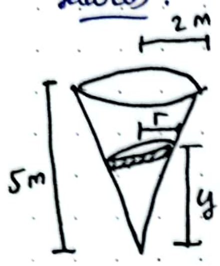
$= 150 \ (0{,}05)^2$

Example 2

Tangki berbentuk kerucut terbalik terisi penuh dgn air. Tinggi tangki 5 m. jari-jari permukaan tangki 2 m. Cari usaha, jika:

a.) Air dipompa sampai permukaan tangki

Jawab:



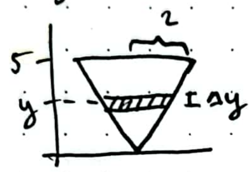
$V = \frac{1}{3}\pi . 4 . 5$

$= \frac{20}{3}\pi \ m^3$

berat jenis air : 10000 N/m^3

berat air : $\frac{20}{3}\pi . 10.000$

$\frac{r}{y} = \frac{2}{5} \rightarrow r = \frac{2y}{5}$



ΔW = berat satu potongan × perpindahan

= volume potongan air × berat jenis × $(5-y)$

= volume cakram × berat jenis × $(5-y)$

$= \pi r^2 \Delta y \times 10000 \times (5-y)$

$= \pi \left(\frac{2}{5}y\right)^2 . 10000 . (5-y) . \Delta y$

$= \pi \frac{4}{25} y^2 . \cancel{10000}^{400} (5-y) . \Delta y$

$= \pi \ 1600 \ y^2 (5-y) . \Delta y$

$W = \int_0^5 \pi \ 1600 \ y^2 (5-y) \, dy$

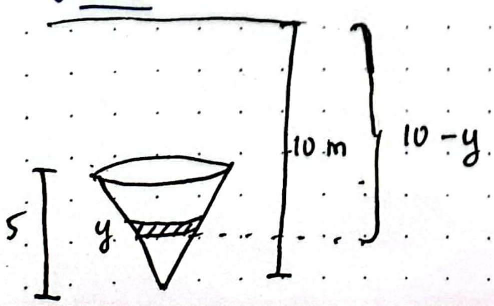
$= 1600\pi \int_0^5 (5y^2 - y^3) \, dy$

$= 1600\pi \left(\left(\frac{5}{3}y^3 - \frac{1}{4}y^4\right) \Big|_0^5 \right)$

$= 1600\pi \left(\frac{5^4}{3} - \frac{5^4}{4} \right)$ //

b.) Air dipompa sampai 10 m diatas tangki.

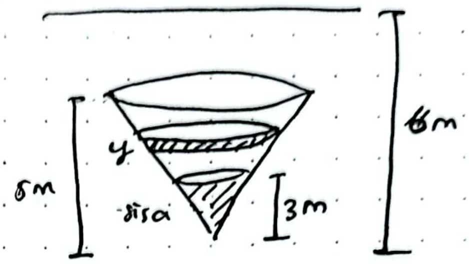
Jawab:



$W = \int_0^5 \pi \ 1600 \ y^2 (10-y) \, dy$

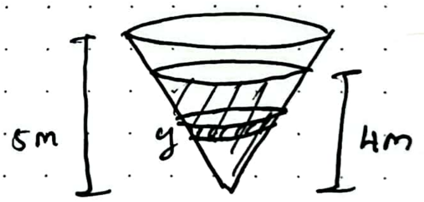
Variasi

Air dipompa sampai ketinggian 1 m di atas tangki dan tersisa 3 m di tangki



$W = \int_3^5 1600 \pi \ y^2 (6-y) \, dy$

Mula-mula tangki berisi air setinggi 4m. Di pompa sampai permukaan tangki.



$W = \int_0^4 1600 \pi \ y^2 (5-y) \, dy$

6.1. Fungsi Logaritma Natural

Definisi

$$\boxed{\ln(x) = \int_1^x \frac{1}{t}\, dt}$$

Turunan Fungsi Logaritma Natural

Berdasarkan TDK I

$$\boxed{\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\, dt = \frac{1}{x}}$$

• Tentukan turunan dari fungsi² berikut:

a.) $x^2 . \ln x$

Jawab: Pakai uv

$$\frac{d}{dx} x^2 \ln x = 2x . \ln(x) + x^2 . \frac{1}{x}$$

$$= 2x \ln(x) + x$$

b.) $\ln(7x)$

Jawab:

$$\frac{d}{dx}\ln(7x) = \frac{1}{7x} . 7 = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}\ln(kx) = \frac{1}{x}}$$

c.) $\ln(x^3)$

Jawab:

$$\frac{d}{dx}\ln(x^3) = \frac{1}{x^3} . 3x^2 = \frac{3}{x}$$

d.) $\ln(-x)$

Jawab:

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x} . (-1) = \frac{1}{x}$$

$$\boxed{\int \frac{1}{x}\, dx = \ln|x| + C}$$

$$\int_{-5}^{-1} \frac{1}{x}\, dx = \ln|x| \Big|_{-5}^{-1}$$

$$= \ln|-1| - \ln|-5|$$

$$= \ln 1 - \ln 5$$

$$= -\ln 5$$

Contoh: Tentukan

• $\int_0^1 \frac{1}{2x-3}\, dx$

Jawab: misal $u = 2x - 3$
$du = 2\, dx$

$$\int_0^1 \frac{1}{u}\frac{du}{2}$$

$$= \frac{1}{2}\int_{-3}^{-1} \frac{1}{u}\, du$$

$$= \frac{1}{2}\left(\ln|u| \Big|_{-3}^{-1}\right)$$

$$= \frac{1}{2}\left(\ln|-1| - \ln|-3|\right)$$

$$= \frac{1}{2}\left(\ln(1) - \ln 3\right)$$

$$= -\frac{1}{2}\ln 3$$

• $\int \frac{x}{x^2+1}\, dx$

Jawab: misal $u = x^2 + 1$
$du = 2x\, dx$

$$\int \frac{\cancel{x}}{u} . \frac{du}{2\cancel{x}}$$

$$= \int \frac{1}{u}\, du = \ln|u| + C$$

• $\int \tan x\, dx$ dan $\int \cot x\, dx$

Jawab:

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

misal, $u = \cos x$

$du = -\sin x \, dx$

$$\int \frac{\sin x}{\cos x} dx = \int \frac{\cancel{\sin x}}{u} \cdot \frac{du}{-\cancel{\sin x}}$$

$$= \int -\frac{1}{u} du$$

$$= -\int \frac{1}{u} du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

Sifat-sifat Fungsi Logaritma Natural

a.) $\ln(ab) = \ln(a) + \ln(b)$

b.) $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

c.) $\ln(a^n) = n \ln(a)$, n : bil. bulat

d.) $\ln(a^{1/n}) = \frac{1}{n} \ln(a)$, n : bil. asli

e.) $\ln(a^r) = r \ln(a)$; r = rasional

Turunan logaritmik

$$\frac{d}{dx} \ln y = \frac{1}{y} \frac{dy}{dx}$$

$$\frac{dy}{dx} = y \cdot \frac{d}{dx} \ln y$$

Contoh:

Tentukan turunan dari $f(x) = \frac{\sqrt{1-x^2}}{(x^2+1)^{2/3}}$

Jawab:

$$y = \frac{\sqrt{1-x^2}}{(x^2+1)^{2/3}}$$

$$\ln y = \ln\left(\frac{\sqrt{1-x^2}}{(x^2+1)^{2/3}}\right)$$

$$\ln y = \ln(\sqrt{1-x^2}) - \ln(x^2+1)^{2/3}$$

$$\ln y = \ln(1-x^2)^{1/2} - \ln(x^2+1)^{2/3}$$

$$\ln y = \frac{1}{2} \ln(1-x^2) - \frac{2}{3} \ln(x^2+1)$$

$$\frac{d}{dx} \ln y = \frac{1}{2} \cdot \frac{1}{1-x^2} \cdot (-2x) - \frac{2}{3} \cdot \frac{1}{x^2+1} \cdot 2x$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{-x}{1-x^2} - \frac{4x}{3(x^2+1)}$$

$$\frac{dy}{dx} = y\left(\frac{-x}{1-x^2} - \frac{4x}{3(x^2+1)}\right)$$

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2}}{(x^2+1)^{2/3}}\left(\frac{-x}{1-x^2} - \frac{4x}{3(x^2+1)}\right)$$

$$\boxed{\begin{aligned} &\lim_{x \to \infty} \ln(x) = \infty \\ &\lim_{x \to 0^+} \ln(x) = -\infty \end{aligned}}$$

6.2 Fungsi Invers dan Turunannya

Fungsi g disebut sebagai invers dari fungsi f jika memenuhi persamaan

$$f(g(x)) = x \text{ dan } g(f(x)) = x$$

berlaku:

$$\boxed{f(x) = y \iff g(y) = x}$$

Catatan :

$D_{f^{-1}} = R_f$

$R_{f^{-1}} = D_f$

Contoh :

a.) $f(x) = 3x-2$

Jawab :

$y = 3x-2 \iff 3x = y+2$

$x = \frac{y+2}{3} = f^{-1}(y)$

$\therefore f^{-1}(x) = \frac{x+2}{3}$ //

b.) $f(x) = x^2$ dengan $D_f = (0, \infty)$

Jawab :

$y = x^2 \iff x = \sqrt{y}$

$\therefore f^{-1}(x) = \sqrt{x}$ //

c.) $g(x) = x^3+1$ dengan $D_g = (-\infty, \infty)$

Jawab :

$y = x^3+1 \iff x^3 = y-1$

$x = \sqrt[3]{y-1}$

$\therefore g^{-1}(x) = \sqrt[3]{x-1}$ //

d.) $h(x) = \frac{2x+5}{x-1}$

Jawab :

$y = \frac{2x+5}{x-1}$

$xy - y = 2x+5$

$xy - 2x = 5+y$

$x(y-2) = 5+y$

$x = \frac{y+5}{y-2}$

$\therefore h^{-1}(x) = \frac{x+5}{x-2}$ //

e.) $p(x) = x - \frac{1}{x}$ pada $(-\infty, 0)$

Jawab :

$y = x - \frac{1}{x}$

$y = \frac{x^2-1}{x}$

$xy = x^2-1$

$x^2 - xy - 1 = 0$ (pake rumus abc)

Syarat Cukup Keberadaan Invers

Jika f monoton naik atau monoton turun, pada I, maka f memiliki invers.

a.) $f'(x) = 5x^4+2$ pada $(-\infty, \infty)$

$\therefore f'(x) > 0$ untuk $x \in (-\infty, \infty)$

↳ f monoton naik pada $(-\infty, \infty)$

Berdasarkan teorema: f memiliki invers.

c.) $h(x) = 2\sin x - 3x$ pada $(-\infty, \infty)$

$h'(x) = 2\cos x - 3$

karena $-1 \le \cos x \le 1$

$\Rightarrow 2\cos x \le 2$

$2\cos x - 3 \le 2-3 = -1 < 0$

$\therefore h'(x) < 0 \Rightarrow h$ monoton turun pada $(-\infty, \infty)$

$\Rightarrow h$ memiliki invers

Turunan Fungsi Invers

$$\boxed{(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}}$$

Jika $f(b) = a$, maka $(f^{-1})'(a) = \dfrac{1}{f'(b)}$

Contoh: Tebaklah nilai $f^{-1}(a)$ dan hitunglah $(f^{-1})'(a)$.

a) $f(x) = x^3 + 2x + 5$ dan $a = 5$

Jawab: cari $f^{-1}(5) = ?$

Misal $f^{-1}(5) = b$

$\rightarrow f(b) = 5$

$b^3 + 2b + 5 = 5$

$b = 0$

$\therefore f^{-1}(5) = 0$

Jadi, $(f^{-1})'(5) = \dfrac{1}{f'(0)}$

$f(x) = x^3 + 2x + 5$

$f'(x) = 3x^2 + 2$

$f'(0) = 2$

$\therefore (f^{-1})'(5) = \dfrac{1}{f'(0)} = \dfrac{1}{2}$ //

c) $f(x) = 3x - \sin x$ dan $a = \dfrac{3\pi}{2} - 1$

Jawab:

$(f^{-1})'\left(\dfrac{3\pi}{2} - 1\right) = ?$

$f^{-1}\left(\dfrac{3\pi}{2} - 1\right) = b$

$f(b) = \dfrac{3\pi}{2} - 1$

$3b - \sin b = \dfrac{3\pi}{2} - 1$

Tebak: $b = \dfrac{\pi}{2}$

$f^{-1}\left(\dfrac{3\pi}{2} - 1\right) = \dfrac{\pi}{2}$

$\Rightarrow (f^{-1})'\left(\dfrac{3\pi}{2} - 1\right) = \dfrac{1}{f'\left(\frac{\pi}{2}\right)}$

$f'(x) = 3 - \cos x$

$f'\left(\dfrac{\pi}{2}\right) = 3 - \cos\left(\dfrac{\pi}{2}\right) = 3$

$\therefore (f^{-1})'\left(\dfrac{3\pi}{2} - 1\right) = \dfrac{1}{f'\left(\frac{\pi}{2}\right)} = \dfrac{1}{3}$ //

6.3. Fungsi Eksponen Natural

Definisi

Fungsi eksponen natural = exp
exp = invers dari fungsi ln

- Hubungan :
  - $\rightarrow \ln(\exp(a)) = a$
  - $\rightarrow \exp(\ln(b)) = b$

Daerah asal : $D_{exp} = (-\infty, \infty)$
Daerah hasil : $R_{exp} = (0, \infty)$

Bilangan Euler

bilangan euler = e
$e = \exp(1)$
$\therefore \ln(e) = \ln(\exp(1)) = 1$

Sifat-sifat Fungsi EXP

1.) $\exp(a) = b \Longleftrightarrow \ln(b) = a$

2.) $\exp(a+b) = \exp(a) \cdot \exp(b)$

3.) $\exp(a-b) = \dfrac{\exp(a)}{\exp(b)}$

4.) $\exp(n) = e^n$, n = bil. bulat

5.) $\exp(r) = e^r$, r = bil. rasional

$\rightarrow \exp(3) = \exp(1+1+1)$
$= \exp(1) \cdot \exp(1) \cdot \exp(1)$
$= e \cdot e \cdot e$
$\exp(3) = e^3$

6.) $e^{bc} = (e^b)^c = (e^c)^b$

Pangkat Irasional

Jika x bilangan irasional, $e^x = ?$

$\cdot$) Untuk sebarang bil. irasional s, berlaku:
$$e^s = \exp(s)$$

$\rightarrow$ Dapat disimpulkan
$$e^x = \exp(x), \ \forall x \in \mathbb{R}$$

$\therefore \ln(e^x) = \ln(\exp(x)) = x$
$e^{\ln x} = \exp(\ln(x)) = x$

Turunan dan Integral Fungsi EXP

$$\boxed{\frac{d}{dx} e^x = e^x \text{ dan } \int e^x \, dx = e^x + C}$$

Contoh:

1.) Tentukan turunan dari fungsi $g(x) = xe^x$
Jawab:
$g'(x) = 1 \cdot e^x + xe^x = (1+x)e^x$ //

2.) Tentukan PGS kurva $y = e^{2x}$ di titik (0,1)
Jawab: $y = e^{2x}$
$\frac{dy}{dx} = e^{2x} \cdot 2$
$m = e^0 \cdot 2 = 2$

$e^0 = \exp(0) = 1$ krn $\ln(1) = 0$

PGS : $y - y_1 = m(x - x_1)$
$y - 1 = 2(x - 0)$
$y = 2x + 1$ //

4.) Tentukan $\int \left(e^{5x} + \frac{1}{e^x}\right) dx$

Jawab: $\int e^{5x}\,dx + \int \frac{1}{e^x}\,dx$

$= \int e^{5x}\,dx + \int e^{-x}\,dx$

$= e^{5x} . 5 + e^{-x} (-1) + C$

$= 5e^{5x} - \frac{1}{e^x} + C$

5.) Hitunglah $\int_0^1 x e^{x^2+1}\,dx$

Jawab: misal $u = x^2+1$

$du = 2x\,dx$

$\int_1^2 \cancel{x} e^u . \frac{du}{2\cancel{x}} = \int_1^2 \frac{1}{2} e^u\,du$

$= \frac{1}{2} \int_1^2 e^u\,du$

$= \frac{1}{2} \left(e^u \Big|_1^2\right)$

$= \frac{1}{2} (e^2 - e)$

6.) Tentukan $\int \frac{e^x}{e^x+1}\,dx$

Jawab: misal, $u = e^x + 1$

$du = e^x\,dx$

$\int \frac{\cancel{e^x}}{u} . \frac{du}{\cancel{e^x}} = \int \frac{1}{u}\,du$

$= \ln |u| + C$

$= \ln |e^x + 1| + C$

8.) Selesaikan PD $\frac{dy}{dx} = 3y$ dan

$y = 5$ untuk $x = 0$; $y > 0$

Jawab:

$\frac{dy}{dx} = 3y$

$\frac{1}{y}\,dy = 3\,dx$

$\int \frac{1}{y}\,dy = \int 3\,dx$

$\ln |y| = 3x + C$

$\ln (y) = 3x + C$

$y = e^{3x+C}$

$y = e^{3x} . e^{c}$

$y = e^{3x} . C$

$y = Ce^{3x}$

$y = 5$ untuk $x = 0$

$5 = Ce^{3(0)}$

$5 = Ce^{0} = C$

$\Rightarrow y = 5e^{3x}$

6.4. Fungsi Eksponen Umum dan Logaritma Umum

Fungsi Eksponen Umum

Setiap bilangan real b dapat dituliskan sebagai $b = e^a$ dgn $a = \ln b$

- $5 = e^{\ln 5}$
- $5^{2/3} = e^{\ln 5^{2/3}} = e^{\frac{2}{3}\ln 5}$
- $\pi^{-7/8} = e^{-7/8 \ln \pi}$

Definisi

$$\boxed{a^x = e^{x \ln(a)}}$$

Turunan dan Integral

$$\boxed{\frac{d}{dx} a^x = a^x . \ln(a) \quad \text{dan} \quad \int a^x \, dx = \frac{a^x}{\ln(a)} + C}$$

$2^x = e^{x \ln 2}$

$\frac{d}{dx} e^{x \ln 2} = e^{x \ln 2} . \ln 2$

$\Rightarrow \frac{d}{dx} 2^x = 2^x . \ln 2$

$\frac{d}{dx} e^{5x} = e^{5x} . 5$

$\int e^{5x} \, dx = \int e^u \frac{du}{5} = \frac{1}{5} e^u + C$

misal $u = 5x$

$du = 5 dx$

$= \frac{1}{5} e^{5x} + C$

$\int e^{x \ln 2} \, dx = \frac{1}{\ln 2} . e^{x \ln 2} + C$

$\Rightarrow \int 2^x = \frac{1}{\ln 2} . 2^x + C$

- $\frac{d}{dx}(x^\pi + \pi^x) = \pi x^{\pi - 1} + \pi^x . \ln \pi$
- $\int (x^\pi + \pi^x) = \frac{1}{\pi + 1} x^{\pi + 1} + \frac{\pi^x}{\ln \pi} + C$

Contoh: pakai turunan logaritmik

- $f(x) = x^x$, $f'(x) = ?$

Jawab:

$y = f(x) = x^x$

$\ln y = \ln x^x = x \ln x$

$\frac{d}{dx} \ln y = \frac{d}{dx}(x \ln x)$

$\frac{1}{y} \frac{dy}{dx} = \ln x + x . \frac{1}{x}$

$\frac{dy}{dx} = y(\ln x + 1)$

$\frac{d}{dx} x^x = x^x (\ln(x) + 1)$ //

$$\boxed{a^{\log_a(b)} = b}$$

Fungsi Logaritma Umum

$$y = \log_a x \iff x = a^y$$

Sifat, $\boxed{\log_a x = \dfrac{\ln x}{\ln a}}$

Contoh :

$\ln(2) \approx 0{,}693$

$\ln(3) \approx 1{,}0986$

$\Rightarrow \log_4 (27 \times 32)$

$= \dfrac{\ln (27 \times 32)}{\ln (4)}$

$= \ln 27 + \ln 32 - \ln 4$

$= \ln (3)^3 + \ln (2)^5 - \ln (2)^2$

$= 3 \ln (3) + 5 \ln (2) - 2 \ln (2)$

$= 3 .\ 1{,}0986 + (5-2)\ 0{,}693$

Sifat - sifat

a.) $\log_a (bc) = \log_a b + \log_a c$

b) $\log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c$

c) $\log_a (b^c) = c \log_a (b)$

Turunan

$\boxed{\dfrac{d}{dx} (\log_a x) = \dfrac{1}{x \ln a}}$ → dikali turunan x

Soal !

1.) Tentukan turunan dari $f(x) = \log_3 (2 + \cos x)$

Jawab : misal $u = 2 + \cos x$

$$\frac{d}{du} (\log_3 u) = \frac{1}{u \ln 3}$$

$$\frac{d}{dx} \log_3 (2 + \cos x) = \frac{1}{(2+\cos x) \ln 3} (-\sin x)$$

$$= \frac{-\sin x}{(2 + \cos x) \cdot \ln 3}$$

2.) Tentukan turunan dari $g(x) = \log_{2x} (x^2+1)^5$

Jawab : $g(x) = \log_{2x} (x^2+1)^5$

$$\log_{2x} (x^2+1)^5 = \frac{\ln (x^2+1)^5}{\ln 2x}$$

$$y = \frac{5 \ln (x^2+1)}{\ln 2x}$$

$$\frac{dy}{dx} = \frac{\left(5 \cdot \frac{1}{(x^2+1)} \cdot 2x\right) \ln 2x + 5 \ln (x^2+1) \cdot \frac{1}{2x} \cdot 2}{(\ln 2x)^2}$$

$$\frac{dy}{dx} = \frac{\frac{10x}{(x^2+1)} \cdot \ln (2x) + \frac{5}{x} \cdot \ln (x^2+1)}{(\ln (2x))^2}$$

6.5. Pertumbuhan dan Peluruhan Eksponensial

Pertumbuhan Eksponensial

Laju pertumbuhan populasi di suatu tempat sebanding dgn banyaknya populasi pada saat itu.

- Misalkan $y(t)$ menyatakan banyak populasi pada waktu t, maka:

$$\boxed{\frac{dy}{dt} = ky}$$

k suatu konstanta (+)

Solusi PD:

$$\boxed{y = Ce^{kt}}$$

Example 2

Jumlah bakteri bertambah dengan cepat, dari 10.000 saat siang dan 40.000 2 jam setelahnya. Prediksikan banyak bakteri saat jam 5.PM

Jawab: 10.000 → $t = 0$

40.000 → $t = 2$

? → $t = 5$

saat $t = 0$:

$y = Ce^{kt}$

$10.000 = Ce^{0}$

$C = 10.000$

→ $y = 10.000\, e^{kt}$

saat $t = 2$:

$40.000 = 10.000\, e^{2k}$

$4 = e^{2k}$

$\ln e^{2k} = \ln 4$

$2k \ln e = 2 \ln 2$

$k = \ln 2$

$\Rightarrow y = 10.000\, e^{(\ln 2)t} = 10.000 \cdot 2^{t}$

saat $t = 5$

$y = 10.000\, e^{(\ln 2)5}$

$= 10.000 \cdot 2^{5}$

$= 320.000$

$$\boxed{e^{\ln a} = a}$$

$$\boxed{e^{bc} = (e^{b})^{c}}$$

Peluruhan Eksponensial

Laju perubahan suatu zat radioaktif pada suatu saat sebanding dengan massa zat pada saat itu

- Misalkan $y(t)$ menyatakan massa zat pada saat t, maka berlaku:

$$\boxed{\frac{dy}{dt} = ky}$$

k suatu konstanta (−)

Solusi PD:

$$\boxed{y = Ce^{kt}}$$ k = negatif

Contoh:

13. Suatu radioaktif mempunyai waktu paruh 700 tahun. Jika ada 10 gram,

a) berapa banyak yg tersisa setelah 300 tahun?

Jawab: $10 \xrightarrow{700 \text{ thn}} 5 \xrightarrow{700 \text{ thn}} 2{,}5 \ldots$

$y = Ce^{kt}$

saat $t=0 \rightsquigarrow y=10$

$10 = Ce^{k(0)}$

$C = 10$

$\rightarrow y = 10e^{kt}$

saat $t = 700 \rightarrow y = 5$

$5 = 10e^{700k}$

$\boxed{\begin{array}{l} e^a = b \\ \ln b = a \end{array}}$

$e^{700k} = \frac{1}{2}$

$\ln \frac{1}{2} = 700k$

$\ln 1 - \ln 2 = 700k$

$-\ln 2 = 700k$

$k = -\frac{1}{700} \ln 2$

$\Rightarrow y = 10e^{(-\frac{1}{700}\ln 2)t} = 10 \; 2^{-\frac{t}{700}}$

saat $t = 300$

$y = 10 . 2^{-\frac{300}{700}} = 10(2^{-3/7})$ gram

b.) Perlu waktu berapa lama agar tersisa 3 gram?

Jawab: $y = 3 \rightarrow t = ?$

$3 = 10e^{(-\frac{1}{700}\ln 2)t}$

$e^{(-\frac{1}{700}\ln 2)t} = \frac{3}{10}$

$(-\frac{1}{700}\ln 2)t = \ln(\frac{3}{10})$

$- \,\text{''}\, - = \ln 3 - \ln 10$

$t = \dfrac{\ln 3 - \ln 10}{-\frac{1}{700}\ln 2}$

$t = \dfrac{700(\ln 10 - \ln 3)}{\ln 2}$ tahun

Hukum Pendinginan Newton

Laju perubahan suhu sebanding dengan selisih antara suhu benda dengan suhu ruangan

$$\boxed{\frac{dT}{dt} = k(T - T_1)}$$

k = konstanta negatif

Solusi PD:

$$\boxed{|T - T_1| = Ce^{kt}}$$

Kasus 1: $T > T_1$ (suhu benda di atas suhu ruang)

$\Rightarrow T - T_1 = Ce^{kt}$

$\boxed{T = T_1 + Ce^{kt}}$ $T > T_1$

Kasus 2: $T < T_1$ (suhu benda di bawah suhu ruang)

$\Rightarrow T_1 - T = Ce^{kt}$

$\boxed{T = T_1 - Ce^{kt}}$ $T < T_1$

Example 4

Suatu objek dikeluarkan dari oven di 350°F dan didiamkan hingga dingin di suhu ruang 70°F. Suhu objek turun hingga 250°F selama 1 jam. Berapa suhu objek setelah 3 jam keluar dari oven?

Jawab: $T = T_1 + Ce^{kt}$

$= 70 + Ce^{kt}$

saat $t=0$, $T = 350$

$350 = 70 + Ce^0$

$350 = 70 + C$

$C = 280$

$\rightarrow T = 70 + 280\, e^{kt}$

saat $t = 1 \rightarrow T = 250$

$250 = 70 + 280\, e^{k}$

$280\, e^{k} = 180$

$e^{k} = \frac{18}{28} = \frac{9}{14}$

$k = \ln\left(\frac{9}{14}\right)$

$\Rightarrow T = 70 + 280\, e^{\ln\left(\frac{9}{14}\right)\cdot t}$

$T = 70 + 280 \cdot \left(\frac{9}{14}\right)^{t}$

$t = 3$

$T = 70 + 280 \left(\frac{9}{14}\right)^{3}$

$= 70 + 280 \left(\frac{243}{196}\right)$ //

21. Benda 26°C diletakkan di air dgn suhu 90°C. Jika suhu benda meningkat menjadi 70°C dlm 5 mnt, Berapa suhu benda setelah 10 menit?

Jawab: T < suhu sekitar

$T = T_1 - Ce^{kt}$

$T = 90 - Ce^{kt}$

saat $t = 0 \rightarrow T = 26$

$26 = 90 - Ce^{0}$

$C = 64$

$\rightarrow T = 90 - 64\, e^{kt}$

saat $t = 5 \rightarrow T = 70$

$70 = 90 - 64\, e^{5k}$

$64\, e^{5k} = 20$

$e^{5k} = \frac{20}{64} = \frac{5}{16}$

$\ln\left(\frac{5}{16}\right) = 5k$

$k = \frac{1}{5}\ln\left(\frac{5}{16}\right)$

$\Rightarrow T = 90 - 64\, e^{\frac{1}{5}\ln\left(\frac{5}{16}\right)\cdot t}$

$= 90 - 64 \left(\frac{5}{16}\right)^{\frac{t}{5}}$

$t = 10$

$T = 90 - 64\left(\frac{5}{16}\right)^{\frac{10}{5}}$

$= 90 - 64\left(\frac{5}{16}\right)^{2}$

$= 90 - \cancel{64} \cdot \frac{25}{\cancel{256}\,4}$

$= 90 - \frac{25}{4}$

$= \frac{360 - 25}{4}$

$= \frac{335}{4}$ °C //